ON THE CLOSURE OF POLYNOMIALS IN WEIGHTED SPACES OF FUNCTIONS ON THE REAL LINE

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Abstract. We describe the closure of polynomials in weighted L^p spaces of functions on the real line for even log-convex non-quasianalytic weights: a function in the space can be approximated by polynomials if and only if it extends to an entire function of zero exponential type. This completes the series of investigations by Akhiezer, Mergelyan, Khachatryan, Koosis, and Levinson–McKean.

1. Introduction

Let $W : \mathbb{R} \to [1, +\infty)$ be an even continuous function such that

$$\lim_{|x|\to+\infty} \frac{\log W(x)}{\log |x|} = +\infty. \tag{1.1}$$

Consider the Banach spaces

$$L^p_W = \Big\{ f : \int_{-\infty}^{+\infty} \frac{|f(x)|^p}{\big(W(x)\big)^p}\, dx < +\infty \Big\}, \qquad 1 \le p < +\infty,$$

$$L^\infty_W = \Big\{ f : \operatorname*{ess\,sup}_{x\in\mathbb{R}} \frac{|f(x)|}{W(x)} < +\infty \Big\},$$

$$C^0_W = \Big\{ f \in C(\mathbb{R}) : \lim_{|x|\to+\infty} \frac{f(x)}{W(x)} = 0 \Big\}.$$

Then C^0_W is a closed subspace of L^∞_W. Denote by $\mathcal{P}$ the set of all polynomials. Our condition (1.1) guarantees that $\mathcal{P} \subset C^0_W$, $\mathcal{P} \subset L^p_W$, $1 \le p \le +\infty$. Denote by $\mathcal{P}^p_W$ the closure of $\mathcal{P}$ in L^p_W, $1 \le p \le +\infty$. Clearly, the closure of $\mathcal{P}$ in C^0_W coincides with $\mathcal{P}^\infty_W$. Let $\mathcal{E}$ be the set of entire functions of zero exponential type, $\mathcal{E}^p_W = \mathcal{E} \cap L^p_W$, $1 \le p < +\infty$, $\mathcal{E}^\infty_W = \mathcal{E} \cap C^0_W$.

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The classical Bernstein approximation problem is to find out whether the polynomials are dense in C^0_W. Suppose that an even function W satisfies (1.1) and is log-convex, that is the function $x \mapsto \log W(\exp x)$ is convex on $\mathbb{R}$. It is well-known (see [5], [9, Section VI D]) that if

$$\int_{-\infty}^{+\infty} \frac{\log W(x)}{x^2+1}\,dx = +\infty,$$

then $\mathcal{P}^\infty_W = C^0_W$, $\mathcal{P}^p_W = L^p_W$, $1 \le p < +\infty$. Otherwise, if

$$\int_{-\infty}^{+\infty} \frac{\log W(x)}{x^2+1}\,dx < +\infty, \tag{1.2}$$

then the polynomials are not dense in C^0_W, L^p_W, $1 \le p < +\infty$, and a well-known theorem going back to M. Riesz ([14, Sections 16,17], see also [1, Section 4], [13, Theorem 6], [9, Section VI B], [10, Theorem 1.5]) claims that $\mathcal{P}^p_W \subset \mathcal{E}^p_W$, $1 \le p \le +\infty$. An important problem is to describe $\mathcal{P}^p_W$. For related results and discussions see [6, 7, 4, 8, 11], [9, Section VI H].

The aim of this paper is to prove the equality

$$\mathcal{P}^p_W = \mathcal{E}^p_W, \qquad 1 \le p \le +\infty, \tag{1.3}$$

for even log-convex W satisfying conditions (1.1) and (1.2). As a consequence, for each log-convex W satisfying just condition (1.1), we obtain that every entire function of zero exponential type belonging to one of the spaces L^p_W, $1 \le p < +\infty$, C^0_W, can be approximated in this space by polynomials.

Earlier, equalities (1.3) were obtained for $p = 2, +\infty$ under more restrictive assumptions on the weight function W. I. O. Khachatryan ([6, 7], see also [9, Section VI H.2]) proved that $\mathcal{P}^\infty_W = \mathcal{E}^\infty_W$ for W satisfying (1.1) and (1.2), such that

$$W(x) = \sum_{n\ge 0} a_n x^{2n}, \qquad a_0 \ge 1,\ \ a_k \ge 0,\ k \ge 1. \tag{1.4}$$

P. Koosis [8, Theorem IV] proved that $\mathcal{P}^\infty_W = \mathcal{E}^\infty_W$ for log-convex W satisfying (1.1) and (1.2), such that for every $\eta > 1$ there exists C_η with $x^2 W(x) \le C_\eta W(\eta x)$, $x \ge 0$. N. Levinson and H. P. McKean [11, Section 10a] proved that $\mathcal{P}^2_W = \mathcal{E}^2_W$ (i) for W of the form (1.4), satisfying (1.1) and (1.2), and (ii) for C^1-smooth log-convex W satisfying (1.2), such that

$$\lim_{x\to+\infty} \frac{xW'(x)}{\log x \cdot W(x)} = +\infty.$$

We should also mention here (a special case of) the theorem of L. de Branges (see [3, Theorem 1], [12, Theorem 8], [9, p.215]) on weighted approximation by the linear combinations of the exponentials. Suppose that W satisfies conditions (1.1) and (1.2). For $A > 0$, $1 \le p \le +\infty$, denote by $L^p_W(A)$ the closure in L^p_W of the linear combinations of $e^{i\lambda x}$, $-A \le \lambda \le A$. Then $\mathcal{E}^p_W = \bigcap_{A>0} L^p_W(A)$.

