# Examples of terminology and notation used in the literature for multidimensional differentiation

Alexey Muranov

18th April 2025

## Contents

# 1 Euler, Leonhard Euler (1707–1783)

## 1.1 “Introductio in analysin infinitorum” [Introduction to analysis of the infinite] (1748)

1988 English translation of Volume 1: “Introduction to analysis of the infinite – Book I” [17].
[...]

    [...]

# 2 Le Rond d'Alembert, Jean Le Rond d'Alembert (1717–1783)

## 2.1 "Recherches sur la courbe que forme une corde tendue mise en vibration" (1747)

[...]

[...]

# 3 Lagrange, Joseph-Louis Lagrange (1736–1813)

## 3.1 "Théorie des fonctions analytiques" (1797)

1797 original: [23].
[...]

[...]

# 4 Legendre, Adrien-Marie Legendre (1752–1833)

[...]

[...]

# 5 Arbogast, Louis François Antoine Arbogast (1759–1803)

## 5.1 "Du calcul des dérivations" [The calculus of derivations] (1800)

1800 original: [1].
[...]

[...]

# 6 Fourier, Joseph Fourier (1768–1830)

[...]

[...]

# 7 Gauss, Johann Carl Friedrich Gauss (1777–1855)

## 7.1 "Disquisitiones generales circa superficies curvas" [General investigations of curved surfaces] (1827)

```
[...]
```

```
        [...]
```

# 8 Cauchy, Augustin-Louis Cauchy (1789–1857)

## 8.1 "Cours d'analyse de l'École Royale Polytechnique" [...] (1821)

1821 original: [10]. 2009 reprint: [11]. 2009 annotated English translation: [3].

```
[...]
```

```
        [...]
```

# 9 Lejeune Dirichlet, Johann Peter Gustav Lejeune Dirichlet (1805–1859)

## 9.1 "Ueber die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinusreihen" [About the representation of completely arbitrary functions using sine and cosine series] (1837)

1837 original: [26]. Appears in *G. Lejeune Dirichlet's Werke* Volume 1 [24, 25]. Quoted in [28] and in [2].

```
[...]
```

```
        [...]
```

# 10 Jacobi, Carl Gustav Jacob Jacobi (1814–1851)

```
See Florian Cajori, "A history of mathematical notation, Volume 2" (1929),
page 237.
[...]
```

```
        [...]
```

# 11 Weierstrass, Karl Theodor Wilhelm Weierstrass (1815–1897)

[...]

[...]

# 12 Hermite, Charles Hermite (1822–1901)

[...]

[...]

# 13 Riemann, Georg Friedrich Bernhard Riemann (1826–1866)

## 13.1 "Grundlagen für eine allgemeine Theorie der Functioneneiner veränderlichen complexen Grösse" [Foundations for a general theory of functions of a complex variable] (1851)

2004 English translation of the 1867 (2nd) printing: "Foundations for a general theory of functions of a complex variable" [18].

[...]

[...]

# 14 Dedekind, Julius Wilhelm Richard Dedekind (1831–1916)

## 14.1 "Was sind und was sollen die Zahlen?" [What are numbers and what should they be?] (1888)

[...]

Quoted in [28].

[...]

[...]

# 15 Lipschitz, Rudolf Otto Sigismund Lipschitz (1832–1903)

[...]

# 16 Jordan, Camille Jordan (1838–1922)

## 16.1 "Cours d'analyse de l'École Polytechnique – Tome I" (1909, 3rd edition)

1909 third edition: [22].
[...]

[...]

# 17 Poincaré, Henri Poincaré (1854–1912)

## 17.1 "La notation différentielle et l'enseignement" (1899)

[...]

[...]

# 18 Goursat, Édouard Jean-Baptiste Goursat (1858–1936)

## 18.1 "Cours d'analyse mathématique – Tome I" (1933, 5th edition)

1933 5th edition: [20].
[...]

[...]

Page 91, Exemple :

[...]

# 19 Hadamard, Jacques Salomon Hadamard (1865–1963)

## 19.1 "Leçons sur le calcul des variations – Tome I" (1910)

1910 original: [21].
[...]

[...]

## 19.2 “La notion de différentielle dans l’enseignement” (1935)

Page 341:

Poincaré, dans sa conférence prononcée au Musée pédagogique de Paris en 1904, déclarait déjà qu’il y avait lieu de penser en dérivées et non en différentielles. Il me semble utile pour l’enseignement de se conformer résolument à ce principe et d’abandonner les explications assez compliquées qui sont classiquement données sur le symbole $d$. Pour la différentielle première, passe encore : Je puis comprendre l’égalité

$$dy = f'(x)\,dx \tag{1}$$

ou

$$dz = pdx + qdy \tag{1'}$$

en relation avec l’égalité approchée

$$\Delta y = f'(x)\,\Delta x \tag{2}$$

ou

$$\Delta z = p\Delta x + q\Delta y \tag{2'}$$

dans laquelle $\Delta x$, $\Delta y$, $\Delta z$ sont des accroissements infiniment petits. Mais la différentielle seconde ! J’ai lu, comme tout le monde, l’histoire de la différentielle de la variable indépendante qui doit être constante (et qui est d’ailleurs forcément variable puisque infiniment petite). Si je me suis décidé à ne pas exposer ces considérations dans les cours que j’ai eu l’occasion de professer sur les débuts du Calcul différentiel, c’est que j’avoue ne les avoir qu’à moitié comprises moi-même.

Page 342:

Que signifiera l’égalité

$$d^2y = f'(x)\,d^2x + f''(x)\,dx^2 \tag{4}$$

ou

$$d^2z = pd^2x + qd^2y + rdx^2 + 2s\,dx\,dy + tdy^2 \ ? \tag{4'}$$

Uniquement que l’on a

$$\frac{d^2y}{du^2} = f'(x)\frac{d^2x}{du^2} + f''(x)\left(\frac{dx}{du}\right)^2 \tag{5}$$

ou

$$(5') \qquad \frac{d^2z}{du^2} = p\frac{d^2x}{du^2} + q\frac{d^2y}{du^2} + r\left(\frac{dx}{du}\right)^2 + 2s\frac{dx}{du}\frac{dy}{du} + t\left(\frac{dy}{du}\right)^2$$

de quelque manière que les variables aient été exprimées en fonction du paramètre $u$, pourvu que l'on ne cesse pas d'avoir $y = f(x)$ ou $z = f(x, y)$.

Enfin, que signifie ou que représente l'égalité

$$(6) \qquad d^2z = rdx^2 + 2s\,dxdy + tdy^2 \ ?$$

A mon avis, rien du tout. On peut, si l'on veut, noter que les deux premiers termes disparaissent au second membre de (5′) lorsque $x$ et $y$ sont fonctions *linéaires* de $u$, comme il arrive dans les démonstrations du théorème de Taylor [seule question, à ma connaissance, qui nécessite l'intervention de l'expression (6)]. En dehors de ce cas, je ne vois pas ce qu'on peut tirer de (6), … si ce n'est peut-être une ou deux idées fausses.

# 20 De la Vallée Poussin, Charles-Jean de la Vallée Poussin (1866–1962)

## 20.1 "Cours d'analyse infinitésimale – Tome I" [Course of infinitesimal analysis – Volume I] (1923, 5th Edition)

[. . .]

[. . .]

# 21 Cartan, Élie Joseph Cartan (1869–1951)

## 21.1 "La géométrie des espaces de Riemann" [...] (1925)

1925 original: [7].
[. . .]

[. . .]

## 21.2 "Les systèmes différentiels extérieurs et leurs applications géométriques" [...] (1945)

1945 original: [8].
[. . .]

[. . .]

# 22 Levi-Civita, Tullio Levi-Civita (1873–1941)

## 22.1 "Lezioni di calcolo differenziale assoluto" [Lessons of the absolute differential calculus] (1925)

[...]

> [...]

# 23 Carathéodory, Constantin Carathéodory (1873–1950)

## 23.1 "Variationsrechnung und partielle Differentialgleichungen erster Ordnung" [Calculus of variations and partial differential equations of the first order] (1935)

1935 original: [6]. 1965–1967 English translation: "Calculus of variations and partial differential equations of the first order" [5, 4].
[...]

