SHIFT INVARIANT SUBSPACES WITH ARBITRARY INDICES IN ℓ^p SPACES

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ABSTRACT. We construct right shift invariant subspaces of index n, $1 \le n \le \infty$, in ℓ^p spaces, $2 < p < \infty$, and in weighted ℓ^p spaces.

0. **Introduction.** We take as our starting point Beurling's 1949 Acta Mathematica paper [6], where a complete description of the (right) shift invariant subspaces of ℓ^2 was found. The space ℓ^2 consists of all square summable sequences $(a_0, a_1, a_2, \dots)$, and the shift is the operator $\mathbf{S}$,

$$\mathbf{S}(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots),$$

which acts isometrically on the Hilbert space ℓ^2. A linear subspace M of ℓ^2 is said to be *shift invariant* provided that it is closed, and $\mathbf{S}a \in M$ whenever $a \in M$. To a sequence $a = (a_0, a_1, a_2, \dots)$ we may associate a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < 1,$$

and a may be recovered from the holomorphic function f as the sequence of its Taylor coefficients. In this fashion, we may identify ℓ^2 with the Hardy space H^2 of functions holomorphic in the open unit disc $\mathbb{D}$ having square summable Taylor coefficients. The action of the shift operator $\mathbf{S}$ carries over to H^2: $\mathbf{S}f(z) = zf(z)$, the operation of multiplication by the independent variable. Beurling's theorem states that each shift invariant subspace M of ℓ^2 is either the zero subspace $\{0\}$, or corresponds to a subspace of H^2 of the form φH^2, where φ is an *inner* function. An inner function is by definition a function in H^2 whose radial boundary values have modulus 1 almost everywhere on the unit circle $\mathbb{T}$. At the end of his paper, Beurling mentions that it would be desirable to have a similar understanding of the lattice of shift invariant subspaces of the spaces ℓ^p, which consist of p-th power

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summable sequences, for p other than 2. To this day no complete characterization of the shift invariant subspaces in ℓ^p, $p \neq 2$, is known; the classical case $p = 1$, in which case ℓ^p is a convolution algebra, is particularly difficult.

In this paper, we demonstrate that for $2 < p < \infty$, the lattice of invariant subspaces in ℓ^p has a very different flavor than for $p = 2$. We first need some terminology. Let M be a shift invariant subspace in ℓ^p, where p is fixed in the interval $0 < p < \infty$ (for $0 < p < 1$, ℓ^p is only a quasi-Banach space, and not a Banach space). The operator $\mathbf{S}$ being an isometry on ℓ^p, the image of M under $\mathbf{S}$ is a closed subspace of M. The dimension of the quotient space $M/\mathbf{S}M$ is called the *index* of M. It is clear from Beurling's classification that each shift invariant subspace of ℓ^2 has either index 0 (if the invariant subspace is the zero subspace) or 1 (in all other cases). The same is true for ℓ^p if $0 < p \leq 1$, as follows from standard arguments based on the fact that these spaces are convolution algebras (quasi-Banach algebras for $0 < p < 1$, and Banach algebra for $p = 1$). Probably, it is so for $1 < p < 2$ as well; however, that appears to be unknown up to this point.

Our main result is the following one: *for $2 < p < \infty$, and for a given number $n = 2, 3, \dots, \infty$, there is a shift invariant subspace M of ℓ^p having index n.* This means that the shift on ℓ^p, with $2 < p < \infty$, shares some characteristics of the Bergman shift, that is the right shift on a weighted ℓ^2 space, the Bergman space, which consists of all sequences $(a_0, a_1, a_2, \dots)$ with $\sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty$. An important difference, however, is that the shift is isometric on ℓ^p, whereas it is contractive on the Bergman space. The existence of invariant subspaces of arbitrary index in a family of weighted ℓ^2 spaces including the Bergman space was first proved by Apostol–Bercovici–Foiaş–Pearcy [4] in 1985. For other results in this direction see [10], [7] and the references listed there.

For a subset N of the Bergman space (or any of the other spaces encountered so far), we consider the smallest shift invariant subspace M containing N, and we say that N *generates* M. A general argument (see, for instance, [12]) shows that the *cyclic multiplicity* of a shift invariant subspace — defined as the smallest cardinality of a generating subset — cannot be less than the index. Furthermore, every invariant subspace of index n contains invariant subspaces of index k with cyclic multiplicity k for $1 \leq k \leq n$. The Aleman–Richter–Sundberg generalization [3] of Beurling's theorem states that every proper shift invariant subspace M of the Bergman space is generated by $M \ominus \mathbf{S}M$, which in this case can have any dimension $n = 1, 2, \dots, \infty$. It follows that the cyclic multiplicity equals the index for shift invariant subspaces of the Bergman space. A recent result of Atzmon [5] shows that in ℓ^1 the story is very much different from what is happening in the Bergman or Hardy space: although all proper shift invariant subspaces have index 1, they can have arbitrarily large cyclic multiplicity $n = 1, 2, \dots, \infty$. Our results imply that ℓ^p spaces, $2 < p < \infty$, contain invariant subspaces of arbitrary cyclic multiplicity.

The shift invariant subspaces constructed in this paper are the solution sets for convolution equations determined by lacunary series in the dual spaces. The method used is related to an investigation of the approximation ability of lacunary series given in [1], and the results improve upon those of [4] and [7].

In addition to the spaces mentioned so far, we shall consider weighted spaces ℓ^p of sequences, and weighted versions of the limit case c_0; c_0 is the space of the sequences converging to 0.

We may consider the elements of the sequence spaces mentioned earlier as holomorphic functions on the unit disc via power series. It is worth mentioning that the shift invariant subspaces of index $2, 3, \dots, \infty$ we construct in the paper have no common zeros in the unit disc $\mathbb{D}$. Nikolskiĭ conjectured in [11] that every Banach space of analytic functions in the unit disc satisfying certain natural conditions, contains shift invariant subspaces that are not determined by the common zeros of their elements. Here we prove this conjecture in our scale of weighted ℓ^p spaces. Note that the existence of proper zero-free invariant subspaces of index 1 is of special interest because of recent results of Esterle–Volberg on shift biinvariant subspaces in weighted ℓ^2 spaces on $\mathbb{Z}$ [9]. However, the method we use does not permit us to produce zero-free invariant subspaces of index 1.

The plan of the paper is as follows. The indices of the solution sets of (systems of) convolution equations are calculated in Section 1. Our construction of invariant subspaces of arbitrary index in weighted ℓ^p spaces is described in Section 4, and the proof is given in Sections 5–7. However, before that, in Sections 2 and 3, we provide a simplified version of our argument for the case of invariant subspaces of index 2 in standard ℓ^p spaces.

Precise conditions on weights and exponents p are discussed in Section 8. The absence of common zeros for the elements of the subspaces of big index we construct is established in Section 9. Our main theorem is given in Section 10. Finally, in Section 11, we discuss some related results and formulate several open questions.

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1. **Sets of solutions of convolution equations.** Let B be a reflexive Banach space of (one-sided) sequences, and let B^* be its dual space. We assume that the functionals $\delta_n^* : (a_0, a_1, \dots) \to a_n$, $n \ge 0$, belong to the dual space B^*, and that the right shift operator $\mathbf{S}$ acts continuously on B. We do not suppose that $\mathbf{S}$ is bounded below on B. Therefore, we modify somewhat the definition of index given in the introduction. Given an invariant subspace E of B, ind E is the dimension of the quotient space $E/\operatorname{Clos} \mathbf{S}E$.

For a sequence $a = (a_0, a_1, \dots)$ denote by supp (a) the set of $n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ such that $a_n \ne 0$. For subsets X in a linear topological space B and Y in its dual space B^* denote by $X^\perp$ and $Y_\perp$ their annihilators in B^* and in B respectively.

Given a family f_k, $k \in \mathcal{K}$, of elements in B^*, denote

$$[f_k, k \in \mathcal{K}] = \operatorname{Clos} \operatorname{Lin} \{\mathbf{S}^{*i} f_k : i \ge 0,\ k \in \mathcal{K}\}.$$

Every element $x \in B^*$ generates a convolution equation in B,

$$\langle x, \mathbf{S}^n a \rangle = 0, \quad n \ge 0.$$

Its set of solutions is an $\mathbf{S}$-invariant subspace of B,

$$[x]_\perp = \{a \in B : \langle x, \mathbf{S}^n a \rangle = 0, n \ge 0\}.$$

By the Hahn–Banach theorem, every $\mathbf{S}$-invariant subspace of B is an intersection of subspaces $[x]_\perp$. Note that for a wide class of spaces B a result of Richter ([12], Theorem 3.16) claims that the intersection of any family of $\mathbf{S}$-invariant subspaces of B of index 1 has index less than or equal to 1. Therefore, when trying to find $\mathbf{S}$-invariant subspaces of higher indices we can start with subspaces $[x]_\perp$.

