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STRONGLY SEQUENTIALLY CONTINUOUS FUNCTIONS

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Abstract. *Given a topological abelian group G, we study the class of strongly sequentially continuous functions on G. Strong sequential continuity is intermediate between sequential continuity and uniform sequential continuity, and apart from trivial cases, the class of strongly sequentially continuous functions is strictly smaller than the class of sequentially continuous functions. On the other hand, the "gap" between strong sequential continuity and uniform sequential continuity heavily depends on the group G: if G has some "completeness" property, then all strongly sequentially continuous functions on G are uniformly sequentially continuous, but we exhibit several examples for which the two notions differ; these examples include a large class of "small" dense subgroups of $\mathbb{R}$, and all groups of the form (X, w), where X is a separable Banach space failing the Schur property.*

Introduction

This paper is in some sense a continuation of **[DM]**, which was itself motivated by recent results of P. Hàjek. In **[H]**, Hàjek proves that if $f : c_0 \to \mathbb{R}$ is a C^1-smooth function whose derivative is uniformly continuous on the unit ball B_{c_0}, then f is uniformly continuous on B_{c_0} for the *weak* topology. One purpose of **[DM]** was to give a short proof of this rather intriguing result, and in that context, the notion of *strong sequential continuity* arose in a surprisingly natural way. In view of its usefulness in seemingly unrelated matters, we believe that this notion deserves a more detailed study.

Definition 1. Let G be a Hausdorff topological abelian group, let (Y, δ) be a metric space, and let A be a subset of G. We say that a function $f : G \to Y$ is *strongly sequentially continuous on* A if, for every sequence $(h_i) \subseteq G$ converging to 0,

$$\inf_{i \le n} \delta\big(f(x + h_i), f(x)\big) \to 0 \text{ as } n \to \infty, \text{ uniformly for } x \in A.$$

If $A = G$, we just say that f is strongly sequentially continuous.

Remarks.

(1) If $f : G \to Y$ is strongly sequentially continuous on $A \subseteq G$, then f is sequentially continuous at each point $x \in A$, hence it is continuous at each point of A if G is metrizable.

(2) We say that a function $f : G \to Y$ is *uniformly sequentially continuous* on a set $A \subseteq G$ if for all sequences $(x_n), (y_n) \subseteq A$ such that $x_n - y_n \to 0$, one has $\lim_{n\to\infty} \delta\big(f(x_n), f(y_n)\big) = 0$. Clearly, uniform sequential continuity on G implies strong sequential continuity on G.

(3) Strong sequential continuity can be viewed as an "iterated limit" property: a function $f : G \to Y$ is strongly sequentially continuous on $A \subseteq G$ if and only if, for each sequence $(x_n) \subseteq A$ and each null sequence (that is, converging to 0) $(h_i) \subseteq G$,

$$\lim_{i\to\infty} \liminf_{n\to\infty} \delta\big(f(x_n + h_i), f(x_n)\big) = 0\,.$$

This is the definition originally given in **[DM]**. Notice that one gets sequential continuity at points of A by considering only *constant* sequences (x_n); and for $A = G$, uniform sequential continuity amounts to replacing the lim inf by a lim sup (as one checks easily).

(4) Uniformly continuous functions "operate" on strongly sequentially continuous functions (if $f : G \to (Y_1, \delta_1)$ is strongly sequentially continuous on $A \subseteq G$ and if $g : (Y_1, \delta_1) \to (Y_2, \delta_2)$ is uniformly continuous, then $g \circ f$ is strongly sequentially continuous on A). In particular, any uniformly equivalent metric on Y gives rise to the same strongly sequentially continuous functions from G into Y. Since any metric space is uniformly homeomorphic to a subspace of some normed linear space, we may therefore assume, if needed, that the range space is a normed space.

Throughout this paper, the letter G will always designate a Hausdorff topological abelian group. If Y is a metric space, we denote by $SSC(G, Y)$ the set of all strongly sequentially continuous functions $f : G \to Y$, and by $BSSC(G, Y)$ the set of all bounded functions in $SSC(G, Y)$; when $Y = \mathbb{R}$, we simply write $SSC(G)$ and $BSSC(G)$.

The following obvious lemma will be used repeatedly.

Lemma 0. *Let (Y, δ) be a metric space, and let $A \subset G$. For a function $f : G \to Y$, the following assertions are equivalent:*

(1) *f is not strongly sequentially continuous on A;*

(2) *there exist a positive number ε, a sequence $(x_n) \subseteq A$ and a null sequence $(h_i) \subseteq G$ such that $\delta\big(f(x_n + h_i), f(x_n)\big) \ge \varepsilon$ whenever $n > i$.*

It is proved in **[DM]** that SSC functions turn Cauchy sequences into Cauchy the main reason for introducing SSC functions. We recall this in the first section of the paper, together with some simple consequences. We also show that $BSSC(G)$ is a closed subalgebra of $l_\infty(G)$, and we clarify the link between strong sequential continuity and what we have called *Cesàro-continuity*, a notion which is implicit in **[DM]**.

Section II is devoted to functions defined on dense subgroups of the real line. The main results are the following: SSC functions on $\mathbb{R}$ are uniformly continuous, but if G is a countable dense subgroup of $\mathbb{R}$, the two notions differ. These results are proved later (Sections III and IV), in a more general form. We also show that SSC functions have a sublinear behaviour at infinity, and that if G is a dense subgroup of $\mathbb{R}$, then any function $f : G \to \mathbb{R}$ which is SSC but is not uniformly continuous has to be highly oscillating.

In Sections III and IV, we investigate "nasty" groups (the ones that distinguish between SSC functions and uniformly sequentially continuous functions) and "nice" groups (the ones that do not).

It turns out that "completeness" assumptions metrizable group is nice. We prove two results in that direction. They both depend on a lemma that allows us incidentally to give a short proof of the Orlicz-Pettis theorem on the equivalence of weak subseries summability and norm subseries summability in Banach spaces.

On the other hand, if G is a dense subgroup of $\mathbb{R}$ which is generated by an increasing sequence of Dirichlet sets, then G is a nasty group; this is to be compared with the fact that nasty subgroups of $\mathbb{R}$ are necessarily both meager and Lebesgue-null. We also prove that if $(X, \|\ \|)$ is a normed space with a countable algebraic basis, then $G = (X, \|\ \|)$ is a nasty group, and that if X is a separable Banach space, then $G = (X, w)$ is nasty if and only if X fails the Schur property (recall that a Banach space X is said to have the Schur property if every weakly convergent sequence in X is actually norm-convergent; apart from finite-dimensional spaces, a typical example is the space l_1). Finally, we give an example of a function $f : c_0 \times l_1 \to \mathbb{R}$ which is SSC for the weak topology, but not uniformly sequentially continuous (for the weak topology) on the unit ball of $c_0 \times l_1$. Such a function can be produced neither in c_0, nor in l_1.

I - General facts

Our first result provides some information on the structure of the space of strongly sequentially continuous functions.

Proposition 1. *For any metric space Y, $SSC(G, Y)$ is closed under uniform convergence. Moreover, if Y is a normed space, then $SSC(G, Y)$ is a vector space and $BSSC(G)$ is a (closed) subalgebra of $l_\infty(G)$.*

Corollary. *The set of all bounded real-valued strongly sequentially continuous functions that are not uniformly sequentially continuous is an open subset of $(BSSC(G), \|\ \|_\infty)$, which is either empty or dense (we shall see that both cases do actually happen).*

The only nontrivial point in Proposition 1 is the stability under sum and product. Now, addition is uniformly continuous on $Y \times Y$ (for any normed space Y), and multiplication is uniformly continuous on bounded subsets of $\mathbb{R} \times \mathbb{R}$. Since uniformly continuous functions operate on SSC functions, it is therefore enough to prove

Lemma 1. *If $f_0 : G \to (Y_0, \delta_0)$ and $f_1 : G \to (Y_1, \delta_1)$ are strongly sequentially continuous, then the pair $(f_0, f_1) : G \to Y_0 \times Y_1$ is also strongly sequentially continuous.*

Proof.

Of course, $Y_0 \times Y_1$ is endowed with any of the usual product metrics.