Combining this result with equality (1.3), for log-convex W satisfying (1.1), we obtain that $\mathcal{P}^p_W = \bigcap_{A>0} L^p_W(A)$; in other words, every function in L^p_W that can be approximated by finite linear combinations of $e^{i\lambda x}$, $-A \le \lambda \le A$, with arbitrary small $A > 0$, can also be approximated by polynomials.

Examples of (non-monotonic) even weights W such that $\mathcal{P}^\infty_W \subsetneqq \mathcal{E}^\infty_W \subsetneqq C^0_W$, $\mathcal{P}^2_W \subsetneqq \mathcal{E}^2_W \subsetneqq L^2_W$ are given in [7], [11, Section 11], [9, Section VI H.3]. An open question remains: is it true that the equality (1.3) holds for even weight functions W just increasing on $[0, +\infty)$ and satisfying (1.1) and (1.2)?

In the next section we give a proof of equality (1.3) for $p = +\infty$ using a classical result of de Branges. Another proof of this fact (using a result by Khachatryan) and the proof of (1.3) for $1 \le p < +\infty$ are given in Section 3. Section 4 contains the proofs of auxiliary statements.

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2. Polynomial approximation in C^0_W

Theorem 2.1. *Suppose that W is an even* log*-convex function on the real line satisfying* (1.1), (1.2). *Then* $\mathcal{P}^\infty_W = \mathcal{E}^\infty_W$.

Proof. We can identify elements of $(C^0_W)^*$ with elements μ of the space $M(\mathbb{R})$ of finite complex Borel measures $\mathbb{R}$, with the duality being

$$\langle \mu, f\rangle_{C^0_W} = \int_{-\infty}^{+\infty} \frac{f(x)}{W(x)}\, d\mu(x).$$

Fix $F \in \mathcal{E}^\infty_W$. By the Hahn–Banach theorem, to prove that $F \in \mathcal{P}^\infty_W$, we need to verify that $\langle \mu, F\rangle_{C^0_W} = 0$ for every finite (complex) measure μ vanishing on the polynomials. By polarization, it suffices to prove that $\langle \mu, F\rangle_{C^0_W} = 0$ only for real measures μ of variation less than or equal to 1, such that $\langle \mu, \mathcal{P}\rangle_{C^0_W} = 0$. Denote the set of such measures by $\Sigma(W)$.

For every measure μ denote by $\widetilde{\mu}$ the measure defined by $\widetilde{\mu}(E) = \mu\big(\{x \in \mathbb{R} : -x \in E\}\big)$, $E \subset \mathbb{R}$, and put

$$\mu_\pm = \frac{\mu \pm \widetilde{\mu}}{2}.$$

Since the weight W is even,

$$\mu \in \Sigma(W) \implies \widetilde{\mu} \in \Sigma(W) \implies \mu_\pm \in \Sigma(W).$$

Denote

$$\Sigma_\pm(W) = \big\{\mu_\pm : \mu \in \Sigma(W)\big\} = \big\{\mu \in \Sigma(W) : \mu = \mu_\pm\big\}.$$

For every measure μ we have $\mu = \mu_+ + \mu_-$, hence $\Sigma(W) \subset \Sigma_+(W) + \Sigma_-(W)$, and we need only to verify that $\langle \mu, F\rangle_{C^0_W} = 0$ for $\mu \in \Sigma_+(W) \cup \Sigma_-(W)$.

The sets $\Sigma_\pm(W)$ are convex and weak* compact subsets of the space $M(\mathbb{R})$ dual to C^0_W, and by the Krein–Milman theorem, it suffices to verify the equality $\langle \mu, F\rangle_{C^0_W} = 0$ only for the extreme points μ of $\Sigma_\pm(W)$. Also, we note that for some entire functions F_1, F_2, we have $F(z) = F_1(z^2) + zF_2(z^2)$, with both summands in the right-hand side belonging to $\mathcal{E}^\infty_W$.

Consider the auxiliary weight

$$U(x) = \begin{cases} x^{-1/2}W(x^{1/2}), & x \ge 0, \\ +\infty, & x < 0, \end{cases}$$

and define a map

$$I : \big[M(\mathbb{R})\big]_- \to M\big((0,+\infty)\big)$$

by the relation

$$\langle I\mu, f\rangle_{C^0_U} = \langle \mu, x \mapsto xf(x^2)\rangle_{C^0_W}, \qquad f \in C^0_U,$$

where C^0_U is defined by analogy with C^0_W.

Since the functions $x \mapsto h_1(x^2) + xh_2(x^2)$, for continuous h_1, h_2 with finite support on $\mathbb{R}$, form a dense subset of C^0_W, we see that I is an isometrical isomorphism. Furthermore, $\mu \in [M(\mathbb{R})]_-$ belongs to $\Sigma_-(W)$ if and only if $I\mu$ belongs to $\Sigma(U)$. Hence, the fact that μ is an extreme point of $\Sigma_-(W)$ implies that $I\mu$ is an extreme point of $\Sigma(U)$.

