Tychonoff's Theorem minus the Axiom of Choice

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The goal of this note is to state and prove a theorem in Zermelo-Fraenkel set theory without the Axiom of Choice that allows to easily deduce Tychonoff's theorem from the Axiom of Choice and to easily prove that the product of two compacts is compact without using the Axiom of Choice. (Usually Tychonoff's theorem and the compactness of the product of two compacts are proved by quite different arguments.) This theorem also shows that the Hilbert cube $[0,1]^{\mathbf{Z}}$ is compact without using the Axiom of Choice. (Another way to see this is to observe that $[0,1]^{\mathbf{Z}}$ is a continuous image of the Cantor set.)

Apparently this or a very similar result was published in [1].

Definition. A set-theoretic *tree* is a partially ordered set $(T,<)$ such that for every element $t \in T$, the set $\{\, s \in T \mid s < t \,\}$ is well-ordered. A *branch* of a tree $(T,<)$ is a maximal chaine in the tree. A *subtree* of a tree $(T,<)$ is an ordered subset $(S,<)$ with the property that for every s in S and every t in T such that $t < s$, t is in S.

Note that according to this definition, a tree can be empty or have multiple "roots." Probably *forest* would be a better term. In this note, however, all trees will be nonempty with a single "root."

Recall that a *basic open set* in a Cartesian product $\prod_{i\in I} X_i$ of topological spaces is a set of the form $\prod_{i\in I} U_i$, where each U_i is a nonempty open subset of X_i and $U_i = X_i$ for all but finitely many $i \in I$.

The word "collection" will be used to mean "set of sets."

Theorem (Tychonoff's theorem without the Axiom of Choice). *Let $(I,<)$ be a well-ordered set*[1] *and $(X_i)_{i\in I}$ be a family of compact topological spaces. Denote $X = \prod_{i\in I} X_i$. Let F be the tree $\left(\bigcup_{i\in I}\prod_{j<i} X_j\right) \cup X$ of functions defined on initial intervals of I ordered by inclusion. Consider the set of all nonempty subtrees $T \subset F$ with the following property: for every $i \in I$ and every $f \in T \cap \prod_{j<i} X_j$, the set $\{\, g(i) \mid g \in T,\ f \subsetneq g \,\}$ is closed in X_i; suppose that every such tree T has a branch.*[2] *Then X is compact. Moreover, if all X_i are nonempty, then X is nonempty.*

[1] The Axiom of Choice implies that every set is well-orderable.

[2] The Axiom of Choice in the form of Zorn's Lemma or Hausdorff Maximal Principle implies easily that every nonempty tree has a branch.

Proof. To prove that every open cover of X has a finite subcover, it is enough to prove that every open cover by basic open sets has a finite subcover.

Let $\mathcal{O}$ be a collection of basic open subsets of X such that no finite subcollection of $\mathcal{O}$ covers X. It is enough to prove that $\mathcal{O}$ does not cover X.

Let $T_{\mathcal{O}}$ be the set of all elements $f \in F$ such that the set $\{ x \in X \mid f \subset x \}$ is not covered by any finite subcollection of $\mathcal{O}$. Then $T_{\mathcal{O}}$ is a subtree of F. For every $i \in I$ and $f \in T_{\mathcal{O}} \cap \prod_{j<i} X_j$, denote

$$C_{\mathcal{O}}(f) = \{ g(i) \mid f \subsetneqq g \in T_{\mathcal{O}} \}.$$

Step 1. For every $i \in I$ and every $f \in T_{\mathcal{O}} \cap \prod_{j<i} X_j$, $C_{\mathcal{O}}(f)$ is closed in X_i and nonempty. To prove this, let $\mathcal{W}$ be the collection of all open subsets U of X_i such that there exist a finite subcollection $\mathcal{P} \subset \mathcal{O}$ such that

$$\{ x \in X \mid f \subset x,\ x(i) \in U \ \} \subset \bigcup \mathcal{P}.$$

Then $X_i \setminus C_{\mathcal{O}}(f) = \bigcup \mathcal{W}$ and hence $C_{\mathcal{O}}(f)$ is closed. It also follow that $C_{\mathcal{O}}(f)$ is nonempty, because otherwise, by compactness of X_i, $\mathcal{W}$ would have a finite subcover for X_i, which would yield a finite subcollection of $\mathcal{O}$ that covers $\{ x \in X \mid f \subset x \ \}$ in contradiction with the fact that $f \in T_{\mathcal{O}}$. (Here the compactness of X_i is used similarly to the usual proof that the product of two compact spaces is compact.) It is left to show that indeed $X_i \setminus C_{\mathcal{O}}(f) = \bigcup \mathcal{W}$. Consider an arbitrary $a \in X_i \setminus C_{\mathcal{O}}(f)$ and define $g \in \prod_{j \le i} X_j$ by: $f \subset g$ and $g(i) = a$. Then $g \notin T_{\mathcal{O}}$, and therefore there is a finite collection $\mathcal{P} \subset \mathcal{O}$ such that

$$\{ x \in X \mid f \subset x,\ x(i) = a \} = \{ x \in X \mid g \subset x \} \subset \bigcup \mathcal{P}$$

and

$$\{ x \in X \mid f \subset x,\ x(i) = a \} \cap V \neq \varnothing \quad \text{for every} \quad V \in \mathcal{P}.$$

Let U be the intersection of the ith projections of all elements of $\mathcal{P}$. Then U is an open subset of X_i, $a \in U$, and

$$\{ x \in X \mid f \subset x,\ x(i) \in U \} \subset \bigcup \mathcal{P}.$$

Therefore $U \cap C_{\mathcal{O}}(f) = \varnothing$ and $a \in U \in \mathcal{W}$.