Fix $f_j : G \to Y_j$ $(j = 0, 1)$ and assume that (f_0, f_1) is not strongly sequentially continuous. By lemma 0, this means that there exist a positive number ε and two sequences (h_i), $(x_n) \subseteq G$ such that $h_i \to 0$ and $\mathrm{Max}\,\big\{\delta_0\big(f_0(x_n + h_i), f_0(x_n)\big), \delta_1\big(f_1(x_n + h_i), f_1(x_n)\big)\big\} \geq \varepsilon$ for $n > i$.

For each infinite set $I \subseteq \mathbb{N}$, let us denote by $I^{(2)}$ the set of all ordered pairs of elements of I, that is, $I^{(2)} = \{(i, n) \in I^2;\ i < n\}$. Now, put

$$A_0 = \{(i, n) \in \mathbb{N}^{(2)};\ \delta_0\big(f_0(x_n + h_i), f_0(x_n)\big) \geq \varepsilon\},$$
$$A_1 = \{(i, n) \in \mathbb{N}^{(2)};\ \delta_1\big(f_1(x_n + h_i), f_1(x_n)\big) \geq \varepsilon\}.$$

By assumption, one has $A_0 \cup A_1 = \mathbb{N}^{(2)}$. Hence, by Ramsey's theorem for pairs of integers, there is an infinite set $I \subseteq \mathbb{N}$ such that either $I^{(2)} \subseteq A_0$ or $I^{(2)} \subseteq A_1$. Thus, we may assume for instance that $\delta_0\big(f_0(x_n + h_i), f_0(x_n)\big) \geq \varepsilon$ whenever $n > i$. This means that f_0 is not SSC.

The following result is Lemma 3 of **[DM]**. The proof given in **[DM]** is a simple application of Ramsey's theorem for triples of integers. For completeness, we include an "elementary" proof.

Proposition 2. *Let (Y, δ) be a metric space, let $A \subseteq G$, and let $f : G \to Y$.*

(a) *If f is strongly sequentially continuous on A, then f turns Cauchy sequences in A into Cauchy sequences in Y.*

(b) *If, in addition, every sequence in A has a Cauchy subsequence, then f is uniformly sequentially continuous on A.*

Proof.

To prove (a), assume that there exists a Cauchy sequence $(x_j) \subseteq A$ such that $\big(f(x_j)\big)$ is not a Cauchy sequence in Y. Then one can find a positive number ε and two subsequences (y_k), (z_i) of (x_j) such that $\delta\big(f(y_k), f(z_i)\big) \geq 2\varepsilon$ for all $k, i \geq 0$.

Now, f is sequentially continuous at all points z_i, and (y_k) is a Cauchy sequence; hence, for each fixed i, there exists an integer $k(i)$ such that $\delta\big(f[(y_k - y_{k(i)}) + z_i], f(z_i)\big) < \varepsilon$ whenever $k \geq k(i)$. Moreover, we can assume that the sequence $k(i)$ is increasing. Put $h_i = -y_{k(i)} + z_i$. Since (x_j) is a Cauchy sequence, the sequence (h_i) converges to 0. Therefore, by strong sequential continuity, $\inf\big\{\delta(f(x), f(x + h_i)) :\ i \leq n\big\}$ tends to 0 uniformly on A. In particular, one can find an integer i_0 such that for every k, there exists $i \in \{0, 1, \ldots, i_0\}$ for which $\delta(f(y_k), f(y_k + h_i)) < \varepsilon$.

Now, let $k = k(i_0)$, and choose an integer $i \leq i_0$ such that $\delta(f(y_k), f(y_k + h_i)) < \varepsilon$. Since $i \leq i_0$, we also have $\delta\big(f(y_k + h_i), f(z_i)\big) < \varepsilon$, whence $\delta\big(f(y_k), f(z_i)\big) < 2\varepsilon$, which is a contradiction.

Part (b) follows easily from (a).

Corollary 1. *Assume that G is metrizable.*

If (Y, δ) is a complete metric space and if $f : G \to Y$ is strongly sequentially continuous on $A \subseteq G$, then f has a continuous extension $F : \overline{A} \to Y$, where $\overline{A}$ denotes the closure of A in the completion of G. In particular, if G is a subgroup of $\mathbb{R}$, SSC functions on G are uniformly continuous on bounded sets; and if G is totally bounded, then any SSC function $f; G \to \mathbb{R}$ is uniformly continuous.

As we shall see later on, SSC functions on $\mathbb{R}$ are uniformly continuous, but there exist SSC functions on $\mathbb{Q}$ which are not uniformly continuous. Hence, the continuous extension of an SSC function $f : G \to Y$ needs not be strongly sequentially continuous on $\overline{G}$.

Corollary 2. *Let X be a Banach space not containing ℓ_1.*

If $f : X \to \mathbb{R}$ is strongly sequentially continuous on some bounded set $A \subseteq X$ for the weak topology, then f is uniformly sequentially continuous on A (for the weak topology); and if X^* is separable, then f is uniformly continuous on A (for the weak topology).

Proof.

This follows from Rosenthal's ℓ_1 theorem (see [**Di**]).

Corollary 3. *If G is metrizable, noncompact and nondiscrete, then there exist continuous, real-valued functions on G which are not SSC.*

Proof.

If G is totally bounded, then SSC functions on G are uniformly continuous, by Corollary 1. Thus, in that case, it is enough to find a continuous function $f : G \to \mathbb{R}$ which is not uniformly continuous. This is an easy exercice, using the Tietze extension theorem.

Now, assume that G is not totally bounded. Then one can find a sequence $(x_n) \subseteq G$ and a neighbourhood of 0, $V \subseteq G$, such that $(x_n + V) \cap (x_m + V) = \emptyset$ if $n \neq m$. Choose a null sequence $(h_i) \subseteq G$ such that $h_i \in V$ and $h_i \neq 0$ for all i. Put $F_n = \{x_n; x_n{+}h_0; \ldots; x_n{+}h_n\}$ $(n \geq 0)$, and $F = \bigcup_n F_n$. The set F is a closed, discrete subset of G, hence there exists a continuous function $f : G \to \mathbb{R}$ such that $f(x_n) = 0$ and $f(x_n + h_i) = 1$ whenever $0 \leq i \leq n$. The function f is not SSC.

Remark.

The statement of Corollary 3 is far from optimal; in particular, the metrizability assumption is too restrictive. For example, the same proof gives the existence of continuous functions on G which are not SSC provided that G is a normal topological space, is not totally bounded, and contains a null sequence which is not eventually constant.

The last result of this section is closely related in spirit to Hàjek's work and to [**DM**]. If Y is a normed space, let us say that a function $f : G \to Y$ is (uniformly) *Cesàro-continuous* on $A \subseteq G$ if every null sequence $(h_i) \subseteq G$ has a subsequence (h'_i) such that

$\big(f(x+h'_i)\big)$ converges in the sense of Cesàro to $f(x)$, uniformly for $x \in A$. It is shown in **[DM]** (though not stated explicitely) that if Y is a Banach space with finite cotype and if $f : c_0 \to Y$ is a C^1-smooth function whose derivative is uniformly continuous on bounded sets (for the norm topology of c_0), then f is Cesàro continuous on bounded sets for the weak topology. Yet, the link between Cesàro-continuity and strong sequential continuity is not made very clear in that paper.

Proposition 3. *If $f : G \to \mathbb{R}$ is Cesàro-continuous on $A \subseteq G$, then f is strongly sequentially continuous on A.*

Proof.

Assume that $f : G \to \mathbb{R}$ is not SSC on $A \subseteq G$. Choose $\varepsilon > 0$, $(h_i) \to 0$ and $(x_n) \subseteq A$ such that $|f(x_n + h_i) - f(x_n)| \geq \varepsilon$ whenever $n > i$.

Putting $A^+ = \{(i,n) \in \mathbb{N}^{(2)} : \ f(x_n + h_i) - f(x_n) \geq \varepsilon\}$, $A^- = \{(i,n) \in \mathbb{N}^{(2)} : \ f(x_n + h_i) - f(x_n) \leq -\varepsilon\}$ and applying Ramsey's theorem, we may assume for instance that $f(x_n + h_i) - f(x_n) \geq \varepsilon$ whenever $n > i$.

This implies that f is not Cesàro-continuous on A: indeed, if $(h'_i) = (h_{p_i})$ is any subsequence of (h_i), then for every $n \geq 1$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\Big(f(x_{p_n} + h'_i) - f(x_{p_n})\Big) \geq \varepsilon.$$

II - Functions on dense subgroups of the real line.

