ON THE MINIMUM OF HARMONIC FUNCTIONS

ALEXANDER BORICHEV

ABSTRACT. Let u be a function harmonic in the unit disc or in the plane, and let $u(z) \le M(|z|)$ for a majorant M. We formulate conditions on M that guarantee that $u(z) \ge -(1+o(1))M(|z|)$ for $|z| \to 1$ in the disc and for $|z| \to \infty$ in the plane.

1. **Introduction.**

Let M be a non-decreasing function on $(0,1)$, let u be harmonic in the unit disc, $u(0)=0$, and

$$B(r,u) \stackrel{\text{def}}{=} \max_{|z|=r} u(z) \le M(r), \qquad 0 \le r < 1.$$

The question we are interested in is how to estimate the function

$$A(r,u) \stackrel{\text{def}}{=} \max_{|z|=r} \bigl[-u(z)\bigr].$$

The Carathéodory inequality yields that if $M \equiv 1$, then $A(r,u) \le 2/(1-r)$. A theorem of M. Cartwright [3] claims that if $M(r) = (1-r)^{-a}$, for some $a > 1$, then $A(r,u) \le C(a)(1-r)^{-a}$, for some $C(a) > 1$ independent of u. N. Nikolskiĭ proved in [13, Section 1.3, Theorem 2] that for sufficiently regular M such that

$$\lim_{r\to 1} \frac{\log M(r)}{\log\bigl[1/(1-r)\bigr]} = \infty, \tag{1.1}$$

we have

$$A(r,u) \le C_1 + CM(r), \qquad 0 \le r < 1, \tag{1.2}$$

with C_1 independent of u and a large absolute constant C. For other results in this direction see [13, Section 1.3], [10].

Replacing u by $-u$ we obtain that the estimate (1.2) cannot hold with $C < 1$ for unbounded M. The aim of this paper is to improve

Key words and phrases. Harmonic functions, minimum and maximum values.
2000 *Mathematical Subject Classification* 31A05, 30D15, 30H05.

the above mentioned result by Nikolskiĭ: we prove that for sufficiently regular M satisfying (1.1) we have

$$A(r,u) \le (1+o(1))M(r), \qquad r \to 1. \tag{1.3}$$

Thus, the maximum and the minimum values of harmonic functions of rapid growth are quite close. One immediate application of this result is that if f is a function analytic in the unit disc without zeros,

$$\limsup_{|z|\to 1} \frac{\log|f(z)|}{M(|z|)} = 1,$$

where M is sufficiently regular and satisfies (1.1), then

$$\liminf_{|z|\to 1} \frac{\log|f(z)|}{M(|z|)} = -1.$$

In particular, this shows that the functions in the weighted Bergman spaces constructed in [2] do have extremal growth and decay; to establish this fact was the initial motivation for writing this paper.

Also, we consider the majorants M of moderate growth,

$$1 < \lim_{r\to 1} \frac{\log M(r)}{\log\bigl[1/(1-r)\bigr]} = \Delta < \infty.$$

In this case, the best possible analog of (1.3) is

$$A(r,u) \le \Bigl[\Bigl(\cos\frac{\pi}{\Delta+1}\Bigr)^{-(\Delta+1)} + o(1)\Bigr] M(r), \qquad r \to 1.$$

This result improves the estimates by M. Cartwright mentioned above.

For results on the analogous problems for functions harmonic in the plane see, for instance, [7]. We mention here a result by A. Wiman [15]: if a non-constant u is harmonic in the plane, then

$$A(r,u) = (1+o(1))B(r,u),$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Furthermore, it follows from the results of S. Apresyan [1] that for sufficiently regular M we have

$$A(r,u) \le C_1 + CM(r), \qquad 0 \le r < \infty,$$

with C_1 independent of u and large absolute constant C.

We prove, as in the case of the disc, that for sufficiently regular M, if u is harmonic in the plane, $u(0) = 0$, and

$$B(r,u) \le M(r), \qquad 0 \le r < \infty,$$

then

$$A(r,u) \le (1+o(1))M(r), \qquad r \to \infty.$$

Our method is close to that used in [13]. The main difference consists in using the following result that seems to be of independent interest: *If v is a function harmonic in the closed upper half-plane,*

$$v(z) \le |z|^2, \tag{1.4}$$

then

$$v(iy) \ge -y^2 + y\big(v(i)+1\big), \qquad y \ge 1. \tag{1.5}$$

The plan of the paper is as follows. We give two (quite different) proofs of the estimate (1.5) for harmonic functions satisfying (1.4) in Section 2. The first one uses the Ahlfors–Carleman estimates of the harmonic measure; the second one, due to M. Sodin, uses the Nevanlinna–Poisson representation for harmonic functions of finite order in the half-plane. In Section 3, we obtain the main results on the minimum of functions harmonic in the disc and in the plane. Finally, in Section 4, we produce an elementary example showing that the estimate (1.3) may fail if M is just supposed to be log-convex (that is the function $x \mapsto M(\exp(-x))$ is convex).

