ON THE BEKOLLE–BONAMI CONDITION

ALEXANDER BORICHEV

Consider the system $\mathfrak{A}$ of the Carleson "squares"

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - |Q| \le r \le 1,\ |\theta - \theta_0| \le |Q|/2\}$$

in the unit disc $\mathbb{D}$. Given a non-negative function w on $\mathbb{D}$, and a subset E of $\mathbb{D}$, we denote

$$\langle w\rangle_E = \frac{1}{m_2(E)} \int_E w(z)\, dm_2(z),$$

where dm_2 is Lebesgue area measure. The classes $\mathcal{B}_{p,q}$, $0 < p, q < \infty$, consist of w such that

$$\sup_{Q\in\mathfrak{A}} \langle w^p\rangle_Q^{1/p} \langle w^{-q}\rangle_Q^{1/q} < \infty. \tag{1}$$

The classes $B_p = \mathcal{B}_{1,1/(p-1)}$, $1 < p < \infty$, were introduced by D. Bekollé and A. Bonami in [2]. They proved that for locally integrable non-negative weights w on $\mathbb{D}$, and for $1 < p < \infty$, the Bergman projection operator $T : f \mapsto Tf$,

$$Tf(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dm_2(\zeta),$$

acts continuously on $L^p(\mathbb{D}, w\, dm_2)$ if and only if $w \in B_p$. This result is similar to the Hunt–Muckenhoupt–Wheeden theorem (see [11], [7, Chapter 6]) that claims that the Hilbert transform is bounded on $L^p(\mathbb{R}, w\, dm)$ if and only if w satisfies the condition (A_p). The class A_p consisting of functions w satisfying (A_p) is analogous to the class B_p, with squares $Q \subset \mathbb{D}$ in the definition (1) replaced by intervals of $\mathbb{R}$.

By the Hölder inequality, we have

$$A_{p_1} \subset A_p, \qquad \mathcal{B}_{p,q} \subset \mathcal{B}_{p_1,q_1}, \qquad p_1 \le p,\ q_1 \le q.$$

It is known (see [13]) that

$$A_p \subset \bigcup_{\varepsilon>0} A_{p-\varepsilon}.$$

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On the other hand,

$$\mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q+\varepsilon}.$$

The aim of this article is to study what additional conditions on the weight $w \in \mathcal{B}_{p,q}$ imply that $w \in \cup_{\varepsilon>0}\mathcal{B}_{p+\varepsilon,q+\varepsilon}$. In other words, we ask when the fact that T acts continuously on $L^p(w\,dm_2)$ implies that T acts continuously on $L^{p-\varepsilon}(w^{1+\varepsilon}\,dm_2)$, for small $\varepsilon = \varepsilon(w)$.

Note that in applications of the Bekollé–Bonami theorem [1, 3, 10] the weight w is frequently equal to $|\varphi'|^\alpha$ for a univalent function φ and for real α.

Denote by $\mathcal{A}$ the class of functions $|f|^\alpha$, $\alpha \in \mathbb{R}$, for f analytic in $\mathbb{D}$, by $\mathcal{M}$ the class of functions $|f|$ for f meromorphic in $\mathbb{D}$, by $\mathcal{ES}$ the class of functions $\exp u$ for u subharmonic in $\mathbb{D}$, and by $\mathcal{S}$ the class of functions non-negative and subharmonic in $\mathbb{D}$. Note that $\mathcal{M}\cap\mathcal{B}_{p,q} \subset \mathcal{A}$, $p \ge 2$.

Theorem. *For $0 < p, q < \infty$ we have*

(I) $\mathcal{A} \cap \mathcal{B}_{p,q} \subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q+\varepsilon}$,
(II) $\mathcal{ES} \cap \mathcal{B}_{p,q} \subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q+\varepsilon}$,
(III) $\mathcal{S} \cap \mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q+\varepsilon}$.

For $0 < p, q < 2$ we have

(IV) $\mathcal{M} \cap \mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q+\varepsilon}$.

Proof. (I) follows from (II).