In this section, G is a dense subgroup of $\mathbb{R}$, and we study real-valued strongly sequentially continuous functions on G.

To begin with, we introduce another class of functions.

Definition 2. A function $f : G \to \mathbb{R}$ is said to be *eventually regular* if for every $\varepsilon > 0$ and every null sequence $(h_i) \subseteq G$, there exist an integer i_0 and a bounded set $B \subseteq G$ such that

$$\sup_{x \notin B} |f(x + h_{i_0}) - f(x)| < \varepsilon.$$

In other words, f is eventually regular if and only if

$$\lim_{h \to 0}\left[\limsup_{|x| \to \infty} |f(x+h) - f(x)|\right] = 0\,.$$

The following result is a useful criterion for recognizing a strongly sequentially continuous function.

Lemma 2. *If $f : G \to \mathbb{R}$ is uniformly continuous on bounded sets and eventually regular, then f is strongly sequentially continuous.*

Proof.

Let $f : G \to \mathbb{R}$ be uniformly continuous on bounded sets and eventually regular. In order to prove that f is strongly sequentially continuous, let us fix a null sequence $(h_i) \subseteq G$ and $\varepsilon > 0$.

Since f is eventually regular, there exist i_0 and a bounded set $B \subseteq G$ such that

$$\sup_{x \notin B} |f(x + h_{i_0}) - f(x)| < \varepsilon\,.$$

Since f is uniformly continuous in a neighborhood of B, there exists i_1 such that

$$\sup_{x \in B} |f(x + h_{i_1}) - f(x)| < \varepsilon\,.$$

Therefore, for all $x \in G$ we have

$$\inf \Big\{ |f(x + h_{i_0}) - f(x)|, |f(x + h_{i_1}) - f(x)| \Big\} < \varepsilon.$$

Proposition 4.

(a) *If* $f : \mathbb{R} \to \mathbb{R}$ *is strongly sequentially continuous, then* f *is uniformly continuous.*

(b) *If* G *is countable, then there exists a function* $f : G \to \mathbb{R}$ *which is eventually regular and uniformly continuous on bounded sets (hence strongly sequentially continuous on* G*), but which is not uniformly continuous on* G*.*

(c) *If* G *is countable, then there exists a function* $f : G \to \mathbb{R}$ *which is strongly sequentially continuous, but not eventually regular.*

Proposition 4 follows from more general results to be established in Sections III and IV. The results stated here are actually not easier to prove than the more general ones.

To motivate the next definition, let us give, however, some indications about the proofs of (b) and (c). Let (μ_n) be a sequence of positive numbers. Define $f : G \to \mathbb{R}$ by $f(x) = \sin(\pi x) \sin(\mu_n(x - 3n))$ if $x \in [3n - 1, 3n + 1]$ for some $n \geq 0$, and $f(x) = 0$ otherwise. If each μ_n is a term of some fixed sequence (λ_j) such that $\lambda_j \to +\infty$ and $\operatorname{dist}(\lambda_j x, \mathbb{Z}) \to 0$ pointwise on G, then f turns out to be strongly sequentially continuous. Moreover, if $\mu_n = \lambda_n$ for all n, then f is eventually regular; but if each λ_j is repeated infinitely many times in the sequence (μ_n), then f is not eventually regular. This will be proved in Section IV. In any case, if the sequence (μ_n) is unbounded, then f is not uniformly continuous; in fact, it is “oscillating at infinity”, in the following sense.

Definition 3. We say that a function $f : G \to \mathbb{R}$ is *oscillating at infinity* if there exists $a > 0$ such that, for all positive numbers K, ε and for every integer N, there exist $x_1, \ldots, x_{2N} \in G$ such that $K < x_1 < x_2 < \ldots < x_{2N}$ and

(i) *for every* $i \in \{1, \ldots, 2N - 1\}$, $x_{i+1} - x_i < \varepsilon$;

(ii) *for every* $k, l \in \{1, \ldots, N\}$, $f(x_{2k}) - f(x_{2l-1}) \geq a$.

Clearly, such a function cannot be uniformly continuous. The following result is a kind of converse, and shows that the oscillating behaviour of the function f above is by no means accidental.

Proposition 5. *Let $f : G \to \mathbb{R}$ be a strongly sequentially continuous function. Then either f is uniformly continuous, or f is oscillating at infinity.*

Corollary. *A monotonic function $f : G \to \mathbb{R}$ is strongly sequentially continuous if and only if it is uniformly continuous.*

Proof of Proposition 5.

Assume that f is not uniformly continuous, and fix $K, \varepsilon > 0$ and a positive integer N.

By assumption, there exist $a > 0$ and two sequences $(u_p), (v_p) \subseteq G$ such that $u_p < v_p$, $v_p - u_p \to 0$ and $|f(v_p) - f(u_p)| > 3a$. Since f is uniformly continuous on bounded sets, we may assume that $u_p > K$ for all p; and by extracting subsequences, we may also assume for instance that $f(v_p) - f(u_p) > 3a$, $p \geq 0$.

Let $(h_i) \subseteq G$ be a null sequence such that $0 < h_i < \varepsilon$ for all i.

Since f is strongly sequentially continuous, we can find an integer m such that, for all p, $\inf_{i \leq m} \big|f(u_p + h_i) - f(u_p)\big| < a/N$. Therefore, $\big|f(u_p + h_{i_0}) - f(u_p)\big| < a/N$. In particular, there exists p_0 such that $u_{p_0} < v_{p_0} < u_{p_0} + h_{i_0}$ and $\big|f(u_{p_0} + h_{i_0}) - f(u_{p_0})\big| < a/N$. Repeating this argument, we can produce two increasing sequences of integers (p_n) and (i_n) such that $u_{p_n} < v_{p_n} < u_{p_n} + h_{i_n}$ and $\big|f(u_{p_n} + h_{i_n}) - f(u_{p_n})\big| < a/N$ for all n. Thus, we may assume that there exists a null sequence $(k_n) \subseteq G$ such that $u_n < v_n < u_n + k_n$ and $\big|f(u_n + k_n) - f(u_n)\big| < a/N$, $n \geq 0$. Similarly, we may assume that there exists another null sequence $(\ell_n) \subseteq G$ such that $v_n < u_n + k_n < v_n + \ell_n$ and $\big|f(v_n + \ell_n) - f(v_n)\big| < a/N$ for all n. At this point, each quadruple $(u_n, v_n, u_n + k_n, v_n + l_n)$ satisfies conditions (i),(ii) of Definition 3, with $a(3 - 2/N)$ instead of a. Proposition 5 follows by iterating this procedure N times.

The last result of this section asserts that SSC functions have a "sublinear" behaviour at infinity.

Proposition 6. *If $f : G \to \mathbb{R}$ is strongly sequentially continuous, then there exists a constant C such that for all $x \in G$,*

$$|f(x)| \leq C(1 + |x|) \,.$$

Proof.

Since f is continuous, there exists $\delta > 0$ and $M < +\infty$ such that $|f(y)| \leq M$ if $|y| \leq \delta$. Choose $h_1, h_2, \ldots, h_p \in G$ in such a way that $0 < h_i < \delta$ and for all $x \in \mathbb{R}$,

$$\inf \big\{ |f(x - h_i) - f(x)|,\ 1 \leq i \leq p \big\} \leq 1.$$

Let $a = \min\{h_1, \ldots, h_p\} > 0$. Given any positive $x \in G$, there exists a sequence (i_n), $i_n \in \{1, \ldots, p\}$, such that, if we put $x_n = x - \big(h_{i_1} + h_{i_2} + \ldots + h_{i_n}\big)$, then $|f(x_{n+1}) - f(x_n)| \leq 1$

for all $n \geq 1$. The sequence (x_n) decreases to $-\infty$, and $x_n - x_{n+1} < \delta$ for all n. Therefore, one can find an integer n_0 such that $x_{n_0} \in [0, \delta]$. Then $|f(x_{n_0})| \leq M$ and $x \geq n_0 a$, so $|f(x)| \leq |f(x_{n_0})| + n_0 \leq M + \dfrac{x}{a}$. The proof for $x < 0$ is similar.

III - Nice groups.

The purpose of this section is to show that if the group G satisfies some reasonable "completeness" assumptions, then strong sequential continuity on G is equivalent to uniform sequential continuity.