The author is grateful to Misha Sodin for helpful discussions.

2. **An auxiliary lemma.**

Lemma 2.1. *Let u be a function harmonic in the upper half-plane $\mathbb{C}_+$, and continuous in the closed upper half-plane $\overline{\mathbb{C}}_+$, such that*

$$\begin{aligned} &u(x) \le x^2, && x \in \mathbb{R}, \\ &u(z) \le o(|z|^3), && z \in \mathbb{C},\ |z| \to \infty. \end{aligned} \tag{2.1}$$

Then the function $y \mapsto \dfrac{u(iy)+y^2}{y}$ does not decrease on $[0,+\infty)$.

As a matter of fact, the assertion of Lemma 2.1 is somewhat stronger than that used later on, in Section 3, where (2.1) is replaced by (1.4).

The first proof of Lemma 2.1. Put

$$v(z) = \frac{u(x+iy)+u(-x+iy)}{2} - \operatorname{Re} z^2.$$

Then

$$v(x+iy) = v(-x+iy), \qquad x+iy \in \overline{\mathbb{C}}_+,$$

$$v(x) \le 0, \qquad x \in \mathbb{R}, \tag{2.2}$$

$$v(z) \le o(|z|^3), \qquad z \in \mathbb{C}_+,\ |z| \to \infty, \tag{2.3}$$

and we need to verify that $v(iy) \ge yv(i)$, $y \ge 1$.

Otherwise, if $v(iy) < yv(i)$ for some $y > 1$, then, changing, if necessary, $v(z)$ by $v(z) + cy$, we get $v(i) > 0$ and

$$\inf_{1<y<\infty} v(iy) < 0. \tag{2.4}$$

Fix a connected component Ω of the set $\{z \in \mathbb{C}_+ : v(z) > 0\}$ containing the point i. Clearly, Ω is unbounded and symmetric with respect to $i\mathbb{R}_+$, $v\big|\partial\Omega = 0$. If $\Omega \cap i\mathbb{R}_+$ were unbounded, then by the maximum principle we would get that $i[1, +\infty) \subset \Omega$ which is impossible because of (2.4). Hence, for some $r_0 > 0$ we have

$$\Omega \setminus \bigl\{z \in \mathbb{C}_+ : |z| \le r_0\bigr\} = \Omega_+ \cup \Omega_-,$$
$$\Omega_\pm \subset \bigl\{z \in \mathbb{C}_+ : \operatorname{Re} z \gtrless 0\bigr\}.$$

Next, we fix a (symmetric) connected component Ω_0 of the set $\{z \in \mathbb{C}_+ : v(z) < 0\}$ such that $\Omega_0 \cap i[1, +\infty) \ne \emptyset$. If $\partial\Omega_0 \cap \mathbb{R} \ne \emptyset$, then by the maximum principle, we would get $v(i) \le 0$, which is impossible. Hence, Ω_0 is unbounded, $u\big|\partial\Omega_0 = 0$. Denote

$$m(r) = \max\bigl\{v(z) : z \in \Omega_\pm,\ |z| = r\bigr\},$$
$$m_0(r) = \max\bigl\{-v(z) : z \in \Omega_0,\ |z| = r\bigr\}.$$

Applying an argument essentially due to A. Beurling (see [4, Lemma 4]) that uses the Ahlfors–Carleman estimate of the harmonic measure, we obtain for some C and sufficiently large r that

$$\frac{2}{\log m(r)} + \frac{1}{\log m_0(r)} \le \frac{1}{\log r - C}.$$

Together with (2.3) this yields

$$\lim_{r\to\infty} \frac{m_0(r)}{r^3} = +\infty.$$

For sufficiently large $r > 0$ denote $\Omega_r = \bigl\{z \in \Omega_0 : v(z) \le -m_0(r)\bigr\}$. Then, by the maximum principle, for every $t \ge r$ we have

$$\bigl\{z \in \Omega_r : |z| = t\bigr\} \ne \emptyset.$$

Applying the theorem on two constants to v in the domains $O_r = \{z \in \mathbb{C}_+ : r < |z| < 3r\}$, and using the fact that by the Hall lemma (see, for instance, [5, p.367]),

$$\omega(2r\gamma, \Omega_r \cap O_r, O_r) \ge c > 0, \qquad \gamma \in \Gamma = \bigl\{e^{it} : \pi/4 \le t \le 3\pi/4\bigr\},$$

for an absolute constant c, we conclude that for sufficiently large r,

$$v(2r\gamma) \le \sup_{z\in O_r} v(z) - cm_0(r) \le -r^3, \qquad \gamma \in \Gamma.$$

One way to complete the proof is now to use Theorem 2 of [9, Lecture 26] to get a contradiction.