> [...]

# 24 Fréchet, René Maurice Fréchet (1878–1973)

## 24.1 "Sur la notion de différentielle totale" [...] (1912)

[...]

> [...]

# 25 Wilson, Edwin Bidwell Wilson (1879–1964)

## 25.1 "Vector analysis" founded upon the lectures of Willard Gibbs (1901)

1901 original: [19]. Ninth printing: 1947.
[...]

> [...]

# 26 Weyl, Hermann Klaus Hugo Weyl (1885–1955)

## 26.1 “Reine Infinitesimalgeometrie” [Pure infinitesimal geometry] (1918)

1918 original: [29].
`[...]`

> `[...]`

# 27 Courant, Richard Courant (1888–1972)

## 27.1 “Differential and integral calculus – Volume II” (1936, English edition)

1988 reprint of the 1936 English edition: [13].
`[...]`

> `[...]`

In [13, II.4.4, p. 66]:

> `[...]`

## 27.2 “Differential and integral calculus – Volume I” (1937, 2nd English edition)

1988 reprint of the 1937 (2nd) English edition: [12].
`[...]`

> `[...]`

## 27.3 With Fritz John: “Introduction to calculus and analysis – Volume II” (1974)

1989 reprint of the 1974 edition: [14].
`[...]`

> `[...]`

In [14, 1.5.d, pp. 49–51]:

### *d. The Differential of a Function*

As for functions of one variable, it is often convenient to have a special name and symbol for the linear part of the increment of a differentiable function $u = f(x, y)$ which occurs in formula (14),

$$\Delta u = f(x+h, y+k) - f(x, y) = hf_x(x, y) + kf_y(x, y) + \varepsilon\sqrt{h^2 + k^2}.$$

We call this linear part the *differential* of the function, and write

$$du = df(x, y) = \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y. \tag{15a}$$

The differential, sometimes called the *total differential*, is a function of *four* independent variables, namely, the coordinates $x$ and $y$ of the point under consideration and the increments $h$ and $k$ of the independent variables. We emphasize again that this has nothing to do with the vague concept of "infinitely small quantities." It simply means that $du$ approximates to the increment $\Delta u = f(x+h, y+k) - f(x, y)$ of the function, with an error that is an arbitrarily small fraction $\varepsilon$ of $\sqrt{h^2 + k^2}$, provided that $h$ and $k$ are sufficiently small quantities. For the independent variables $x$ and $y$ we find from (15a) that

$$dx = \frac{\partial x}{\partial x}\Delta x + \frac{\partial x}{\partial y}\Delta y = \Delta x \quad \text{and} \quad dy = \frac{\partial y}{\partial x}\Delta x + \frac{\partial y}{\partial y}\Delta y = \Delta y.$$

Hence, the differential $df(x, y)$ is written more commonly

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy. \tag{15b}$$

Incidentally, the differential completely determines the first partial derivatives of $f$. For example, we obtain the partial derivative $\partial f/\partial x$ from $df$, by putting $dy = 0$ and $dx = 1$.

We emphasize that the total differential of a function $f(x, y)$ as the linear approximation to $\Delta f$ has no meaning unless the function is differentiable in the sense defined above (for which the continuity, but not the mere existence, of the two partial derivatives suffices).

If the function $f(x, y)$ also has continuous partial derivatives of higher order, we can form the differential of the differential $df(x, y)$; that is, we can multiply its partial derivatives with respect to $x$ and $y$ by $h = dx$ and $k = dy$, respectively, and then add these products. In this differentiation, we regard $h$ and $k$ as constants, corresponding to the fact that the differential $df = hf_x(x, y) + kf_y(x, y)$ is a function of the four independent variables $x$,

$y$, $h$, and $k$. We thus obtain the *second differential*[1] of the function,

$$\begin{aligned} d^2f = d(df) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k\right)h + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k\right)k \\ &= \frac{\partial^2 f}{\partial x^2}h^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}hk + \frac{\partial^2 f}{\partial y^2}k^2 \\ &= \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2. \end{aligned}$$ [2]

Similarly, we may form the *higher differentials*

$$\begin{aligned} d^3f = d(d^2f) &= \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\,\partial y}dx^2\,dy + 3\frac{\partial^3 f}{\partial x\,\partial y^2}dx\,dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3, \\ d^4f = \frac{\partial^4 f}{\partial x^4}dx^4 &+ 4\frac{\partial^4 f}{\partial x^3\,\partial y}dx^3\,dy + 6\frac{\partial^4 f}{\partial x^2\,\partial y^2}dx^2\,dy^2 \\ &+ 4\frac{\partial^4 f}{\partial x\,\partial y^3}dx\,dy^3 + \frac{\partial^4 f}{\partial y^4}dy^4, \end{aligned}$$

and, as is easily shown by induction, in general

$$\begin{aligned} d^nf = \frac{\partial^n f}{\partial x^n}dx^n &+ \binom{n}{1}\frac{\partial^n f}{\partial x^{n-1}\,\partial y}dx^{n-1}\,dy + \cdots \\ &\cdots + \binom{n}{k}\frac{\partial^n f}{\partial x^{n-k}\,\partial y^k}dx^{n-k}\,dy^k + \cdots + \frac{\partial^n f}{\partial y^n}dy^n. \end{aligned}$$

The last formula can be expressed symbolically by the equation

$$d^nf = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n f$$

where the expression on the right is first to be expanded formally by the binomial theorem, and then the terms

$$\frac{\partial^n f}{\partial x^n}dx^n,\ \frac{\partial^n f}{\partial x^{n-1}\,\partial y}dx^{n-1}\,dy,\ \ldots,\ \frac{\partial^n f}{\partial y^n}dy^n$$

are to be substituted for

$$\left(\frac{\partial}{\partial x}dx\right)^n f,\ \left(\frac{\partial}{\partial x}dx\right)^{n-1}\left(\frac{\partial}{\partial y}dy\right)f,\ \ldots,\ \left(\frac{\partial}{\partial y}dy\right)^n f.$$

For calculations with differentials the rule

$$d(fg) = f\,dg + g\,df$$

holds good; this follows immediately from the rule for the differentiation of a product.

In conclusion, we remark that the discussion in this section can immediately be extended to functions of more than two independent variables.

[1] We shall later see (p. 68) that the differentials of higher order introduced formally here correspond exactly to the terms of the same order in the expansion of the function.
[2] Traditionally, one writes the powers $(dx)^2$, $(dx)^3$, $(dy)^2$, $(dy)^3$ of differentials simply as $dx^2$, $dx^3$, $dy^2$, $dy^3$. This is, of course, somewhat misleading, since they might be confused with $d(x^2) = 2x\,dx$, $d(x^3) = 3x^2\,dx$, and so on.

*Remark.* In the above quotation, two obvious (typographic?) errors were fixed:

(1) one "$\frac{\partial y}{\partial x}$" had to be replaced with "$\frac{\partial y}{\partial y}$,"

(2) one "$d^2f$" had to be replaced with "$d^3f$."

# 28 Ince, Edward Lindsay Ince (1891–1941)

## 28.1 "Ordinary differential equations" (1926)

[...]

[...]

# 29 Struik, Dirk Jan Struik (1894–2000)

## 29.1 "Lectures on classical differential geometry" (1988, 2nd Edition)

[...]

[...]

# 30 Menger, Karl Menger (1902–1985)

## 30.1 "Calculus – A modern approach" (1955, new edition)

1955 new edition: [27].
[...]

[...]

# 31 Cartan, Henri Paul Cartan (1904–2008)

## 31.1 “Calcul différentiel” (1967)

1967 original: [9].