(II) Let $f = \exp u$, for u subharmonic in $\mathbb{D}$, and let $f \in \mathcal{B}_{p,q}$, that is

$$\sup_{Q\in\mathfrak{A}} \langle f^p\rangle_Q^{1/p} \langle f^{-q}\rangle_Q^{1/q} < \infty. \tag{2}$$

Since the function f^p is subharmonic, by the mean value inequality, for every $Q \in \mathfrak{A}$,

$$\langle f^p\rangle_Q \ge c\Big(\frac{\operatorname{dist}(z, \partial Q)}{|Q|}\Big)^2 f(z)^p, \qquad z \in Q. \tag{3}$$

For every Carleson square Q denote by TQ the set $Q\setminus(Q_1\cup Q_2)$, where $Q_1, Q_2 \in \mathfrak{A}$, $|Q_1| = |Q_2| = |Q|/2$, $(Q_1 \cup Q_2) \cap \mathbb{T} = Q \cap \mathbb{T}$. Put

$$F(TQ) = \sup_{z\in TQ} f(z). \tag{4}$$

Fix $\gamma > 0$. If a square $Q \in \mathfrak{A}$ contains two subsets E_1 and E_2 with $m_2(E_1) \ge \gamma m_2(Q)$, $m_2(E_2) \ge \gamma m_2(Q)$, $\langle f^p\rangle_{E_1} \ge \lambda \sup_{E_2} f$, then (2) implies that λ is bounded uniformly in $Q \in \mathfrak{A}$. Therefore, the following claim is proved:

Claim. *If two squares $Q_1, Q_2 \in \mathfrak{A}$ are of comparable sidelength, and if the distance between them is bounded by a constant times the sidelength of Q_1, then $F(TQ_1)$ is comparable to $F(TQ_2)$.*

Fix a dyadic system of Carleson squares $Q_{j,k}$, say

$$Q_{j,k} = \{re^{i\theta} \in \mathbb{D} : 1 - 2\pi \cdot 2^{-k} \le r \le 1,\ |\theta - 2\pi \cdot 2^{-k} j| \le \pi \cdot 2^{-k}\},$$

with trivial modification for $k = 1$. Put

$$g(z) = F(TQ_{j,k}), \qquad z \in TQ_{j,k}. \tag{5}$$

Then $f \le g$, and

$$\begin{aligned} \langle g^s \rangle_{TQ_{j,k}} &\le c(s) \langle f^s \rangle_{TQ_{j,k}}, \qquad 0 < s < \infty, \\ \langle g^s \rangle_Q &\le c(s) \langle f^s \rangle_Q, \qquad Q \in \mathfrak{A},\ 0 < s < \infty. \end{aligned} \tag{6}$$

Next we verify that for some positive ε, c independent of j, k,

$$\frac{1}{m_2(TQ_{j,k})} \int_{TQ_{j,k}} f(z)^{-q-\varepsilon} dm_2(z) \le cF(TQ_{j,k})^{-q-\varepsilon}. \tag{7}$$

For every $TQ_{j,k}$ we consider a rectangle $\Omega_{j,k}$ containing $TQ_{j,k}$ with $\operatorname{dist}(z, TQ_{j,k} \cup \mathbb{T}) \asymp |Q_{j,k}|$, $z \in \partial\Omega_{j,k}$, and the conformal map $\omega_{j,k} : \mathbb{D} \to \Omega_{j,k}$ such that $\omega_{j,k}^{-1}(TQ_{j,k}) \subset r^2\mathbb{D}$ for a constant $r < 1$. We write

$$u \circ \omega_{j,k} = u_{j,k} + \log F(TQ_{j,k}).$$

Then $u_{j,k}$ are subharmonic in $\mathbb{D}$, $u_{j,k}(z) \le c$, with c independent of j, k.