Lemma 1.1. *Let $x \in B^*$. If $\delta_0^* \notin [\mathbf{S}^* x] \ne [\mathbf{S}^{*2} x]$, then* $\operatorname{ind} [\mathbf{S}^* x]_\perp \ge 2$.

Proof. Denote $E = [\mathbf{S}^* x]_\perp$. Then E is a closed $\mathbf{S}$-invariant subspace of B, $E^\perp = [\mathbf{S}^* x]$. Furthermore, $\mathbb{C}\delta_0^* + [x] \subset (\operatorname{Clos} \mathbf{S}E)^\perp$. Using general relations $(X/Y)^* \cong Y^\perp / X^\perp$, $\dim X/Y = \dim Y^\perp / X^\perp$, where $Y \subset X$ are closed subspaces of a locally convex topological linear space, we get

$$\operatorname{ind} E = \dim E / \operatorname{Clos} \mathbf{S}E = \dim (\operatorname{Clos} \mathbf{S}E)^\perp / E^\perp \ge \\ \dim(\mathbb{C}\delta_0^* + [x]) / [\mathbf{S}^* x] = \dim(\mathbb{C}\delta_0^* + \mathbb{C}x + [\mathbf{S}^* x]) / [\mathbf{S}^* x].$$

The last dimension is less than 2 if and only if a non-trivial linear combination of δ_0^* and x belongs to $[\mathbf{S}^* x]$, and then either $\delta_0^* \in [\mathbf{S}^* x]$ or $\mathbf{S}^* x \in \mathbf{S}^* [\mathbf{S}^* x] \subset [\mathbf{S}^{*2} x]$, and hence $[\mathbf{S}^* x] = [\mathbf{S}^{*2} x]$. □

From now on we assume that $(\mathbf{S}B)^\perp = \mathbb{C}\delta_0^*$, or, equivalently, that $\ker \mathbf{S}^* = \mathbb{C}\delta_0^*$.

Lemma 1.2. *Consider an element* $x \in B^* \setminus \{0\}$ *such that* $\delta_0^* \notin [\mathbf{S}^* x]$. *Suppose that* $\mathbf{S}^*$ *is bounded below on* $[x]$. *If* $[\mathbf{S}^* x] = [\mathbf{S}^{*2} x]$, *then* $\operatorname{ind}[\mathbf{S}^* x]_\perp = 1$, *otherwise* $\operatorname{ind}[\mathbf{S}^* x]_\perp = 2$.

Proof. Our condition on $\mathbf{S}^*$ implies that $[\mathbf{S}^* x] = \mathbf{S}^*[x]$. Therefore, defining as above $E = [\mathbf{S}^* x]_\perp$ we have $(\text{Clos } \mathbf{S}E)^\perp = \mathbb{C}\delta_0^* + [x]$. Indeed, if $y \in B^*$, $\langle y, \mathbf{S}E \rangle = 0$, then $\langle \mathbf{S}^* y, [\mathbf{S}^* x]_\perp \rangle = 0$, and $\mathbf{S}^* y \in [\mathbf{S}^* x] = \mathbf{S}^*[x]$, $y \in \mathbb{C}\delta_0^* + [x]$. As in the proof of Lemma 1.1, we get

$$\text{ind } E = \dim(\mathbb{C}\delta_0^* + \mathbb{C}x + [\mathbf{S}^* x])/[\mathbf{S}^* x] \leq 2.$$

Finally, the equality $[\mathbf{S}^* x] = [\mathbf{S}^{*2} x]$ holds if and only if a non-trivial linear combination of δ_0^* and x belongs to $[\mathbf{S}^* x]$. □

Proposition 5.9 in [2] provides results similar to those of Lemmas 1.1 and 1.2; however, our assumptions on $\mathbf{S}$ and B are somewhat weaker. In particular, we do not assume $\mathbf{S}$ to be bounded below on B.

Lemma 1.3. *For some* $2 \leq n \leq \infty$ *consider* $x_k \in B^*$, $1 \leq k < n$. *If* $\mathbf{S}^*$ *is bounded below on* $[x_k, 1 \leq k < n]$, *and no (non-trivial) finite linear combination of* δ_0^* *and* x_k, $1 \leq k < n$, *belongs to* $[\mathbf{S}^* x_k, 1 \leq k < n]$, *then*

$$\operatorname{ind}[\mathbf{S}^* x_k, 1 \leq k < n]_\perp = n.$$

Proof. We argue as in the proof of Lemma 1.2. □

Finally, suppose that $\mathbf{S}$ is bounded below on B. Then $\mathbf{S}B$ is closed, $\mathbf{S}^*$ is a bijection between the Banach spaces $B^*/\mathbb{C}\delta_0^*$ and B^*, and $\mathbf{S}^* B^* = B^*$. Given an $\mathbf{S}^*$-invariant subspace F of B^* we define its index $\operatorname{ind}^* F$ as $\dim F/\mathbf{S}^* F$. The following result describes a relation between the index of an $\mathbf{S}$-invariant subspace in B and that of its annihilator.

Proposition 1.4. *Let* E *be a closed* $\mathbf{S}$*-invariant subspace of* B. *Then*

$$\text{ind } E = \begin{cases} \operatorname{ind}^* E^\perp & \text{if} \quad E \subset \mathbf{S}B \\ 1 + \operatorname{ind}^* E^\perp & \text{if} \quad E \not\subset \mathbf{S}B. \end{cases}$$

Proof. Put $G = \{x \in B^* : \mathbf{S}^* x \in E^\perp\} = (\mathbf{S}E)^\perp \ni \delta_0^*$. Then G and $\mathbf{S}^* G = E^\perp$ are closed $\mathbf{S}^*$-invariant subspaces of B^*, and

$$\text{ind } E = \dim(\mathbf{S}E)^\perp/E^\perp = \dim G/\mathbf{S}^* G.$$

Since $\mathbf{S}^*$ is a bijection between the Banach spaces $G \subset B^*/\mathbb{C}\delta_0^*$ and $\mathbf{S}^* G$, we obtain that $\dim G/\mathbf{S}^* G = \dim \mathbf{S}^* G/\mathbf{S}^{*2} G$ if $\delta_0^* \in \mathbf{S}^* G$ or, equivalently, if $E \subset \mathbf{S}B$, and that $\dim G/\mathbf{S}^* G = 1 + \dim \mathbf{S}^* G/\mathbf{S}^{*2} G$ if $\delta_0^* \notin \mathbf{S}^* G$. □

2. **Model case: a lacunary series in ℓ^p, $p>2$.** To apply the criteria obtained in the previous section to weighted ℓ^p spaces we produce lacunary series with good control on the approximation ability of their left shifts. To explain our approach in a simpler case we start with producing invariant subspaces of index 2 in the standard spaces ℓ^p.

The operator $\mathbf{S}$ acts isometrically on ℓ^p. The space dual to ℓ^p is ℓ^q, $1/p+1/q=1$, with the Cauchy duality

$$\left\langle \{b_n\}_{n\ge 0}, \{a_n\}_{n\ge 0}\right\rangle = \sum_{n\ge 0} a_n b_n. \tag{2.1}$$

By Lemma 1.3, to produce an invariant subspace of index 2 in ℓ^p it suffices to find an element $f\in\ell^q$ such that no (non-trivial) finite linear combination of δ_0^* and f can be approximated by finite linear combinations of $\mathbf{S}^{*i}f$, $i\ge 1$. This last condition is reformulated as follows:

$$\inf \operatorname*{dist}_{\ell^q}\Big(f, \sum_{i\ge 1} c_i \mathbf{S}^{*i} f\Big) > 0, \tag{2.2}$$

and

$$\inf \operatorname*{dist}_{\ell^q}\Big(\delta_0^*, \sum_{i\ge 0} c_i \mathbf{S}^{*i} f\Big) > 0, \tag{2.3}$$

where the infima are taken over all finite sums.