In [9, I.2.1, p. 28]:

**Définition d’une application différentiable 2.1**

Dans ce qui suit, on se donne deux espaces de Banach $\mathsf{E}$ et $\mathsf{F}$, et un ouvert non vide $\mathsf{U} \subset \mathsf{E}$. On considère des applications $\mathsf{f} : \mathsf{U} \to \mathsf{F}$. Chaque point $\mathsf{a} \in \mathsf{U}$ définit une *relation d’équivalence* dans l’ensemble de ces fonctions, comme suit :

DÉFINITION. On dit que $\mathsf{f}_1 : \mathsf{U} \to \mathsf{F}$ et $\mathsf{f}_2 : \mathsf{U} \to \mathsf{F}$ sont *tangentes* en un point $\mathsf{a} \in \mathsf{U}$ si la quantité

$$\mathsf{m}(\mathsf{r}) = \sup_{\|\mathsf{x}-1\| \leqslant \mathsf{r}} \|\mathsf{f}_1(\mathsf{x}) - \mathsf{f}_2(\mathsf{x})\|,$$

qui est définie pour $\mathsf{r} > 0$ assez petit (puisque $\mathsf{U}$ est ouvert), satisfait à la condition

(2.1.1) $$\lim_{\substack{r\to 0\\ r>0}} \frac{\mathsf{m}(\mathsf{r})}{\mathsf{r}} = 0,$$

condition qu’on écrit aussi

(2.1.2) $$\mathsf{m}(\mathsf{r}) = \mathsf{o}(\mathsf{r}).$$

Le lecteur vérifiera que la relation : « $\mathsf{f}_1$ et $\mathsf{f}_2$ sont tangentes en $\mathsf{a}$ » est bien une relation d’équivalence. On a, en particulier, la notion d’une $\mathsf{f}$ *tangente à* $0$ au point $\mathsf{a}$.

La condition (2.1.2) implique que la fonction $\mathsf{f}_1 - \mathsf{f}_2$ est continue au point $\mathsf{a}$, et prend la valeur $0$ au point $\mathsf{a}$. Ainsi : deux fonctions tangentes en $\mathsf{a}$ prennent la même valeur au point $\mathsf{a}$, et si l’une d’elles est continue en $\mathsf{a}$, l’autre est aussi continue en $\mathsf{a}$.

[...]

In [9, I.2.1, p. 29]:

DÉFINITION. On dit que $\mathsf{f} : \mathsf{U} \to \mathsf{F}$ est *dijfférentiable* au point $\mathsf{a} \in \mathsf{U}$ si les conditions suivantes sont vérifiées :

(i) $\mathsf{f}$ est continue au point $\mathsf{a}$ ;

(ii) il existe une $\mathsf{g}$ linéaire $\mathsf{E} \to \mathsf{F}$ telle que les applications $\mathsf{x} \mapsto \mathsf{f}(\mathsf{x}) - \mathsf{f}(\mathsf{a})$ et $\mathsf{x} \mapsto \mathsf{g}(\mathsf{x} - \mathsf{a})$ soient tangentes au point $\mathsf{a}$.

Cette condition s'exprime ainsi :

$$\|f(x) - f(a) - g(x - a)\| = o(\|x - a\|). \tag{2.1.3}$$

Si $f$ est différentiable au point $a$, l'unique $g$ linéaire qu'elle définit est *continue*, d'après la remarque ci-dessus. C'est un élément de $\mathscr{L}(E\,;F)$, qu'on notera $f'(a)$, et qu'on appellera la *dérivée* de l'application $f$ au point $a$.

Une définition équivalente est celle-ci : $f$ est *différentiable* au point $a \in U$ *s'il existe une* $g \in \mathscr{L}(E\,;F)$ *telle que* (2.1.3) *ait lieu*. En effet, la continuité de $g$ entraîne alors la continuité de $f$ au point $a$.

Nous récrivons (2.1.3) avec la notation $f'(a)$ :

$$\|f(x) - f(a) - f'(a)(x - a)\| = o(\|x - a\|). \tag{2.1.4}$$

In [9, I.2.6, p. 38]:

On suppose maintenant que $E = E_1 \times \ldots \times E_n$, et que $U$ est un ouvert de $E$. Soit $f : U \to F$ une application continue. Pour chaque $a = (a_1, \ldots, a_n) \in U$, considérons l'injection $\lambda_i : E_i \to E$ définie par

$$\lambda_i(x_i) = (a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n).$$

L'application composée $f \circ \lambda_i$ est définie dans l'ouvert $(\lambda_i)^{-1}(U) \subset E_i$, qui contient $a_i \in E_i$ ; on l'appelle la $i$-ième application partielle au point $a$.

PROPOSITION 2.6.1 ET DÉFINITION. Avec les notations précédentes, si $f$ est différentiable au point $a$, alors, pour chaque entier $i$ $(1 \leqslant i \leqslant n)$, l'application partielle $f \circ \lambda_i$ est différentiable au point $a_i$. On note $f'_{x_i}(a)$, ou $\partial f/\partial x_i(a)$, ou $f'_{x_i}(a_1, \ldots, a_n)$, ou $\partial f/\partial x_i(a_1, \ldots, a_n)$ la dérivée de cette application partielle au point $a$ ; c'est un élément de $\mathscr{L}(E_i\,;F)$, qu'on appelle aussi la *dérivée partielle* de $f$ par rapport à $x_i$. En outre, on a

$$f'(a) \cdot (h_1, \ldots, h_n) = \sum_{i=1}^{n} f'_{x_i}(a) \cdot h_i, \quad \text{pour } h_1 \in E_1, \ldots, h_n \in E_n. \tag{2.6.1}$$

In [9, I.5.1, p. 64]:

[. . .]

In [9, II.1.1, p. 109]:

[. . .]

# 32 Ehresmann, Charles Ehresmann (1905–1979)

[. . .]

[. . .]

# 33 Dieudonné, Jean Alexandre Eugène Dieudonné (1906–1992)

## 33.1 "Foundations of modern analysis" (1969, enlarged and corrected printing)

1969 enlarged and corrected printing of the 1960 edition: [15].

In [15, VIII.1, p. 149]:

> We say that a continuous mapping $f$ of A into F is *differentiable* at the point $x_0 \in$ A if there is a linear mapping $u$ of E into F such that $x \to f(x_0) + u(x - x_0)$ is tangent to $f$ at $x_0$. We have just seen that this mapping is then *unique*; it is called the *derivative* (or *total derivative*) of $f$ at the point $x_0$, and written $f'(x_0)$ or $\mathrm{D}f(x_0)$.

In [15, VIII.10, p. 176]:

> We mention here the usual notations $f'_{\xi_i}(\xi_1, \ldots, \xi_n)$, $\dfrac{\partial}{\partial \xi_i} f(\xi_1, \ldots, \xi_n)$, for $D_i f(\xi_1, \ldots, \xi_n)$, which unfortunately lead to hopeless confusion when substitutions are made (what does $f'_y(y, x)$ or $f'_x(x, x)$ mean?); the jacobian $\det(D_j \varphi_i(\xi_1, \ldots, \xi_n))$ is also written $D(\varphi_1, \ldots, \varphi_n)/D(\xi_1, \ldots, \xi_n)$ or $\partial(\varphi_1, \ldots, \varphi_n)/\partial(\xi_1, \ldots, \xi_n)$.

## 33.2 "Treatise on analysis – Volume III" (1972)

1972 translation from French original: [16].

In [16, Chapter XVI, p. 2]:

> The property, for two mappings of $\mathbf{R}$ into $\mathbf{R}^n$, of being *tangent* at a point $t_0 \in \mathbf{R}$ (8.1) is invariant under a differentiable change of variables in $\mathbf{R}$ or in $\mathbf{R}^n$. This fact makes it possible to define, for two differentiable mappings $f, g$ of $\mathbf{R}$ into a differential manifold M of dimension $n$, the relation "$f$ and $g$ are tangent at a point $t_0 \in \mathbf{R}$" (which implies that $f(t_0) = g(t_0) = x_0 \in$ M). The equivalence classes for this relation (with $t_0$ and $x_0$ fixed) are no longer called "derivatives" but *tangent vectors to* M *at the point* $x_0$. They form in a natural way a real vector space $\mathrm{T}_{x_0}(\mathrm{M})$ of dimension $n$, called the *tangent vector space to* M *at the point* $x_0$ (16.5). The fundamental difference, compared with analysis in vector spaces, is that the space $\mathrm{T}_x(\mathrm{M})$ *varies with*

$x$ (whereas all the values of the derivative of a function with values in $\mathbf{R}^n$ are considered as belonging to the same vector space). This variation of the tangent space may seem familiar, from the example of surfaces in $\mathbf{R}^3$; unfortunately, geometric intuition here is misleading, because it suggests that the "tangent plane" also is embedded in $\mathbf{R}^3$. To see that one cannot arrive in this way at a correct conception of tangent vectors, it is enough to remark that a tangent vector at a point $x$ of a surface S depends firstly on the two parameters which determine $x$, and then for each $x$ on two more parameters which fix the vector in the tangent plane $\mathrm{T}_x(\mathrm{S})$. The tangent vectors to S must therefore be considered as forming a *four*-dimensional manifold, which clearly cannot be embedded in $\mathbf{R}^3$. The notion which is appropriate here, and which enables us to "pull" the tangent vectors out of the ambient space, is the second fundamental idea in this chapter, that of a *fiber bundle*. In its various forms it dominates nowadays not only differential geometry, but all of topology (see [43] and [49]).