The extreme points of $\Sigma(U)$ are described by L. de Branges [2], [9, Section VI F]. (One considers there continuous weight functions; in the situation under consideration, the description is the same.) For every such measure $I\mu$ there exists a transcendental entire function E of zero exponential type having only simple real zeros $0 < x_1^2 < x_2^2 < \dots$ such that

$$\sum_{k\ge1} \frac{U(x_k^2)}{|E'(x_k^2)|} < +\infty,$$

$$\langle I\mu, f\rangle_{C^0_U} = \sum_{k\ge1} \frac{f(x_k^2)}{E'(x_k^2)}, \qquad f \in C^0_U.$$

Put $B(z) = E(z^2)$. Then

$$\left.\begin{aligned} &\sum_{k\in\mathbb{Z}} \frac{W(x_k)}{|B'(x_k)|} < +\infty, \\ \langle \mu, F\rangle_{C^0_W} = \langle \mu, x \mapsto xF_2(x^2)\rangle_{C^0_W} &= \sum_{k\in\mathbb{Z}} \frac{F(x_k)}{B'(x_k)}, \end{aligned}\right\} \tag{2.1}$$

where $x_k = -x_{1-k}$, $k \le 0$.

Next, consider the auxiliary weight

$$V(x) = \begin{cases} W(x^{1/2}), & x \ge 0, \\ +\infty, & x < 0, \end{cases}$$

and define an isometrical isomorphism

$$J : [M(\mathbb{R})]_+ \to M((0, +\infty))$$

by the relation

$$\langle J\mu, f\rangle_{C^0_V} = \langle \mu, x \mapsto f(x^2)\rangle_{C^0_W}, \qquad f \in C^0_V.$$

For every extreme measure μ of $\Sigma_+(W)$, $J\mu$ is an extreme point of $\Sigma(V)$, and there exists a transcendental entire function E of zero exponential type having only simple real zeros $0 \le x_1^2 < x_2^2 < \dots$ such that

$$\sum_{k\ge 1} \frac{V(x_k^2)}{|E'(x_k^2)|} < +\infty,$$
$$\langle J\mu, f\rangle_{C^0_V} = \sum_{k\ge 1} \frac{f(x_k^2)}{E'(x_k^2)}, \qquad f \in C^0_V.$$

We consider two cases: $E(0) = 0$ and $E(0) \ne 0$. In the first case, put $B(z) = E(z^2)/z$. Then

$$\left.\begin{aligned} &\sum_{k\in\mathbb{Z}} \frac{W(x_k)}{|B'(x_k)|} < +\infty, \\ \langle \mu, F\rangle_{C^0_W} = \langle \mu, x \mapsto F_1(x^2)\rangle_{C^0_W} &= \sum_{k\in\mathbb{Z}} \frac{F(x_k)}{B'(x_k)}, \end{aligned}\right\} \tag{2.2}$$

where $x_k = -x_{2-k}$, $k \le 0$.

In the second case, put $B(z) = zE(z^2)$. Then

$$\left.\begin{aligned} &\sum_{k\in\mathbb{Z}} \frac{x_k^2 W(x_k)}{|B'(x_k)|} < +\infty, \\ \langle \mu, F\rangle_{C^0_W} = \langle \mu, x\mapsto F_1(x^2)\rangle_{C^0_W} &= \sum_{k\in\mathbb{Z}} \frac{x_k^2 F(x_k)}{B'(x_k)}, \end{aligned}\right\} \tag{2.3}$$

where $x_0 = 0$, $x_k = -x_{-k}$, $k < 0$.

As a result of (2.1)–(2.3), to prove the theorem it remains to verify the following statement. *For every even log-convex function W satisfying* (1.1), (1.2), *for every $F \in \mathcal{E}^\infty_W$, and for every transcendental entire function B of zero exponential type having only simple real zeros x_k, $k \in \mathbb{Z}$, such that for an entire function E,*

$$\textit{either}\quad B(z) = E(z^2) \quad \textit{or}\quad B(z) = zE(z^2), \tag{2.4}$$

and such that

$$\sum_{k\in\mathbb{Z}} \frac{W(x_k)}{|B'(x_k)|} < +\infty,$$

we have

$$\sum_{k\in\mathbb{Z}} \frac{F(x_k)}{B'(x_k)} = 0. \tag{2.5}$$

An essential part of the proof consists in the use of

Lemma 2.2. *For W and B as above, for some real number $C = C(W, B)$ and for every $y \ge 1$,*

$$\int_{-\infty}^{+\infty} \frac{y \log W(x)}{y^2 + x^2}\, dx \le C + \int_{-\infty}^{+\infty} \frac{y \log |B(x)/x|}{y^2 + x^2}\, dx.$$

Using this lemma, to be proved below, and the facts that $F \in \mathcal{E}^\infty_W$ is of zero exponential type, and that the function $z \mapsto B(z)/z$ is outer in the upper half-plane, we get

$$\begin{aligned} |F(iy)| &\le \exp \int_{-\infty}^{+\infty} \frac{|y| \log |F(x)|}{y^2 + x^2}\, dx = o(1) \exp \int_{-\infty}^{+\infty} \frac{|y| \log W(x)}{y^2 + x^2}\, dx \\ &= o(1) \exp \int_{-\infty}^{+\infty} \frac{|y| \log |B(x)/x|}{y^2 + x^2}\, dx = o(1) \Big|\frac{B(iy)}{y}\Big|, \qquad |y| \to +\infty. \end{aligned}$$

Therefore, we can write the interpolation formula

$$R(z) = \frac{zF(z)}{B(z)} - \sum_{k\in\mathbb{Z}} \frac{x_k F(x_k)}{B'(x_k)(z - x_k)} = 0$$

(R is an entire function of zero exponential type tending to 0 along the imaginary axis, hence $R=0$). Setting $z=0$, we deduce

$$\sum_{k\in\mathbb{Z}}\frac{F(x_k)}{B'(x_k)}=0.$$

Relation (2.5) is proved, and the proof of the theorem is completed. □

Proof of Lemma 2.2. **A**. By (2.4), $|B(z)|=|B(-z)|$, $z\in\mathbb{C}$, and to prove the lemma, it suffices to verify that for some C and for every $y\ge 1$,

$$\int_0^{+\infty}\frac{y\log W(x)}{y^2+x^2}\,dx\le C+\int_0^{+\infty}\frac{y\log|B(x)/x|}{y^2+x^2}\,dx. \tag{2.6}$$