Relations (2), (4), and (6) show that

$$\langle e^{-qu} \rangle_{Q_{j,k}} \le cF(TQ_{j,k})^{-q}.$$

Moreover, by the Claim,

$$\langle e^{-qu} \rangle_{\Omega_{j,k}} \le cF(TQ_{j,k})^{-q}.$$

Hence,

$$\int_{r\mathbb{D}} e^{-qu_{j,k}(z)} dm_2(z) \le c.$$

If (7) is false, then for some c and for every n there exists a function u_n subharmonic in $\mathbb{D}$, such that

$$u_n(z) \le c, \qquad z \in \mathbb{D}, \tag{8}$$

$$\int_{r\mathbb{D}} e^{-qu_n(z)} dm_2(z) \le c, \tag{9}$$

and

$$\int_{r^2\mathbb{D}} e^{-(q+\frac{1}{n})u_n(z)} dm_2(z) \to \infty, \qquad n \to \infty. \tag{10}$$

Consider the measures $\mu_n = \triangle u_n$. By (8) and (9), there exist $\varepsilon, \delta > 0$ such that for every disc D of radius 2δ centered at a point of $r\mathbb{D}$, and for every n,

$$\mu_n(D) \le \frac{2}{q+2\varepsilon}.$$

We cover $r^2\mathbb{D}$ by a finite union of small discs D_s with

$$\max_{n,s} \mu_n(2D_s) \le \frac{2}{q+2\varepsilon},$$

where $2D_s \subset r\mathbb{D}$ are the discs concentric with D_s with radii twice those of D_s.

Using the Riesz representation, for every n, s we obtain

$$u_n(z) = u_{n,s}(z) + v_{n,s}(z) = u_{n,s}(z) + \int_{2D_s} \log|z-\zeta|\, d\mu_n(\zeta), \qquad (11)$$

where $u_{n,s}$ is harmonic in $2D_s$. Since $v_{n,s}(z)$ is non-positive in $2D_s$, condition (9) implies that

$$\int_{2D_s} e^{-qu_{n,s}(z)} dm_2(z) \le c.$$

By the mean value property,

$$u_{n,s}(z) \ge c, \qquad z \in D_s. \qquad (12)$$

Denote

$$w_{n,s}(z) = \exp\Bigl[-(q+\varepsilon)\int_{2D_s} \log|z-\zeta|\, d\mu_n(\zeta)\Bigr].$$

By Cartan's lemma (see [4, Chapitre II], [9, Lemma 6.17]),

$$m_2\bigl\{z \in D_s : w_{n,s}(z) > t\bigr\} \le Ct^{-(q+2\varepsilon)/(q+\varepsilon)}, \qquad t > 1,$$

with an absolute constant C, and hence,

$$\int_{D_s} w_{n,s}(z)\, dm_2(z) \le c(\varepsilon).$$

By (11) and (12),

$$\int_{D_s} e^{-(q+\varepsilon)u_n(z)} dm_2(z) \le c(\varepsilon),$$

and

$$\int_{r^2\mathbb{D}} e^{-(q+\varepsilon)u_n(z)} dm_2(z) < \infty,$$

that contradicts to (10). Thus, (7) is proved.

By (2), (5), (6), (7), we have $g \in \mathcal{B}_{p,q}$, and for some $c, \varepsilon > 0$,

$$\langle f^{p+\varepsilon}\rangle_Q^{1/(p+\varepsilon)} \langle f^{-q-\varepsilon}\rangle_Q^{1/(q+\varepsilon)} \le c \langle g^{p+\varepsilon}\rangle_Q^{1/(p+\varepsilon)} \langle g^{-q-\varepsilon}\rangle_Q^{1/(q+\varepsilon)}, \quad Q \in \mathfrak{A}.$$

To complete the proof of (II) it remains to verify that for every positive $g \in \mathcal{B}_{p,q}$ which is constant on each $TQ_{j,k}$ we have $g \in \mathcal{B}_{p+\varepsilon,q}$ for some $\varepsilon = \varepsilon(g,p,q)$; after that, repeating the argument, we obtain $g \in \mathcal{B}_{p+\varepsilon,q+\varepsilon_1}$ for some $\varepsilon_1 = \varepsilon_1(g,p,q,\varepsilon)$.

