Uniqueness theorems for Korenblum type spaces

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(joint work with Yuri Lyubarskii)

Given a topological space X of analytic functions in the unit disc $\mathbb{D}$ and a class $\mathcal{E}$ of subsets E of $\mathbb{D}$, we call a non-decreasing positive function $M : [0,1) \to (0,\infty)$ a minorant for the pair $(X,\mathcal{E})$ and write $M \in \mathcal{M}(X,\mathcal{E})$ if

$$\begin{aligned} f \in X, &\qquad E \in \mathcal{E}, \\ \log|f(z)| \le -M(|z|), &\qquad z \in E, \end{aligned} \tag{1}$$

imply that $f = 0$.

Clearly, $\mathcal{M}(X,\mathcal{E}) \neq \emptyset$ implies that $\mathcal{E} \subset \mathcal{U}(X)$, where $\mathcal{U}(X)$ is the family of the uniqueness subsets E for the space X: $E \in \mathcal{U}(X)$ if and only if

$$f \in X, \quad f\big|E = 0 \qquad \Longrightarrow \qquad f = 0.$$

Suppose that $H^\infty \subset X \subset A(\lambda)$, for some λ, where

$$A(\lambda) = \bigl\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)|/\lambda(|z|) < \infty \bigr\}.$$

Then a simple argument shows that the class $\mathcal{M}(X,\mathcal{U}(X))$ is empty. The reason is that the class $\mathcal{U}(X)$ contains subsets $E \subset \mathbb{D}$ which are not massive enough: E may be the union of clusters E_j of nearby points in such a way that the estimate (1) on $x \in E_j$ implies a similar estimate (with M replaced by $M/2$) on the whole E_j. That is why we need to consider only elements in the family $\mathcal{U}(X)$ which are sufficiently separated. In [3], the authors deal with the case $X = H^\infty$, and consider the class $\mathcal{SU}(H^\infty)$ of hyperbolically separated subsets E of $\mathbb{D}$ that are uniqueness subsets for H^∞. They prove that

$$M \in \mathcal{M}(H^\infty, \mathcal{SU}(H^\infty)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

Here we work with the scale of spaces

$$\mathcal{A}_s^r = \bigl\{ f \in \mathrm{Hol}(\mathbb{D}) : \log|f(z)| \le r \log^s \frac{1}{1-|z|} + c_f \bigr\}, \qquad r,s > 0,$$

$$\mathcal{A}_s = \bigcup_{r<\infty} \mathcal{A}_s^r.$$

We have $H^\infty \subset \mathcal{A}_s \subset \mathcal{A}_1 \subset \mathcal{A}_t$, $0 < s < 1 < t$, where $\mathcal{A}_1$ is the so called Korenblum space,

$$\mathcal{A}_1 = \bigl\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \le \frac{c_f}{(1-|z|)^{c'_f}} \bigr\}.$$

The uniqueness subsets for $\mathcal{A}_s$ are described by Korenblum [2] (1975, $s = 1$) and Seip [5] (1995, $s > 0$). For $0 < s < 1$, we define $\mathcal{SU}(\mathcal{A}_s)$ as the class of hyperbolically separated subsets E of $\mathbb{D}$ that are uniqueness subsets for $\mathcal{A}_s$.

Theorem 1. *For regular M, $0 < s < 1$,*

$$M \in \mathcal{M}(\mathcal{A}_s, \mathcal{SU}(\mathcal{A}_s)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

For $s = 1$, no hyperbolically separated subset of $\mathbb{D}$ belongs to $\mathcal{U}(\mathcal{A}_1)$. The results of Korenblum and Seip do not give a complete description of $\mathcal{U}(\mathcal{A}_1^r)$, $r < \infty$. However, there is only a small gap between necessary conditions and sufficient conditions. In particular, it is known that every $\mathcal{U}(\mathcal{A}_1^r)$ contains hyperbolically separated subsets. We define $\mathcal{SU}(\mathcal{A}_1)$ as the class of $E \subset \mathbb{D}$ such that for every r there exists a hyperbolically separated subset E_r of E such that $E_r \in \mathcal{U}(\mathcal{A}_1^r)$.

Theorem 2. *For regular M,*

$$M \in \mathcal{M}(\mathcal{A}_1, \mathcal{SU}(\mathcal{A}_1)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

For $s > 1$, we introduce

$$\rho_s(z) = (1-|z|)\Big(\log\frac{1}{1-|z|}\Big)^{(1-s)/2},$$

and say that E is s-separated if for some $\varepsilon > 0$,

$$|\lambda - \mu| \geq \varepsilon\rho_s(\lambda), \qquad \lambda, \mu \in E, \quad \lambda \neq \mu.$$

We define $\mathcal{SU}(\mathcal{A}_s)$, $s > 1$, as the class of $E \subset \mathbb{D}$ such that for every r there exists an s-separated subset E_r of E such that $E_r \in \mathcal{U}(\mathcal{A}_s^r)$.

Theorem 3. *For regular M, $s > 1$,*

$$M \in \mathcal{M}(\mathcal{A}_s, \mathcal{SU}(\mathcal{A}_s)) \iff \int_0 \Big(\log\frac{1}{t}\Big)^{s-1} \frac{dt}{tM(1-t)} < \infty.$$

Remarks. 1. In Theorems 1–3, when the integrals diverge, we can find $E \in \mathcal{SU}(\mathcal{A}_s)$ and $f \in H^\infty \setminus \{0\}$ satisfying the estimate (1).

2. Our result should be compared to that by Pau and Thomas [4] concerning $\mathcal{M}(H^\infty, \mathcal{E})$, for some special classes $\mathcal{E} \subset \mathcal{SU}(H^\infty)$.

3. By duality, using a method of Havinson [5], we can deduce from Theorem 2 a result on approximation by simple fractions with restrictions on coefficients in the space $C_A^\infty = C^\infty(\mathbb{T}) \cap H^\infty$.

Question. How to get analogous results for the Bergman space (no description of uniqueness subsets is known yet), for the spaces $\mathcal{A}_s^r$, $0 < s < 1$?

References

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