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ON CONVOLUTION EQUATIONS WITH RESTRICTIONS ON SUPPORTS

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Let X be a Banach space of sequences $\{a_n\}_{n\in\mathbb{Z}}$, and let X^* be its dual space. Denote

$$X_\pm = \bigl\{\{a_n\}_{n\in\mathbb{Z}} \in X : a_n = 0 \text{ for } n \lessgtr 0\bigr\}.$$

Suppose that the shift operator S, $S\{a_n\}_{n\in\mathbb{Z}} = \{a_{n-1}\}_{n\in\mathbb{Z}}$, and its inverse S^{-1} act continuously on X. We are interested in the structure of (closed) biinvariant (that is, S, S^{-1}-invariant) subspaces of X. By the Hahn–Banach theorem, X has proper biinvariant subspaces if and only if the convolution equation

$$u * v = \{\langle S^n u, v\rangle\}_{n\in\mathbb{Z}} = 0 \tag{1}$$

has solutions $u \in X\setminus\{0\}$, $v \in X^*\setminus\{0\}$. Finding solutions of this equation becomes much more difficult if some restrictions are imposed on the supports of u and v. For example, an interesting question is whether (1) has solutions $u \in X_+ \setminus \{0\}$, $v \in (X^*)_- \setminus \{0\}$. In this case, for every Banach space of sequences $\widetilde{X}$ with $\widetilde{X}_+ = X_+$, u generates a proper biinvariant subspace $\widetilde{E}$ of $\widetilde{X}$, and $\widetilde{E} \cap X_+$ is a proper S-invariant subspace of X_+.

Consider a weight σ, that is, a function $\sigma\colon \mathbb{Z}_+ \to (0,+\infty)$ such that

$$0 < \inf_{n\ge 0} \frac{\sigma(n+1)}{\sigma(n)} \le \sup_{n\ge 0} \frac{\sigma(n+1)}{\sigma(n)} < \infty,$$

and $\sigma(0) = 1$. We set $\sigma(-n) = 1/\sigma(n)$, $n > 0$, and define

$$\ell^2_\sigma(\mathbb{Z}) = \Bigl\{\{a_n\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n|^2\sigma^2(n) < \infty\Bigr\}, \quad \ell^2_\sigma(\mathbb{Z}_\pm) = \bigl(\ell^2_\sigma(\mathbb{Z})\bigr)_\pm.$$

In [12], Jean Esterle produced solutions of the convolution equation

$$u * v = 0, \quad u \in \ell^2_\sigma(\mathbb{Z}_+)\setminus\{0\},\ \ v \in \ell^2_\sigma(\mathbb{Z}_-)\setminus\{0\}, \tag{2}$$

for some weights σ of arbitrarily slow growth. These weights σ are monotonic, but not much more regular than that.

A. In the first part of the present paper we show that some mild regularity conditions on σ guarantee the absence of solutions to equation (2).

Clearly, (2) has no solutions in the usual ℓ^2 case where $\sigma \equiv 1$.

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A1. *Suppose that $\ell^2_\sigma(\mathbb{Z}_+)$ is a convolution Banach algebra, that is, for some $c>0$ we have $\|u*v\|\le c\|u\|\|v\|$, $u,v\in\ell^2_\sigma(\mathbb{Z}_+)$. (This is so, for instance, if the weight σ satisfies the condition*

$$\sup_{n\ge 0}\sum_{0\le k\le n}\left(\frac{\sigma(n)}{\sigma(k)\sigma(n-k)}\right)^2<\infty.)$$

Then equation (2) has no solutions.

Proof. Clearly, the weight function σ satisfies the inequality

$$\sigma(n)\le c\sigma(k)\sigma(n-k),\quad 0\le k\le n,\ \ n\ge 0.$$

Hence, $\sigma(n)\le\|S^n\|_{\ell^2_\sigma(\mathbb{Z}_+)}\le c\sigma(n)$, and the following limit exists:

$$0<\delta=\lim_{n\to+\infty}\sigma^{-1/n}(n)<\infty.$$

Without loss of generality, we assume that $\delta=1$. For every $u\in\ell^2_\sigma(\mathbb{Z}_+)$, the function $u(z)=\sum_{n\ge 0}u_nz^n$ is analytic in the unit disk $\mathbb{D}$, and is continuous up to the boundary of $\mathbb{D}$. The space of maximal ideals of $\ell^2_\sigma(\mathbb{Z}_+)$ coincides with $\overline{\mathbb{D}}$ (see, e.g., [20, Corollary 1, p. 94]).