In [16, XVI.5.1, pp. 22–23]:

[...]

In [16, XVI.5.1, p. 24]:

[...]

In [16, XVI.5.2, p. 24]:

[...]

In [16, XVI.5.3, pp. 24–25]:

[...]

In [16, XVI.5.7, p. 27]:

[...]

# 34 Schwartz, Laurent-Moïse Schwartz (1915–2002)

## 34.1 "Cours d'analyse" (1967)

Volume 1, pages (192 quart)-193:

### § 3 DÉRIVÉE D'UNE APPLICATION D'UN ESPACE AFFINE DANS UN AUTRE. VECTEUR DÉRIVÉ D'UNE FONCTION D'UNE VARIABLE SCALAIRE

Considérons une application d'un ouvert $\Omega$ du corps des scalaires $\mathbb{K}$ dans un espace affine normé F *. On peut alors donner un sens, pour $a \in \Omega$, à la formule

$$\text{(III, 3 ; 1)} \qquad \overrightarrow{f'(a)} = \lim_{h \neq 0, h \to 0, a+h \in \Omega} \frac{\overrightarrow{f(a+h) - f(a)}}{h} \in \vec{\mathrm{F}}.$$

Dans le deuxième membre, nous avons d'abord la différence $\overrightarrow{f(a+h) - f(a)}$ de deux points de F, qui est un vecteur de l'espace vectoriel associé $\vec{\mathrm{F}}$. On peut diviser ce vecteur par le scalaire $h \neq 0$, et l'on peut chercher la limite de ce vecteur dans $\vec{\mathrm{F}}$ lorsque $h$ tend vers 0, puisque l'espace vectoriel $\vec{\mathrm{F}}$ est supposé normé. Si $\overrightarrow{f'(a)}$ existe, on l'appelle le vecteur dérivé ou la dérivée de $f$ en $a$. L 'existence de la dérivée et sa valeur ne dépendent pas de la norme, mais seulement de la topologie de F, puisqu'il en est ainsi de la notion de limite. On peut dé même parler de dérivée à gauche et de dérivée à droite, si $\mathbb{K} = \mathbb{R}$. On peut ensuite considérer la fonction dérivée $\vec{f'} : x \to f'(x)$, si la dérivée existe partout dans $\Omega$. C'est une application de $\Omega$ dans l'espace vectoriel normé $\vec{\mathrm{F}}$. On peut ensuite prendre les dérivées ultérieures, dans les mêmes conditions qu'au § 2 ; elles se noteront de la même manière que pour les fonctions réelles (à savoir : $\vec{f''}, \ldots, \overrightarrow{f^{(m)}}, \ldots$ etc…) ; ce sont toutes, si elles existent, des applications de $\Omega$ dans $\vec{\mathrm{F}}$. Notons que $f$ prend ses valeurs dans l'espace *affine* F, et que ses dérivées $\vec{f'}, \vec{f''}, \ldots$ prennent leurs valeurs dans l'espace *vectoriel* associé $\vec{\mathrm{F}}$. Si E = F = $\mathbb{R}$, on retombe sur la dérivée usuelle d'une fonction réelle d'une variable réelle.

* $\mathbb{K}$ est $\mathbb{R}$ ou $\mathbb{C}$ ; F est supposé affine sur $\mathbb{K}$. Si $\mathbb{K} = \mathbb{R}$, et si F est donné comme affine sur $\mathbb{C}$, on se bornera à considérer F comme affine sur $\mathbb{R}$.

[…]

`Volume 1, page 195:`

**Matrice dérivée, déterminant jacobien**

`[...]`

`Volume 1, page 197:`

**Dérivée totale ou application dérivée**

Soit $f$ une application d'un ouvert d'un espace affine normé E dans un espace affine normé F ; on dit que *$f$ admet, au point $a$ de $\Omega$, une application dérivée ou dérivée totale ou différentielle ou différentielle totale* L, *si* L *est une application linéaire continue de* $\vec{\mathrm{E}}$ *dans* $\vec{\mathrm{F}}$*, et si l'on a, pour* $a + \vec{h} \in \Omega$ :

$$\text{(III, 3 ; 13)} \qquad f(a + \vec{h}) = f(a) + \mathrm{L}.\vec{h} + \varphi(\vec{h})\|\vec{h}\|,$$

*où $\varphi(\vec{h})$ tend vers $\vec{0}$ lorsque $\vec{h} \neq \vec{0}$ tend vers $\vec{0}$.* […]

`Volume 1, page 198:`

> <u>Notation</u> $f$ étant une application de $\Omega \subset \mathrm{E}$ dans F, on pourra noter par $f'(a)$ ou $\frac{df}{dx}(a)$ l'application dérivée de $f$ au point $a$ ; on a donc $f'(a) \in \mathscr{L}(\vec{\mathrm{E}}, \vec{\mathrm{F}})$. Si alors $\vec{\mathrm{X}}$ est un vecteur de $\vec{\mathrm{E}}$, on pourra noter par $f'(a) \cdot \vec{\mathrm{X}}$, ou $\frac{df}{dx}(a) \cdot \vec{\mathrm{X}} \in \vec{\mathrm{F}}$, la valeur de cette application dérivée sur le vecteur $\vec{\mathrm{X}}$.
>
> On a donc la formule :
>
> (III, 3 ; $14^{\text{bis}}$) $$\overrightarrow{\mathrm{D}_{\vec{\mathrm{X}}} f}(a) = f'(a) \cdot \vec{\mathrm{X}} \in \vec{\mathrm{F}}.$$

# 35 Federer, Herbert Federer (1920–2010)

## 35.1 "Geometric measure theory" (1969)

`3.1.1, page 209:`

> `[...]`

`3.1.2, page 211:`

> `[...]`

`3.1.11, pages 218-219:`

> `[...]`

`[...]`

# 36 Rudin, Walter Rudin (1921–2010)

## 36.1 "Principles of mathematical analysis" (1976, 3rd Edition)

`Page 212:`

> **9.11 Definition** Suppose $E$ is an open set in $R^n$, $\mathbf{f}$ maps $E$ into $R^m$, and $\mathbf{x} \in E$. If there exists a linear transformation $A$ of $R^n$ into $R^m$ such that
>
> (14) $$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}|}{|\mathbf{h}|} = 0,$$
>
> then we say that $\mathbf{f}$ is *differentiable* at $\mathbf{x}$, and we write
>
> (15) $$\mathbf{f}'(\mathbf{x}) = A.$$
>
> If $\mathbf{f}$ is differentiable at every $\mathbf{x} \in E$, we say that $\mathbf{f}$ is *differentiable* in $E$.