Step 2. Every branch of $T_{\mathcal{O}}$ has the greatest element. To prove this, suppose that B is a branch of $T_{\mathcal{O}}$ without the greatest element. Let $f = \bigcup B$. Let i be the least element of I that is not in the domain of any element of B; then $f \in \prod_{j<i} X_j$. Since B has no greatest element, $f \notin B$, and since B is a maximal chain in $T_{\mathcal{O}}$, $f \notin T_{\mathcal{O}}$. Let $\mathcal{P} \subset \mathcal{O}$ be a finite collection such that

$$\{ x \in X \mid f \subset x \} \subset \bigcup \mathcal{P}.$$

Let m be the greatest element of the finite set of all $j \in I$ such that $j < i$ and for some $V \in \mathcal{P}$, the jth projection of V is not the whole X_j. Let g be any element of B that is defined on m. Consider an arbitrary $x \in X$ such that $g \subset x$. Let $y \in X$ be defined by $f \subset y$ and $y(j) = x(j)$ for every $j \geq i$, and choose $V \in \mathcal{P}$ such that $y \in V$. Then $x \in V$ (because V does not "take into account" the values of $x(j)$ for $m < j < i$). Thus

$$\{\, x \in X \mid g \subset x \,\} \subset \bigcup \mathcal{P},$$

in contradicts with the fact that $g \in T_{\mathcal{O}}$.

Step 3. Every maximal element of $T_{\mathcal{O}}$ is an element of X: an element $f \in T_{\mathcal{O}} \setminus X$ cannot be maximal in $T_{\mathcal{O}}$ because $C_{\mathcal{O}}(f) \neq \varnothing$.

Now it can be shown that there is $f \in X$ such that $f \notin \bigcup \mathcal{O}$. Indeed, according to the hypotheses, $T_{\mathcal{O}}$ has a branch B. Let f be the greatest element of B. Then f is a maximal element of $T_{\mathcal{O}}$. Therefore $f \in X$. Therefore the set $\{f\} = \{\, x \in X \mid f \subset x \,\}$ is not covered by any finite subcollection of $\mathcal{O}$, and hence $f \notin \bigcup \mathcal{O}$. By the choice of $\mathcal{O}$, the compactness of X is thus proved.

Suppose now that all X_i are nonempty. Then F is nonempty, and, by one of the assumptions, it has a branch. The union of this branch is an element of X, thus X is nonempty. □

Corollary (without the Axiom of Choice)**.** *The Cartesian product of a finite family of compact topological spaces is compact.*

Corollary (without the Axiom of Choice)**.** *Let I be a well-orderable set and $(X_i)_{i \in I}$ a family of compact topological spaces. Suppose that the Cartesian product of all nonempty closed subsets of all X_i is nonempty:*

$$\prod \{\, C \mid C \textit{ is nonempty and closed in } X_i \textit{ for some } i \in I \,\} \neq \varnothing.$$

Then $\prod_{i \in I} X_i$ is compact.

Outline of a proof. Let $<$ be a well-order relation on I. Denote $X = \prod_{i \in I} X_i$. Let F be the tree $\left(\bigcup_{i \in I} \prod_{j < i} X_j\right) \cup X$ ordered by inclusion. To apply the theorem, it is enough to verify that if T is a nonempty subtree of F such that for every $i \in I$ and every $f \in T \cap \prod_{j<i} X_j$, the set $\{\, g(i) \mid g \in T,\ f \subsetneqq g \,\}$ is closed in X_i, then T has a branch.

Let

$$e \in \prod \{\, C \mid C \text{ is nonempty and closed in } X_i \text{ for some } i \in I \,\}$$

be a "choice function." Let B_e be the minimal tree among all subtrees S of T with the property that for every $i \in I$ and every $f \in S \cap \prod_{j<i} X_j$,

$$e(\{\, g(i) \mid g \in T,\ f \subsetneqq g \,\}) \in \{\, g(i) \mid g \in S,\ f \subsetneqq g \,\}$$

unless

$$\{\, g(i) \mid g \in T,\ f \subsetneqq g \,\} = \varnothing.$$

Such subtrees of T exist because T itself is such, and the minimal such subtree is the intersection of all such subtrees. It can be shown that B_e is a branch by assuming that it is not, considering the minimal $i \in I$ where it "branches," and arriving at a contradiction with its minimality. □

Example. The Hilbert cube $[0,1]^{\mathbf{Z}}$ is compact independently of the Axiom of Choice, because the product of all nonempty closed subsets of $[0,1]$ contains, for example, the function min that to every nonempty closed subset of $[0,1]$ associate its minimal element.

Tichonoff's theorem is an easy corollary of the *Alexander subbase theorem* (in the presence of the Axiom of Choice):

Theorem (Alexander subbase theorem)**.** *If a topological space X has a subbase such that every cover of X by elements of this subbase has a finite subcover, then X is compact.*

It remains to be seen how the hypotheses of the Alexander subbase theorem could be modified to eliminate the need for the Axiom of Choice in its proof.

References

[1] Peter A. Loeb, *A new proof of the Tychonoff Theorem*, The American Mathematical Monthly **72** (1965), no. 7, 711–717.