Otherwise, we apply once again the theorem on two constants to v in the domains $U_r = \{z \in \mathbb{C}_+ : |z| < 2r\}$, together with the standard estimate

$$\omega(i, 2r\Gamma, U_r) \geq \frac{c}{r}, \qquad r > 1,$$

to get finally for some $c > 0$,

$$v(i) \leq -cr^2,$$

which is impossible for large r. □

The second proof of Lemma 2.1. We start with v harmonic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, satisfying conditions (2.2) and (2.3). Then we use the Nevanlinna–Poisson representation for harmonic functions of order 3 in the half-plane (see [6, Part I, Theorems 1.1 and 3.3]) to get

$$v(z) = \operatorname{Im}\Bigg[az + bz^2 + cz^3 + \frac{z^4}{\pi}\int_{|t|\geq 1} \frac{v(t)}{t^4(t-z)}dt + \frac{1}{\pi}\int_{|t|\leq 1} \frac{v(t)}{t-z}dt\Bigg], \qquad z \in \mathbb{C}_+, \quad (2.5)$$

for some real a, b, c. Then,

$$v(iy) = ay - cy^3 + \frac{y^5}{\pi}\int_{|t|\geq 1} \frac{v(t)}{t^4(t^2+y^2)}dt + \frac{y}{\pi}\int_{|t|\leq 1} \frac{v(t)}{t^2+y^2}dt, \quad y > 0.$$

If

$$\int_{|t|\geq 1} \frac{v(t)}{t^4}dt = -\infty,$$

then

$$\lim_{y\to\infty} \frac{y^2}{\pi}\int_{|t|\geq 1} \frac{v(t)}{t^4(t^2+y^2)}dt = -\infty,$$

and

$$\lim_{y\to\infty} \frac{v(iy)}{y^3} = -\infty,$$

which is impossible by the Phragmén–Lindelöf theorem applied to v in $\{z \in \mathbb{C}_+ : \pi/6 \leq \arg z \leq \pi/2\}$ and by (2.3).

Hence,

$$v(iy) = ay + dy^3 - \frac{y^3}{\pi}\int_{|t|\geq 1} \frac{v(t)}{t^2(t^2+y^2)}dt + \frac{y}{\pi}\int_{|t|\leq 1} \frac{v(t)}{t^2+y^2}dt, \quad y > 0,$$

where

$$d = -c + \frac{1}{\pi}\int_{|t|\geq 1} \frac{v(t)}{t^4}dt,$$

and

$$\lim_{y\to\infty}\frac{v(iy)}{y^3}=d.$$

Again by the Phragmén–Lindelöf theorem and by (2.3) we get $d=0$. Therefore,

$$v(iy)=ay-\frac{y^3}{\pi}\int_{|t|\ge 1}\frac{v(t)}{t^2(t^2+y^2)}dt+\frac{y}{\pi}\int_{|t|\le 1}\frac{v(t)}{t^2+y^2}dt,\quad y>0,$$

and

$$\frac{v(iy)}{y}=a-\frac{1}{\pi}\int_{|t|\ge 1}\frac{v(t)}{t^2(1+t^2/y^2)}dt+\frac{1}{\pi}\int_{|t|\le 1}\frac{v(t)}{t^2+y^2}dt,\quad y>0.$$

Since every term in the right-hand side of the last equation does not decrease in y, we get the assertion of the lemma. □

3. **Main results.**

We suppose that M is a positive C^2-smooth increasing function on $(0,1)$, denote $\psi(t)=\log M(1-e^{-t})$, and assume that

$$\lim_{t\to\infty}\psi'(t)=+\infty, \tag{3.1}$$

and that for some $\delta>0$,

$$|\psi''(t)|=O\big((\psi'(t))^{2-\delta}\big),\qquad t\to\infty. \tag{3.2}$$

Theorem 3.1. *If u is harmonic in the unit disc, $u(0)=0$, and*

$$B(r,u)\le M(r),\qquad 0\le r<1,$$

then for every $\varepsilon>0$ there exists $K(\varepsilon,M)$ independent of u such that

$$A(r,u)\le K(\varepsilon,M)+(1+\varepsilon)M(r),\qquad 0\le r<1. \tag{3.3}$$

Proof. We follow the argument in [13, Section 1.2] with minor changes. It suffices to prove that for arbitrary $\varepsilon>0$,

$$u(r)\le K(\varepsilon,M)+(1+\varepsilon)M(r),\qquad 0\le r<1,$$

where $K(\varepsilon,M)$ does not depend on u. Applying this to the functions $z\mapsto u(ze^{i\theta})$, $0\le\theta<2\pi$, we get (3.3).