Take an Hadamard lacunary sequence (of order 2) $D=\{d_n\}_{n\ge 1}$ of $\mathbb{Z}_+$, that is $d_1=0$, $d_{n+1}>2d_n$, $n\ge 1$. Fix a sequence of positive numbers $\{a_n\}_{n\ge 1}$ in ℓ^q, such that

$$\sum_{n\ge 1} R_n^p \le 2^{-p}, \tag{2.4}$$

where

$$R_n = \Big[a_n^q \big/ \sum_{m>n} a_m^q\Big]^{1/q}.$$

For a possibility to find such a sequence $\{a_n\}$ see Section 8 (this is the place where we use the condition $p>2>q$).

Denote by $e(j)$, $j\ge 0$, the standard basis in ℓ^q, and set

$$f=\sum_{n\ge 1} a_n e(d_n).$$

For simplicity, we restrict ourselves to proving only relation (2.2). This will be done in the next section. Relation (2.3) is proved in a similar manner.

To prove (2.2) and analogous inequalities in the general weighted ℓ^p case we need to establish some properties (partially contained in [1], Section 1) of the left shifts of D.

Lemma 2.1. (a) *If $a, b \in \mathbb{Z}_+$, $a \neq b$, then*

$$\operatorname{card}\big((D-a)\cap(D-b)\cap\mathbb{Z}_+\big) \le 1.$$

(b) *If $a, b \in \mathbb{Z}_+$, $a < b$, $\{x\} = (D-a)\cap(D-b)\cap\mathbb{Z}_+$, then*

$$(D-b)\cap[0, x-1] = \emptyset.$$

(c) *If $a, b, c \in \mathbb{Z}_+$, $\{x\} = (D-a)\cap(D-b)\cap\mathbb{Z}_+$, $\{y\} = (D-b)\cap(D-c)\cap\mathbb{Z}_+$, $a < b$, $b \neq c$, $x \neq y$, then $b < c$ and $x < y$.*

(d) *If $a_0, \dots, a_{n-1}$ are pairwise different elements of $\mathbb{Z}_+$, $a_n = a_0$, and $(D-a_{k-1})\cap(D-a_k)\cap\mathbb{Z}_+ \neq \emptyset$, $1 \le k \le n$, then*

$$\bigcap_{1\le k\le n} (D-a_k)\cap\mathbb{Z}_+ \neq \emptyset.$$

Put $\Delta = \cup_{s\ge 0}\big([d_{s+1}-d_s, d_{s+1}]\cap\mathbb{Z}_+\big)$.

(e) *Suppose that $a_0, \dots, a_n$ are pairwise different elements of $\mathbb{Z}_+$, $(D-a_{k-1})\cap(D-a_k)\cap\mathbb{Z}_+ = \{x_k\}$, $1 \le k \le n$, $x_0 = \min\cup_{0\le k\le n}((D-a_k)\cap\mathbb{Z}_+) \in (D-a_0)\cap\mathbb{Z}_+$, and that the points x_k, $0 \le k \le n$, are pairwise different. Then $a_0 < \ldots < a_n$, $x_0 < \ldots < x_n$, $x_k = \min((D-a_k)\cap\mathbb{Z}_+)$. Furthermore, $(D-a_k)\cap\mathbb{Z}_+ \subset \Delta$, $1 \le k \le n$, $(D-a_0)\cap\mathbb{Z}_+ \subset \Delta\cup\{x_0\}$.*

Proof. (a) If $x, y \in (D-a)\cap(D-b)$, $0 \le x < y$, $a < b$, then $b+y, a+y, b+x \in D$, and we obtain three inequalities $b+y > a+y$, $b+y > b+x$, $b+y \le a+y+b+x$; together they contradict to the lacunarity condition on D.

(b) If $x+a, x+b \in D$, $b > a$, $b+s \in D$, for some $0 \le s < x$, then as in (a) we obtain three inequalities $x+b \le x+a+b+s$, $x+b > x+a$, $x+b > b+s$; together they contradict to the lacunarity condition on D.

(c) If $b > a$, $b > c$, then by (b), both x and y are minimal elements of $(D-b)\cap\mathbb{Z}_+$. Since $x \neq y$, we get a contradiction. The inequality $x < y$ follows from (b).

(d) Denote $\{x_k\} = (D-a_{k-1})\cap(D-a_k)\cap\mathbb{Z}_+$. If $x_k = x_{k+1}$ for some k, then we can just remove the corresponding a_k. Thererefore, without loss of generality we assume that every two consecutive x_k are different. Furthermore, we may suppose that $a_0 < a_1$. Applying part (c) to a_0, a_1, and a_2, we get $a_0 < a_1 < a_2$; iterating, we obtain $a_0 < a_n$. This contradiction shows that all x_k coincide.

(e) By (b) we have $a_0 < a_1$. By (c) we have $a_1 < \ldots < a_n$ and $x_0 < \ldots < x_n$. Since $x_0 = \min((D-a_0)\cap\mathbb{Z}_+)$, we get $(D-a_0)\cap\mathbb{Z}_+ \subset \Delta\cup\{x_0\}$. By (b), $(D-a_k)\cap[0,x_k-1] = \emptyset$, $x_k = \min((D-a_k)\cap\mathbb{Z}_+)$, and an induction argument shows that $(D-a_k)\cap\mathbb{Z}_+ \subset \Delta$, $1 \le k \le n$. □

3. **The proof of relation (2.2).** We assume that for a finite subset Q of $\mathbb{Z}_+$, $0 \in Q$, and for some coefficients c_i, $i \in Q$, with $c_0 = 1$, we have

$$\Big\|\sum_{i\in Q} c_i \mathbf{S}^{*i} f\Big\| < \frac{1}{2}\|f\|, \tag{3.1}$$

and try to reach a contradiction. The norm in this section is the norm in ℓ^q.

Construction. We are going to introduce a structure on the set Q describing the intersection relations for the sets

$$\operatorname{supp}(\mathbf{S}^{*i} f) = (D-i)\cap\mathbb{Z}_+.$$

Given $i \in Q$, consider the minimal n such that for some

$$i_0 = 0, i_1, \ldots, i_n = i \in Q,$$

we have

$$(D-i_t)\cap(D-i_{t+1})\cap\mathbb{Z}_+ \ne \emptyset, \qquad 0 \le t < n,$$

and denote $\theta(i) = n$; if there is no such n, put $\theta(i) = \infty$. Denote by Q_0 the set of $i \in Q$ with $\theta(i) < \infty$.

In an inductive process, at the step s, $s \ge 1$, consider all $i \in Q$ with $\theta(i) = s$. Clearly, for every such i there exists $i' \in Q$ with $\theta(i') = s-1$ such that

$$(D-i)\cap(D-i')\cap\mathbb{Z}_+ \ne \emptyset.$$

By Lemma 2.1 (c), we prove by induction that $i > i'$.

Furthermore, put $\varphi(i) = i'$. By Lemma 2.1 (d), $\varphi(i)$ is uniquely defined. Otherwise, if

$$(D-i)\cap(D-i')\cap\mathbb{Z}_+ \ne \emptyset, \quad (D-i)\cap(D-i'')\cap\mathbb{Z}_+ \ne \emptyset,$$

with $\theta(i') = \theta(i'') = s-1$, $i' \ne i''$, then by the definition of θ, we have two sequences $\{i'_t\}, \{i''_t\}$ of elements of Q with $i'_0 = i''_0 = 0$, $i'_{s-1} = i'$, $i''_{s-1} = i''$, such that

$$\left.\begin{aligned} (D-i'_t)\cap(D-i'_{t+1})\cap\mathbb{Z}_+ \ne \emptyset, \\ (D-i''_t)\cap(D-i''_{t+1})\cap\mathbb{Z}_+ \ne \emptyset, \end{aligned}\right\} \qquad 0 \le t < s-1.$$

If $i'_t = i''_t$, and $i'_{t+1} \neq i''_{t+1}$ for some $0 \le t < s-1$, then the system

$$\{i\} \cup \{i'_u\}_{t \le u \le s-1} \cup \{i''_u\}_{t+1 \le u \le s-1}$$

would satisfy the conditions of Lemma 2.1 (d), and we would get a contradiction.

If

$$(D-i) \cap (D-i') \cap \mathbb{Z}_+ = \{x\},$$

then we put $\psi(i) = x$. By Lemma 2.1 (a), $\psi(i)$ is well-defined.

As a result of the construction, we define φ and ψ on $Q_0 \setminus \{0\}$. Again by Lemma 2.1 (d), if for some $i, i' \in Q_0$, $i \neq i'$, $i \neq 0$, we have

$$(D-i) \cap (D-i') \cap \mathbb{Z}_+ \neq \emptyset,$$

then either $\theta(i) = \theta(i') + 1$, $\varphi(i) = i'$ or $\theta(i') = \theta(i) + 1$, $\varphi(i') = i$, or $\theta(i) = \theta(i')$, $\varphi(i) = \varphi(i')$ and $\psi(i) = \psi(i')$.