First, we choose s such that $0 < 1/s < \min(p,q)$, and define $h = g^{1/s} \in \mathcal{B}_{ps,qs} \subset \mathcal{B}_{ps,1}$. Then, using the Cauchy-Schwarz inequality, we get

$$\langle h\rangle_Q \le \langle h^{ps}\rangle_Q^{1/ps} \le K\langle h\rangle_Q, \qquad Q \in \mathfrak{A}. \tag{13}$$

Next we use a reverse Hölder inequality (cf. [8], [5]): for some $\varepsilon, c > 0$ depending only on K, p, s,

$$\langle h^{ps+\varepsilon s}\rangle_Q^{1/(ps+\varepsilon s)} \le c\langle h\rangle_Q, \qquad Q \in \mathfrak{A}. \tag{14}$$

Inequalities (13), (14) imply that

$$\langle g^{p+\varepsilon}\rangle_Q^{1/(p+\varepsilon)} \le c\langle g^p\rangle_Q^{1/p}, \qquad Q \in \mathfrak{A},$$
$$\sup_{Q\in\mathfrak{A}}\langle g^{p+\varepsilon}\rangle_Q^{1/(p+\varepsilon)}\langle g^{-q}\rangle_Q^{1/q} \le \sup_{Q\in\mathfrak{A}}\langle g^p\rangle_Q^{1/p}\langle g^{-q}\rangle_Q^{1/q} < \infty,$$

and hence, $g \in \mathcal{B}_{p+\varepsilon,q}$. Thus, (II) is proved modulo (14).

Finally, we verify that (13) implies (14) for h which are constant on each $TQ_{j,k}$. Denote $t = ps > 1$. Fix a dyadic square $Q = Q_{j,k}$, and, without loss of generality, assume that $\langle h\rangle_Q = 1$. Next, we fix a large N, and modify h by making it equal to $\langle h\rangle_{Q_{jN}}$ on $Q_{jN} \subset Q$. Inequality (13) still holds, and we need to verify that for small $\gamma > 0$, $\langle h^{t+\gamma}\rangle_Q$ is bounded uniformly in N.

We use the standard Calderon–Zygmund decomposition. For every $\lambda \ge 1$ denote by $H(\lambda)$ the set of all $z \in Q$ such that $h(z) \ge \lambda$, and consider the set $\mathfrak{A}(\lambda)$ of maximal dyadic squares $Q' \subset Q$ such that $\langle h\rangle_{Q'} \ge \lambda$. Denote the union of these squares by $\mathbb{H}(\lambda)$. Then

$$\langle h\rangle_{Q'} \le 5\lambda, \qquad Q' \in \mathfrak{A}(\lambda), \tag{15}$$

and

$$H(4\lambda) \subset \mathbb{H}(\lambda) \tag{16}$$

(here we use that h is constant on $TQ_{j,k}$). By (13), (15) and (16) we get

$$\int_{H(4\lambda)} h(z)^t dm_2(z) \le \int_{\mathbb{H}(\lambda)} h(z)^t dm_2(z) = \sum_{Q'\in\mathfrak{A}(\lambda)} \int_{Q'} h(z)^t dm_2(z)$$
$$\le \sum_{Q'\in\mathfrak{A}(\lambda)} (5\lambda)^{t-1}K^t \int_{Q'} h(z)\, dm_2(z), \qquad \lambda \ge 1.$$

Furthermore, for every $Q'\in\mathfrak{A}(\lambda)$,

$$\int_{Q'\setminus H(\lambda/2)} h(z)\,dm_2(z)\le \frac{\lambda}{2}m_2(Q'),$$

and hence,

$$\int_{Q'} h(z)\,dm_2(z)\le 2\int_{Q'\cap H(\lambda/2)} h(z)\,dm_2(z).$$

Thus,

$$\int_{H(4\lambda)} h(z)^t dm_2(z)\le 2\cdot(5\lambda)^{t-1}K^t\int_{H(\lambda/2)} h(z)dm_2(z),\qquad \lambda\ge 1. \tag{17}$$