To verify (2.6), we write down all the zeros of B on $(0,+\infty)$ as

$$0<x_1=\exp t_1<x_2=\exp t_2<\ldots<x_k=\exp t_k<\ldots,1\le k<+\infty,$$

and denote

$$K=\sum_{k\ge 1}\frac{W(x_k)}{|B'(x_k)|}.$$

Furthermore, put

$$p'_k=\frac{W(x_k)}{|B'(x_k)|},$$
$$p_k=\max(p'_k,p'_{k+1}).$$

Then

$$\sum_{k\ge 1}p_k\le 2K. \tag{2.7}$$

For every $k\ge 1$ denote

$$B_k(x)=\frac{xB(x)}{\Bigl(1-\dfrac{x^2}{x_k^2}\Bigr)\Bigl(1-\dfrac{x^2}{x_{k+1}^2}\Bigr)}. \tag{2.8}$$

Then

$$B'(x_k)=-\frac{2B_k(x_k)}{x_k^2}\Bigl(1-\frac{x_k^2}{x_{k+1}^2}\Bigr)=2B_k(x_k)\Bigl(\frac{1}{x_{k+1}^2}-\frac{1}{x_k^2}\Bigr),$$
$$B'(x_{k+1})=-\frac{2B_k(x_{k+1})}{x_{k+1}^2}\Bigl(1-\frac{x_{k+1}^2}{x_k^2}\Bigr)=2B_k(x_{k+1})\Bigl(\frac{1}{x_k^2}-\frac{1}{x_{k+1}^2}\Bigr),$$

and we get

$$\log \frac{W(\exp t_k)}{|B_k(\exp t_k)|} = \log \frac{W(\exp t_k)}{|B'(\exp t_k)|} + \log 2 + \log(e^{-2t_k} - e^{-2t_{k+1}})$$
$$= \log(2p'_k) + \log(e^{-2t_k} - e^{-2t_{k+1}}),$$
$$\log \frac{W(\exp t_{k+1})}{|B_k(\exp t_{k+1})|} = \log \frac{W(\exp t_{k+1})}{|B'(\exp t_{k+1})|} + \log 2 + \log(e^{-2t_k} - e^{-2t_{k+1}})$$
$$= \log(2p'_{k+1}) + \log(e^{-2t_k} - e^{-2t_{k+1}}).$$

B. Recall the relation (2.4). By the Hadamard factorization theorem we have

$$E(x^2) = \prod_{k\ge 1}\Bigl(1 - \frac{x^2}{x_k^2}\Bigr).$$

Since the function $t \mapsto \log|1-e^{2t-2s}|$ is concave on each of the intervals $(-\infty, s)$, $(s, +\infty)$, using (2.4) and (2.8) we obtain that the function $t \mapsto \log|B_k(\exp t)|$ is concave on $[t_k, t_{k+1}]$. Furthermore, the function $t \mapsto \log|W(\exp t)|$ is convex on $(-\infty, +\infty)$. Therefore,

$$\log \frac{W(\exp t)}{|B_k(\exp t)|} \le \log(2p_k) + \log(e^{-2t_k} - e^{-2t_{k+1}}), \qquad t_k \le t \le t_{k+1}.$$

Put

$$R(t) = \log \frac{W(\exp t)\exp(t - A)}{|B(\exp t)|},$$

for some parameter $A > 0$ to be defined later on. Then

$$R(t) \le \log(e^{-2t_k} - e^{-2t_{k+1}}) + \log(2p_k) + 2t - A$$
$$- \log(e^{2(t-t_k)} - 1) - \log(1 - e^{2(t-t_{k+1})}), \qquad t_k \le t \le t_{k+1}.$$

Since

$$t + \log(e^{-t_k} + e^{-t_{k+1}}) - \log(e^{t-t_k} + 1) - \log(1 + e^{t-t_{k+1}})$$
$$= \log(1 + e^{t_k - t_{k+1}}) - \log(1 + e^{t_k - t}) - \log(1 + e^{t-t_{k+1}})$$
$$\le 0, \qquad t_k \le t \le t_{k+1},$$

we get

$$R(t) \le \log(e^{-t_k} - e^{-t_{k+1}}) + \log(2p_k) + t - A$$
$$- \log(e^{t-t_k} - 1) - \log(1 - e^{t-t_{k+1}}), \qquad t_k \le t \le t_{k+1}. \tag{2.9}$$

To prove (2.6), we need to verify that

$$\sum_{k\ge 1} \int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x)\, dx \le O(1), \qquad y \to +\infty. \tag{2.10}$$

C. Put $\beta_k = t_{k+1} - t_k$. Suppose that $\beta_k \le \log 2$.

Then

$$\int_{\exp t_k}^{\exp t_{k+1}} \log(xe^{-t_k}-1)\,dx = \int_{t_k}^{t_{k+1}} \log(e^{t-t_k}-1)e^t\,dt$$
$$= e^{t_k}\int_0^{\beta_k} \log(e^s-1)e^s\,ds \ge -e^{t_k}\beta_k\big(\log\frac{1}{\beta_k}+C\big),$$

for some absolute constant C. Analogously,

$$\int_{\exp t_k}^{\exp t_{k+1}} \log(1-xe^{-t_{k+1}})\,dx \ge -e^{t_k}\beta_k\big(\log\frac{1}{\beta_k}+C\big).$$