We start with inequality (3.1). Since

$$\Big(\bigcup_{i \in Q_0} \operatorname{supp}(\mathbf{S}^{*i} f)\Big) \cap \Big(\bigcup_{i \in Q \setminus Q_0} \operatorname{supp}(\mathbf{S}^{*i} f)\Big) = \emptyset,$$

we have

$$\Big\|\sum_{i \in Q_0} c_i \mathbf{S}^{*i} f\Big\| \le \Big\|\sum_{i \in Q} c_i \mathbf{S}^{*i} f\Big\|.$$

Given a subset Q_j of Q, define a sequence $\{\omega_r^{(j)}\}_{r \ge 0}$ as follows:

$$\omega_r^{(j)} = \begin{cases} 1 \text{ if } \operatorname{card}\{i \in Q_j : r \in D - i\} > 1, \\ 1/2 \text{ otherwise.} \end{cases}$$

Clearly,

$$\Big\|\omega^{(0)} \sum_{i \in Q_0} c_i \mathbf{S}^{*i} f\Big\| \le \Big\|\sum_{i \in Q_0} c_i \mathbf{S}^{*i} f\Big\|,$$

where the sequences in the left-hand side are multiplied elementwise.

Now, in an inductive process, for every $m \ge 0$ such that $Q_m \neq \{0\}$, we pick an element $i \in Q_m$ with maximal value $\theta(i)$. Then $i \in Q_m \setminus \varphi(Q_m \setminus \{0\})$, we denote $i_0 = \varphi(i)$, $y = \psi(i)$, consider all the elements $i_j \in Q_m$, $1 \le j \le r$, such that $\psi(i_j) = y$, and put $Q_{m+1} = Q_m \setminus \{i_j : 1 \le j \le r\}$. Let us verify that

$$\Big\|\omega^{(m+1)} \sum_{i \in Q_{m+1}} c_i \mathbf{S}^{*i} f\Big\| \le \Big\|\omega^{(m)} \sum_{i \in Q_m} c_i \mathbf{S}^{*i} f\Big\|, \qquad m \ge 0. \tag{3.2}$$

Define t_j by $y+i_j=d_{t_j}$, and denote $A_j=a_{t_j}$, $C_j=c_{i_j}$, $0\le j\le r$. Inequality (3.2) is a consequence of the following one:

$$2^{-q}|C_0A_0|^q\le\Bigl|C_0A_0+\sum_{1\le j\le r}C_jA_j\Bigr|^q+2^{-q}\sum_{1\le j\le r}\sum_{t>t_j}|C_ja_t|^q. \tag{3.3}$$

Indeed, passing from the sequence $Z=\omega^{(m)}\sum_{i\in Q_m}c_i\mathbf{S}^{*i}f$ to $Z'=\omega^{(m+1)}\sum_{i\in Q_{m+1}}c_i\mathbf{S}^{*i}f$ we drop the part $Z''=\omega^{(m)}\sum_{1\le j\le r}\bigl(C_j\mathbf{S}^{*i_j}f-C_jA_je(y)\bigr)$ such that $\operatorname{supp}(Z')\cap\operatorname{supp}(Z'')=\emptyset$, and change the y-th element of Z equal to $C_0A_0+\sum_{1\le j\le r}C_jA_j$ to C_0A_0. It remains to note that $\omega^{(m+1)}_y=1/2$, $\omega^{(m)}_y=1$, and by Lemma 2.1 (b),

$$\mathbf{S}^{*i_j}f-A_je(y)=\sum_{t>t_j}a_te(d_t-i_j),\qquad 1\le j\le r.$$

To prove inequality (3.3), we consider two cases. If $|\sum_{1\le j\le r}C_jA_j|\le|C_0A_0|/2$, then (3.3) follows immediately. Otherwise,

$$\frac12|C_0A_0|<\Bigl|\sum_{1\le j\le r}C_jA_j\Bigr|. \tag{3.4}$$

Recall the definition of R_n and denote $\mathcal{R}_j=R_{t_j}$. We get

$$\frac{A_j^q}{\mathcal{R}_j^q}=\sum_{t>t_j}a_t^q,$$

$$\sum_{1\le j\le r}\Bigl|\frac{C_jA_j}{\mathcal{R}_j}\Bigr|^q=\sum_{1\le j\le r}\sum_{t>t_j}|C_ja_t|^q. \tag{3.5}$$

It remains to note that by (3.4), by the Hölder inequality, and by property (2.4),

$$|C_0A_0|^q<2^q\Bigl|\sum_{1\le j\le r}C_jA_j\Bigr|^q\le$$
$$2^q\Bigl(\sum_{1\le j\le r}\Bigl|\frac{C_jA_j}{\mathcal{R}_j}\Bigr|^q\Bigr)\bigl(\sum_{1\le j\le r}\mathcal{R}_j^p\bigr)^{q/p}\le\sum_{1\le j\le r}\Bigl|\frac{C_jA_j}{\mathcal{R}_j}\Bigr|^q.$$

Taking into account (3.5), we obtain (3.3), and, as a consequence, (3.2). At the end of the inductive process, $Q_m=\{0\}$, and we get the inequality

$$\|f/2\|=\|\omega^{(m)}f\|\le\Bigl\|\sum_{i\in Q}c_i\mathbf{S}^{*i}f\Bigr\|,$$

that contradicts to (3.1). Relation (2.2) is proved.

4. **Lacunary series in weighted ℓ^p spaces.** In this section we deal with the Banach spaces of sequences

$$\ell^p(\gamma) = \big\{\{a_n\}_{n\ge 0} : \sum_{n\ge 0} \gamma(n)^p |a_n|^p < \infty\big\}, \qquad 1 < p < \infty,$$

with weights $\gamma : \mathbb{Z}_+ \to]0,\infty[$ such that $\gamma(0) = 1$. The space dual to $\ell^p(\gamma)$ is $\ell^q(1/\gamma)$, $1/p + 1/q = 1$, with the Cauchy duality (2.1). The operator $\mathbf{S}$ acts continuously on $\ell^p(\gamma)$ if and only if $\sup_n \gamma(n+1)/\gamma(n) < \infty$.

Under some conditions on γ and p to be specified in Section 8, we produce $\mathbf{S}$-invariant subspaces of indices $2, 3, \dots, \infty$ in the spaces $\ell^p(\gamma)$. By Lemma 1.3, it suffices to find such elements $f_n \in \ell^q(1/\gamma)$, $1 \le n < \infty$, that $\mathbf{S}^*$ is bounded below on $[f_n, 1 \le n < \infty]$, and no (non-trivial) finite linear combination of δ_0^* and f_n, $1 \le n < \infty$, can be approximated by finite linear combinations of $\mathbf{S}^{*i} f_k$, $k \ge 1$, $i \ge 1$. This last condition is reformulated as follows: for every $n \ge 1$,

$$\inf \operatorname*{dist}_{\ell^q(1/\gamma)} \Big(f_n, \sum_{(k,i)\ne(n,0)} c_{k,i} \mathbf{S}^{*i} f_k \Big) > 0, \tag{4.1}$$

and

$$\inf \operatorname*{dist}_{\ell^q(1/\gamma)} \Big(\delta_0^*, \sum c_{k,i} \mathbf{S}^{*i} f_k \Big) > 0, \tag{4.2}$$

where the infima are taken over all finite sums.

As in Section 2, take an Hadamard lacunary sequence $D = \{d_n\}_{n\ge 1}$ of $\mathbb{Z}_+$. We suppose that for some $\eta \ge 1$ we have

(4.i) $\gamma(d_{n+1})/\eta \le \gamma(l) \le \eta\gamma(d_{n+1})$ for $d_{n+1} - d_n - 1 \le l \le d_{n+1}$, $n \ge 1$,

then we divide D into the union of disjoint infinite sequences $D_n = \{d_{m,n}\}_{m\ge 1}$, $n \ge 1$, such that $d_{1,1} = 0$, $d_{m+1,n} \ge d_{m,n}$, $m \ge 1$, $n \ge 1$, and additionally suppose that for a sequence of positive numbers $\{a_{m,n}\}_{m\ge 1, n\ge 1}$, we have

(4.ii) $\sum_{d_{m,n}\ge x} \gamma(d_{m,n})^p R_{m,n}^p \le (2\eta)^{-p} \gamma(x)^p$ for every $x \in \Delta$,

$$\sum_{m\ge 1} a_{m,n}^q \gamma(d_{m,n})^{-q} < \infty, \qquad n \ge 1,$$

where

$$R_{m,n} = \Big[a_{m,n}^q \gamma(d_{m,n})^{-q} / (\sum_{l>m} a_{l,n}^q \gamma(d_{l,n})^{-q}) \Big]^{1/q}, \qquad 1/p + 1/q = 1,$$

and Δ is defined in Lemma 2.1.