Therefore, for every $0<\gamma<1/2$,

$$\begin{aligned}\int_Q h(z)^{t+\gamma}dm_2(z)&\le\sum_{n\in\mathbb{Z}}2^{n\gamma+\gamma}\int_{2^n\le h(z)<2^{n+1},\,z\in Q}h(z)^t dm_2(z)\\ &\le c\,m_2(Q)+\gamma\sum_{n\ge3}2^{n\gamma}\int_{H(2^n)}h(z)^t dm_2(z)\\ &\overset{(17)}{\le} c\,m_2(Q)+c(K,t)\gamma\sum_{n\ge3}2^{n\gamma+nt-n}\int_{H(2^{n-3})}h(z)\,dm_2(z)\\ &\le c\,m_2(Q)+\frac{c(K,t)\gamma}{2^{t-1+\gamma}-1}\int_Q h(z)^{t+\gamma}dm_2(z).\end{aligned}$$

For sufficiently small γ, $0<\gamma\le\gamma_0(K,t)$, we get

$$\int_Q h(z)^{t+\gamma}dm_2(z)\le c(K,t)m_2(Q),$$

and (14) is proved for $\varepsilon\le\gamma_0(K,ps)/s$.

(III) Just consider the function f, $f(z)=|z|^{2/q}(\log(A/|z|))^{2/q}$ for $A>1$ to be determined later on. Then $f\in\mathcal{B}_{p,q}\setminus\bigcup_{\varepsilon>0}\mathcal{B}_{p,q+\varepsilon}$, $0<p<\infty$. (In fact, $f\in C(\overline{\mathbb{D}})$, $1/f\in L^q(\mathbb{D})\cap C(\overline{\mathbb{D}}\setminus\{0\})$, $1/f\notin\bigcup_{\varepsilon>0}L^{q+\varepsilon}(\mathbb{D})$.) It remains to verify that for sufficiently large A, the function f is subharmonic in $\mathbb{D}$, or, what is equivalent, the function f_1, $f_1(z)=(\log r)^s r^{-s}$, $r=|z|$, with $0<s<\infty$, is subharmonic for sufficiently large r, which is equivalent, in its turn, to the fact that the function f_2, $f_2(r)=f_1(\exp r)=r^s e^{-rs}$ is convex for large r:

$$\begin{aligned}&f_2'(r)=sr^{s-1}e^{-rs}-sr^se^{-rs},\\ &f_2''(r)=s(s-1)r^{s-2}e^{-rs}-2s^2r^{s-1}e^{-rs}+s^2r^se^{-rs}\ge0,\qquad r>r(s).\end{aligned}$$

(IV) We start with the following elementary calculation. Take small positive x, y such that $0 < 2x < y$. Given $0 < p, q < 2$, we fix a natural number N such that $1 \le Np < 2$, and estimate the integrals

$$I(p+\varepsilon) = \int_{\mathbb{D}} \Big| \frac{z^{3N} - y^{3N}}{(z^3 - x^3)^N} \Big|^{p+\varepsilon} dm_2(z), \qquad 0 < \varepsilon < \frac{2}{N} - p,$$

$$J = \int_{\mathbb{D}} \Big| \frac{(z^3 - x^3)^N}{z^{3N} - y^{3N}} \Big|^{q} dm_2(z).$$

We have

$$I(p+\varepsilon) = \int_{|z| \le x/2} + \int_{x/2 < |z| \le 3x/2} + \int_{3x/2 < |z| \le 2y} + \int_{|z| > 2y} = I_1 + I_2 + I_3 + I_4,$$

$$I_1 \asymp \Big(\frac{y}{x}\Big)^{3N(p+\varepsilon)} x^2,$$
$$I_2 \le c\Big(\frac{y}{x}\Big)^{3N(p+\varepsilon)} x^2,$$
$$I_3 \le c\Big(\frac{y}{x}\Big)^{3N(p+\varepsilon)} x^2,$$
$$I_4 \asymp 1.$$