Let us say that G is a *nice group* if for any metric space Y, all strongly sequentially continuous functions $f : G \to Y$ are uniformly sequentially continuous. For example, metrizable totally bounded groups are nice, by Corollary 1 of Proposition 2. Of course, sequentially compact groups are nice as well.

All the results proved in this section will depend on the following lemma. Recall that a subset A of some topogical space X is said to have the Baire property in X if one can write $A = U \triangle M$, where U is open and M is meager in X; and that a series $\sum k_n$ in G is said to be *unconditionally convergent* if $\sum \alpha_n k_n$ converges in G for any choice of the sequence $\alpha = (\alpha_n) \in \{0,1\}^{\mathbb{N}}$.

Lemma 3. *Let (A_n), (B_n) be two sequences of subsets of G, and let $(k_n) \subseteq G$ be a null sequence. Assume that $A_n \cup (k_n + B_n) = G$ for each $n \geq 0$, and put $Z = \bigcap_{n\geq 0} \Big(\bigcup_{p \geq n} (A_p \cup B_p)\Big)$.*

(a) If G is a Baire space and if all the sets A_n, B_n have the Baire property in G, then Z is comeager in G.

(b) If G is locally compact and all the sets A_n, B_n are Haar-measurable, then $G \setminus Z$ has Haar measure 0.

(c) If the series $\sum k_n$ is unconditionally convergent and if all the sets A_n, B_n are "sequentially open" (which means that their complements are sequentially closed), then Z contains a point of the form $\sum_0^\infty \alpha_n k_n$, where $\alpha_n \in \{0; 1\}$.

Proof.

(a) If all the sets A_n, B_n are open (which will precisely happen in the proof of Theorem 1'), the proof is immediate: the sets $Z_n = \bigcup_{p\geq n} (A_p \cup B_p)$ are open in G, and they are also dense because $k_n \to 0$; hence $Z = \cap_n Z_n$ is comeager

In the general case, we only need to prove that all the Z_n's are comeager in G, and since the Z_n's have the Baire property, it is enough to check that if W is a nonempty open subset of G, then each $Z_n \cap W$ is nonmeager in G. Let us fix n and W.

Pick $a \in W$ together with a neighbourhood of 0, $U \subseteq G$, such that $U + a + U \subseteq W$, and choose $p \geq n$ such that $k_p \in U$. If $A_p \cap W$ and $B_p \cap W$ were both meager, then so would be

$A_p\cap(k_p+a+U)$ (because $k_p+a+U\subseteq W$) and $(k_p+B_p)\cap(k_p+a+U)$ (by translation); this is impossible because $k_p+a+U\subseteq V\subseteq A_p\cup(k_p+B_p)$ and, since G is a Baire space, k_p+a+U is nonmeager. It follows that $(A_p\cup B_p)\cap W$ is nonmeager, and the proof is complete.

(b) Let V be any open subset of G. If L is a compact subset of V, then $m(L)\le m(A_p\cap L)+m((k_p+B_p)\cap L)=m(A_p\cap L)+m(B_p\cap(-k_p+L))$ for all p, where m is the Haar measure on G. If p is large enough, then $-k_p+L\subseteq V$, and we get $m(L)\le m(A_p\cap V)+m(B_p\cap V)$. Since L is an arbitrary compact subset of I, it follows that $m(V\cap\left(\bigcup_{p\ge n}(A_p\cup B_p)\right)\ge m(I)/2$ for all n. Consequently, $m(Z\cap V)\ge m(V)/2$ for each open set $V\subseteq G$.

Now, assume that $m(G\setminus Z)>0$. Let K be a compact subset of $G\setminus Z$ such that $m(K)>0$, and choose an open set V such that $K\subseteq V$ and $m(V\setminus K)\le m(K)/3$. Then $m(V\cap Z)\le m(V\setminus K)\le m(V)/3$, which contradicts the preceding remark.

(c) By the unconditional convergence of $\sum k_n$, the series $\Sigma(\alpha)=\sum\alpha_n k_n$ is uniformly convergent for $\alpha\in\{0,1\}^{\mathbb{N}}$.

Let Δ be the compact abelian group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, and let m be the normalized Haar measure on Δ. Identifying Δ with $\{0,1\}^{\mathbb{N}}$, let $\varphi:\Delta\to G$ be the map defined by $\varphi(\alpha)=\sum_{n\ge 0}\alpha_n k_n$.

The map φ is continuous, and since the sets A_n, B_n are sequentially open, all the sets $A'_n=\varphi^{-1}(A_n)$, $B'_n=\varphi^{-1}(B_n)$ are open in Δ.

We have to show that $\bigcap_n\Big(\bigcup_{p\ge n}A'_p\Big)\neq\emptyset$, and this will be done if we can find an infinite set $\Lambda\subseteq\mathbb{N}$ such that either $\forall n\in\Lambda,\ m(A'_n)\ge c$ or $\forall n\in\Lambda,\ m(B'_n)\ge c$, for some positive constant c.

For each $n\in\mathbb{N}$, put $W_0^{(n)}=\{\alpha\in\Delta;\ \alpha_n=0\}$ and $W_1^{(n)}=\{\alpha\in\Delta;\ \alpha_n=1\}$.

Since $\Delta=\varphi^{-1}(A_n)\cup\varphi^{-1}(B_n+k_n)$ and $m(W_1^{(n)})=1/2$ for all n, there exists an infinite set $\Lambda\subseteq\mathbb{N}$ such that either $m(A'_n\cap W_1^{(n)})\ge 1/4$, $n\in\Lambda$ or $m(\varphi^{-1}(B_n+k_n)\cap W_1^{(n)})\ge 1/4$, $n\in\Lambda$. In the first case we are done, so assume $m(\varphi^{-1}(B_n+k_n)\cap W_1^{(n)})\ge 1/4$ for all $n\in\Lambda$.

Observe that for any $n\in\mathbb{N}$,

$$\varphi^{-1}(B_n+k_n)=\Big\{\alpha:\ \sum_{i\neq n}\alpha_i k_i+(\alpha-\delta^{(n)})_n k_n\in B_n\Big\}$$

where $\delta^{(n)}\in\Delta$ is defined by $\delta_n^{(n)}=1$ and $\delta_i^{(n)}=0$ if $i\neq n$. It follows that

$$\varphi^{-1}(B_n+k_n)\cap W_1^{(n)}=W_1^{(n)}\cap\{\alpha:\ \varphi(\alpha-\delta^{(n)})\in B_n\}=\Big(W_0^{(n)}\cap B'_n\Big)+\delta^{(n)}.$$

Consequently, $m(B'_n\cap W_0^{(n)})=m\big(\varphi^{-1}(B_n+k_n)\cap W_1^{(n)}\big)\ge 1/4$ for all $n\in\Lambda$. This concludes the proof.

Theorem 1. *If G is completely metrizable, then G is a nice group.*

Corollary. *Every Banach space is a nice group. In particular,* $\mathbb{R}$ *is a nice group.*

We shall in fact prove two extensions of Theorem 1. The first one relies on a Baire category argument, the second one is based on a measure-theoretic argument.

Theorem 1′. *If* G *is metrizable and is a Baire space, then* G *is a nice group.*

Proof.

Assume that G is metrizable and is a Baire space.

Since SSC functions on G are continuous (by metrizability of G), we have to show that if $f : G \to (Y, \delta)$ is a continuous function which is not uniformly continuous, then f is not SSC; let us fix such an f.