Denote by $e(j)$, $j \geq 0$, the standard basis in $\ell^q(1/\gamma)$, and set

$$f_n = \sum_{m\geq 1} a_{m,n} e(d_{m,n}), \qquad n \geq 1.$$

Now our plan is as follows. We verify inequality (4.1) with these f_n in Section 5. Furthermore, inequality (4.2) is proved in Section 6, and the fact that $\mathbf{S}^*$ is bounded below on $[f_n, 1 \leq n < \infty]$ is proved in Section 7.

5. **The proof of relation (4.1).** Let us assume that for some n, for a finite set Q of pairs (i,k) with $(0,n) \in Q$, and for some coefficients $c_{i,k}$, $(i,k) \in Q$, with $c_{0,n} = 1$, we have

$$\Big\| \sum_{(i,k)\in Q} c_{i,k} \mathbf{S}^{*i} f_k \Big\| < \frac{1}{2} \|f_n\|, \tag{5.1}$$

and try to reach a contradiction. The norm here and later on is the norm in $\ell^q(1/\gamma)$.

We are going to introduce a structure on (a part of) Q describing the intersection relations for the sets

$$\operatorname{supp}(\mathbf{S}^{*i} f_k) = (D_k - i) \cap \mathbb{Z}_+, \qquad (i,k) \in Q.$$

Construction. We start with $z \in \mathbb{Z}_+$ and a subset U_0 of Q such that

$$\left.\begin{aligned} U_0 = \big\{(i,k) \in Q : z \in D_k - i\big\}, \\ z = \min\big\{\cup_{(i,k)\in U_0} \big[(D_k - i) \cap \mathbb{Z}_+\big]\big\}. \end{aligned}\right\} \tag{5.2}$$

Given $(i,k) \in Q$, consider the minimal n such that for some

$$(i_0, k_0) \in U_0, (i_1, k_1), \dots, (i_n, k_n) = (i,k) \in Q,$$

we have

$$(D_{k_t} - i_t) \cap (D_{k_{t+1}} - i_{t+1}) \cap \mathbb{Z}_+ \neq \emptyset, \qquad 0 \leq t < n,$$

and denote $\theta(i,k) = n$; if there is no such n, put $\theta(i,k) = \infty$. Denote by Q_0 the set of $(i,k) \in Q$ with $\theta(i,k) < \infty$.

In an inductive process, at the step s, $s \geq 1$, consider all pairs $(i,k) \in Q$ with $\theta(i,k) = s$. Clearly, for every such pair there exists a pair $(i', k') \in Q$ with $\theta(i',k') = s-1$ such that

$$(D_k - i) \cap (D_{k'} - i') \cap \mathbb{Z}_+ \neq \emptyset.$$

By condition (5.2) and by Lemma 2.1 (c), we prove by induction that $i > i'$.

Furthermore, put $\varphi(i,k) = (i',k')$. If

$$(D_k - i) \cap (D_{k'} - i') \cap \mathbb{Z}_+ = \{x\},$$

we put $\psi(i,k) = x$. Using the same argument as in Section 3, we prove that $\varphi(i,k)$ and $\psi(i,k)$ are uniquely defined.

As a result of the construction, we define φ and ψ on $Q_0 \setminus U_0$. As in Section 3 we get that the relation

$$(D_k - i) \cap (D_{k'} - i') \cap \mathbb{Z}_+ \neq \emptyset,$$

with $(i,k) \in Q_0$, $(i',k') \in Q_0 \setminus U_0$, $(i,k) \neq (i',k')$, implies that either $\theta(i,k) = \theta(i',k')+1$, $\varphi(i,k) = (i',k')$ or $\theta(i',k') = \theta(i,k)+1$, $\varphi(i',k') = (i,k)$, or $\theta(i,k) = \theta(i',k')$, $\varphi(i,k) = \varphi(i',k')$ and $\psi(i,k) = \psi(i',k')$.

In what follows, the construction needs to be applied several times, for slightly differemt z, $U_0 \subset Q_0 \subset Q$; condition (5.2) is to be verifed for every concrete application.

Here we put $z = \min\{\operatorname{supp}(f_n)\} = d_{1,n}$, apply the construction and get the subsets $U_0 \subset Q_0$ of Q and the maps φ, ψ (condition (5.2) holds because of Lemma 2.1 (b); indeed, if $(D - i) \cap D_n = \{z\}$, then $(D - i) \cap [0, z-1] = \emptyset$).

For every pair $(i,k) \in Q_0 \setminus U_0$, there is a sequence of pairs $(i_j, k_j) \in Q_0$, $(i_0, k_0) \in U_0$, with $(i_t, k_t) = (i,k)$, and a sequence $x_j \in \mathbb{Z}_+$, $x_0 = z$, such that

$$i_j < i_{j+1}, \quad (D_{k_j} - i_j) \cap (D_{k_{j+1}} - i_{j+1}) \cap \mathbb{Z}_+ = \{x_{j+1}\}, \qquad 0 \leq j < t.$$

Then by Lemma 2.1 (e), $x_j < x_{j+1}$, $0 \leq j < t$, and $(D_k - i) \cap \mathbb{Z}_+ \subset \Delta \cup \{z\} = \Delta$. Thus,

$$\bigcup_{(i,k)\in Q_0} \big[(D_k - i) \cap \mathbb{Z}_+\big] \subset \Delta. \tag{5.3}$$

Since

$$\Big(\bigcup_{(i,k)\in Q_0} \operatorname{supp}(\mathbf{S}^{*i} f_k)\Big) \cap \Big(\bigcup_{(i,k)\in Q\setminus Q_0} \operatorname{supp}(\mathbf{S}^{*i} f_k)\Big) = \emptyset,$$

we have

$$\Big\| \sum_{(i,k)\in Q_0} c_{i,k} \mathbf{S}^{*i} f_k \Big\| \leq \Big\| \sum_{(i,k)\in Q} c_{i,k} \mathbf{S}^{*i} f_k \Big\|.$$

Given a subset Q_j of Q, define a sequence $\{\omega_r^{(j)}\}_{r\geq 0}$ as follows:

$$\omega_r^{(j)} = \begin{cases} 1 \text{ if } \operatorname{card}\{(i,k) \in Q_j : r \in D_k - i\} > 1, \\ 1/2 \text{ otherwise.} \end{cases}$$

Clearly,

$$\Big\|\omega^{(0)}\sum_{(i,k)\in Q_0} c_{i,k}\mathbf{S}^{*i}f_k\Big\| \le \Big\|\sum_{(i,k)\in Q_0} c_{i,k}\mathbf{S}^{*i}f_k\Big\|,$$

where the sequences in the left-hand side are multiplied elementwise.

Now, in an inductive process, for every $m\ge 0$ such that $Q_m\ne U_0$, we pick a pair $(i,k)\in Q_m$ with maximal value $\theta(i,k)$. Then $(i,k)\in Q_m\setminus\varphi(Q_m\setminus U_0)$, we denote $(i_0,k_0)=\varphi(i,k)$, $y=\psi(i,k)$, consider all the pairs $(i_j,k_j)\in Q_m$, $1\le j\le r$, such that $\psi(i_j,k_j)=y$, and put $Q_{m+1}=Q_m\setminus\{(i_j,k_j),1\le j\le r\}$. By (5.3), $y\in\Delta$. Let us verify that

$$\Big\|\omega^{(m+1)}\sum_{(i,k)\in Q_{m+1}} c_{i,k}\mathbf{S}^{*i}f_k\Big\| \le \Big\|\omega^{(m)}\sum_{(i,k)\in Q_m} c_{i,k}\mathbf{S}^{*i}f_k\Big\|,\quad m\ge 0. \tag{5.4}$$

Define t_j by $y+i_j=d_{t_j,k_j}$, and denote $A_j=a_{t_j,k_j}$, $C_j=c_{i_j,k_j}$, $0\le j\le r$. As in Section 3, inequality (5.4) is a consequence of the following one:

$$\Big|\frac{C_0A_0}{2\gamma(y)}\Big|^q \le \Big|\frac{C_0A_0+\sum_{1\le j\le r}C_jA_j}{\gamma(y)}\Big|^q + \sum_{1\le j\le r}\sum_{t>t_j}\Big|\frac{C_ja_{t,k_j}}{2\gamma(d_{t,k_j}-i_j)}\Big|^q. \tag{5.5}$$