Thus, if

$$x \le y^{3N(p+\varepsilon)/[3N(p+\varepsilon)-2]},$$

then

$$I(p+\varepsilon) \asymp \Big(\frac{y}{x}\Big)^{3N(p+\varepsilon)} x^2.$$

Analogously,

$$J = \int_{|z| \le y/2} + \int_{y/2 < |z| \le 2y} + \int_{|z| > 2y} = J_1 + J_2 + J_3,$$

$$J_1 \le cy^2,$$
$$J_2 \le cy^2,$$
$$J_3 \asymp 1.$$

Hence,

$$J \asymp 1.$$

Choose sequences $\{x_k\}$, $\{y_k\}$ such that

$$0 < x_k = y_k^{3Np/(3Np-2)}, \qquad (y_k/x_k)^{1/k} \to \infty, \qquad k \to \infty,$$

and define

$$\Phi_k(z) = \frac{z^{3N} - y_k^{3N}}{(z^3 - x_k^3)^N}.$$

Then

$$\int_{\mathbb{D}} |\Phi_k(z)|^p dm_2(z) \asymp \int_{\mathbb{D}} |\Phi_k(z)|^{-q} dm_2(z) \asymp 1,$$
$$\int_{\mathbb{D}} |\Phi_k(z)|^{p+\frac{1}{k}} dm_2(z) \to \infty, \qquad k \to \infty.$$

For $w \in \mathbb{D}$ denore by φ_w the Möbius function $\varphi_w(z) = (z-w)/(1-z\bar{w})$. For $w_k \in [0,1]$ sufficiently rapidly tending to 1 we put

$$\Phi = \prod_k \Phi_k \circ \varphi_{w_k}.$$

Then

$$\langle |\Phi|^p \rangle_Q \asymp \langle |\Phi|^{-q} \rangle_Q \asymp 1, \qquad Q \in \mathfrak{A},$$
$$\langle |\Phi|^{p+\frac{1}{k}} \rangle_{Q_k} \to \infty, \qquad k \to \infty,$$

for squares Q_k such that $w_k \in TQ_k$, $\operatorname{dist}(w_k, \partial TQ_k) \asymp |Q_k|$. Thus, $|\Phi| \in \mathcal{M} \cap \mathcal{B}_{p,q}$, but $|\Phi| \notin \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q}$. Hence, $\mathcal{M} \cap \mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q}$. □

Remark 1. We can refine somewhat assertions (III)–(IV) of Theorem: for $0 < p, q < \infty$ we have

(i) $\mathcal{S} \cap \mathcal{B}_{p,q} \subset \bigcup_{\varepsilon>0} \mathcal{B}_{p+\varepsilon,q}$,
(ii) $\mathcal{S} \cap \mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \mathcal{B}_{p,q+\varepsilon}$;
for $0 < p, q < 2$ we have
(iii) $\mathcal{M} \cap \mathcal{B}_{p,q} \not\subset \bigcup_{\varepsilon>0} \big[\mathcal{B}_{p+\varepsilon,q} \cup \mathcal{B}_{p,q+\varepsilon}\big]$.

The proof of (i) is similar to that of the part (II) of Theorem. Instead of (3) we use that for $0 < p < \infty$, and for functions f, non-negative and subharmonic on the unit disc $\mathbb{D}$, we have $f^p(0) \le c(p)\langle f^p \rangle_{\mathbb{D}}$.

Remark 2. The class B_1 consists of non-negative functions w on $\mathbb{D}$ such that for some $K = K(w)$,

$$\langle w \rangle_Q \le K \cdot w(z), \qquad z \in Q,\ Q \in \mathfrak{A}.$$

Denote by $\mathcal{EH}$ the class of functions $\exp(f)$ for f harmonic in $\mathbb{D}$. J. Rubio de Francia [14] (see also [6]) extended the factorization theorem of P. Jones [12] and obtained that for every $w \in \mathcal{B}_{p,q}$, $0 < p, q < \infty$, there exist $w_1, w_2 \in B_1$ such that