By assumption there exist $\varepsilon > 0$ and two sequences $(u_n), (v_n) \subseteq G$ such that $(u_n - v_n) \to 0$ and

$$\delta\big(f(u_n), f(v_n)\big) > 2\varepsilon, \qquad n \geq 0.$$

We want to construct two sequences (x_n) and (h_i) such that $h_i \to 0$ as $i \to \infty$ and

$$\delta\big(f(x_n + h_i), f(x_n)\big) > \varepsilon \quad \text{whenever} \quad i \leq n\,.$$

For $n \geq 0$, consider the open sets:

$$A_n = \big\{h \in G : \ \delta\big(f(u_n + h), f(u_n)\big) > \varepsilon\big\}\,,$$

$$B_n = \big\{h \in G : \ \delta\big(f(v_n + h), f(v_n)\big) > \varepsilon\big\}\,.$$

For any $h \in G$ and each $n \geq 0$, it follows from the triangle inequality that either $\delta\big(f(u_n), f(u_n + h)\big) > \varepsilon$ or $\delta\big(f[v_n + (-v_n + u_n + h)], f(v_n)\big) > \varepsilon$. Thus, if we put $k_n = -u_n + v_n$, then

$$A_n \cup \big(k_n + B_n\big) \;=\; G \;\text{ for all }\; n \geq 0.$$

According to Lemma 3 (part (a)), for every neighbourhood V of 0 in G and for any infinite set $\Lambda \subseteq \mathbb{N}$, there exists an infinite set $\Lambda' \subseteq \Lambda$ such that either $V \cap \bigcap_{n\in\Lambda'} A_n \neq \emptyset$ or $V \cap \bigcap_{n\in\Lambda'} B_n \neq \emptyset$. Since G is metrizable, we can construct by induction a null sequence $(h_i) \subseteq G$ and a decreasing sequence (Λ_i) of infinite subsets of $\mathbb{N}$ such that, for each $i \geq 0$, either $h_i \in \bigcap_{n\in\Lambda_i} A_n$ or $h_i \in \bigcap_{n\in\Lambda_i} B_n$. Extracting if necessary a subsequence of (Λ_i), we may assume that either for all i and for all $n \in \Lambda_i$,

$$\delta\big(f(u_n + h_i), f(u_n)\big) > \varepsilon\,,$$

or for all i and for all $n \in \Lambda_i$,

$$\delta\big(f(v_n + h_i), f(v_n)\big) > \varepsilon\,.$$

Put $x_n = u_{\min \Lambda_n}$ in the first case, and $x_n = v_{\min \Lambda_n}$ in the second case. The sequences (x_n), (h_i) have the required property.

Remark.

The proof of Theorem $1'$ gives in fact a more general result (under the assumptions that G is metrizable and is a Baire space): *if $f : G \to Y$ is a continuous function which is SSC on some set $A \subseteq G$, then f is uniformly continuous on A.*

Before giving a second extension of Theorem 1, we need another definition.

Definition 4. We shall say that G has the *unconditional property (K)* if each null sequence $(k_n) \subseteq G$ has a subsequence (k'_n) such that $\sum k'_n$ is unconditionally convergent.

As may be guessed, this terminology refers to another one: by dropping the unconditionality condition, one gets the so-called *property (K)*, which seems to have been introduced by C. Kliś ([**Kli**]). Actually, as observed in [**BKL**], property *(K)* had already been isolated by S. Mazur and W. Orlicz ([**MzO**] p. 169), who pointed out that it could be used as a substitute for completeness (see [**S**] for precise results in that direction). Property *(K)* has also been studied (among others) by J. Burzyk, I. Labuda and Z. Lipecki (see e.g. [**BKL**], [**Kli**] or [**LaLi**]).

Clearly, any completely metrizable group has the unconditional property *(K)*: if d is a translation-invariant metric compatible with the topology of G, then each null sequence $(k_n) \subseteq G$ has a subsequence (k'_n) such that $d(0, k'_n) < 2^{-n}$, $n \geq 0$, and this entails the unconditional convergence of $\sum k'_n$ because (G, d) is necessarily complete (see [**Kle**]).

We should also add that a *metrizable* group with property *(K)* is necessarily a Baire space; and if, in addition, the group has the Baire property in its completion, then it is actually complete ([**BKL**]). For example, a Borel (or even analytic) subgroup of a Banach space has property *(K)* if and only if it is closed, a result obtained independently of [**BKL**] by A.R.D. Mathias (see [**T**] p. 31). On the other hand (if the Axiom of Choice is allowed), there exist normed spaces with property *(K)* which are not complete (see [**Kli**] and [**LaLi**]). Banach space with the Schur property, then $G = (X, w)$ has the unconditional property *(K)*, but it is not a Baire space.

We can now state our second extension of Theorem 1. In view of the preceding remarks, it is of interest in the nonmetrizable case only.

Theorem $1''$. *If G has the unconditional property (K), then G is a nice group.*

Proof.

Assume that G has the unconditional property *(K)*. Keeping the same notation as in the proof of Theorem $1'$, let $f : G \to (Y, \delta)$ be a function which is not uniformly sequentially continuous, with "witnesses" $(u_n), (v_n)$ and 2ε. We have to show that f is not SSC, and we may assume that it is sequentially continuous (otherwise, there is nothing to prove).

As above, put $A_n = \{z \in G : \delta(f(u_n + z), f(u_n)) > \varepsilon\}$, $B_n = \{z \in G : \delta(f(v_n + z), f(v_n)) > \varepsilon\}$, and $k_n = -u_n + v_n$. Then $k_n \to 0$ and $G = A_n \cup (B_n - k_n)$ for all n.

By the unconditional property *(K)*, we may assume that the series $\sum k_n$ is unconditionally convergent. Put $\Delta = \{0;1\}^{\mathbb{N}}$, and, for $i \geq 0$, define $\varphi_i : \Delta \to G$ by $\varphi_i(\alpha) = \sum_{n \geq i} \alpha_n k_n$. By the unconditional convergence of $\sum k_n$, the sequence (φ_i) converges to 0 uniformly on

Δ. Keeping in mind the proof of Theorem $1'$, it is therefore enough to show that for each $i \in \mathbb{N}$ and each infinite set $\Lambda \subseteq \mathbb{N}$, there exists an infinite subset $\Lambda' \subseteq \Lambda$ such that either $\bigcap_{n\in\Lambda'} \varphi_i^{-1}(A_n) \neq \emptyset$ or $\bigcap_{n\in\Lambda'} \varphi_i^{-1}(B_n) \neq \emptyset$. from Lemma 3 (part (c)).

At this point, some additional remarks about *(K)*-properties seem to be necessary.

(1) For any infinite set $I \subseteq \mathbb{N}$, let us denote by $[I]^\omega$ the family of all infinite subsets of I. A subset $\mathcal{A}$ of $[\mathbb{N}]^\omega$ is said to be a *Ramsey subset* if there exists an infinite set $I \subseteq \mathbb{N}$ such that either $[I]^\omega \subseteq \mathcal{A}$ or $[I]^\omega \cap \mathcal{A} = \emptyset$. Using the axiom of choice, it is not difficult to exhibit non-Ramsey subsets of $[\mathbb{N}]^\omega$ (see **[Bo]** p. 159). On the other hand, it is known (see **[CS]**) that if the theory (*ZFC + there exist inaccessible cardinals*) is consistent, then the theory (*ZF + Dependent Choice + every subset of* $[\mathbb{N}]^\omega$ *is a Ramsey subset*) is also consistent.

Now, let (h_i) be a null sequence in the topological abelian group G. Put $\mathcal{A} = \{I \in [\mathbb{N}]^\omega :$ *the series* $\sum_{i\in I} h_i$ *converges in* $G\}$. If I is an infinite subset of $\mathbb{N}$ such that $[I]^\omega \subseteq \mathcal{A}$, then the series $\sum_{i\in I} h_i$ is unconditionally convergent; and if $[I]^\omega \cap \mathcal{A} = \emptyset$, then the sequence $(h_i)_{i\in I}$ shows that G does not have property *(K)*. Therefore, in view of the above metamathematical result, property *(K)* and unconditional property *(K)* may be equivalent. Furthermore, the proof of the aforementioned result of Mathias given in **[T]** shows that if all subsets of $[\mathbb{N}]^\omega$ were Ramsey subsets, then all metrizable abelian groups with property *(K)* would be complete.

(2) It has already been observed that if X is a Banach space with the Schur property, then $G = (X, w)$ has the unconditional property *(K)*. It turns out that the converse implication is true. This follows easily from the classical

Orlicz-Pettis Theorem. *If X is a Banach space, then a series $\sum x_n$ in X is unconditionally convergent for the weak topology if and only if it is unconditionally convergent for the norm topology.*

Using Lemma 3, we can give a very short proof of the Orlicz-Pettis theorem.

By standard arguments, it is enough to prove that if X is separable and if $\sum x_n$ is unconditionally convergent in X for the weak topology, then $||x_n|| \to 0$. So, assume that $\sum x_n$ is unconditionally convergent for the weak topology, and that $||x_n|| > 3$ for all n.