To prove inequality (5.5), we consider two cases. If $|\sum_{1\le j\le r}C_jA_j|\le |C_0A_0|/2$, then (5.5) follows immediately. Otherwise,

$$\frac12|C_0A_0| < \Big|\sum_{1\le j\le r}C_jA_j\Big|. \tag{5.6}$$

Since $i_j\le d_{t_j,k_j}$, by property (4.i) we have $\gamma(d_{t,k_j}-i_j)\le\eta\gamma(d_{t,k_j})$, $t>t_j$. Recall the definition of $R_{m,n}$ and denote $\mathcal{R}_j=R_{t_j,k_j}$. We get

$$\frac{1}{\mathcal{R}_j^q}\Big(\frac{A_j}{\eta\gamma(d_{t_j,k_j})}\Big)^q \le \sum_{t>t_j}\Big(\frac{a_{t,k_j}}{\gamma(d_{t,k_j}-i_j)}\Big)^q,$$

$$\sum_{1\le j\le r}\frac{1}{\mathcal{R}_j^q}\Big|\frac{C_jA_j}{\eta\gamma(d_{t_j,k_j})}\Big|^q \le \sum_{1\le j\le r}\sum_{t>t_j}\Big|\frac{C_ja_{t,k_j}}{\gamma(d_{t,k_j}-i_j)}\Big|^q. \tag{5.7}$$

It remains to note that by (5.6), by the Hölder inequality, and by property (4.ii),

$$|C_0A_0|^q < \Big|2\sum_{1\le j\le r} C_jA_j\Big|^q \le$$

$$(2\eta\gamma(y))^q\Big(\sum_{1\le j\le r}\frac{1}{\mathcal{R}_j^q}\Big|\frac{C_jA_j}{\eta\gamma(d_{t_j,k_j})}\Big|^q\Big)\Big(\sum_{1\le j\le r}\mathcal{R}_j^p\frac{\gamma(d_{t_j,k_j})^p}{\gamma(y)^p}\Big)^{q/p} \le$$

$$(\gamma(y))^q\sum_{1\le j\le r}\frac{1}{\mathcal{R}_j^q}\Big|\frac{C_jA_j}{\eta\gamma(d_{t_j,k_j})}\Big|^q.$$

Taking into account (5.7), we obtain (5.5), and as a consequence, (5.4). At the end of the inductive process, we have $Q_m = U_0$. Next, making one more step just as it was done above, we remove all the elements of $U_0 \setminus \{(0,n)\}$, and arrive at the inequality

$$\|f_n/2\| \le \Big\|\sum_{(i,k)\in Q} c_{i,k}\mathbf{S}^{*i}f_k\Big\|,$$

that contradicts to (5.1). Relation (4.1) is proved.

6. **The proof of relation (4.2).** Together with (4.2) we prove its generalization. This will be needed in Section 9. Suppose that $Y = (1, y, y^2, \dots) \in \ell^q(1/\gamma)$ for some $y \in \mathbb{C}$, $|y| < 1$. We claim that

$$\inf \operatorname*{dist}_{\ell^q(1/\gamma)}\big(Y, \sum c_{i,k}\mathbf{S}^{*i}f_k\big) = c(y) > 0, \tag{6.1}$$

where the infimum is taken by all finite sums.

Consider first the case $y \ne 0$. If (6.1) does not hold, then we can first find K such that

$$\sum_{i>K}\gamma(i)^{-q}|y|^{iq} \le \frac{1}{4}, \tag{6.2}$$

and then find a set of pairs Q' and a finite family $\{c'_{i,k}\}_{(i,k)\in Q'}$ such that

$$\Big\|Y + \sum_{(i,k)\in Q'} c'_{i,k}\mathbf{S}^{*i}f_k\Big\| < \frac{1}{4}|y|^{K+1}\|(\mathbf{S}^*)^{K+1}\|^{-1}.$$

Applying the operator $(\mathbf{S}^*)^{K+1}$ to the sum in the left-hand side, and using the fact that $\mathbf{S}^*Y = yY$, we get

$$\Big\|Y + \sum_{(i,k)\in Q} c_{i,k}\mathbf{S}^{*i}f_k\Big\| < \frac{1}{4}. \tag{6.3}$$

for a set of pairs Q and some finite family $\{c_{i,k}\}$, $(i,k) \in Q$; furthermore, $i > K$ for every pair $(i,k) \in Q$.

Now we put $z = 0$, and use Construction from Section 5 to define $U_0 \subset Q_0$ and the maps φ and ψ on $Q_0 \setminus U_0$ (if U_0 is empty, then Q_0 is also empty). For $(i,k) \in U_0$ we have $i > K$, $0 \in D_k - i$, and by the lacunarity condition on D,

$$(D_k - i) \cap [1, K] = \emptyset.$$

Using Lemma 2.1 (e) we get that

$$(D_k - i) \cap [1, K] = \emptyset, \qquad (i,k) \in Q_0. \tag{6.4}$$

Put $F = \sum_{(i,k)\in Q_0} c_{i,k} \mathbf{S}^{*i} f_k$,

$$\omega_r = \begin{cases} 1, & r \in \operatorname{supp}(F), \\ 0, & r \notin \operatorname{supp}(F). \end{cases}$$

Then

$$Y + \sum_{(i,k)\in Q} c_{i,k} \mathbf{S}^{*i} f_k =$$

$$\Big[\delta_0^* + F + \omega(Y - \delta_0^*)\Big] + \Big[\sum_{(i,k)\in Q\setminus Q_0} c_{i,k} \mathbf{S}^{*i} f_k + (1-\omega)(Y - \delta_0^*)\Big].$$

Since the spectra of the expressions in the two square brackets are disjoint, we obtain

$$\Big\|Y + \sum_{(i,k)\in Q} c_{i,k} \mathbf{S}^{*i} f_k\Big\| \ge \big\|\delta_0^* + F + \omega(Y - \delta_0^*)\big\|.$$

Using (6.2) and (6.4), we get

$$\big\|\omega(Y - \delta_0^*)\big\| \le \frac{1}{4}.$$

By (6.3), we obtain that

$$\frac{1}{4} > \Big\|Y + \sum_{(i,k)\in Q} c_{i,k} \mathbf{S}^{*i} f_k\Big\| \ge \big\|\delta_0^* + F\big\| - \frac{1}{4},$$

and hence,

$$\Big\|\delta_0^* + \sum_{(i,k)\in Q_0} c_{i,k} \mathbf{S}^{*i} f_k\Big\| < \frac{1}{2}. \tag{6.5}$$

If (6.1) does not hold for $y = 0$, then we immediately get (6.5) for some Q_0, $\{c_{i,k}\}$. Starting with (6.5) we put $z = 0$, define $U_0 \subset Q_0$, and arguing as in Section 5, we arrive at

$$\|\delta_0^*/2\| \le \Big\|\delta_0^* + \sum_{(i,k)\in Q_0} c_{i,k} \mathbf{S}^{*i} f_k\Big\| < \frac{1}{2},$$

which is impossible. Relations (6.1) and (4.2) are proved.

7. **An estimate on $\mathbf{S}^*$.** It remains to verify that $\mathbf{S}^*$ is bounded below on $[f_n, 1 \le n < \infty]$. We consider a finite set Q of pairs (i,k), and real coefficients $c_{i,k}$, $(i,k) \in Q$, and prove that

$$\Big\| \sum_{(i,k)\in Q} c_{i,k}\mathbf{S}^{*i} f_k \Big\| \le 3\eta^2 \Big\| \mathbf{S}^*\Big(\sum_{(i,k)\in Q} c_{i,k}\mathbf{S}^{*i} f_k \Big) \Big\|. \tag{7.1}$$

In an inductive process we start with $Q^0 = Q$, and at the step $j \ge 1$ put

$$z^{[j]} = \min \bigcup_{(i,k)\in Q^{j-1}} \operatorname{supp}(S^{*i} f_k).$$

Then, using Construction of Section 5, and starting with $z^{[j]}$, we define the subsets $U_0^j \subset Q_0^j$ of Q^{j-1} and the maps $\varphi^{[j]}$ and $\psi^{[j]}$ acting on $Q_0^j \setminus U_0^j$. Denote $Q^{[j]} = Q_0^j$, $Q^j = Q^{j-1} \setminus Q_0^j$. Our process stops when $Q^j = \emptyset$.