The space $\ell^2_\sigma(\mathbb{Z}_+)$ has the (restricted) division property: for some ρ with $0<\rho=\rho(\sigma)\le 1$, if $u\in\ell^2_\sigma(\mathbb{Z}_+)$ and $u(\lambda)=0$ for some $\lambda\in\rho\mathbb{D}$, then there exists $u_\lambda\in\ell^2_\sigma(\mathbb{Z}_+)$ such that $(z-\lambda)u_\lambda(z)=u(z)$. Furthermore, if u and v satisfy (2), and $u(\lambda)=0$ for some $\lambda\in\rho\mathbb{D}$, then $u_\lambda*v=0$. Indeed, the relation $(S-\lambda)u_\lambda*v=0$ implies that $u_\lambda*v=\{c\lambda^{-n}\}_{n\in\mathbb{Z}}$. Suppose that $c\ne 0$. Since

$$|(u_\lambda*v)_n|=|\langle S^nu_\lambda,v\rangle|\le\|S\|^n\|u_\lambda\|\|v\|,$$

we get $|c\lambda^{-n}|\le\|S\|^n\le c\sigma(n)$, which is impossible for large n because $|\lambda|<1$.

Now we fix u and v satisfying (2) and consider the S-invariant subspace (ideal) I of $\ell^2_\sigma(\mathbb{Z}_+)$ consisting of all w such that $w*v=0$. Denote by $Z(I)$ the set of common zeros of I in $\overline{\mathbb{D}}$. Our previous remark shows that $Z(I)\cap\rho\mathbb{D}=\varnothing$. Furthermore, $Z(I)\cap\mathbb{T}$ is a proper closed subset of zero Lebesgue measure of $\mathbb{T}$, where $\mathbb{T}=\partial\mathbb{D}$.

We denote by $I^\perp$ the set of all $w\in\ell^2_\sigma(\mathbb{Z}_-)$ vanishing on I. For every element $w\in I^\perp$ we define its analytic transform

$$\widehat{w}(\lambda)=\langle(z-\lambda+I)^{-1},w\rangle,\quad\lambda\in\widehat{\mathbb{C}}\setminus Z(I),$$

where $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, $\widehat{w}(\infty)=0$. By [10, Theorem 2.4], $\widehat{w}$ is well defined and analytic in the (connected) domain $\widehat{\mathbb{C}}\setminus Z(I)$. If $w\ne 0$, then $\widehat{w}\not\equiv 0$. Furthermore,

$$\begin{aligned}&S^{-k}v\in I^\perp, && k\ge 0,\\ &\widehat{S^{-k}v}(\lambda)=\widehat{v}(\lambda)\lambda^{-k}, && k\ge 0,\ \ \lambda\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}.\end{aligned}$$

Hence, $\lambda^k\widehat{S^{-k}v}(\lambda)=\widehat{v}(\lambda)$, $k\ge 0$, $\lambda\in\mathbb{D}\setminus Z(I)$, and $\widehat{v}$ vanishes at zero with all its derivatives. Thus, $v=0$. Our assertion is proved. □

A2. *Suppose that σ is logarithmically concave, i.e., $\sigma(n-1)\sigma(n+1)\le\sigma^2(n)$, and that $\lim_{n\to+\infty}\sigma(n)=\infty$. Then equation (2) has no solutions.*

Proof. Fix u and v satisfying (2), $u(0)\ne 0$, and consider the S^{-1}-invariant subspace E of $\ell^2_\sigma(\mathbb{Z}_-)$ consisting of all w such that $u*w=0$. Next, consider the compression T of S on $\ell^2_\sigma(\mathbb{Z}_-)$. If $w\in E$, then $\langle u,T^n w\rangle=0$. Hence, E generates a proper T-invariant subspace E_1 of $\ell^2_\sigma(\mathbb{Z}_-)$. Finally, $u(0)\ne 0$ implies that $e_0=\{\delta_{0n}\}_{n\in\mathbb{Z}}\notin E_1$, where $\delta_{0n}=0$ if $n\ne 0$, $\delta_{0n}=1$ if $n=0$.