Page 213:

**9.13 Remarks**
(a) The relation (14) can be rewritten in the form

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{h} + \mathbf{r}(\mathbf{h}) \tag{17}$$

where the remainder $\mathbf{r}(\mathbf{h})$ satisfies

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\mathbf{r}(\mathbf{h})|}{|\mathbf{h}|} = 0. \tag{18}$$

We may interpret (17), as in Sec. 9.10, by saying that for fixed $\mathbf{x}$ and small $\mathbf{h}$, the left side of (17) is approximately equal to $\mathbf{f}'(\mathbf{x})$, that is, to the value of a linear transformation applied to $\mathbf{h}$.
(b) Suppose $\mathbf{f}$ and $E$ are as in Definition 9.11, and $\mathbf{f}$ is differentiable in $E$. For every $\mathbf{x} \in E$, $\mathbf{f}'(\mathbf{x})$ is then a function, namely, a linear transformation of $R^n$ into $R^m$. But $\mathbf{f}'$ is also a function: $\mathbf{f}'$ maps $E$ into $L(R^n, R^m)$.
(c) A glance at (17) shows that $\mathbf{f}$ is continuous at any point at which $\mathbf{f}$ is differentiable.
(d) The derivative defined by (14) or (17) is often called the *differential* of $\mathbf{f}$ at $\mathbf{x}$, or the *total derivative* of $\mathbf{f}$ at $\mathbf{x}$, to distinguish it from the partial derivatives that will occur later.

## 36.2 "Real and complex analysis" (1987, 3rd Edition)

Chapter 7, page 150:

**7.22 Definitions** Suppose $V$ is an open set in $R^k$, $T$ maps $V$ into $R^k$, and $x \in V$. If there exists a linear operator $A$ on $R^k$ (i.e., a linear mapping of $R^k$ into $R^k$, as in Definition 2.1) such that

$$\lim_{h\to 0} \frac{|T(x+h) - T(x) - Ah|}{|h|} = 0 \tag{1}$$

(where, of course, $h \in R^k$), then we say that $T$ is *differentiable* at $x$, and define

$$T'(x) = A. \tag{2}$$

The linear operator $T'(x)$ is called the *derivative* of $T$ at $x$. (One shows easily that there is at most one linear $A$ that satisfies the preceding requirements; thus it is legitimate to talk about *the* derivative of $T$.) The term *differential* is also often used for $T'(x)$.

The point of (1) is of course that the difference $T(x+h) - T(x)$ is approximated by $T'(x)h$, a *linear* function of $h$.

Since every real number $\alpha$ gives rise to a linear operator on $R^1$ (mapping $h$ to $\alpha h$), our definition of $T'(x)$ coincides with the usual one when $k = 1$.

# 37 Nomizu, Katsumi Nomizu (1924–2008)

## 37.1 “Lie groups and differential geometry” (1956)

`Page 2:`

> Given a differentiable mapping $\varphi$ of $V$ into $V'$, we define the *differential* of $\varphi$ at $p \in V$. Let $X \in T_p$. Then $f' \to X' \bullet f' = X(f' \circ \varphi)$ is a tangent vector at $\phi(p)$. By mapping $X \in T_p$ into $X' \in T_{\varphi(p)}$, we get a linear mapping of $T_p$, into $T_{\varphi(p)}$ induced by $\varphi$, which is called the differential of $\varphi$ at $p$. We denote it by $d\varphi_p$, or by $\varphi'_p$. Later on we shall use the same letter $\varphi$ in case there is no danger of confusion.

*Remark.* The sense of the last phrase is unclear.

## 37.2 With Shoshichi Kobayashi: “Foundations of differential geometry” (1963)

`Page 6:`

> For a point $p$ of $M$, the dual vector space $T_p^*(M)$ of the tangent space $T_p(M)$ is called the space of *covectors* at $p$. An assignment of a covector at each point $p$ is called a 1-*form* (*differential form of degree* 1). For each function $f$ on $M$, the *total differential* $(df)_p$ of $f$ at $p$ is defined by
>
> $$\langle (df)_p,\ X\rangle = Xf \qquad \text{for } X \in T_p(M),$$
>
> where $\langle\, ,\, \rangle$ denotes the value of the first entry on the second entry as a linear functional on $T_p(M)$. If $u^1, \ldots, u^n$ is a local coordinate system in a neighborhood of $p$, then the total differentials $(du^1)_p, \ldots, (du^n)_p$ form a basis for $T_p^*(M)$. In fact, they form the dual basis of the basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$ for $T_p(M)$. [...]

# 38 Klingenberg, Wilhelm Paul Albert Klingenberg (1924–2010)

## 38.1 “A course in differential geometry” (1978)

`Page 3:`

> Let $U \subset \mathbb{R}^n$ be an open set, and suppose $F\colon U \to \mathbb{R}^m$ is any continuous map. $F$ is said to be *differentiable at* $x_0 \in U$ if there exists a linear mapping $L = L(F, x_0) \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that
>
> $$\lim_{x \to x_0} \frac{|Fx - Fx_0 - L(x - x_0)|}{|x - x_0|} = 0.$$

It will be convenient to denote by $\mathrm{o}(x)$ an arbitrary function with

$$\lim_{x\to 0}\frac{\mathrm{o}(x)}{|x|}=0.$$

In terms of this notation, the equation above may be rewritten as

$$|Fx - Fx_0 - L(x-x_0)| = \mathrm{o}(x-x_0).$$

If such an $L = L(F, x_0)$ exists, it is unique. [...]

Page 4:

The unique linear map $L = L(F, x_0)$ is called the *differential of $F$ at $x_0$*, which will also be denoted by $dF_{x_0}$, or simply $dF$.

# 39 Serre, Jean-Pierre Serre (1926–)

## 39.1 "Lie algebras and Lie groups" (1965, 2nd Edition)

Page 82:

Let $f \in \underline{H}_x$. Then $f - f(x) \in \mathfrak{m}_x$. The image of $f - f(x)$ in $\mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^* X$ is called the *differential* of $f$ at $x$ and is denoted by $df_x$. Let $v \in T_x X$. Then $v$ applied to $df_x$ is called the derivative of $f$ in the direction $v$ and is denoted by $\langle v, df_x\rangle$ or $v \cdot f_x$; we may think of $df_x$ as a linear form on $T_x X$.

# 40 Lang, Serge Lang (1927–2005)

## 40.1 "Calculus of several variables" (1987, 3rd Edition)

Page 80:

[...]

Pages 438-439:

[...]

## 40.2 "Differential and Riemannian manifolds" (1995, 3rd Edition)

[...]

[...]

### 40.3 “Fundamentals of differential geometry” (2001)

Pages 8-9:

> Let $\mathbf{E}$, $\mathbf{F}$ be two topological vector spaces and $U$ open in $\mathbf{E}$. Let $f: U \to \mathbf{F}$ be a continuous map. We shall say that $f$ is **differentiable** at a point $x_0 \in U$ if there exists a continuous linear map $\lambda$ of $\mathbf{E}$ into $\mathbf{F}$ such that, if we let
>
> $$f(x_0 + y) = f(x_0) + \lambda y + \varphi(y)$$
>
> for small $y$, then $\varphi$ is tangent to 0. It then follows trivially that $\lambda$ is uniquely determined, and we say that it is the **derivative** of $f$ at $x_0$. We denote the derivative by $Df(x_0)$ or $f'(x_0)$. It is an element of $L(\mathbf{E}, \mathbf{F})$. If $f$ is differentiable at every point of $U$, then $f'$ is a map
>
> $$f': U \to L(\mathbf{E}, \mathbf{F}).$$
>
> It is easy to verify the chain rule.

### 40.4 “Introduction to differential manifolds” (2002, 2nd Edition)

Page 6:

> Let $\mathbf{E}$, $\mathbf{F}$ be two vector spaces and $U$ open in $\mathbf{E}$. Let $f: U \to \mathbf{F}$ be a continuous map. We shall say that $f$ is **differentiable** at a point $x_0 \in U$ if there exists a continuous linear map $\lambda$ of $\mathbf{E}$ into $\mathbf{F}$ such that, if we let
>
> $$f(x_0 + y) = f(x_0) + \lambda y + \varphi(y)$$
>
> for small $y$, then $\varphi$ is tangent to 0. It then follows trivially that $\lambda$ is uniquely determined, and we say that it is the **derivative** of $f$ at $x_0$. We denote the derivative by $Df(x_0)$ or $f'(x_0)$. It is an element of $L(\mathbf{E}, \mathbf{F})$. If $f$ is differentiable at every point of $U$, then $f'$ is a map
>
> $$f': U \to L(\mathbf{E}, \mathbf{F}).$$
>
> It is easy to verify the chain rule.