First, we observe that one can find a subsequence (k_n) of (x_n) and a w^*-null sequence $(k_n^*) \subseteq X^*$ such that $\langle k_n^*, k_n \rangle > 1$ for all n. To see this, fix a norm-dense sequence $(d_j) \subseteq X$. Since (x_n) is weakly null, one can construct by induction a subsequence (k_n) of (x_n) such that $\text{dist}\big(k_n, \text{span}\,\{d_j : j < n\}\big) > 1$ for all n. By the Hahn-Banach theorem, one can find a normalized sequence $(k_n^*) \subseteq X^*$ such that $\langle k_n^*, k_n \rangle > 1$, $n \geq 0$, and $\langle k_n^*, d_j \rangle = 0$ whenever $j < n$. Since (d_j) is norm-dense in X, the sequence (k_n^*) is w^*-null.

Now, put $A_n = \{x \in X;\ \langle k_n^*, x \rangle > 1/2\}$, and $A'_n = \{x \in X;\ \langle k_n^*, k_n - x \rangle > 1/2\}$. The sets A_n are weakly open in X, and $X = A_n \cup A'_n = A_n \cup (-A_n + k_n)$ for all n. Since the series $\sum k_n$ is unconditionally convergent for the weak topology, it follows from Lemma 3 (part (c)) that one can find a point $a \in X$ such that a or $-a$ belongs to infinitely many $A'_n s$. In other words, $\langle k_n^*, a \rangle > 1/2$ or $\langle k_n^*, a \rangle < -1/2$ for infinitely many n's, which is impossible since (k_n^*) is w^*-null.

To conclude this section, we show that nice groups can be quite ugly: nonmeasurable subgroups of $\mathbb{R}$ and subgroups of $\mathbb{R}$ without the Baire property are nice!

Proposition 7 *Let G be a dense subgroup of $\mathbb{R}$. If G is nonmeager or not Lebesgue-null, then G is nice.*

Proof.

If G is nonmeager, then it is nonmeager in any nonempty open subset of $\mathbb{R}$ (by translation). It follows easily that G is a Baire space, hence we may apply Theorem $1'$.

Now, assume that G is not Lebesgue-null. Since SSC functions on G extend continuously to $\mathbb{R}$, we have to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous but not uniformly continuous, then $f_{|G}$ is not SSC on G. By the proof of Theorem $1'$, it is enough to check that if (k_n) is a null sequence of real numbers and if (A_n), (B_n) are two sequences of open subsets of $\mathbb{R}$ such that $\mathbb{R} = A_n \cup (k_n + B_n)$ for all n, then, for each nonempty open set $V \subseteq \mathbb{R}$, there exists an infinite set $\Lambda \subseteq \mathbb{N}$ such that either $V \cap G \cap \bigcap_{n \in \Lambda} A_n \neq \emptyset$ or $V \cap G \cap \bigcap_{n \in \Lambda} B_n \neq \emptyset$. This follows from Lemma 3 (part (b)).

Remark. We are unable to decide whether every compact group is nice.

IV - Nasty groups

In this last section, we exhibit several examples of groups where strong sequential continuity does not imply uniform sequential continuity.

Below, we shall say that G is *nasty* for some metric space Y if there exists a function $f \in SSC(G, Y)$ which is not uniformly sequentially continuous, and that G is a *nasty group* if it is nasty for $Y = \mathbb{R}$.

To produce examples of nasty groups, we need some simple way of constructing non-trivial SSC functions. To this end, we introduce again some terminology.

Definition 5. Let (Y, δ) be a metric space, and let (f_k) be a sequence of functions from G into Y. We say that the sequence (f_k) is *equi-SSC* if, for any null sequence $(h_i) \subseteq G$,

$$\sup_k \Big[\inf_{i \leq n} \delta\big(f_k(x + h_i), f_k(x)\big) \Big] \to 0 \text{ as } n \to \infty, \text{ uniformly on } G.$$

Basic example. *If (g_k) is a sequence of SSC functions from G into $\mathbb{R}$ such that, for each $h \in G$ we have $g_k(x + h) - g_k(x) \to 0$ as $k \to \infty$, uniformly on G, then the sequence (g_k) is equi-SSC. This happens, for example, if (g_k) is a sequence of continuous homomorphisms from G into $\mathbb{R}$ which converges pointwise to 0.*

Proof.

Let (g_k) be as above, and fix a null sequence $(h_i) \subseteq G$ and a positive number ε.

Choose a positive integer K_0 such that $\sup_{x\in G} |g_k(x+h_0)-g_k(x)| < \varepsilon$ if $k > K_0$. Since $g_0,\dots,g_{K_0}$ are SSC, there exists an integer N such that for each $k \le K_0$ and all $x \in G$, we have $\inf_{i\le N} |g_k(x+h_i)-g_k(x)| < \varepsilon$. Then $\inf_{i\le N} |g_k(x+h_i)-g_k(x)| < \varepsilon$ for each $x \in G$ and for all k.

Lemma 4. *Let $(Y, |\ |)$ be a normed space, let $(g_k) \subseteq Y^G$ be a uniformly bounded, equi-SSC sequence of functions, and let $b : G \to \mathbb{R}$ be bounded and uniformly sequentially continuous. Let also $(u_k), (v_k) \subseteq G$, and define $f_k : G \to Y$ by $f_k(x) = b(u_k+x)g_k(v_k+x)$. Then (f_k) is equi-SSC.*

Proof.

One can write $|f_k(x+h)-f_k(x)| \le C\big(|b(u_k+x+h)-b(u_k+x)|+|g_k(v_k+x+h)-g_k(v_k+x)|\big)$, for some constant C.

Definition 6. Let (F_k) be a sequence of subsets of G. We say that (F_k) is *separated* if there exists a neighbourhood of 0, $V \subseteq G$, such that $(F_k+V)\cap F_l = \emptyset$ if $k \ne l$. More generally, we say that (F_k) is *quasi-separated* if there exists a set $V \subseteq G$ satisfying the following properties:

(i) V "absorbs" every null sequence (which means that for every null sequence (h_i), there exists an integer i_0 such that $h_i \in V$ for all $i \ge i_0$); in particular, $0 \in V$.

(ii) $(F_k+V)\cap F_l = \emptyset$ if $k \ne l$.

Clearly, the two notions coincide if G is metrizable. On the other hand, if $G = (l_1, w)$ (the space l_1 with the weak topology) and if (e_k) is the canonical basis of l_1, then the sequence $\big(B(e_k, 1/3)\big)$ is quasi-separated because it is separated for the norm topology and l_1 has the Schur property, but it is not separated.

Of course, if (F_k) is quasi-separated, then the F_k's are pairwise disjoint.

Lemma 5. *Let $(Y, |\ |)$ be a normed space, and let (f_k) be an equi-SSC sequence of functions from G into Y. Assume that the f_k's have quasi-separated supports, and put $f = \sum_{k\ge 0} f_k$. Then f is strongly sequentially continuous.*

Proof.

Put $F_k = \operatorname{supp} f_k$, $k \ge 0$, and choose V according to Definition 6. Clearly, we may assume that V is symmetric. For any $x \in G$, there is at most one integer k such that $x \in F_k + V$. Let us denote this integer by $k(x)$ if it exists, and put $k(x) = 0$ if there is no such k.

For each null sequence $(h_i) \subseteq G$, there is an integer i_0 such that $h_i \in V$ if $i \ge i_0$. Then, for any $x \in G$, $f_k(x) = 0 = f_k(x+h_i)$ if $i \ge i_0$ and $k \ne k(x)$. Thus, for $i \ge i_0$, one can write $|f(x+h_i)-f(x)| = |f_{k(x)}(x+h_i)-f_{k(x)}(x)|$ for all $x \in G$. The lemma follows immediately.

Proposition 8. *Assume that G is not totally bounded. For any normed space $(Y, |\ |)$, the following assertions are equivalent.*

(1) *G is nasty for Y.*

(2) *There exist an equi-SSC sequence* $(g_k) \subseteq Y^G$, *a null sequence* $(h_k) \subseteq G$ *and a positive number* ε *such that* $g_k(0) = 0$ *and* $|g_k(h_k)| \geq \varepsilon$ *for all* k.

Proof.

Assume that (2) holds for some sequence (g_k). Replacing g_k by $g_k/1+|g_k|$, we may assume that the sequence (g_k) is uniformly bounded.