Hence,

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2}\Big[\log(xe^{-t_k}-1)+\log(1-xe^{-t_{k+1}})\Big]\,dx$$
$$\ge -\frac{ye^{t_k}}{y^2+e^{2t_k}}\cdot 2\beta_k\big(\log\frac{1}{\beta_k}+C\big). \tag{2.11}$$

Since

$$\log(e^{-t_k}-e^{-t_{k+1}})+t = t-t_{k+1}+\log(e^{\beta_k}-1) \le \log(2\beta_k), \quad t_k \le t \le t_{k+1},$$

we obtain

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2}\Big[\log(e^{-t_k}-e^{-t_{k+1}})+\log x+\log(2p_k)-A\Big]\,dx$$
$$\le \int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2}[\log\beta_k+\log(4p_k)-A]\,dx. \tag{2.12}$$

Furthermore,

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2}\,dx \ge \frac{ye^{t_k}(e^{\beta_k}-1)}{y^2+e^{2t_k+2\beta_k}} \ge e^{-2\beta_k}\beta_k\frac{ye^{t_k}}{y^2+e^{2t_k}}. \tag{2.13}$$

Finally, by (2.7),

$$\log\beta_k+\log(4p_k) \le \log(\log 2)+\log(8K).$$

Therefore, by (2.9), (2.11)–(2.13), for $A \ge \log(\log 2)+\log(8K)$, we get

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2}R(\log x)\,dx$$
$$\le \frac{ye^{t_k}}{y^2+e^{2t_k}}\Big[\beta_k e^{-2\beta_k}\big(\log\beta_k+\log(4p_k)-A\big)+2\beta_k\big(\log\frac{1}{\beta_k}+C\big)\Big]$$
$$\le \frac{1}{2}\big[\beta_k\log\frac{p_k}{\beta_k}+\beta_k(C_1-A)\big],$$

for some C_1 independent of k, A.

D. If $\beta_k > \log 2$, then by (2.7), (2.9), we get

$$R(t) + A$$
$$\le \log(1-e^{-\beta_k}) + \log(4K) + t - t_k - \log(e^{t-t_k}-1) - \log(1-e^{t-t_{k+1}})$$
$$\le \log(4K) - \log(1-e^{t_k-t}) - \log(1-e^{t-t_{k+1}})$$
$$\le \log(4K) - 2\log(1-2^{-1/2}) = C_2, \qquad t \in \Big[t_k + \frac{\log 2}{2}, t_{k+1} - \frac{\log 2}{2}\Big],$$

with C_2 independent of k, A. Arguing as in part C, we obtain

$$\int_{\exp t_k}^{\sqrt{2}\exp t_k} \frac{y}{y^2+x^2} R(\log x)\, dx \le (C_3 - C_4 A)\frac{y e^{t_k}}{y^2+e^{2t_k}},$$
$$\int_{(1/\sqrt{2})\exp t_{k+1}}^{\exp t_{k+1}} \frac{y}{y^2+x^2} R(\log x)\, dx \le (C_3 - C_4 A)\frac{y e^{t_{k+1}}}{y^2+e^{2t_{k+1}}},$$

for some $C_3, C_4 > 0$ independent of k, A.

Hence, for sufficiently big A independent of k, we have

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2} R(\log x)\, dx \le \frac{\beta_k}{2}\log\frac{p_k}{\beta_k}, \tag{2.14}$$

if $\beta_k \le \log 2$, and

$$\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2} R(\log x)\, dx \le 0, \tag{2.15}$$

if $\beta_k > \log 2$. As a result of (2.14) and (2.15),

$$\sum_{k\ge1}\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2+x^2} R(\log x)\, dx \le \sum_{k\ge1}\frac{\beta_k}{2}\log^+\frac{p_k}{\beta_k}$$
$$= \sum_{k\ge1}\frac{p_k}{2}\cdot\frac{\beta_k}{p_k}\log^+\frac{p_k}{\beta_k} \le \frac{1}{2e}\sum_{k\ge1}p_k \le \frac{K}{e},$$

because $x\log^+\frac{1}{x} \le \frac{1}{e}$, $0 < x < +\infty$. This proves (2.10) and the whole lemma. □

3. Polynomial approximation in L^p_W, $1 \le p < +\infty$

Theorem 3.1. *Suppose that W is an even* log*-convex function on the real line satisfying* (1.1), (1.2). *Then $\mathcal{P}^p_W = \mathcal{E}^p_W$, $1 \le p < +\infty$.*

Notice that as a consequence of condition (1.2), $\mathcal{E}^p_W$ is a closed subspace of L^p_W, and the point evaluations $F \mapsto F(z)$, $z \in \mathbb{C}$, are bounded linear functionals on $\mathcal{E}^p_W$.

For the proof of Theorem 3.1, we need the following three results, whose proofs are deferred to Section 4.

Lemma 3.2. *Under conditions of Theorem* 3.1 *we have*

$$\lim_{|x|\to+\infty} \frac{W(x+1)}{W(x)} = 1. \tag{3.1}$$

Lemma 3.3. *Let $F \in \mathcal{E}$ and either $1 \le p < r < +\infty$ or $1 < p < r \le +\infty$. Suppose that $\mathcal{P}^r_W = \mathcal{E}^r_W$ and W satisfies* (3.1)*. If the function $z \mapsto zF(z)$ belongs to $\mathcal{E}^p_W$, then $F \in \mathcal{P}^p_W$.*

Lemma 3.4. *Let W be monotone increasing on $\mathbb{R}_+$ and monotone decreasing on $\mathbb{R}_-$, and satisfy* (1.1)*. Suppose that $F \in \mathcal{P}^p_W$, $1 \le p \le +\infty$. If the function $z \mapsto zF(z)$ belongs to $\mathcal{E}^p_W$, then it belongs to $\mathcal{P}^p_W$ as well.*

The last lemma together with a result of Khachatryan [7, Corollary 2] formulated below gives another proof of Theorem 2.1.