Without loss of generality, we assume that $\lim_{n\to\infty}\sigma^{1/n}(n)=1$. Using a discrete version of [5, Proposition B.1] (see also [11, Lemma 5.2]) and replacing σ by an equivalent weight $\widetilde\sigma$, $0<c_1\le\widetilde\sigma(n)/\sigma(n)\le c_2<\infty$, we find a continuous positive integrable function φ on $[0,1)$ such that

$$2\int_0^1 r^{2n+1}\varphi(r)\,dr=\widetilde\sigma^2(-n),\quad n\ge 0.$$

Then $\ell^2_{\widetilde\sigma}(\mathbb{Z}_-)$ is isometrically isomorphic to the weighted Bergman space B,

$$B=\Big\{f\in\mathrm{Hol}(\mathbb{D}):\int_{\mathbb{D}}|f(z)|^2\varphi(|z|)\,dm_2(z)<\infty\Big\}.$$

Furthermore, E becomes a subspace of B invariant under multiplication by z, and E_1 becomes a subspace of B invariant under the backward shift operator $f\mapsto(f-f(0))/z$, $1\notin E_1$.

Applying [1, Theorem 4.8], we conclude that E_1 is a subset of the Nevanlinna class. Take $f\in E\setminus\{0\}$, $f=f_1/f_2$, $f_1,f_2\in H^\infty$. Then $f_1\in E\setminus\{0\}$. Since $\|z^n\|_B\to 0$ as $n\to\infty$, there exists a function $g\in B$ with nontangential boundary values nowhere on $\mathbb{T}$. Then $f_1 g\in E\subset E_1$ has nontangential boundary values almost nowhere on $\mathbb{T}$, and, hence, does not belong to the Nevanlinna class. This completes the proof. □

A3. In the situations described in subsections A and B, the space $X=\ell^2_\sigma(\mathbb{Z}_+)$ satisfies the following *index* 1 *property*: for every proper S-invariant subspace E of X, the index of E (that is, the dimension of E/SE) is equal to 1. On the other hand, in the situation considered by Esterle in [12, Theorem 4.10] (see also Theorem 4.2 there), both $\ell^2_\sigma(\mathbb{Z}_+)$ and $\ell^2_\sigma(\mathbb{Z}_-)$ do not satisfy this property.

Question: Suppose that X_+ satisfies the index 1 property. Does equation (1) have solutions $u\in X_+\setminus\{0\}$, $v\in(X^*)_-\setminus\{0\}$?

B. In the second part of this paper we consider the equation

$$u*v=0,\quad u\in X_+\setminus\{0\},\ v\in X^*\setminus\{0\}. \tag{3}$$

If u and v satisfy (3), then u generates a biinvariant subspace $E\subset X$ such that $E_1=E\cap X_+$ is a proper S-invariant subspace of X, and E is generated by E_+. In the terminology of [13], such subspaces are called *analytic* subspaces.

For some weighted ℓ^2 spaces of sequences with asymmetric weights, every biinvariant subspace is analytic [13, 14]. For a short historical survey of related results on translation invariant subspaces, see [19].

Suppose that $X=\ell^2_\sigma(\mathbb{Z})$ for a weight function σ (and, consequently, $X=X^*$). If $\sigma\equiv 1$, then equation (3) has no solutions. If σ decays sufficiently fast,

$$\frac{n}{\log^\alpha n}\le\log\frac{1}{\sigma(n)}=o(n),\quad n\to+\infty,$$

for some $0<\alpha<\infty$, then equation (3) has solutions (see [6, Theorem 1.3]). For weights σ decreasing polynomially, $\sigma(n)=(n+1)^{-A}$, $n\ge 0$, for some $A>0$, the existence of solutions of (3) is an open problem (see, e.g., [15, Section 8.8.11]).

Finally, we consider growing weights σ. As in part A1, we deal with the case where $\ell^2_\sigma(\mathbb{Z}_+)$ is a Banach algebra.