Since G is not totally bounded, there exist a sequence $(x_k) \subseteq G$ and a neighbourhood of 0, $U \subseteq G$, such that $-x_l + x_k \notin U$ if $k \neq l$. Choose a symmetric neighbourhood of 0, $V \subseteq G$, such that $V + V + V \subseteq U$, and let d be a uniformly continuous pseudometric on G such that $\{x \in G : \ d(0,x) \leq 1\} \subseteq V$. Finally, let $b_0 : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function such that $b_0(0) = 1$ and $\operatorname{supp} b_0 \subseteq [0,1]$. Define $b : G \to \mathbb{R}$ by $b(x) = b_0\big(d(0,x)\big)$. The function b is bounded, uniformly continuous, and $b(0) = 1$. Moreover, $\operatorname{supp} b \subseteq V$, so that $(x_k + \operatorname{supp} b + V) \cap (x_l + \operatorname{supp} b) = \emptyset$ if $k \neq l$.

For each $k \geq 0$, define $f_k : G \to Y$ by

$$f_k(x) = b(-x_k + x)\, g_k(-x_k + x)\,.$$

By Lemma 4, the sequence (f_k) is equi-SSC, and since $\operatorname{supp} f_k \subseteq x_k + \operatorname{supp} b$ for all k, the f_k's have separated supports. Hence, by Lemma 5, the function $f = \sum f_k$ is SSC. On the other hand, this function is not uniformly sequentially continuous because $f(x_k + h_k) - f(x_k) = b(h_k)g_k(h_k)$ if k is large enough, whence $f(x_k + h_k) - f(x_k)$ does not tend to 0. Thus, G is nasty for Y.

Conversely, assume that G is nasty for Y, and let $f : G \to Y$ be an SSC function which is not uniformly sequentially continuous. Choose $(x_k) \subseteq G$, $(h_k) \to 0$ and $\varepsilon > 0$ such that $|f(x_k + h_k) - f(x_k)| \geq \varepsilon$ for all k. The sequence $(g_k) \subseteq Y^G$ defined by $g_k(x) = f(x_k + x) - f(x_k)$ is equi-SSC, and $(g_k), (h_k)$ satisfy (2).

Corollary. *If* G *is metrizable, or if* G *is not totally bounded, then* G *is a nasty group if and only if it is nasty for some metric space* Y.

Proof.

We know that metrizable, totally bounded groups are nice (Corollary 2 of Proposition 2). Hence we may assume that G is not totally bounded.

If G is nasty for some metric space, it is also nasty for some normed space, by Remark 4 following the definition of SSC functions. For this normed space $(Y, |\ \ |)$, property (2) is satisfied. Now, the sequence $(|g_k|)$ shows that property (2) is also true for $\mathbb{R}$; hence G is a nasty group.

Remark. We do not know if Proposition 8 is still true when G is totally bounded (and of course, nonmetrizable).

Recall that if H is a locally compact abelian group, a compact set $L \subseteq H$ is said to be a *Dirichlet set* if there exists a sequence of characters (γ_k) tending to ∞ such that $\gamma_k(x) \to 1$ uniformly on L. The classical Dirichlet theorem asserts that finite subsets of $\mathbb{T}$ are Dirichlet sets, and the same is true in any second-countable nondiscrete LCA group.

Moreover, it is well known that there exist uncountable Dirichlet sets; actually, in the sense of Baire category, "most" compact subsets of any second-countable nondiscrete LCA group are Dirichlet (see [**KeLo**]).

Definition 7. If G is a subgroup of $\mathbb{R}$, we say that G is *Dirichlet-like* if there exists a sequence of positive numbers (λ_k) tending to infinity such that $e^{i\lambda_k x} \to 1$ for all $x \in G$.

For example, the group generated by an increasing sequence of Dirichlet sets is Dirichlet-like, by a standard diagonal argument. In particular, any countable subgroup of $\mathbb{R}$ is Dirichlet-like, and $\mathbb{R}$ has many uncoutable Dirichlet-like subgroups.

If $G \subseteq \mathbb{R}$ is Dirichlet-like with "witness" (λ_k), then, for each fixed $h \in G$, $\sin(\lambda_k(x+h)) - \sin(\lambda_k x) \to 0$ uniformly on G: this follows from the inequality $|\sin u - \sin v| \leq |1 - e^{i(v-u)}|$. Thus (according to the basic example), the sequence $(g_k) \subseteq \mathbb{R}^G$ defined by $g_k(x) = \sin(\lambda_k x)$ is equi-SSC. Moreover, if G is dense in $\mathbb{R}$, then one can find a null sequence $(h_k) \subseteq G$ such that $\sin(\lambda_k h_k) \to 1$ as $k \to \infty$. By Proposition 8, we obtain

Proposition 9. *Every Dirichlet-like, dense subgroup of* $\mathbb{R}$ *is a nasty group.*

Actually, as announced in Section II (Proposition 4), a bit more can be said. To show this, we first need another lemma.

Lemma 6. *Let G be a subgroup of $\mathbb{R}$ and let $(g_k) \subseteq \mathbb{R}^G$ be a uniformly bounded sequence of functions such that, for each fixed $h \in G$, $g_k(x+h) - g_k(x) \to 0$ uniformly on G (as $k \to \infty$). Let also $b : \mathbb{R} \to \mathbb{R}$ be a continuous function with support in $[-1,1]$. Put $f_k(x) = b(x-3k)g_k(x)$, $k \geq 0$, $x \in G$, and let $f = \sum_{k\geq 0} f_k$. Then f is eventually regular.*

Proof.

The proof is very similar to that of Lemma 5, but we give the details anyway.

If $x \in \bigcup_{k\in\mathbb{N}} [3k-4/3, 3k+4/3]$, let us denote by $k(x)$ the unique integer $k \geq 0$ such that $x \in [3k-4/3, 3k+4/3]$; otherwise, let $k(x)$ be the integral part of $|x|$. Clearly, $k(x) \to +\infty$ as $|x| \to +\infty$. Consequently, one has $\lim_{|x|\to\infty} |g_{k(x)}(x+h) - g_{k(x)}(x)| = 0$ for each fixed $h \in G$.

Now, if $|h| < 1/3$, then $f(x+h) - f(x) = f_{k(x)}(x+h) - f_{k(x)}(x)$ for all $x \in G$. Therefore, one can write

$$|f(x+h) - f(x)| \leq C\left(\left|g_{k(x)}(x+h) - g_{k(x)}(x)\right| + \left|b(x-3k(x)+h) - b(x-3k(x))\right| \right)$$

for some absolute constant C.

Hence,

$$\limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq C\,||b - b_h||_\infty \quad \text{for all} \quad h \in G\cap]-1/3, 1/3[.$$

This completes the proof.

Theorem 2. *Assume that G is a Dirichlet-like dense subgroup of* $\mathbb{R}$.

(a) *There exists a function $f : G \to \mathbb{R}$ which is uniformly continuous on bounded sets and eventually regular, but not uniformly continuous*

(b) *There exists a function $f : G \to \mathbb{R}$ which is SSC but not eventually regular.*

Proof.

(a) Let (λ_k) be a sequence of positive numbers "showing" that G is Dirichlet-like. Then, for each $h \in G$,

$$\sin(\lambda_k(x+h-3k)) - \sin(\lambda_k(x-3k)) \to 0 \text{ as } k \to \infty, \text{ uniformly on } G.$$

Now, let $b : \mathbb{R} \to \mathbb{R}$ be any bounded, continuous function such that $b(0) = 1$ and $\operatorname{supp} b \subseteq [-1, 1]$. For each $k \in \mathbb{N}$, define $f_k : G \to \mathbb{R}$ by

$$f_k(x) = b(x-3k)\,\sin(\lambda_k(x-3k)).$$

By Lemma 6, the function $f = \sum f_k$ is eventually regular. Moreover, f is uniformly continuous on bounded sets (the same formula defines a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$), and it is not uniformly continuous because $\lambda_k \to \infty$.

(b) We keep the notation of (a).

Let (μ_k) be a sequence of positive numbers such that each μ_k belongs to the set $\{\lambda_i,\ i \geq 0\}$ and each λ_i is repeated infinitely many times in the sequence (μ_k). As before, put $f_k(x) = b(x-3k)\sin(\mu_k(x-3k))$, and $f = \sum f_k$. Put also $g_k(x) = \sin(\lambda_k(x-3k))$ and $\tilde{f}_k(x) = \sin(\mu_k(x-3k))$, $x \in G$.