Proposition 3.5 (Khachatryan)**.** *In the conditions of Theorem* 3.1*, suppose that $F \in \mathcal{E}$. If the function $z \mapsto z^2F(z)$ belongs to $\mathcal{E}^\infty_W$, then $F \in \mathcal{P}^\infty_W$.*

Since [7] is rather inaccessible, in Section 4 we show how to derive this propostion from some results given in [9].

The second proof of Theorem 2.1*.* We start with $G \in \mathcal{E}^\infty_W$, and define

$$G_1(z) = \frac{G(z) - G(0)}{z}, \qquad G_2(z) = \frac{G_1(z) - G_1(0)}{z}. \tag{3.2}$$

By Proposition 3.5, $G_2 \in \mathcal{P}^\infty_W$. Furthermore, applying Lemma 3.4 twice, we obtain $G_1 \in \mathcal{P}^\infty_W$, $G \in \mathcal{P}^\infty_W$. Thus, $\mathcal{E}^\infty_W = \mathcal{P}^\infty_W$. □

Furthermore, Lemma 3.4 together with Theorem 2.1 gives a proof of Theorem 3.1.

Proof of Theorem 3.1*.* We start with the case $1 < p < +\infty$. Take $G \in \mathcal{E}^p_W$, and define G_1 as in (3.2). Applying Lemma 3.3 with $r = +\infty$ and Theorem 2.1, we obtain that $G_1 \in \mathcal{P}^p_W$. By Lemma 3.4, $G \in \mathcal{P}^p_W$. Thus, $\mathcal{E}^p_W = \mathcal{P}^p_W$, $1 < p < +\infty$.

If $p = 1$, $G \in \mathcal{E}^\infty_W$, then we apply Lemma 3.3 with $r = 2$ and the (already proved) equality $\mathcal{P}^2_W = \mathcal{E}^2_W$ to get $G_1 \in \mathcal{P}^1_W$. Again by Lemma 3.4, $G \in \mathcal{P}^1_W$. Thus, $\mathcal{E}^1_W = \mathcal{P}^1_W$. □

4. The proofs of auxiliary statements

Proof of Lemma 3.2. Without loss of generality, assume that W is C^1-smooth. Put $\varphi = \log W$. Since W is log-convex, the function $x \mapsto x\varphi'(x)$ increases, and elementary inequalities

$$\varphi'(x) \ge \frac{t\varphi'(t)}{x} \ge \frac{\varphi'(t)}{3}, \qquad t \le x \le 3t,$$

$$\varphi(x) \ge \int_t^x \varphi'(y)\,dy \ge \frac{(x-t)\varphi'(t)}{3} \ge \frac{t\varphi'(t)}{3}, \qquad 2t \le x \le 3t,$$

$$\int_{2t}^{3t} \frac{\varphi(x)}{x^2}\,dx \ge \frac{t\cdot t\varphi'(t)}{9t^2\cdot 3} \ge \frac{\varphi'(t)}{27}, \qquad t \ge 0,$$

together with (1.2) show that

$$\lim_{x\to+\infty} \varphi'(x) = 0.$$

Therefore,

$$\lim_{|x|\to+\infty} \frac{W(x+1)}{W(x)} = 1.$$

□

Proof of Lemma 3.3. Since

$$\int_{-\infty}^{+\infty} \frac{|F(x)|^p}{\big(W(x)\big)^p}\,dx \le \int_{|x|\le 1} \frac{|F(x)|^p}{\big(W(x)\big)^p}\,dx + \int_{|x|>1} \frac{|x|^p\cdot|F(x)|^p}{\big(W(x)\big)^p}\,dx < +\infty,$$

we obtain $F \in \mathcal{E}^p_W$. For $\varepsilon > 0$ put

$$F_\varepsilon(z) = \frac{1}{\varepsilon}\int_0^\varepsilon F(z+t)\,dt.$$

By (3.1) it follows easily that the shift operator is continuous in L^s_W, for $1 \le s < +\infty$, and in C^0_W, and therefore $F_\varepsilon \to F$ in L^p_W as $\varepsilon \to 0$. Furthermore, the functions $z \mapsto (z+i)F_\varepsilon(z)$, $\varepsilon > 0$, belong to both C^0_W and L^p_W, because again by (3.1),

$$\begin{aligned}|x+i|\cdot|F_\varepsilon(x)| &\le \frac{CW(x)}{\varepsilon}\int_0^\varepsilon \frac{|x+t+i|\cdot|F(x+t)|}{W(x+t)}\,dt \\ &\le \frac{CW(x)}{\varepsilon^{1/p}}\Big(\int_0^\varepsilon \frac{|x+t+i|^p\cdot|F(x+t)|^p}{\big(W(x+t)\big)^p}\,dt\Big)^{1/p} \\ &= \frac{CW(x)}{\varepsilon^{1/p}}\Big(\int_x^{x+\varepsilon} \frac{|t+i|^p\cdot|F(t)|^p}{\big(W(t)\big)^p}\,dt\Big)^{1/p} = o(W(x)), \qquad |x| \to +\infty,\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{|x+i|^p\cdot|F_\varepsilon(x)|^p}{\big(W(x)\big)^p}\,dx &= \frac{1}{\varepsilon^p}\int_{-\infty}^{\infty}\Big(\int_0^\varepsilon |F(x+t)|\,dt\Big)^p\frac{|x+i|^p}{\big(W(x)\big)^p}\,dx\\ &\le \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Big(\int_0^\varepsilon |F(x+t)|^p\,dt\Big)\frac{|x+i|^p}{\big(W(x)\big)^p}\,dx\\ &= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Big(\int_0^\varepsilon \frac{|x-t+i|^p}{\big(W(x-t)\big)^p}\,dt\Big)|F(x)|^p\,dx\\ &\le C\int_{-\infty}^{\infty}\frac{|x+i|^p\cdot|F(x)|^p}{\big(W(x)\big)^p}\,dx<+\infty.\end{aligned}$$