Without loss of generality we assume that $\psi'(x) > 2$, $x \geq 0$. Set

$$\begin{aligned} P(z) &= \log \frac{1}{1-z}, \\ s(x) &= \frac{\pi}{\psi'\big(\max\{x,0\}\big)}, \qquad x \in \mathbb{R}, \\ \Omega &= \big\{x+iy : x > -\beta,\ |y| < s(x)\big\}, \\ \Pi &= \big\{x+iy : x > -1,\ |y| < \pi/2\big\}, \end{aligned}$$

where $\beta > 0$ is so small that $\operatorname{Clos} P^{-1}(\Omega) \subset \mathbb{D} \cup \{1\}$.

Let Z be the conformal map $\Omega \to \Pi$ such that $Z(+\infty) = +\infty$, $Z(0) = 0$. Then

$$\Phi = P^{-1} \circ Z^{-1} \circ \log$$

is a conformal map from $\exp \Pi$ onto $P^{-1}(\Omega)$ (see Figure 1), and the function $v = u \circ \Phi$ is harmonic on $\exp \Pi$, $v(1) = u(0) = 0$. Our next step is to estimate v from above.

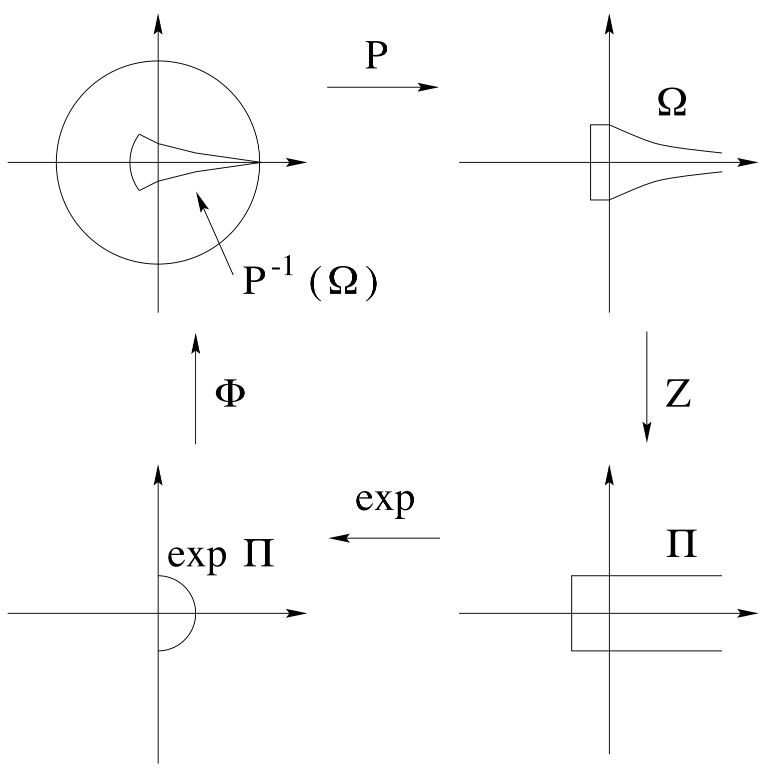

Figure 1.

Let $t+i\gamma\in\Omega$. Then $|\gamma|<s(t)\to 0$ as $t\to\infty$, and we have

$$|P^{-1}(t+i\gamma)|=|1-\exp(-t-i\gamma)|=\sqrt{(1-e^{-t}\cos\gamma)^2+(e^{-t}\sin\gamma)^2}$$
$$\le 1-\frac{1}{2}e^{-t}\cos\gamma\le 1-\exp[-t-C\gamma^2]\le 1-\exp[-t-C(s(t))^2],$$

for an absolute constant C, and

$$M\big(|P^{-1}(t+i\gamma)|\big)\le M\big(1-\exp[-t-C(s(t))^2]\big)$$
$$=\exp\psi\big(t+\pi^2C(\psi'(t))^{-2}\big)\le\exp[\psi(t)+o(1)],\qquad t\to\infty.\quad(3.4)$$

Here we use that if $x>t$, $\psi'(x)=2\psi'(t)$, and $\psi'(y)<2\psi'(t)$ for all $y\in(t,x)$, then by (3.2) we have $x-t\ge c[\psi'(t)]^{\delta-1}$. Hence, $\psi'(p)<2\psi'(t)$, $t\le p\le t+\pi^2C/(\psi'(t))^2$, and the last inequality in (3.4) follows.