According to the basic example, the sequence (g_k) is equi-SSC, hence the sequence $(\tilde{f}_k)$ is equi-SSC as well (because $(\tilde{f}_k)$ is "contained" in (g_k)). Therefore, by Lemmas 4 and 5, the function f is strongly sequentially continuous.

On the other hand, if $|h| \leq 1$, then $f(3k+h) - f(3k) = b(h)\,\sin(\mu_k h)$ for all $k \in \mathbb{N}$, hence

$$\begin{aligned}\limsup_{x\to+\infty} |f(x+h) - f(x)| &\geq |b(h)| \limsup_k |\sin(\mu_k h)| \\ &\geq |b(h)| \sup_i |\sin(\lambda_i h)|\end{aligned}$$

because each λ_i is repeated infinitely many times in the sequence (μ_k).

Since $\lambda_i \to +\infty$ and G is dense in $\mathbb{R}$, one can find a null sequence $(h_i) \subseteq G$ such that $\lambda_i h_i \to \pi/2$, and we get $\limsup_{x\to+\infty} |f(x+h_i) - f(x)| \geq 1/2$ for sufficiently large i. Thus, f is not eventually regular.

Remark.

Denote by $\mathcal{D}$, $\mathbf{N}$, $\mathcal{I}$ respectively the families of Dirichlet-like, nasty, (meager+Lebesgue-null) dense subgroups of $\mathbb{R}$. By Theorem 2 and Proposition 7 (section II), $\mathcal{D} \subseteq \mathbf{N} \subseteq \mathcal{I}$. Moreover, it is clear that a dense subgroup of a nasty group is itself nasty. Thus, nastyness is a "smallness" property which lies between $\mathcal{D}$ and $\mathcal{I}$. We don't know if both inclusions are proper.

Now, we turn to topological vector spaces.

Theorem 3. *Let (X, τ) be a topological vector space, and denote by X^* the dual space of X. Assume that there exist a null sequence $(x_k) \subseteq X$ and a w^*-null sequence $(x_k^*) \subseteq X^*$ such that $\langle x_k^*, x_k \rangle$ does not tend to 0 as $k \to \infty$. Then $G = (X, \tau)$ is a nasty group.*

Proof.

According to the basic example, the sequence $(x_k^*) \subseteq \mathbb{R}^X$ is equi-SSC. Hence, the result follows from Proposition 8.

Corollary 1. *Let $(X, ||\ ||)$ be a normed space, and assume that X has a countable algebraic basis. Then $G = (X, ||\ ||)$ is a nasty group.*

Proof.

Let (e_k) be a countable algebraic basis for X. By the Hahn-Banach theorem, we can choose a sequence $(x_k^*) \subseteq X^*$ such that $||x_k^*|| \to \infty$ and $\langle x_k^*, e_j \rangle = 0$ whenever $j < k$. Since (e_k) is an algebraic basis for X, the sequence (x_k^*) is w^*-null; and since $||x_k^*|| \to \infty$, one can find a null sequence $(x_k) \subseteq X$ such that $\langle x_k^*, x_k \rangle = 1$ for all k. Hence we may apply Theorem 3.

Corollary 2. *If X is a separable Banach space, then (with the notation (X, w) for the space X equipped with the weak topology) the group $G = (X, w)$ is a nasty group if and only if X fails the Schur property.*

Proof.

The "only if" part is a consequence of Theorem 1 in Section III: if X has the Schur property, then strong (resp. uniform) sequential continuity for the weak and the norm topology of X are equivalent, hence (X, w) is a nice group, by Theorem 1. Alternatively, one can apply Theorem $1''$.

To prove the converse, assume that X fails the Schur property. Then one can find a weakly null sequence $(x_k) \subseteq X$ and a w^*-null sequence $(x_k^*) \subseteq X^*$ such that $\langle x_k^*, x_k \rangle > 1$ for all k (see the proof of the Orlicz-Pettis theorem given in section III). Thus, Theorem 3 applies again.

Notice that the same proof gives that if X is a separable Banach space failing the Schur property, then (X^*, w^*) is also a nasty group.

Remarks.

(1) In view of the Josefson-Nissenzweig theorem (see [**Di**]), Corollary 2 may be true even if X is nonseparable. In any case, the implication (*Schur property*) $\Rightarrow$ ((X, w) *is nice*) is always true.

(2) One can construct very "explicit" functions which are SSC but not uniformly sequentially continuous. Here is an example on $G = (c_0, w)$. Let $b : \mathbb{R} \to \mathbb{R}$ be any "bump" function supported on $[-1, 1]$. Define $f : c_0 \to \mathbb{R}$ by $f(x) = b(x_0 - 3k) \min(1, |x_k|)$ if $|x_0 - 3k| \leq 1$, and $f(x) = 0$ otherwise.

(3) The definition of "eventual regularity" makes sense on groups of the form (X, w), where X is a Banach space. In that context, the analogue of Theorem 2 (b) is true: if X (is

separable and) fails the Schur property, then there exist SSC functions on $G = (X, w)$ which are not eventually regular.

Our very last result calls for some preliminary comments.

We have just seen that if X is a Banach space with the Schur property (for example, if $X = l_1$), then SSC functions on $G = (X, w)$ are uniformly sequentially continuous. Besides, by Corollary 3 of Proposition 2, we also know that if X does not contain l_1 (for example, if $X = c_0$), then SSC functions on (X, w) are uniformly sequentially continuous on bounded sets. It is therefore natural to wonder if there exist a Banach space X and an SSC function on (X, w) which is not uniformly sequentially continuous on bounded sets. Surprisingly, an example can indeed be constructed on $X = c_0 \times l_1$ (with norm $||(x, y)|| = \mathrm{Max}\,(||x||_{c_0}, ||y||_{l_1})$).

Proposition 10. *There exists a function $f : c_0 \times l_1 \to \mathbb{R}$ which is strongly sequentially continuous on $c_0 \times l_1$ for the weak topology, but not uniformly sequentially continuous on the unit ball of $c_0 \times l_1$ (for the weak topology).*

Proof.

Let $b : l_1 \to \mathbb{R}$ be defined by $b(y) = 1 - 3 \min(1/3, ||y||_1)$. By Schur's theorem, b is uniformly sequentially continuous on (l_1, w). Moreover, $\operatorname{supp} b$ is the ball $B(0, 1/3)$. Hence, again by Schur's theorem, if we denote by (e_k) the canonical basis of l_1, the sequence $(e_k + \operatorname{supp} b) = \big(B(e_k, 1/3)\big)$ is quasi-separated. Now, define $f : c_0 \times l_1 \to \mathbb{R}$ by

$$f(x, y) = \sum_{k \geq 0} \tilde{x}_k \, b(y - e_k),$$

where $\tilde{x}_k = \min(1, |x_k|)$. The function f has the required properties: it is SSC on $c_0 \times l_1$ by Lemmas 4, 5, and it is not uniformly sequentially continuous on the unit ball because $f(h_k, e_k) - f(0, e_k) \equiv 1$ (where (h_k) is the canonical basis of c_0).

To conclude this paper, let us say a few words about noncommutative groups.

On a nonabelian group G, one can obviously define two kinds of strongly sequentially continuous functions: "left-SSC" functions, and "right-SSC" functions. These two notions should correspond respectively to left and right uniform (sequential) continuity. Since the left-uniformity of G is usually defined by the sets $\{(x, y) : \; x^{-1}y \in V\}$ (where V ranges over all neighbourhoods of the identity), left-SSC functions on a set $A \subseteq G$ will be those for which $\inf\limits_{i \leq n} \delta\big(f(xh_i), f(x)\big) \to 0$ uniformly on A (for each null sequence (h_i)).

With this convention, the "left" (or "right") analogues of Proposition 1, Proposition 3 (Section I), Theorem 1$'$ (Section III) and Proposition 8 (Section IV) are true, with exactly the same proofs. Moreover, Proposition 2 (Section I) holds provided one considers "two-sided" Cauchy sequences. Finally, if G is metrizable and totally bounded, then each sequence in G contains a two-sided Cauchy subsequence. Therefore, totally bounded metrizable groups are nice, whence Corollary 3 of Proposition 2 is still valid in the noncommutative case.

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