Since $r>p$, the functions $z\mapsto(z+i)F_\varepsilon(z)$, $\varepsilon>0$, belong also to L^r_W. Fix $\varepsilon>0$. By conditions of the lemma, $\mathcal{P}^r_W=\mathcal{E}^r_W$, and we can find a sequence of polynomials Q_n approximating the function $z\mapsto(z+i)F_\varepsilon(z)$ in $\mathcal{E}^r_W$. Since the point evaluation $F\mapsto F(-i)$ is bounded on $\mathcal{E}^r_W$, we get $Q_n(-i)\to 0$, $n\to\infty$. Now, without loss of generality we may assume that $Q_n(-i)=0$, and hence, $Q_n(z)=(z+i)P_n(z)$ for some polynomials P_n. Thus,

$$\int_{-\infty}^{+\infty}\frac{|(x+i)F_\varepsilon(x)-(x+i)P_n(x)|^r}{\big(W(x)\big)^r}\,dx\to 0,\qquad n\to+\infty,$$

if $r<+\infty$, and

$$\sup_{x\in\mathbb{R}}\frac{|(x+i)F_\varepsilon(x)-(x+i)P_n(x)|}{W(x)}\to 0,\qquad n\to+\infty.$$

if $r=+\infty$. By Hölder's inequality,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{|F_\varepsilon(x)-P_n(x)|^p}{\big(W(x)\big)^p}\,dx\\ &\le\Big(\int_{-\infty}^{+\infty}\frac{|(x+i)F_\varepsilon(x)-(x+i)P_n(x)|^r}{\big(W(x)\big)^r}\,dx\Big)^{p/r}\\ &\times\Big(\int_{-\infty}^{+\infty}\frac{dx}{|x+i|^{pr/(r-p)}}\Big)^{(r-p)/r}\to 0,\qquad n\to+\infty,\end{aligned}$$

if $r<+\infty$, and

$$\int_{-\infty}^{+\infty}\frac{|F_\varepsilon(x)-P_n(x)|^p}{\big(W(x)\big)^p}\,dx$$

$$\le\Big(\sup_{x\in\mathbb{R}}\frac{|(x+i)F_\varepsilon(x)-(x+i)P_n(x)|}{W(x)}\Big)^p\int_{-\infty}^{+\infty}\frac{dx}{|x+i|^p}\to 0,\ n\to+\infty,$$

if $r=\infty$. Here we use that either $r<+\infty$ and $pr>r-p$, or $r=+\infty$ and $p>1$.

Thus, $F_\varepsilon \in \mathcal{P}^p_W$, $\varepsilon > 0$, and as a consequence, $F \in \mathcal{P}^p_W$. □

Proof of Lemma 3.4. Denote $G(z) = zF(z)$. For every $\lambda \in \mathbb{C}$, the function $z \mapsto \dfrac{G(z) - G(\lambda)}{z-\lambda}$ belongs to $\mathcal{P}^p_W$. Suppose that $G \in \mathcal{E}^p_W \setminus \mathcal{P}^p_W$.

Define

$$\mathfrak{A} = \operatorname*{clos}_{\mathcal{E}^p_W} \operatorname{Lin} \Bigl[\bigl\{G(\eta z), 0 < \eta \le 1\bigr\} \cup \mathcal{P}\Bigr].$$

Then,

$$\left.\begin{array}{c} \text{for every } \lambda \in \mathbb{C},\ H \in \mathfrak{A}, \\ \text{the function } z \mapsto \dfrac{H(z) - H(\lambda)}{z-\lambda} \text{ belongs to } \mathcal{P}^p_W. \end{array}\right\} \tag{4.1}$$

We can identify bounded linear functionals on L^p_W, $1 \le p < +\infty$, with elements of $L^q_{1/W}$, $1/p + 1/q = 1$,

$$L^q_{1/W} = \Bigl\{ f : \int_{-\infty}^{+\infty} |f(x)|^q \bigl(W(x)\bigr)^q \, dx < +\infty \Bigr\}, \qquad 1 < q < +\infty,$$
$$L^\infty_{1/W} = \Bigl\{ f : \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| W(x) < +\infty \Bigr\},$$

the duality being

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x) g(x)\, dx, \qquad f \in L^q_{1/W},\ g \in L^p_W.$$

From now on we restrict ourselves to the case $1 \le p < +\infty$. In the case $p = \infty$ we identify bounded linear functionals on C^0_W with complex Borel measures μ such that

$$\int_{-\infty}^{+\infty} W(x)\, d\mu(x) < +\infty,$$

and the rest of the proof is similar to that for the case $1 \le p < +\infty$.