B1. Theorem. *Let $\ell^2_\sigma(\mathbb{Z}_+)$ be a convolution Banach algebra, and let* $\lim_{n\to+\infty}\sigma(n)^{1/n}=1$*. Suppose that either*

$$1<\liminf_{n\to+\infty}\frac{\log\sigma(n)}{\log n}\le\limsup_{n\to+\infty}\frac{\log\sigma(n)}{\log n}<\infty, \tag{I}$$

or

$$\lim_{n\to+\infty}\frac{\log\sigma(n)}{\log n}=\infty. \tag{II}$$

In the second case we assume that σ extends to a smooth function on $\mathbb{R}_+$ such that the functions φ_k, $\varphi_k(t)=\log[\sigma(t)/t^k]$, $k\ge 0$, are concave, and the function ψ, $\psi(t)=\log\sigma(\exp t)$, is convex for large t.

Then equation (3) *has no solutions.*

Proof. Arguing as before, we fix u and v satisfying (3), and consider the S-invariant subspace (ideal) I of $\ell^2_\sigma(\mathbb{Z}_+)$ consisting of all w such that $w*v=0$. As in Subsection A1, $Z(I)\cap\mathbb{T}$ is a proper closed subset of $\mathbb{T}$. Denote by v^k the element of $\ell^2_\sigma(\mathbb{Z}_-)$ given by

$$(v^k)_n=v_{k+n},\quad n\le 0.$$

Then $v^k\in I^\perp$, $k\in\mathbb{Z}$. We define the analytic transform as in A1, obtaining

$$\widehat{v^k}(\lambda)=\widehat{v^{k-1}}(\lambda)\lambda^{-1}+v_k\lambda^{-1},\quad k\in\mathbb{Z},\ \lambda\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}.$$

Let

$$\widehat{v^0}(\lambda)=\sum_{n\ge0}a_n\lambda^n,\quad \lambda\in\rho\mathbb{D}.$$

Then by induction we obtain

$$\widehat{v^k}(\lambda)=\sum_{n\ge0}a_{n+k}\lambda^n,\quad \lambda\in\rho\mathbb{D},$$
$$a_k=-v_{k+1},$$

and, as a result,

$$\widehat{v^0}(\lambda)=-\sum_{n\ge0}v_{n+1}\lambda^n,\quad \lambda\in\rho\mathbb{D}.$$

Since the power series on the right converges in $\mathbb{D}$, we see that $\widehat{v^0}$ is analytic in $\mathbb{D}$. Note that by definition,

$$\widehat{v^0}(\lambda)=-\sum_{n\ge0}v_{-n}\lambda^{-n-1},\quad \lambda\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}.$$

If the class $\ell^2_\sigma(\mathbb{Z}_+)$ is quasianalytic (i.e., the conditions $u\in\ell^2_\sigma(\mathbb{Z}_+)$ and $u^{(k)}(\lambda)=0$ $(k=0,1,\dots)$ for some $\lambda\in\overline{\mathbb{D}}$ imply $u=0$), then $Z(I)$ is a finite subset of $\mathbb{D}$, and the results of Domar [9] show that the ideal I is determined by its zero set if the multiplicities are taken into account. Accordingly, $I^\perp$ is finite-dimensional. Hence, $v^0|\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ is a finite linear combination of elementary fractions $1/(z-\lambda)^k$, $\lambda\in\mathbb{T}$, and v^0 cannot be smooth on $\overline{\mathbb{D}}$.

Consider

$$\mathcal{H}=\Big\{\sum_{n\ge0}a_nz^n,\ z\in\overline{\mathbb{D}}:\{a_n\}\in\ell^2_\sigma(\mathbb{Z}_+)\Big\},$$
$$\mathcal{G}=\Big\{\sum_{n\ge0}a_{-n}z^{-n-1},\ z\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}:\{a_n\}\in\ell^2_\sigma(\mathbb{Z}_-)\Big\}.$$

To complete the proof of our claim, it remains to establish the following result.

Proposition. *Suppose that the class $\mathcal{H}$ is nonquasianalytic. Then no function $h_1 \in \mathcal{H} \setminus \{0\}$ extends analytically to a function $g \in \mathcal{G}$ across a subset $\mathbb{T} \setminus F$ of the unit circle such that $h_2 \big| F = 0$ for some $h_2 \in \mathcal{H} \setminus \{0\}$.*

Proof. We start with a rather standard argument relating the growth of functions in $\mathcal{G}$ with the rate of decay of functions in $\mathcal{H}$ near their zeros of infinite order in case (II).