Thus, we divide Q into the union of disjoint subsets $Q^{[j]}$. By construction, the sets

$$A^{[j]} = \bigcup_{(i,k)\in Q^{[j]}} \operatorname{supp}(\mathbf{S}^{*i} f_k)$$

are disjoint. Therefore, to verify (7.1) it is sufficient to prove for each j the inequality

$$\Big\| \sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i} f_k \Big\| \le 3\eta^2 \Big\| \mathbf{S}^*\Big(\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i} f_k \Big) \Big\|. \tag{7.2}$$

By Lemma 2.1 (e),

$$A^{[j]} \setminus \{z^{[j]}\} \subset \Delta. \tag{7.3}$$

Furthermore, by property (4.i),

$$\gamma(n-1) \le \eta^2 \gamma(n), \qquad n \in \Delta. \tag{7.4}$$

Arguing as in Section 5 we obtain that for every $c \ge 0$,

$$\Big\| \frac{c}{2} \delta^*_{z^{[j]}} \Big\| \le \Big\| c\delta^*_{z^{[j]}} + \sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i} f_k \Big\|.$$

This implies that

$$\Big\| \delta^*_{z^{[j]}} \sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i} f_k \Big\| \le 2 \Big\| (I - \delta^*_{z^{[j]}}) \sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i} f_k \Big\|,$$

where the sequences are multiplied elementwise, $I = (1, 1, \dots)$. The properties (7.3), (7.4) imply that

$$\Big\|(I-\delta^*_{z[j]})\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big\| \le \eta^2\Big\|\mathbf{S}^*\Big((I-\delta^*_{z[j]})\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big)\Big\|.$$

Now, we have

$$\Big\|\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big\| \le 3\Big\|(I-\delta^*_{z[j]})\Big(\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big)\Big\| \le$$

$$3\eta^2\Big\|\mathbf{S}^*\Big((I-\delta^*_{z[j]})\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big)\Big\| \le 3\eta^2\Big\|\mathbf{S}^*\Big(\sum_{(i,k)\in Q^{[j]}} c_{i,k}\mathbf{S}^{*i}f_k\Big)\Big\|.$$

This proves (7.2), and, as a consequence, (7.1).

8. **Sufficient conditions.** Here we formulate two sets of conditions on p and γ; each of them guarantees a possibility to find D_n, $\{a_{n,k}\}$ satisfying properties (4.i), (4.ii).

(a) $2 < p < \infty$,

$$\left.\begin{array}{c}\text{for some } \eta \ge 1, \text{ and for a sequence } s_k \to \infty,\\ \dfrac{1}{\eta} \le \gamma(l) \le \eta, \qquad s_k - k \le l \le s_k.\end{array}\right\} \tag{8.1}$$

(b) $1 < p < \infty$,

$$\left.\begin{array}{c}\text{for some } \eta \ge 1, \text{ and for a sequence } s_k \to \infty,\\ \lim_{k\to\infty}\gamma(s_k) = 0, \qquad \dfrac{1}{\eta} \le \dfrac{\gamma(l)}{\gamma(s_k)} \le \eta, \quad s_k - k \le l \le s_k.\end{array}\right\} \tag{8.2}$$

In both cases (a), (b) property (4.i) holds for sufficiently rapidly growing $d_n \in \{s_k\}$. To find D_n, $\{a_{n,k}\}$ satisfying property (4.ii), we use the following simple fact (see [1], Section 1). For any divergent series of positive numbers $\{H_n\}$, $\sum H_n = \infty$, there exists a (unique up to a multiplicative constant) convergent sequence of positive numbers $\{a_n\}$, $\sum a_n < \infty$, with $H_n = a_n / \sum_{k>n} a_k$.

In the case (a) take sequences $R_{m,n}$ with

$$\sum_{m\ge 1} R^q_{m,n} = \infty, \quad n \ge 1, \qquad \sum_{m\ge 1, n\ge 1} R^p_{m,n} < (2\eta^3)^{-p}. \tag{8.3}$$

In the case (b) put $R^p_{m,n} \equiv (2\eta^2)^{-p}/2$, and take a subsequence $D \subset \{s_k\}$, so lacunary as

$$\sum_{k>n}\gamma(d_k)^p < \gamma(d_n)^p, \qquad n \ge 0.$$

Furthermore, in an analogous way, we can consider the spaces $B = \ell^1(\gamma)$ (with γ satisfying condition (8.2)) and $B = c_0(\gamma)$ (corresponding to $p = \infty$, with γ satisfying one of the conditions (8.1), (8.2)),

$$c_0(\gamma) = \{\{a_n\}_{n\geq 0} : \lim_{n\to\infty} \gamma(n)a_n = 0\}.$$

In these two cases the space B is not reflexive, and we should modify somewhat the statements in Section 1. Lemma 1.3 is replaced by

Lemma 8.1. *For some $2 \leq n \leq \infty$ consider $x_k \in B^*$, $1 \leq k < n$. If $\mathbf{S}^*$ is bounded below on $[x_k, 1 \leq k < n]$, the subspace $[\mathbf{S}^* x_k, 1 \leq k < n]$ is weak-star closed, and no (non-trivial) finite linear combination of δ_0^* and x_k, $1 \leq k < n$, belongs to $[\mathbf{S}^* x_k, 1 \leq k < n]$, then*

$$\operatorname{ind}[\mathbf{S}^* x_k, 1 \leq k < n]_\perp = n.$$

Correspondingly, our constructions in Section 4 extend to this set-up; we need only to verify that $[\mathbf{S}^* f_k, 1 \leq k < n]$ is really weak-star closed. We make use of the fact that linear subspaces in B^* are weak-star closed if and only if they are weak-star sequentially closed under the condition that B is separable. This fact is a consequence of standard functional analysis arguments.

Let us sketch our argument just for the space $c_0 = c_0(1)$ in the case of index 2. Fix $R_n = \delta/(n+1)$, with small positive δ, such that the natural analog of (8.3) with $q = 1$ and $p = \infty$ holds with this choice of R_n, choose a lacunary sequence D and define f as in Section 2.

Suppose that for some $F \in \ell^1$, for a sequence of finite sets $Q^n \subset \mathbb{Z}_+$, $n \geq 1$, and for coefficients c_i^n, $i \in Q^n$, we have

$$F_n = \sum_{i\in Q^n} c_i^n \mathbf{S}^{*i} f \overset{\text{weak-star}}{\longrightarrow} F, \qquad n \to \infty.$$

Then

$$\|F_n\| \leq A < \infty, \qquad n \geq 1.$$

For $a < b \in \mathbb{Z}_+$ define

$$X_r^{a,b} = \begin{cases} 0, & r < a, \\ 1, & a \leq r \leq b, \\ 0, & r > b. \end{cases}$$

Fix $\varepsilon > 0$, and find a such that

$$\|X^{a,\infty} F\| \leq \varepsilon,$$

where once again the sequences are multiplied elementwise. Furtermore, for every b we find $n(b)$ such that

$$\|X^{0,b}(F - F_{n(b)})\| \leq \varepsilon.$$

Next, fix $I(\varepsilon, a)$ such that for $i \geq I(\varepsilon, a)$ the first non-zero element of the sequence $\mathbf{S}^{*i} f$ does not exceed ε/A times the sum of the other elements, and such that

$$\operatorname{card}\big(\operatorname{supp}(S^{*i} f) \cap [0, a]\big) \leq 1.$$

Acting as in Section 7, we divide $Q^{n(b)}$ into the union of disjoint subsets $Q^{n(b),[j]}$. Then, as in Section 5, we delete from each $Q^{n(b),[j]}$ one by one all $i \geq I(\varepsilon, a)$. In this way we replace $F_{n(b)}$ by

$$\widetilde{F}_{n(b)} = \sum_{i < I(\varepsilon, a)} c_i^{n(b)} \mathbf{S}^{*i} f$$

with

$$\|X^{0,a}(F_{n(b)} - \widetilde{F}_{n(b)})\| \leq 2\varepsilon,$$
$$\|X^{a,b} \widetilde{F}_{n(b)}\| \leq \|X^{a,b} F_{n(b)}\| + 2\varepsilon,$$
$$\|X^{a,\infty} \widetilde{F}_{n(b)}\| \leq \|X^{a,\infty} F_{n(b)}\| + 2\varepsilon.$$