Furthermore, the estimates of conformal maps of infinite strips by S. Warschawski ([14], see also [11, Theorem 8]) show that

$$|\exp Z(t+i\gamma)|=\exp\operatorname{Re}\,Z(t+i\gamma)=\exp\left[\frac{\pi}{2}\int_0^t\frac{dx}{s(x)}+C_1+o(1)\right]$$
$$=(C_2+o(1))\exp[\psi(t)/2],\qquad t\to\infty,\quad(3.5)$$

where C_1,C_2 depend only on ψ. We need only to check that by (3.1), (3.2),

$$\int^\infty\frac{(s'(x))^2\,dx}{s(x)}=\pi\int^\infty\frac{(\psi''(x))^2\,dx}{(\psi'(x))^3}\le C\int^\infty\frac{\psi''(x)\,dx}{(\psi'(x))^{1+\delta}}<\infty.$$

Now, by (3.4) and (3.5) we get

$$M\big(|\Phi(\exp Z(t+i\gamma))|\big)=M\big(|P^{-1}(t+i\gamma)|\big)\le\exp[\psi(t)+o(1)]$$
$$=(C_2^{-2}+o(1))|\exp Z(t+i\gamma)|^2,\qquad t+i\gamma\in\Omega,\ t\to\infty,$$
$$M(|\Phi(w)|)\le(C_2^{-2}+o(1))|w|^2,\qquad w\in\exp\Pi,\ |w|\to\infty.$$

Hence, for every $\varepsilon>0$ there exists $K_1(\varepsilon,M)$ independent of v such that

$$v(w)\le K_1(\varepsilon,M)+(1+\varepsilon)C_2^{-2}|w|^2,\qquad w\in\exp\Pi.$$

Now we diverge from the argument in [13], and apply Lemma 2.1 to v, (or, rather to the function $z\mapsto v(-iz+1/2)$) using that $v(1)=0$. As a result, we obtain that for some $K_2(\varepsilon,M)$ independent of v,

$$v(x)\ge K_2(\varepsilon,M)-(1+2\varepsilon)C_2^{-2}x^2,\qquad x\ge 1.\quad(3.6)$$

Furthermore,

$$M(|P^{-1}(t)|)=\exp\psi(t),\qquad t\ge 0,$$

and by (3.5), we have

$$|\exp Z(t)|=(C_2+o(1))\exp[\psi(t)/2],\qquad t\to\infty,$$

hence

$$M(|\Phi(x)|) \geq K_3(\varepsilon, M) + (1-\varepsilon)C_2^{-2}x^2, \qquad x \geq 1. \tag{3.7}$$

Finally, by (3.6) and (3.7) we obtain

$$u(r) \geq K_4(\varepsilon, M) - (1+4\varepsilon)M(r), \qquad 0 \leq r < 1,$$

and the theorem is proved. □

A. Atzmon noticed that the assertion of Theorem 3.1 fails for majorants M of moderate growth (for which (3.1) does not hold), with $u(z) = -\operatorname{Re}(1-z)^{-a}$.

Now we suppose that M is a positive C^1-smooth increasing function on $(0,1)$, denote $\psi(t) = \log M(1-e^{-t})$, and assume that the function ψ' has bounded variation and

$$1 < \lim_{t\to\infty} \psi'(t) = \Delta < +\infty.$$

Theorem 3.2. (A) *If u is harmonic in the unit disc, $u(0)=0$, and*

$$B(r,u) \leq M(r), \qquad 0 \leq r < 1,$$

then for every $\varepsilon > 0$ there exists $K(\varepsilon, M)$ independent of u such that

$$A(r,u) \leq K(\varepsilon, M) + \Big[\Big(\cos\frac{\pi}{\Delta+1}\Big)^{-(\Delta+1)} + \varepsilon\Big]M(r), \qquad 0 \leq r < 1.$$