Since $\log \sigma$ is concave, for every small $\varepsilon > 0$ the function $t \mapsto \sigma(t)e^{-\varepsilon t}$ attains its maximal value $M(\varepsilon)$ at a unique point t_0 such that $\varepsilon = \sigma'(t_0)/\sigma(t_0)$. Then $M(\varepsilon) \nearrow +\infty$, $\varepsilon t_0 \nearrow +\infty$ as $\varepsilon \to 0$.

First, we prove that

$$\sum_{n\ge 0} \sigma^2(n)e^{-2\varepsilon n} \le ct_0M^2(\varepsilon), \quad \varepsilon > 0, \tag{4}$$

$$t_0^k \le c(k)M(\varepsilon), \quad k \ge 1. \tag{5}$$

Indeed, since the φ_k are concave, it follows that for every k, for sufficiently large t_0, and for $n \ge t_0$ we have

$$\varphi_k(n) - \varphi_k(t_0) \le (n - t_0)\varphi_k'(t_0) = (n - t_0)\Big(\varepsilon - \frac{k}{t_0}\Big),$$

and for $k = 1$ we obtain

$$\sigma(n)e^{-\varepsilon n} = ne^{\varphi_1(n)-\varepsilon n} \le ne^{\varphi_1(t_0)-\varepsilon t_0}e^{-(n-t_0)/t_0},$$

$$\sum_{n\ge t_0} \sigma^2(n)e^{-2\varepsilon n} \le \sum_{n\ge t_0} \Big(\frac{n}{t_0}\Big)^2 M^2(\varepsilon)e^{-2(n-t_0)/t_0} \le ct_0M^2(\varepsilon).$$

In a similar way, for $n < t_0$ we have

$$\varphi_k(t_0) - \varphi_k(n) \ge (t_0 - n)\varphi_k'(t_0),$$

and we conclude that

$$\sum_{n<t_0} \sigma^2(n)e^{-2\varepsilon n} \le ct_0M^2(\varepsilon)$$

and that

$$\log \sigma(t_0) \ge c(k) + k \log t_0 + t_0(\log \sigma)'(t_0) = c(k) + k \log t_0 + t_0\varepsilon,$$

$$M(\varepsilon) = \sigma(t_0)e^{-\varepsilon t_0} \ge c(k)t_0^k.$$

This proves relations (4) and (5).

Since ψ is convex, for large a the function $t \mapsto t^a/\sigma(t)$ attains its maximal value at a unique point t_1 such that $a = t_1\sigma'(t_1)/\sigma(t_1)$. Therefore,

$$\sum_{n\ge 1} \frac{n^{2a-2}}{\sigma^2(n)} \le c\frac{t_1^{2a}}{\sigma^2(t_1)}.$$

Put $a = \varepsilon t_0$. Then $t_1 = t_0$. Choosing $s \in \mathbb{Z}_+$ such that $s \le a - 1 < s + 1$ and using the Stirling formula, we obtain

$$\begin{aligned}\min_{k\in\mathbb{Z}_+}\Bigg[\frac{\varepsilon^k}{k!}\Bigg(\sum_{n\ge 1}\frac{n^{2k}}{\sigma^2(n)}\Bigg)^{1/2}\Bigg] &\le \frac{\varepsilon^s}{s!}\Bigg(\sum_{n\ge 1}\frac{n^{2a-2}}{\sigma^2(n)}\Bigg)^{1/2}\\ &\le c\frac{e^s\varepsilon^s}{s^s}\frac{t_0^a}{\sigma(t_0)} = c\frac{(\varepsilon e)^{s-a}e^{\varepsilon t_0}(\varepsilon t_0)^a}{s^s\sigma(t_0)} = c\frac{e^{s-a}}{M(\varepsilon)}\Big(\frac{a}{s}\Big)^s\Big(\frac{a}{\varepsilon}\Big)^{a-s}\\ &\le c\frac{t_0^2}{M(\varepsilon)}.\end{aligned} \tag{6}$$