Therefore,

$$\|X^{0,b}(F - \widetilde{F}_{n(b)})\| \leq 8\varepsilon,$$
$$\|\widetilde{F}_{n(b)}\| \leq A + 4\varepsilon.$$

When b tends to ∞, and ε, a are fixed, the elements $\widetilde{F}_{n(b)}$ are bounded and belong to a finite dimensional space; hence, we can find a sequence $\widetilde{F}_{n(b_k)}$ converging to an element

$$\widehat{F}_\varepsilon = \sum_{i < I(\varepsilon, a)} c_{i,\varepsilon} \mathbf{S}^{*i} f,$$

for $b_k \to \infty$. Then

$$\|F - \widehat{F}_\varepsilon\| \leq 8\varepsilon,$$

and since ε is arbitrary, we conclude that

$$F \in \operatorname{Clos\,Lin}\{\mathbf{S}^{*i} f : i \geq 0\}.$$

9. **Absence of common zeros.** Let us suppose that

$$\liminf_{n \to \infty} \gamma(n)^{1/n} = R > 0.$$

We identify each element $a = (a_0, a_1, \dots)$ in $\ell^p(\gamma)$ (in $c_0(\gamma)$) with the corresponding formal power series $a(z) = \sum_{k \geq 0} a_k z^k$, converging uniformly on compact subsets of the disc $R\mathbb{D} = \{z \in \mathbb{C} : |z| < R\}$ to an analytic function. In this way, the spaces $\ell^p(\gamma)$ are considered as spaces of analytic functions. The elements of the subspaces $[\mathbf{S}^* f_1, \mathbf{S}^* f_2, \dots]_\perp$ we construct in Section 4 have no common zeros in the disc $R\mathbb{D}$. To verify this property it suffices to use the fact that no element $Y = (1, y, y^2, \dots)$, $|y| < R$, belongs to $[f_1, f_2, \dots]$, which follows from relation (6.1).

10. **Main result.** Summing up we get our main result. Instead of conditions (8.1)–(8.2) we use here more explicit (and more restrictive) conditions (A), (B), and (C).

Theorem 10.1. *Let* $1 \le p \le \infty$, $B = \ell^p(\gamma)$ $(c_0(\gamma)$ *for* $p = \infty)$. *Suppose that one of the following conditions holds:*

(A) $2 < p \le \infty$,

$$\liminf_{n\to\infty} \gamma(n) < \infty, \qquad \lim_{n\to\infty} \frac{\gamma(n+1)}{\gamma(n)} = 1.$$

(B) $1 \le p \le \infty$,

$$\liminf_{n\to\infty} \gamma(n) = 0, \qquad \lim_{n\to\infty} \frac{\gamma(n+1)}{\gamma(n)} = 1.$$

(C) $1 \le p \le \infty$,

$$\gamma(n) \searrow 0, \quad n \to \infty, \qquad \sup_n \frac{\gamma(n+k)}{\gamma(n)} = 1, \quad k \ge 1.$$

Then B contains $\mathbf{S}$*-invariant subspaces of index* $2, 3, \dots, \infty$.

If $\liminf_{n\to\infty} \gamma(n)^{1/n} = R > 0$, *then we can find such subspaces without common zeros in* $R\,\mathbb{D}$.

Remark 10.2. The spaces $\ell^p = \ell^p(1)$, $2 < p \le \infty$, are covered by the case (A). The statement of Theorem in the case (B) is contained in Theorem 5.4 of [7]. The first statement of Theorem in the case (C), for $p = 2$, gives the result on index obtained in [4], Section 3.

Note, that in the case (C), the operator $\mathbf{S}$ is not necessarily bounded below on B.

Remark 10.3. A careful analysis of conditions (4.i), (4.ii) shows that for every p, $2 < p < \infty$, there exists $\delta > 0$ such that the assertions of our theorem are satisfied for the space $\ell^p\big((\log(n+e))^\delta\big)$.

11. **Related results and open questions.** We use the following notation. Given a Banach space of (one-sided) sequences B such that the right shift operator $\mathbf{S}$ acts continuously on B, we write $B \in \mathcal{I}_1$ if $\operatorname{ind} E \le 1$ for every $\mathbf{S}$-invariant subspace E of B, and $B \in \mathcal{I}_\infty$ if for every $n = 1, 2, \dots, \infty$ there exists an $\mathbf{S}$-invariant subspace E_n of B with $\operatorname{ind} E_n = n$.

We have already mentioned that $\ell^2 \in \mathcal{I}_1$. If $p = 1$ and γ is (weakly) submultiplicative, that is $\gamma(n) \le C\gamma(k)\gamma(n-k)$, $0 \le k \le n$, or if

$1 < p < \infty$ and

$$\sum_{0\le k\le n} \frac{1}{\gamma(k)^q\gamma(n-k)^q} \le \frac{C}{\gamma(n)^q}, \qquad n \ge 0,$$

then $\ell^p(\gamma)$ is a Banach algebra with respect to convolution multiplication. If, additionally, the operator $\mathbf{S}$ is bounded below on $\ell^p(\gamma)$, that is $\inf_n \gamma(n+1)/\gamma(n) > 0$, then (see [12], Section 3)

$$\ell^p(\gamma) \in \mathcal{I}_1.$$

For logarithmically concave γ, $\gamma(n-1)\gamma(n+1) \le \gamma(n)^2$, $n \ge 1$, such that

$$\lim_{n\to\infty} \gamma(n) = \infty, \qquad \lim_{n\to\infty} \gamma(n)^{1/n} = 1,$$

one can deduce from a result of Aleman–Richter–Ross ([2], Corollary 5.10) using a discrete version of Proposition B.1 in [8] (see also Lemma 5.2 in [7]) that

$$\ell^2(\gamma) \in \mathcal{I}_1.$$

Our theorem as well as Theorem 5.4 in [7] provides many examples of $\ell^p(\gamma) \in \mathcal{I}_\infty$. In particular, if $1 < p < \infty$ and

$$0 = \liminf_{n\to\infty} \gamma(n) < \limsup_{n\to\infty} \gamma(n) = \infty, \qquad \lim_{n\to\infty} \frac{\gamma(n+1)}{\gamma(n)} = 1,$$

then for the pair of spaces of sequences $\ell^p(\gamma)$, $\ell^q(1/\gamma)$, dual with respect to the Cauchy duality (2.1), we have

$$\ell^p(\gamma) \in \mathcal{I}_\infty, \qquad \ell^q(1/\gamma) \in \mathcal{I}_\infty.$$

Furthermore, as a consequence of our theorem, we get that if $1 < p < \infty$, $\lim_{n\to\infty} \gamma(n+1)/\gamma(n) = 1$, and

$$\ell^p(\gamma) \in \mathcal{I}_1, \qquad \ell^q(1/\gamma) \in \mathcal{I}_1,$$

then necessarily $p = q = 2$ and $0 < \inf_n \gamma(n) \le \sup_n \gamma(n) < \infty$ and hence, the spaces $\ell^p(\gamma)$ and $\ell^q(1/\gamma)$ are both isomorphic to the space ℓ^2.

The main open question we are interested in is as follows.

Question 11.1. For $1 \le p \le \infty$, and for a weight γ such that $\sup_n \gamma(n+1)/\gamma(n) < \infty$ denote $B(p,\gamma) = \ell^p(\gamma)$ ($c_0(\gamma)$ for $p = \infty$). Characterize the pairs of (p,γ) such that (i) $B(p,\gamma) \in \mathcal{I}_1$, (ii) $B(p,\gamma) \in \mathcal{I}_\infty$.

Note that in the case $p = 2$ this question remains open only for non-regular weights γ. Two following questions are (rather concrete) partial cases of Question 1.

Question 11.2. Suppose that $2 < p \le \infty$, $\gamma(n) = (n+1)^\delta$, $\delta > 0$. Is it true that $B(p,\gamma) \in \mathcal{I}_1$?

Question 11.3. Let $1 < p < 2$. Is it true that $\ell^p \in \mathcal{I}_1$?

One more question concerns the cyclic multiplicity in ℓ^p spaces. Our theorem implies that the spaces ℓ^p, $2 < p < \infty$, and the space c_0, have **S**-invariant subspaces of arbitrary cyclic multiplicity. We have already mentioned that by Beurling's result, every **S**-invariant subspace of ℓ^2 is singly generated. Although $\ell^1 \in \mathcal{I}_1$, ℓ^1 contains **S**-invariant subspaces of arbitrary cyclic multiplicity (see [5]).

Question 11.4. What are the cyclic multiplicities of **S**-invariant subspaces in ℓ^p with $1 < p < 2$?

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