## 41 Berger, Marcel Berger (1927–2016)

### 41.1 “A panoramic view of Riemannian geometry” (2003)

Page 164:

> The notion of a smooth map $f: M \to N$ between two manifolds is just defined by pairs of charts, the coherence of the notion being insured by the definition of a manifold, and again the chain rule. The basic point is that

such a map also has a *derivative* (sometimes called the *differential*). It has many possible notations: $f', df, Tf$. At every point, it can be defined in the same three ways that tangent spaces were defined: by curves, by functions and by charts. [...]

Page 165:

An important special case: when the target manifold is $\mathbb{R}$, in which case $df$ is at every point $m \in M$ a linear map

$$df_m : T_m M \to \mathbb{R}$$

i.e. an element of the dual space $T_m^* M$ of the vector space $T_m M$. [...]

# 42 Do Carmo, Manfredo Perdigão do Carmo (1928–2018)

## 42.1 With Francis Flaherty: "Riemannian geometry" (1992)

Page 9:

2.7 Proposition. *Let $M_1^n$ and $M_2^m$ be differentiable manifolds and let $\varphi\colon M_1 \to M_2$ be a differentiable mapping. For every $p \in M_1$ and for each $v \in T_p M_1$, choose a differentiable curve $\alpha\colon (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p\colon T_p M_1 \to T_{\varphi(p)} M_2$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of $\alpha$ (Fig. 4).*

Page 10:

2.8 Definition. The linear mapping $d\varphi_p$ defined by Proposition 2.7 is called the *differential* of $\varphi$ at $p$.

# 43 Smale, Stephen Smale (1930–)

## 43.1 "Diffeomorphisms of the 2-sphere" (1959)

[...]

[...]

## 43.2 "An infinite dimensional version of Sard's theorem" (1965)

Page 861:

A *Fredholm map* is a $C'$ map $f\colon M \to V$ such that for each $x \in M$, the derivative $Df(x)\colon T_x(M) \to T_{f(x)}(V)$ is a Fredholm operator. The *index* of $f$ is defined to be the index of $Df(x)$ for some $x$. By (1.1), since $f$ is $C'$ and $M$ is connected, the definition doesn't depend on $x$.

# 44 Munkres, James Raymond Munkres (1930–)

## 44.1 "Analysis on manifolds" (1991)

Page 42:

**Definition.** Let $A \subset \mathbf{R}^m$; let $f : A \to \mathbf{R}^n$. Suppose $A$ contains a neighborhood of $\mathbf{a}$. Given $\mathbf{u} \in \mathbf{R}^m$ with $\mathbf{u} \neq \mathbf{0}$, define

$$f'(\mathbf{a}; \mathbf{u}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t},$$

provided the limit exists. This limit depends both on $\mathbf{a}$ and on $\mathbf{u}$; it is called the **directional derivative** of $f$ at $\mathbf{a}$ with respect to the vector $\mathbf{u}$. (In calculus, one usually requires $\mathbf{u}$ to be a unit vector, but that is not necessary.)

Page 43:

**Definition.** Let $A \subset \mathbf{R}^m$; let $f : A \to \mathbf{R}^n$. Suppose $A$ contains a neighborhood of $\mathbf{a}$. We say that $f$ is **differentiable** at $\mathbf{a}$ if there is an $n$ by $m$ matrix $B$ such that

$$\frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - B \cdot \mathbf{h}}{|\mathbf{h}|} \to \mathbf{0} \quad \text{as} \quad \mathbf{h} \to \mathbf{0}.$$

The matrix $B$, which is unique, is called the **derivative** of $f$ at $\mathbf{a}$; it is denoted $Df(\mathbf{a})$.

# 45 Milnor, John Willard Milnor (1931–)

## 45.1 "Topology from the differentiable viewpoint" (1965)

Page 2:

To define the notion of *derivative* $df_x$ for a smooth map $f : M \to N$ of smooth manifolds, we first associate with each $x \,\varepsilon\, M \subset R^k$ a linear subspace $TM_x \subset R^k$ of dimension $m$ called the *tangent space* of $M$ at $x$. Then $df_x$ will be a linear mapping from $TM_x$ to $TN_y$ where $y = f(x)$. Elements of the vector space $TM_x$ are called *tangent vectors* to $M$ at $x$.

# 46 Kobayashi, Shoshichi Kobayashi (1932–2012)

## 46.1 With Katsumi Nomizu: "Foundations of differential geometry" (1963)

See 37.2.

# 47 Hirsch, Morris William Hirsch (1933–)

## 47.1 "Differential topology" (1976)

`Page 11:`

If $f\colon M \to N$ is a $C^r$ map (between submanifolds) and $f(x) = z$, a linear map $Tf_x\colon M_x \to N_z$ is defined as follows. Let $(\varphi, U)$, $(\psi, V)$ be charts for $M$, $N$ at $x$, $z$. Put $\varphi(x) = a$, and define $Tf_x$ by

$$Tf_x\colon (x, y) \mapsto (z, D(\phi f \varphi^{-1})_a y).$$

This is independent of the choice of $(\varphi, U)$ and $(\psi, V)$, thanks to the chain rule.

The union of all the tangent spaces of $M$ is called the "tangent bundle" of $M$. The linear maps $Tf_x$ form a map $Tf\colon TM \to TN$. This map plays the role of a "derivative" of the map $f\colon M \to N$.

# 48 Bourbaki (1935–)

## 48.1 "Variétés différentielles et analytiques" [Differential and analytic manifolds] (1967)

`Page 12:`

1.2.1. Soit $f$ une fonction définie dans un voisinage du point $x_0$ de E et à valeurs dans F. On dit que $f$ est *dérivable en* $x_0$ s'il existe une fonction affine continue $v$ de E dans F ayant en $x_0$ un contact d'ordre $\geqslant 1$ avec $f$. Cette application $v$ est unique ; il existe une application linéaire continue et une seule, notée $\mathrm{D}f(x_0)$, de E dans F telle que :

$$v(x) = v(x_0) + \mathrm{D}f(x_0).(x - x_0).$$

Si l'on choisit une norme sur E, ceci équivaut à :

$$f(x_0 + h) = f(x_0) + \mathrm{D}f(x_0).h \qquad \text{mod } o(\|h\|) \qquad \text{pour } h \text{ tendant vers } 0,$$

ce que l'on peut encore écrire sous la forme :

$$\lim_{h\to 0, h\neq 0} \frac{\|f(x_0 + h) - f(x_0) - \mathrm{D}f(x_0).h\|_\gamma}{\|h\|} = 0$$

pour toute semi-norme $\gamma$ continue sur F.
L'élément $\mathrm{D}f(x_0)$ de $\mathscr{L}(\mathrm{E}, \mathrm{E})$ s'appelle la *dérivée* de $f$ en $x_0$. On écrit parfois $\mathrm{D}_h f(x_0)$ pour $\mathrm{D}f(x_0).h$ ; c'est une élément de F défini par la relation :

$$\mathrm{D}_h f(x_0) = \lim_{t\to 0, t\neq 0} \frac{f(x_0 + th) - f(x_0)}{t}.$$

# 49 Sternberg, Shlomo Zvi Sternberg (1936–)

## 49.1 "Lectures on differential geometry" (1964)

Page 72:

Let $f$ be a differentiable function defined at $p$. Then $f$ determines a linear function on $T_p(M)$ via (5.6). We will denote this function by $(df)_p$; thus

$$\langle X_p, (df)_p \rangle = L_{X_p}(f). \tag{5.9}$$

[...]

# 50 Arnold, Vladimir Igorevich Arnold (1937–2010)

[...]

[...]

# 51 Guillemin, Victor William Guillemin (1937–)

## 51.1 With Alan Pollack: "Differential topology" (2011)

Page 8:

### §2 Derivatives and Tangents

We begin by recalling some facts from calculus. Suppose that $f$ is a smooth map of an open set in $\mathbf{R}^n$ into $\mathbf{R}^m$ and $x$ is any point in its domain. Then for any vector $h \in \mathbf{R}^n$, the derivative of $f$ in the direction $h$, taken at the point $x$, is defined by the conventional limit

$$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

With $x$ fixed, we define a mapping $df_x \colon \mathbf{R}^n \to \mathbf{R}^m$ by assigning to each vector $h \in \mathbf{R}^n$ the directional derivative $df_x(h) \in \mathbf{R}^m$. Note that this map, which we call the *derivative of $f$ at $x$*, is defined on all of $\mathbf{R}^n$, even though $f$ need not be.