Now we deal with elements of the spaces $\mathcal{G}$ and $\mathcal{H}$ in case (II). By the Cauchy–Schwarz inequality and by (4) and (5), we obtain

$$
\begin{aligned}
|g(z)| &\le \|g\|_{\mathcal{G}} \Big(\sum_{n\ge 0} \sigma^2(n)|z|^{-2n-2} \Big)^{1/2} \\
&\le c\|g\|_{\mathcal{G}} M^2(\log|z|), \quad |z|>1,\ g\in\mathcal{G}.
\end{aligned} \tag{7}
$$

If $h\in\mathcal{H}$, $h(z)=\sum_{n\ge 0} a_n z^n$, and h vanishes with all its derivatives at a point $\zeta\in\mathbb{T}$, then, by the Taylor formula,

$$
|h(z)| \le \min_{k\in\mathbb{Z}_+} \Big[\frac{|z-\zeta|^k}{k!} \sum_{n\ge 1} n^k |a_n| \Big], \quad z\in\overline{\mathbb{D}},
$$

and the Cauchy–Schwarz inequality together with (5), (6) yields

$$
\begin{aligned}
|h(z)| &\le \|h\|_{\mathcal{H}} \min_{k\in\mathbb{Z}_+} \Big[\frac{|z-\zeta|^k}{k!} \Big(\sum_{n\ge 1} \frac{n^{2k}}{\sigma^2(n)} \Big)^{1/2} \Big] \\
&\le c\|h\|_{\mathcal{H}} M^{-1/2}(|z-\zeta|), \quad z\in\overline{\mathbb{D}}.
\end{aligned} \tag{8}
$$

Next, we pass to case (I). We get

$$
|g(z)| \le c\|g\|_{\mathcal{G}} \frac{1}{(|z|-1)^c}, \quad 1<|z|<2,\ g\in\mathcal{G}. \tag{9}
$$

Furthermore, $\mathcal{H}$ is continuously embedded in a Lipschitz class on $\overline{\mathbb{D}}$. Therefore, for some $\alpha>0$ and for every $h\in\mathcal{H}$ vanishing at a point $\zeta\in\mathbb{T}$, we get

$$
|h(z)| \le c\|h\|_{\mathcal{H}} |z-\zeta|^\alpha, \quad z\in\overline{\mathbb{D}}. \tag{10}
$$

Now we suppose that $g\in\mathcal{G}$, $h_1,h_2\in\mathcal{H}\setminus\{0\}$, h_2 vanishes on a closed subset F of $\mathbb{T}$, and h_1 extends to g across $\mathbb{T}\setminus F$. Without loss of generality, we assume that $|h_1(z)|\le 1$ and $|h_2(z)|\le 1$ for $z\in\mathbb{T}$.

In case (I), we cover F by a sequence of disjoint arcs J_n, $J_n\cap F\ne\varnothing$, for every arc $J_n=\{e^{i\theta}:|\theta-\theta_n|\le\delta_n\}$ we introduce the "square" $Q_n=\{re^{i\theta}:|\theta-\theta_n|\le\delta_n, 0\le r-1\le\delta_n\}$, and consider the domain $\Omega=\widehat{\mathbb{C}}\setminus\big(\overline{\mathbb{D}}\cup\bigcup Q_n\big)$. The boundary of Ω is a subset of $\mathbb{T}\cup\bigcup\partial Q_n$. Fix $z\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. If $\max|J_n|$ is sufficiently small, say, less than $(|z|-1)/2$, then, by the theorem on two constants and by (9),

$$
\begin{aligned}
\log|g(z)| &\le \int_{\partial\Omega\setminus\mathbb{T}} \log|g(w)|\omega(z,dw,\Omega) = \sum_n \int_{\partial Q_n\setminus\mathbb{T}} \log|g(w)|\omega(z,dw,\Omega) \\
&\le c(z) \sum_n |J_n| \log\frac{1}{|J_n|},
\end{aligned}
$$

where $\omega(z,dw,\Omega)$ is harmonic measure on $\partial\Omega$ with respect to the point z in the domain Ω, and $|J|$ is the length of the arc J. Furthermore, by (10),

$$
\sum_n |J_n| \log\frac{1}{|J_n|} \le -c_1 \int_{\bigcup J_n} \log(c|h_2(z)|)\,dm(z).
$$

Since the integral

$$\int_{\mathbb{T}} \log(|h_2(z)|)\, dm(z)$$

converges, we can find a covering $\{J_n\}$ of F such that

$$-\int_{\bigcup J_n} \log(c|h_2(z)|)\, dm(z)$$

is arbitrarily small. Hence, $|g(z)| \le 1$ for $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and the proof is complete ($g - h_1$ is an $L^2(\mathbb{T})$-function vanishing outside a set of zero measure, hence both g and h_1 are equal to 0).

