Equivalent definitions of Jacobson radical

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Let R be an associative unitary ring (i.e. an associative ring with unity). The *Jacobson radical* of R is the two-sided ideal of R usually defined either as the intersection of all maximal proper left ideals, or as the intersection of all maximal proper right ideals of R. The goal of this note is to explain why these two definitions are equivalent by showing that they are both equivalent to a certain left-right symmetric one. (Based on information found in the Internet.)

Lemma. *The intersection of all maximal proper left ideals of R is the set of all $x \in R$ such that every element of $1 + Rx$ is left-invertible.*

Proof. The following are easy to be seen equivalent:

(1) x belongs to every maximal proper left ideal of R,

(2) for every maximal proper left ideal A of R, $A + Rx \neq R$,

(3) for every proper left ideal A of R, $A + Rx \neq R$,

(4) for every proper left ideal A of R, for every $a \in A$ and every $r \in R$, $a + rx \neq 1$,

(5) for every $a, r \in R$, if $a + rx = 1$, then $Ra = R$,

(6) for every $r \in R$, $1 - rx$ is left-invertible,

(7) every element of $1 + Rx$ is left-invertible.

□

Lemma. *Let x and y be elements of R. If $1 - xy$ is left-invertible, then so is $1 - yx$. More precisely, if $u(1 - xy) = 1$, then $(1 + yux)(1 - yx) = 1$.*

Proof. Suppose $u(1 - xy) = 1$. Then

$$\begin{aligned}(1 + yux)(1 - yx) &= 1 - yx + yux(1 - yx)\\ &= 1 - yx + yu(1 - xy)x = 1 - yx + yx = 1.\end{aligned}$$

□

Corollary. *Let x be an element of R. Then the following are equivalent*:

(1) *every element of $1 + Rx$ is left-invertible*,

(2) *every element of $1 + xR$ is left-invertible*,

(3) *every element of $1 + RxR$ is left-invertible.*

Lemma. *Let x be an element of R. If every element of $1 + Rx$ is left-invertible, then $1 - x$ is invertible* (*i.e. is a* unit).

Proof. Let u be a left inverse of $1 - x$:

$$u(1 - x) = 1.$$

Then u is left-invertible because

$$u = 1 + ux \in 1 + Rx.$$

Hence u and $1 - x$ are both units and are each other's inverses. Indeed, if $vu = 1$, then

$$(1 - x)u = vu(1 - x)u = vu = 1.$$

□

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(2) *every element of $1 + Rx$ is a unit.*

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Corollary. *The intersection of all maximal proper left ideals of R is the set of all $x \in R$ such that every element of $1 + RxR$ is a unit.*

Passing to the *opposite ring*, the following can be deduced.

Corollary. *The intersection of all maximal proper right ideals of R is the set of all $x \in R$ such that every element of $1 + RxR$ is a unit.*

Definition. The *Jacobson radical* of R is the set

$$J(R) \overset{\text{def}}{=} \{\, x \in R \mid 1 + RxR \subset R^\times \,\},$$

where $R^\times$ is the *group of units* of R.

It is not hard to show even directly from this definition that $J(R)$ is a two-sided ideal.

As shown above, $J(R)$ is the intersection of all maximal proper left ideals of R and also the intersection of all maximale proper right ideals of R.

Example. If R is an associative algebra over a field, then $J(R)$ is the set of all $x \in R$ such that the *spectrum* (in the sense of operator algebras) of every element of RxR is $\{0\}$.