[...]

Page 10:

We can now construct the best linear approximation of a smooth map of arbitrary manifolds $f \colon X \to Y$ at a point $x$. If $f(x) = y$, this *derivative* should be a linear transformation of tangent spaces, $df_x \colon T_x(X) \to T_y(Y)$. We require two items of our generalized definition of derivative. First, for maps in Euclidean space, we expect the new derivative to be the same as the usual one. Second, we demand the chain rule. It is easy to convince yourself that there is only one possible definition with this requisites. [...]

# 52 Warner, Frank Wilson Warner (1938–)

## 52.1 “Foundations of differentiable manifolds and Lie groups” (1983)

Pages 16-17:

**1.22 The Differential** Let $\psi\colon M \to N$ be $C^\infty$, and let $m \in M$. The *differential of $\psi$ at $m$* is the linear map

$$d\psi\colon M_m \to N_{\psi(m)} \tag{1}$$

defined as follows. If $v \in M_m$, then $d\psi(v)$ is to be a tangent vector at $\psi(m)$, so we describe how it operates on functions. Let $g$ be a $C^\infty$ function on a neighborhood of $\psi(m)$. Define $d\psi(v)(g)$ by setting

$$d\psi(v)(g) = v(g \circ \psi). \tag{2}$$

It is easily checked that $d\psi$ is a linear map of $M_m$ into $N_{\psi(m)}$. Strictly speaking, this map should be denoted $d\psi \mid M_m$, or simply $d\psi_m$. However, we omit the subscript $m$ when there is no possibility of confusion. The map $\psi$ is called *non-singular at $m$* if $d\psi_m$ is non-singular, that is, if the kernel of (1) consists of 0 alone. The *dual map*

$$\delta\psi\colon N^*_{\psi(m)} \to M^*_m \tag{3}$$

is defined as usual by requiring that

$$\delta\psi(\omega)(v) = \omega(d\psi(v)) \tag{4}$$

whenever $\omega \in N^*_{\psi(m)}$ and $v \in M_m$. In the special case of a $C^\infty$ function $f\colon M \to \mathbb{R}$, if $v \in M_m$ and $f(m) = r_0$, then

$$df(v) = v(f)\left.\frac{d}{dr}\right|_{r_0}. \tag{5}$$

In this case, we usually take $df$ to mean the element of $M^*_m$, defined by

$$df(v) = v(f). \tag{6}$$

That is, we identify $df$ with $\delta f(\omega)$, where $\omega$ is the basis of the 1-dimensional space $\mathbb{R}^*_{r_0}$ dual to $(d/dr)|_{r_0}$. Particular usage will be clear from the context.

Page 20:

**1.26 Higher Order Tangent Vectors and Differentials** [...]

Page 65:

**2.19 Definition** If $f \in C^\infty(M)$, the differential $df$ is a smooth mapping of $T(M)$ into $\mathbb{R}$ which is linear on each tangent space. Thus $df$ can be considered as a 1-form, $df: M \to \Lambda_1^*(M)$. The 1-form $df$ is called the *exterior derivative* of the 0-form $f$, and this *exterior differentiation operator* $d$ has an important extension to $E^*(M)$ given by the following.
**2.20 Theorem (Exterior Differentiation)** *There exists a unique antiderivation* $d: E^*(M) \to E^*(M)$ *of degree* $+1$ *such that*

(1) $d^2 = 0$.

(2) *Whenever* $f \in C^\infty(M) = E^0(M)$, $df$ *is the differential of* $f$.

# 53 Spivak, Michael David Spivak (1940–2020)

## 53.1 "Calculus on manifolds: A modern approach to classical theorems of advanced calculus" (1965)

Page 16:

A function $f: \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** at $a \in \mathbf{R}^n$ if there is a linear transformation $\lambda: \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Note that $h$ is a point of $\mathbf{R}^n$ and $f(a+h) - f(a) - \lambda(h)$ a point of $\mathbf{R}^m$, so the norm signs are essential. The linear transformation $\lambda$ is denoted $Df(a)$ and called the **derivative** of $f$ at $a$. [...]

Page 17:

It is often convenient to consider the matrix of $Df(a): \mathbf{R}^n \to \mathbf{R}^m$ with respect to the usual bases of $\mathbf{R}^n$ and $\mathbf{R}^m$. This $m \times n$ matrix is called the **Jacobian matrix** of $f$ at $a$, and denoted $f'(a)$. [...]

Page 26:

[...]

Pages 44-45:

[...]

Page 89:

[...]

Page 134:

[...]

### 53.2 “A comprehensive introduction to differential geometry – Volume 1” (1999)

Page 65:

> If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable map, and $p \in \mathbb{R}^n$, then the linear transformation $Df(p)\colon \mathbb{R}^n \to \mathbb{R}^m$ may be used to produce a linear map from $\mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ defined by
>
> $$v_p \mapsto [Df(p)(v)]_{f(p)}.$$
>
> This map, whose apparently anomalous features will be soon be justified, is denoted by $f_{*p}$; the symbol $f_*$ denotes the map $f_*\colon T\mathbb{R}^n \to T\mathbb{R}^m$ which is the union of all $f_{*p}$. [...]

Page 109:

> If $f\colon M \to \mathbb{R}$ is a $C^\infty$ function, then a $C^\infty$ section $df$ of $T^*M$ can be defined by
>
> $$df(p)(X) = X(f) \quad \text{for } X \in M_p.$$
>
> The section $df$ is called the **differential** of $f$. [...]

## 54 Jänich, Klaus Werner Jänich (1940–)

### 54.1 “Vektoranalysis” [Vector analysis] (1993, 2nd German edition)

## 55 Michor, Peter Wolfram Michor (1949–)

### 55.1 “Topics in differential geometry” (2008)

Page 9:

> If $f \in C^\infty(M)$, then $Tf : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$. We define the *differential* of $f$ by $df := \mathrm{pr}_2 \circ Tf : TM \to \mathbb{R}$. Let $t$ denote the identity function on $\mathbb{R}$. Then $(Tf.X_x)(t) = X_x(t \circ f) = X_x(f)$, so we have $df(X_x) = X_x(f)$.

## 56 Lee, John Marshall Lee (1950–)

### 56.1 “Introduction to smooth manifolds” (2013, 2nd Edition)

Page 55:

> [...]

# 57 Tu, Loring Wuliang Tu (1952–)

## 57.1 "An introduction to manifolds" (2011, 2nd Edition)

Page 87:

Let $F\colon N \to M$ be a $C^\infty$ map between two manifolds. At each point $p \in N$, the map $F$ induces a linear map of tangent spaces, called its *differential at* $p$,

$$F_*\colon T_pN \to T_{F(p)}M$$

as follows. If $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined by

$$(F_*(X_p))f = X_p(f \circ F) \in \mathbb{R} \quad \text{for } f \in C^\infty_{F(p)}(M). \tag{8.1}$$

Here $f$ is a germ at $F(p)$, represented by a $C^\infty$ function in a neighborhood of $F(p)$. Since (8.1) is independent of the representative of the germ, in practice we can be cavalier about the distinction between a germ and a representative function for the germ.

# 58 Tao, Terence Tao (1975–)

## 58.1 "Analysis II" (2016, 3rd Edition)

Page 135:

**Definition 6.2.2** (Differentiability)**.** Let $E$ be a subset of $\mathbf{R}^n$, $f : E \to \mathbf{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. We say that $f$ is *differentiable at* $x_0$ *with derivative* $L$ if we have

$$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{\|f(x) - (f(x_0) + L(x - x_0))\|}{\|x - x_0\|} = 0.$$

Here $\|x\|$ is the length of $x$ (as measured in the $l^2$ metric):

$$\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

Page 136:

Because of Lemma 6.2.4, we can now talk about *the* derivative of $f$ at interior points $x_0$, and we will denote this derivative by $f'(x_0)$. Thus $f'(x_0)$ is the unique linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that

$$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{\|f(x) - (f(x_0) + f'(x_0)(x - x_0))\|}{\|x - x_0\|} = 0.$$

Page 137:

We will sometimes refer to $f'$ as the *total derivative* of $f$, to distinguish this concept from that of partial and directional derivatives below. The total derivative $f$ is also closely related to the *derivative matrix* $Df$, which we shall define in the next section.