In case (II), first we assume that

$$\int_0 \log M(t)\, dt < \infty. \tag{11}$$

The set F consists of a countable set of points z_k and a perfect set F_0 of zero measure; h_2 has zeros of infinite order at the points of F_0. As before, we cover F_0 by a sequence of disjoint arcs J_n. Estimates (7) and (8) together with an argument similar to that used in case (I) show that J_n can be chosen in such a way that the expression

$$\begin{aligned}\sum_n \int_{\partial Q_n \setminus \mathbb{T}} \log|g(w)|\omega(z, dw, \Omega) &\le c(z) \sum_n \int_0^{|J_n|} \log M(t)\, dt \\ &\le -c(z) \int_{\bigcup J_n} \log(c|h_2(z)|)\, dm(z)\end{aligned}$$

is arbitrarily small. Furthermore, we can cover the (countable) set $\{z_k\} \setminus \bigcup J_n$ by a sequence of disjoint small arcs J'_n such that the sum

$$\sum_n \int_{\partial Q'_n \setminus \mathbb{T}} \log|g(w)|\omega(z, dw, \Omega) \le c(z) \sum_n \int_0^{|J'_n|} \log M(t)\, dt$$

is arbitrarily small. After that, the proof is completed as in case (I).

Finally, if (11) fails, then inequality (8) shows that $\mathcal{H}$ is a quasianalytic class. □

B2. By using an argument similar to that in [21], if an analog of Proposition can be obtained in the case where $\mathcal{H}$ is a quasianalytic class (with $F = \{1\}$).

Note that functions $h_1 \in \mathcal{H}$ may extend analytically across $\mathbb{T} \setminus \{1\}$ to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with growth there slightly more rapid than that permitted in $\mathcal{G}$ (see [7, Example 7.1]). On the other hand, if we consider spaces $\widehat{\mathcal{G}}$ somewhat smaller than $\mathcal{G}$ (and $\mathcal{H}$ still quasianalytic), then the Levinson–Cartwright theorem and a result by the author [3, Theorem 1] imply that $h_1 \in \mathcal{H}$ cannot extend to $g \in \widehat{\mathcal{G}}$ even across a set of positive measure (extension via nontangential boundary values); see also a related result of Beurling [2, Corollary 4.2, p. 407] for $\widehat{\mathcal{G}} = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$.

Under some regularity and growth conditions on M (essentially, if

$$\lim_{\delta \to 0} \frac{\log M(\delta)}{\log(1/\delta)} = \infty, \qquad \int_0^\delta \log M(t)\, dt \asymp \delta \log M(\delta)\,),$$

our Proposition follows from a result by Hruščëv (see [17, Theorem 9.1], where no smoothness conditions were imposed on h_1).

The relationship between the space of smooth functions $\mathcal{H}$, the space $\mathcal{G}$, and the class of zero sets $F \subset \mathbb{T}$ of functions in $\mathcal{H}$ such that the claim of our Proposition is fulfilled deserves additional study. In particular, we may need results similar to that proposition when solving equation (3) for weighted ℓ^2 spaces of sequences with asymmetric weights.

Note that for nonquasianalytic (in the closed unit disk) Gevrey classes, the boundary zero sets are described in a rather complicated fashion [16]. Our elementary estimate (8) should be replaced by a much better estimate from [4, Theorem 2.6]. For even more precise estimates of the decay near a zero of infinite order for elements of nonquasianalytic classes on $\mathbb{T}$, see [18].

In the proof of Theorem B1 (case (II)) we could use an argument from Subsection A2 and reduce our problem to an analog of Proposition with $\mathcal{G}$ replaced by $\mathcal{G} \cap N$, where N is the Nevanlinna class in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. A related problem was considered in [8].

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