(B) *If $u_\Delta(z) = -\operatorname{Re}(1-z)^{-\Delta}$, then*

$$A(r,u_\Delta) = (1-r)^{-\Delta}, \qquad 0 \leq r < 1,$$
$$B(r,u_\Delta) = \Big[\Big(\cos\frac{\pi}{\Delta+1}\Big)^{\Delta+1} + o(1)\Big](1-r)^{-\Delta}, \qquad r \to 1.$$

Proof. (A) We argue as in the proof of Theorem 3.1. We assume that $\psi'(x) > 2\Delta/(\Delta+1)$, $x \geq 0$, and define P, Π, and Ω as before, with

$$s(x) = \frac{\pi}{\psi'\big(\max\{x,0\}\big)} \cdot \frac{\Delta}{\Delta+1}, \qquad x \in \mathbb{R}.$$

Then we define Z, Φ, and v.

Let $t + i\gamma \in \Omega$. Then as before we have

$$|P^{-1}(t+i\gamma)| = 1 - \exp[-t + \log\cos\gamma + o(1)], \qquad t \to \infty,$$

uniformly in $\gamma \in [-s(t), s(t)]$. Therefore,

$$M\big(|P^{-1}(t+i\gamma)|\big) = \exp[\psi(t) - \Delta\log\cos\gamma + o(1)], \qquad t \to \infty.$$

Applying the estimates of conformal maps of infinite strips in [12, Theorem 3], see also [8] (the regularity conditions on s are much weaker here because $\lim_{t\to\infty} s(t) > 0$), we get

$$|\exp Z(t+i\gamma)| = (C_1 + o(1)) \exp\Big[\frac{\Delta+1}{2\Delta}\psi(t)\Big], \qquad t\to\infty,$$

where C_1 depends only on ψ.

Hence,

$$v(w) \le M(|\Phi(w)|) \le C|w|^{2\Delta/(\Delta+1)}, \qquad w\in \exp\Pi,\ |w|\to\infty,$$

$$v(w) \le M(|\Phi(w)|) \le (C_2 + o(1))\Big(\cos\frac{\pi}{\Delta+1}\Big)^{-\Delta}|w|^{2\Delta/(\Delta+1)}, \qquad w\in \partial(\exp\Pi),\ |w|\to\infty,$$

with $C_2 = C_1^{-2\Delta/(\Delta+1)}$. Next, we apply a version of Lemma 2.1:

If $1<q<2$, u is harmonic in the upper half-plane $\mathbb{C}_+$, and continuous in the closed upper half-plane $\overline{\mathbb{C}}_+$, and if

$$\begin{aligned} u(x) &\le x^q, \qquad x\in\mathbb{R},\\ u(z) &\le o(|z|^3), \qquad z\in\mathbb{C},\ |z|\to\infty, \end{aligned} \tag{3.8}$$

then the function

$$y\mapsto \frac{u(iy) - y^q/\cos(\pi q/2)}{y}$$

does not decrease on $[0,+\infty)$.

(The proof is analogous to that of Lemma 2.1).

As a result, for every $\varepsilon>0$ we find $K(\varepsilon, M)$ independent of v such that

$$v(x) \ge K(\varepsilon, M) - (1+\varepsilon)C_2\Big(\cos\frac{\pi}{\Delta+1}\Big)^{-(\Delta+1)} x^{2\Delta/(\Delta+1)}, \quad x\ge 1.$$

Furthermore,

$$M(|\Phi(x)|) \ge K_1(\varepsilon, M) + (1-\varepsilon)C_2 x^{2\Delta/(\Delta+1)}, \qquad x\ge 1,$$

and we get

$$u(r) \ge K_2(\varepsilon, M) - (1+3\varepsilon)\Big(\cos\frac{\pi}{\Delta+1}\Big)^{-(\Delta+1)} M(r), \qquad 0\le r<1.$$

(B) The first equality is evident. To prove the second one we fix $\varepsilon>0$, consider $z = 1 - se^{i\varphi}\in\mathbb{D}$ for small $s>0$, and note that

$$\begin{gathered} 1-|z| = 1 - |1-se^{i\varphi}| = 1 - \sqrt{1+s^2-2s\cos\varphi},\\ 1-|z| \le s(\cos\varphi+\varepsilon), \qquad s\to 0. \end{gathered}$$

Furthermore,

$$u_\Delta(z) = -s^{-\Delta}\,\mathrm{Re}\, e^{-i\Delta\varphi} = s^{-\Delta}\cos(\pi - \Delta\varphi) \tag{3.9}$$
$$\le (1-|z|)^{-\Delta}\big[(\cos\varphi)^\Delta\cos(\pi-\Delta\varphi) + C\varepsilon\big], \qquad s \to 0.$$