`Page 137:`

**Definition 6.3.1** (Directional derivative)**.** Let $E$ be a subset of $\mathbf{R}^n$, $f : E \to \mathbf{R}^m$ be a function, let $x_0 \in E$ be a point, and let $v$ be a vector in $\mathbf{R}^n$. If the limit

$$\lim_{t\to 0; t>0, x_0+tv\in E} \frac{f(x_0+tv) - f(x_0)}{t}$$

exists, we say that $f$ is *differentiable in the direction* $v$ *at* $x_0$, and we denote the above limit by $D_v f(x_0)$:

$$D_v f(x_0) := \lim_{t\to 0; t>0} \frac{f(x_0+tv) - f(x_0)}{t}.$$

## References


[1] Louis François Antoine Arbogast. *Du calcul des dérivations*. French. Strasbourg: Levrault frères, 1800. xxii+404.

[2] Umberto Bottazini. *The higher calculus: a history of real and complex analysis from Euler to Weierstrass*. Translated from Italian. New York, NY: Springer, 24th Sept. 1986. viii+332. URL: `https://link.springer.com/book/9780387963020`.

[3] Robert E. Bradley and C. Edward Sandifer. *Cauchy's Cours d'analyse. An annotated translation*. Sources and Studies in the History of Mathematics and Physical Sciences. New York, NY: Springer, 18th Aug. 2009. xx+411. DOI: `10.1007/978-1-4419-0549-9`. URL: `https://link.springer.com/book/10.1007/978-1-4419-0549-9`.

[4] Constantin Carathéodory. *Calculus of variations and partial differential equations of the first order. Calculus of variations*. Translated from German. Holden-Day, Inc., San Francisco-London-Amsterdam, 1967. xvi+175-398.

[5] Constantin Carathéodory. *Calculus of variations and partial differential equations of the first order. Partial differential equations of the first order*. Translated from German. Holden-Day, Inc., San Francisco-London-Amsterdam, 1965. xvi+171.

[6] Constantin Carathéodory. *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*. German. Leipzig, Berlin: B. G. Teubner, 1935. xi+407.

[7] Élie Cartan. *La géométrie des espaces de Riemann*. French. Paris: Gauthier-Villars, 1925. 61 pp. URL: `https://eudml.org/doc/192543`.

[8] Élie Cartan. *Les systèmes différentiels extérieurs et leurs applications géométriques*. French. Paris: Hermann, 1945. 215 pp. URL: `https://gallica.bnf.fr/ark:/12148/bpt6k3385132c/`.

[9] Henri Cartan. *Calcul différentiel.* French. Paris: Hermann, 1967. 178 pp.

[10] Augustin-Louis Cauchy. *Cours d'analyse de l'École Royale Polytechnique. Première partie : Analyse algébrique.* French. Paris: Imprimerie Royale, Debure frères, 1821. xiv+576.

[11] Augustin-Louis Cauchy. *Cours d'analyse de l'École Royale Polytechnique. Première partie : Analyse algébrique.* French. New York, NY: Cambridge University Press, 2009. xiv+576.

[12] Richard Courant. *Differential and integral calculus.* 2nd ed. Vol. 1. Wiley Classics Library. Translated from German. New York, NY: John Wiley & Sons, Inc., 1988. xiv+616.

[13] Richard Courant. *Differential and integral calculus.* Vol. 2. Wiley Classics Library. Translated from German. New York, NY: John Wiley & Sons, Inc., 1988. x+682.

[14] Richard Courant and Fritz John. *Introduction to calculus and analysis.* Vol. 2. Reprint of the 1974 edition. New York, NY: Springer-Verlag, 1989. xxvi+954.

[15] Jean Alexandre Eugène Dieudonné. *Foundations of modern analysis. Volume I of "Treatise on analysis".* Vol. 10.1. Pure and Applied Mathematics. Enlarged and corrected printing of "Foundations of modern analysis" published in 1960. New York, NY: Academic Press, 1969. xviii+387. URL: `https://www.sciencedirect.com/bookseries/pure-and-applied-mathematics/vol/10/part/P1`.

[16] Jean Alexandre Eugène Dieudonné. *Treatise on analysis. Volume III.* Trans. from the French by Ian Grant Macdonald. Vol. 10.3. Pure and Applied Mathematics. New York, NY: Academic Press, 1972. xvii+388. URL: `https://www.sciencedirect.com/bookseries/pure-and-applied-mathematics/vol/10/part/P3`.

[17] Leonhard Euler. *Introduction to analysis of the infinite. Book I.* Translated from Latin. New York, NY: Springer, 21st Sept. 1988. xv+327. DOI: `10.1007/978-1-4612-1021-4`. URL: `https://link.springer.com/book/978-1-4612-1021-4`.

[18] 'Foundations for a general theory of functions of a complex variable'. In: Inaugural dissertation, Göttingen 1851; second printing (unchanged), Göttingen 1867, pp. 1–41.

[19] Josiah Willard Gibbs and Edwin Bidwell Wilson. *Vector analysis. A text-book for the use of students of mathematics and physics.* Founded upon the lectures of J. Willard Gibbs, Ph.D., LL.D. New Haven, CT: Yale University, 1901. xviii+436. URL: `https://en.wikipedia.org/wiki/Vector_Analysis`.

[20] Édouard Goursat. *Cours d'analyse mathématique.* Vol. 1: *Dérivées et différentielles. Intégrales définies. Développement en série. Applications géométriques.* French. 5th ed. Paris: Gauthier-Villars, 1933. 674 pp. URL: `https://gallica.bnf.fr/ark:/12148/bpt6k9454797`.

[21] Jacques Hadamard. *Leçons sur le calcul des variations*. Vol. 1: *La variation première et les conditions de premier ordre, les conditions de l'extremum libre*. Recueillies par M. Fréchet. French. Paris: Librairie scientifique A. Hermann et fils, 1910. viii+520.

[22] Camille Jordan. *Cours d'analyse de l'École Polytechnique*. Vol. 1: *Calcul différentiel*. French. 3rd ed. Paris: Gauthier-Villars, 1909. xv+620.

[23] Joseph-Louis Lagrange. *Théorie des fonctions analytiques. Contenant les principes du calcul différentiel*. French. Paris: Imprimerie de la République, 1797. xiii+277.

[24] Johann Peter Gustav Lejeune Dirichlet. *G. Lejeune Dirichlet's Werke*. Vol. 1. German. Ed. by Leopold Kronecker. 2 vols. Berlin: Druck und Verlag von Georg Reimer, 1889. x+644. URL: `https://gallica.bnf.fr/ark:/12148/bpt6k99435r`.

[25] Johann Peter Gustav Lejeune Dirichlet. *G. Lejeune Dirichlet's Werke*. Vol. 1. German. Ed. by Leopold Kronecker. 2 vols. Cambridge University Press, 2012. x+644. DOI: `10.1017/CBO9781139237338`.

[26] Johann Peter Gustav Lejeune Dirichlet. 'Ueber die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinusreihen'. German. In: *Repertorium der Physik* 1 (1837), pp. 152–175. URL: `https://doi.org/10.3931/e-rara-55569`.

[27] Karl Menger. *Calculus. A modern approach*. New edition. Boston: Ginn and Company, 1955. xviii+354.

[28] Dieter Rüthing. 'Some definitions of the concept of function from Joh. Bernoulli to N. Bourbaki'. In: *The Mathematical Intelligencer* 6.4 (1984), pp. 72–77. ISSN: 0343-6993,1866-7414. URL: `https://link.springer.com/article/10.1007/BF03026743`.

[29] Hermann Weyl. 'Reine Infinitesimalgeometrie'. German. In: *Mathematische Zeitschrift* (Sept. 1918), pp. 384–411. ISSN: 0025-5874. DOI: `10.1007/BF01199420`. URL: `https://link.springer.com/article/10.1007/BF01199420`.