$$w = w_1^{1/p} w_2^{-1/q}.$$

(It is clear that $w_1^{1/p}w_2^{-1/q} \in \mathcal{B}_{p,q}$ for any $w_1, w_2 \in B_1$). It appears to be unknown whether such a factorization is possible for $w \in \mathcal{EH} \cap \mathcal{B}_{p,q}$ with $w_1, w_2 \in \mathcal{EH} \cap B_1$ (or if an analogous statement holds with $\mathcal{EH}$ replaced by $\mathcal{A}$). If true, this would provide a short proof of the inclusion $\mathcal{EH} \cap \mathcal{B}_{p,q} \subset \cup_{\varepsilon>0}\mathcal{B}_{p+\varepsilon,q+\varepsilon}$. Indeed, we would only need to verify that for any $w \in \mathcal{S} \cap B_1$ there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in B_1$. Fix a dyadic square $Q \in \mathfrak{A}$ and assume that $\langle w \rangle_Q = 1$. For every $n \geq 1$ we consider the maximal dyadic subsquares Q_j^n of Q such that TQ_j^n intersects with the set $\{z : w(z) > 2^n\}$. Since $w \in \mathcal{S} \cap B_1$, for some $c, \delta > 0$,

$$2^n m_2(Q) \asymp \int_{Q_j^n} w(z)\,dm_2(z) \leq c \cdot 2^n \int_{\{z \in Q_j^n : w(z) > 2^n\delta\}} dm_2(z). \tag{18}$$

Then for small $\varepsilon > 0$,

$$\begin{aligned}
\int_Q w(z)^{1+\varepsilon} dm_2(z) &\leq c\,m_2(Q) + \varepsilon \sum_{n\geq 0} 2^{n\varepsilon} \int_{\{z : w(z) > 2^n\}} w(z)\,dm_2(z) \\
&\leq c\,m_2(Q) + \varepsilon \sum_{n\geq 0} 2^{n\varepsilon} \sum_j \int_{Q_j^n} w(z)\,dm_2(z) \\
&\overset{(18)}{\leq} c\,m_2(Q) + c\varepsilon \sum_{n\geq 0} 2^{n\varepsilon+n} \int_{\{z : w(z) > 2^n\delta\}} dm_2(z) \\
&\leq c\,m_2(Q) + \frac{c\varepsilon}{1+\varepsilon} \int_Q w(z)^{1+\varepsilon} dm_2(z),
\end{aligned}$$

and we are done.

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References

[1] D. Bekollé, *Projections sur des espaces de fonctions holomorphes dans des domaines plans,* Canad. J. Math. **38** (1986), no. 1, 127–157.

[2] D. Bekollé, A. Bonami, *Inégalités à poids pour le noyau de Bergman,* C. R. Acad. Sci. Paris **286** (1978), no. 18, 775–778.

[3] J. Burbea, *The Bergman projection on weighted norm spaces,* Canad. J. Math. **32** (1980), no. 4, 979–986.

[4] H. Cartan, *Sur les systèmes de fonctions holomorphes á variétés linéaires lacunaires et leurs applications,* Ann. Sci. École Norm. Sup. (3) **45** (1928), 255–346.

[5] R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 241–250.

[6] R. Coifman, P. Jones, J. Rubio de Francia, *Constructive decomposition of BMO functions and factorization of A_p weights,* Proc. Amer. Math. Soc. **87** (1983), 675–676.

[7] J. B. Garnett, *Bounded analytic functions,* Academic Press, New York–London, 1981.
[8] F. W. Gehring, *The L^p-integrability of the partial derivatives for quasiconformal mapping,* Acta Math. **130** (1973), 265–278.
[9] W. K. Hayman, *Subharmonic functions,* Vol. 2, Academic Press, 1989.
[10] H. Hedenmalm, *The dual of a Bergman space on simply connected domains,* Journ. d'Analyse Math. **88** (2002), 311–335.
[11] R. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for conjugate function and the Hilbert transform,* Trans. Amer. Math. Soc. **176** (1973), 227–251.
[12] P. Jones, *Factorization of A_p weights,* Ann. of Math. **111** (1980), 511–530.
[13] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal functions,* Trans. Amer. Math. Soc. **165** (1972), 207–226.
[14] J. L. Rubio de Francia, *Factorization and extrapolation of weights,* Bull. Amer. Math. Soc. **7** (1982), 393–395.

Alexander Borichev, Department of Mathematics,
University of Bordeaux I, 351, cours de la Libération, 33405 Talence, France
E-mail: borichev@math.u-bordeaux.fr