Since the maximal value of the function $(\cos\varphi)^\Delta\cos(\pi-\Delta\varphi)$ on the interval $[0,\pi/2]$ is attained at the point $\pi/(\Delta+1)$ and equals to

$$\Big(\cos\frac{\pi}{\Delta+1}\Big)^{\Delta+1},$$

we get

$$u_\Delta(z) \le \Big[\Big(\cos\frac{\pi}{\Delta+1}\Big)^{\Delta+1} + C\varepsilon\Big](1-|z|)^{-\Delta}, \qquad |1-z| \to 0.$$

This gives the upper bound for $B(r,u_\Delta)$ we are interested in; the lower one follows immediately from (3.9) with $\varphi = \pi/(\Delta+1)$. □

Suppose now that M is a positive C^2-smooth increasing function on $(0,\infty)$, denote $\psi(t) = \log M(\exp t)$, and assume that

$$\lim_{t\to\infty} \psi'(t) = +\infty,$$

and that for some $\delta > 0$,

$$|\psi''(t)| = O\big((\psi'(t))^{2-\delta}\big), \qquad t \to \infty.$$

Theorem 3.3. *If u is harmonic in the plane, $u(0) = 0$, and*

$$B(r,u) \le M(r), \qquad 0 \le r < \infty,$$

then for every $\varepsilon > 0$ there exists $K(\varepsilon, M)$ independent of u such that

$$A(r,u) \le K(\varepsilon, M) + (1+\varepsilon)M(r), \qquad 0 \le r < \infty.$$

The proof is analogous to that of Theorem 3.1.

Remark 3.4. If $\psi'(t) = O(1)$, $t \to \infty$, $B(r,u) \le M(r)$, $0 \le r < \infty$, then u is the real part of a polynomial, and

$$A(r,u) = (1 + O(1/r))B(r,u), \qquad r \to \infty.$$

4. **An example.**

It is natural to ask whether condition (3.2) is really necessary for the assertions of Theorems 3.1, 3.3 to hold. The following example shows that if we relax our conditions on M, and require just that $x \mapsto M(1-e^{-x})$ be convex (this holds for $B(r,u)$ with harmonic u), then the conclusion (3.3) may fail.

Proposition 4.1. *Let φ be a positive non-decreasing function on $\mathbb{R}_+$, and $0 = r_0 < r_1 < \dots r_k < r_{k+1} < \dots \to 1$. There exists a function u harmonic in the unit disc such that*

$$A(r_k,u) \ge \varphi\big(B(r_k,u)\big) \ge \varphi(k), \qquad k \ge 1. \tag{4.1}$$

Proof. Set $v(z) = \operatorname{Re}(z/(1-z))$, and note that $v(0)=0$, v is real-valued and increases on $(0,1)$, $v(x) \to \infty$ as $x \to 1$, and $\inf_{\mathbb{D}} v = -1/2$.

In an inductive process we are going to produce auxiliary harmonic functions u_m, and non-negative numbers a_m and b_m in the following way. On the step $m = m_0 \ge 1$ we start with the inequalities

$$\max_{|z|=r_k} \sum_{j=1}^{m-1} u_j(z) \ge \varphi(a_k+1) + 2^{-m+1}, \qquad 0 < k \le m-1, \tag{4.2}$$

$$\min_{|z|=r_k} \sum_{j=1}^{m-1} u_j(z) \ge -a_k - 1 + 2^{-m+1}, \qquad 0 < k \le m-1. \tag{4.3}$$

Put

$$a_m = \max_{|z|=r_m} \Big| \sum_{j=1}^{m-1} u_j(z) \Big| + m, \tag{4.4}$$

$$b_m = \max_{|z|=r_{m-1}} \Big| v\Big(\frac{z}{r_m}\Big) \Big| + 1.$$

(For $m=1$ we have $a_m = b_m = 1$.) Choose $r_m < R_m < 1$ such that

$$\frac{1}{2^{m+1} b_m} v\Big(\frac{r_m}{R_m}\Big) \ge \varphi(a_m+1) + a_m + 1.$$

The function $(2^{m+1}b_m)^{-1} v(z/R_m)$ is harmonic in a neighborhood of $r_m\overline{\mathbb{D}}$, and coincides there with the real part of an analytic function F_m. Using the Runge theorem, we approximate F_m by a polynomial P_m on $r_m\overline{\mathbb{D}}$, and obtain for harmonic $u_m = \operatorname{Re} P_m$,

$$\Big| \frac{1}{2^{m+1} b_m} v\Big(\frac{z}{R_m}\Big) - u_m(z) \Big| \le 2^{-m-1}, \qquad |z| \le r_m.$$