Using the Hahn–Banach theorem, choose $g \in L^q_{1/W}$ such that

$$\langle g, G\rangle \ne 0, \qquad \langle g, \mathcal{P}^p_W\rangle = 0.$$

By (4.1), for every $\lambda \in \mathbb{C}$ we get

$$\int_{-\infty}^{+\infty} g(x) \frac{G(x) - G(\lambda)}{x - \lambda}\, dx = 0,$$

$$G(\lambda) \int_{-\infty}^{+\infty} \frac{g(x)}{x-\lambda}\, dx = \int_{-\infty}^{+\infty} \frac{g(x)G(x)}{x-\lambda}\, dx, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.2}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{|t|\to+\infty}\int_{-\infty}^{+\infty} g(x)G(x)\frac{it}{x-it}\,dx = -\int_{-\infty}^{+\infty} g(x)G(x)\,dx \neq 0, \qquad (4.3)$$

and hence,

$$\int_{-\infty}^{+\infty} \frac{g(x)}{x-it}\,dx \neq 0 \qquad (4.4)$$

for t real and $|t|$ large enough.

Next, take $H \in \mathfrak{A}$ such that $\langle g, H\rangle = 0$. By (4.1), for every $\lambda \in \mathbb{C}$ we obtain as before:

$$H(\lambda)\int_{-\infty}^{+\infty} \frac{g(x)}{x-\lambda}\,dx = \int_{-\infty}^{+\infty} \frac{g(x)H(x)}{x-\lambda}\,dx, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}, \qquad (4.5)$$

$$\lim_{|t|\to+\infty}\int_{-\infty}^{+\infty} g(x)H(x)\frac{it}{x-it}\,dx = -\int_{-\infty}^{+\infty} g(x)H(x)\,dx = 0. \qquad (4.6)$$

Then we divide (4.2) by (4.5), taking account of observation (4.4), and use (4.3), (4.6), to get

$$|H(it)| = o(1)|G(it)|, \qquad |t| \to +\infty. \qquad (4.7)$$

One immediate consequence of this relation is that for every $n \geq 0$ we have

$$|t|^n = o(1)|G(it)|, \qquad |t| \to +\infty. \qquad (4.8)$$

Furthermore, if $\dim(\mathfrak{A}/\mathcal{P}^p_W) \geq 2$, then we may take

$$G_1 \in \mathfrak{A}\setminus(\mathbb{C}G + \mathcal{P}^p_W)$$

and $f, f_1 \in L^q_{1/W}$ such that

$$\langle f, G\rangle \neq 0, \qquad \langle f, \mathbb{C}G_1 + \mathcal{P}^p_W\rangle = 0,$$
$$\langle f_1, G_1\rangle \neq 0, \qquad \langle f_1, \mathbb{C}G + \mathcal{P}^p_W\rangle = 0.$$

Applying the above argument we deduce

$$|G_1(it)| = o(1)|G(it)|, \qquad |t| \to +\infty,$$
$$|G(it)| = o(1)|G_1(it)|, \qquad |t| \to +\infty,$$

which is impossible.

Thus, $\dim(\mathfrak{A}/\mathcal{P}^p_W) = 1$, and we have $\mathfrak{A} = \mathbb{C}G + \mathcal{P}^p_W$. Given $0 < \eta < 1$, put $G_\eta(z) = G(\eta z)$. Since $W(x)$ increases with $|x|$, by dominated convergence we obtain $G_\eta \to G$ in $\mathcal{E}^p_W$ as $\eta \to 1$. Since $G \notin \mathcal{P}^p_W$, for some $\eta < 1$ we should have $G_\eta \notin \mathcal{P}^p_W$, and

$$G_\eta = H(\eta) + A(\eta)G$$

for some $H(\eta) \in \mathcal{P}^p_W$, $A(\eta) \in \mathbb{C} \setminus \{0\}$. By (4.7) we obtain

$$|G(i\eta t)| = |G_\eta(it)| = |A(\eta) + o(1)| \cdot |G(it)|, \qquad |t| \to +\infty.$$

Hence, for every $M > \log A(\eta)/\log(1/\eta)$ there exists C_M such that

$$|G(it)| \le C_M(|t|+1)^M, \qquad t \in \mathbb{R},$$

which contradicts (4.8). Thus, $G \in \mathcal{P}^p_W$, and the proof is completed. □

Proof of Proposition 3.5. We make extensive use of the material in [9, Section VI H.2]. First, by an argument in the proof of the theorem on page 226 of [9], for some positive C, for some function S of the form (1.4), and for any η, $0 < \eta < 1$, we have

$$CW(x) \le S(x) \le 1 + \frac{x^2}{1-\eta^2} W\Big(\frac{x}{\eta}\Big), \qquad x \in \mathbb{R}. \tag{4.9}$$

Denote $G(z) = z^2 F(z)$, $W_\eta(x) = W(x/\eta)$, $F_\eta(x) = F(\eta x)$. Since $G \in \mathcal{E}^p_W$, by the first inequality in (4.9), we get $G \in \mathcal{E}^\infty_S$. By the result of Khachatryan mentioned in Section 1 (for the proof see pages 223–226 of [9]), we obtain $G \in \mathcal{P}^\infty_S$. Hence, as in the proof of Lemma 3.3, we can find a sequence of polynomials P_n, such that $z \mapsto z^2 P_n(z)$ approximate G in $\mathcal{E}^\infty_S$. Applying the second inequality in (4.9), we conclude that P_n tend to F in $\mathcal{E}^\infty_{W_\eta}$, and hence, $F \in \mathcal{P}^\infty_{W_\eta}$, $F_\eta \in \mathcal{P}^\infty_W$. Finally, since $\eta < 1$ is arbitrary, and $F_\eta \to F$ as $\eta \to 1$, we obtain that $F \in \mathcal{P}^\infty_W$. □

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