Thus,

$$\max_{|z|=r_{m-1}} |u_m(z)| \le 2^{-m}, \tag{4.5}$$

$$\max_{|z|=r_m} u_m(z) \ge \varphi(a_m+1) + a_m + 2^{-m}, \tag{4.6}$$

$$\min_{|z|=r_m} u_m(z) \ge -2^{-m}. \tag{4.7}$$

Therefore, by (4.2), (4.3), (4.5) we obtain

$$\max_{|z|=r_k} \sum_{j=1}^{m} u_j(z) \ge \varphi(a_k+1) + 2^{-m}, \qquad 0 < k \le m-1,$$

$$\min_{|z|=r_k} \sum_{j=1}^{m} u_j(z) \ge -a_k - 1 + 2^{-m}, \qquad 0 < k \le m-1,$$

and by (4.4), (4.6), (4.7) we obtain

$$\max_{|z|=r_m} \sum_{j=1}^{m} u_j(z) \ge \varphi(a_m+1) + 2^{-m},$$

$$\min_{|z|=r_m} \sum_{j=1}^{m} u_j(z) \ge -a_m - 1 + 2^{-m}.$$

This establishes the inequalities (4.2), (4.3) for $m = m_0 + 1$.

By (4.5), the sum $\sum_{m=1}^{\infty} u_m$ converges to a harmonic function u_* uniformly on compact subsets of the unit disc. By (4.2), (4.3), (4.4) we get

$$\max_{|z|=r_k} u_*(z) \ge \varphi(a_k+1) \ge \varphi(k), \qquad k \ge 1,$$

$$\min_{|z|=r_k} u_*(z) \ge -a_k - 1, \qquad k \ge 1.$$

and (4.1) is proved with $u = -u_*$. □

References

[1] S. A. Apresyan, *All closed ideals of the algebra $A_\varphi(\mathbb{C})$ are divisorial,* Zap. Nauchn. Semin. LOMI **92** (1979), 253–258, in Russian.

[2] A. Borichev, H. Hedenmalm, A. Volberg, *Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index,* preprint, 2000.

[3] M. L. Cartwright, *On analytic functions regular in the unit circle, I,* Quart. J. Math. Oxfors **4** (1933), 246–256.

[4] Y. Domar, *Extensions of the Titchmarsh convolution theorem with applications in the theory of invariant subspaces,* Proc. London Math. Soc. **46** (1983), 288–300.

[5] J. Garnett, *Bounded analytic functions,* Academic Press, 1981.

[6] N. V. Govorov, *Riemann's boundary problem with infinite index,* "Nauka", Moscow, 1986; English translation in Birkhäuser Verlag, 1994.
[7] W. K. Hayman, *The local growth of power series: a survey of the Wiman–Valiron method,* Canad. Math. Bull. **17** (1974), 317–358.
[8] J. Jenkins, K. Oikawa, *On results of Ahlfors and Hayman,* Illinois J. Math. **15** (1971), 664–671.
[9] B. Ya. Levin, *Lectures on entire functions,* Translations of Mathematical Monographs 150, American Mathematical Society, 1996.
[10] V. I. Macaev, E. Z. Mogulskiĭ, *A division theorem for analytic functions with a given majorant and some applications,* Zap. Nauchn. Semin. LOMI **56** (1976), 73–89; English translation in J. Soviet Math. **14** (1980), 1078–1091.
[11] B. Rodin, S. E. Warschawski, *Extremal length and the boundary behavior of conformal mappings,* Ann. Acad. Sci. Fenn. Ser. A I Math. **2** (1976), 467–500.
[12] B. Rodin, S. E. Warschawski, *Extremal length and univalent functions II. Integral estimates of strip mappings,* J. Math. Soc. Japan **31** (1979), 87–99.
[13] N. K. Nikolskiĭ, *Selected problems of weighted approximation and spectral analysis,* Trudy MIAN **120** (1974); English translation in Proc. of the Steklov Institute of Math. **120** (1974), Amer. Math. Soc., Providence, RI (1976), 276 pp.
[14] S. E. Warschawski, *On conformal mapping of infinite strips,* Trans. Amer. Math. Soc. **51** (1942), 280–335.
[15] A. Wiman, *Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Betrage bei gegebenem Argumente der Funktion,* Acta Math. **41** (1916), 1–28.

Alexander Borichev, Department of Mathematics, University of Bordeaux I, 351, cours de la Libération, 33405 Talence, France
E-mail: borichev@math.u-bordeaux.fr