Bochner-Martinelli formulas on singular complex spaces

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Abstract.

Let $\tilde{\mathbf{C}^n}$ be the blowing-up of $\mathbf{C}^n$ at a point P. We prove that the pullback to $\tilde{\mathbf{C}^n}$ of the Bochner-Martinelli form centered at P is logarithmic along the exceptional divisor. It follows that the Bochner-Martinelli integral formula appears as a Leray residue formula. Moreover the Bochner-Martinelli form is logarithmic also at infinity.

Let U be a complex space of complex dimension $n \geq 2$, subject to the following assumption: *there exist a compact complex space X bimeromorphic to a Kähler manifold, and a closed subspace $T \subset X$, such that $X \setminus T = U$.* An affine, or a quasi projective variety U satisfies the above property (X is a projective compactification of U). For a point $P \in U$, the cohomology group $H^{2n-1}(U \setminus \{P\}, \mathbf{C})$, equipped with the weight filtration W_m, carries a mixed Hodge structure. Thus the first graded quotient

$$BM(U \setminus \{P\}) = \frac{W_1 H^{2n-1}(U \setminus \{P\}, \mathbf{C})}{W_0 H^{2n-1}(U \setminus \{P\}, \mathbf{CC})}$$

carries a pure Hodge structure of weight $2n-2$, which turns out to contain only elements of pure type $(n-1, n-1)$. The elements of $BM(U \setminus \{P\})$ are represented by closed forms ω on $U \setminus \{P\}$ of pure type $(n, n-1)$, which are logarithmic in a suitable sense. Thanks to a more general residue formula we prove that the forms ω give rise to an integral formula of Bochner-Martinelli type for holomorphic functions.

We prove that the forms ω can be chosen to depend C^∞ on P, that is, we prove the existence of ($\overline{\partial}$-closed) Bochner-Martinelli Kernels. Such Kernels can be used to prove integral formulas for differential forms (in sense of Grauert) on U.

MSC 32C15, 32A25 .

1. The Bochner-Martinelli formula as a residue formula.

It is well known (see for example [GH]) that the Bochner-Martinelli integral formula can be proved as a consequence of the Grothendieck residue formula. Here we show that it can also be proved from the Leray residue formula [L]. This suggests how to extend such formulas to complex spaces.

Let $w = (w_1, \cdots, w_n)$ be a point in $\mathbb{C}^n$. We consider the differential form

$$\omega'\left(\overline{z-w}, d\overline{z}\right) = \sum_{k=1}^{n} (-1)^{k-1} (\overline{z_k - w_k}) d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_k} \wedge \cdots \wedge d\overline{z}_n \tag{1.1}$$

The form

$$\omega(w,z) = \frac{\omega'\left(\overline{z-w}, d\overline{z}\right) \wedge dz_1 \wedge \cdots \wedge dz_n}{\|z-w\|^{2n}} \tag{1.2}$$

with

$$\|z-w\|^2 = \sum_{k=1}^{n} |z_k - w_k|^2 \tag{1.3}$$

is the *Bochner-Martinelli form.* It is d-closed and $\overline{\partial}$-closed. The Bochner-Martinelli formula reads as

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a relatively compact domain with (piecewise) smooth boundary. Let f be a holomorphic function on Ω, which is continuous on $\bar{\Omega}$. For a point $w \in \Omega$ one has*

$$f(w) = \frac{(n-1)!}{(2i\pi)^n} \int_{\partial\Omega} f(z)\omega(w,z) \tag{1.4}$$

Proof. One can assume that $w = 0$. We shall prove that (1.4) can be deduced from the Leray residue formula. Let $\tilde{\Omega}$ be the manifold obtained by blowing-up the point $0 \in \Omega$. $\tilde{\Omega}$ is the submanifold of $\Omega \times \mathbb{P}^{n-1}$ defined by the equations $Z_l z_k = Z_k z_l$ for $k \neq l$, where $[Z_1, \cdots, Z_n]$ are the homogeneous coordinates in $\mathbb{P}^{n-1}$. We consider a point m of the exceptional divisor $(0) \times \mathbb{P}^{n-1} \subset \tilde{\Omega}$ such that $Z_n \neq 0$ at m. Then, one can choose as complex coordinates in the neighborhood of m, the numbers

$$\zeta_1 = \frac{Z_1}{Z_n}, \cdots, \zeta_k = \frac{Z_k}{Z_n}, \cdots, \zeta_{n-1} = \frac{Z_{n-1}}{Z_n},\ z_n$$

so that $z_k = \zeta_k z_n$ for $k \leq n-1$. Thus $z_n = 0$ is the local equation of the exceptional divisor $(0) \times \mathbb{P}^{n-1}$ in a neighborhood of m. We consider the form

$$\phi = f(z) \frac{\omega'\left(\overline{z}, d\overline{z}\right) \wedge dz^1 \wedge \cdots \wedge dz^n}{\|z\|^{2n}} \tag{1.5}$$

The the pull back of ϕ by the projection map $p : \tilde{\Omega} \to \Omega$ is logarithmic along the exceptional divisor D.

In fact $p^*\phi$ is given by the following formula

$$p^*\phi = \frac{f(\zeta_1 z_n, \cdots, \zeta_{n-1} z_n, z_n)}{|z_n|^{2n}(1+\sum_{k=1}^{n-1}|\zeta_k|^2)^n} \times$$

$$[\sum_{k=1}^{n-1}(-1)^{k-1}\bar{\zeta}_k \bar{z}_n d(\bar{\zeta}_1 \bar{z}_n) \wedge \cdots \wedge d(\widehat{\bar{\zeta}_k \bar{z}_n}) \wedge \cdots \wedge d\bar{z}_n$$

$$+(-1)^{n-1}\bar{z}_n d(\bar{\zeta}_1 \bar{z}_n) \wedge \cdots \wedge d(\bar{\zeta}_{n-1}\bar{z}_n)] \wedge [d(\zeta_1 z_n) \wedge \cdots \wedge d(\zeta_{n-1} z_n) \wedge dz_n]$$

$$\equiv \frac{f(\zeta_1 z_n, \cdots, \zeta_{n-1} z_n, z_n)}{(1+\sum_{k=1}^{n-1}|\zeta_k|^2)^n} \left(d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{n-1} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1} \wedge \frac{dz_n}{z_n} \right)$$

(modulo smooth forms).
So the residue of $p^*\phi$ on the exceptional divisor is

$$Res \quad p^*\phi = f(0) \frac{\left(d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{n-1} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}\right)}{(1+\sum_{k=1}^{n-1}|\zeta_k|^2)^n}$$

Moreover because ϕ is closed, one has

$$\int_{\partial\Omega} \phi = \int_{\partial B(0,\epsilon)} \phi = \int_{S(0,\epsilon)} \phi$$

where $B(0,\epsilon)$ (resp. $S(0,\epsilon)$) is the ball (resp.the sphere) of center at 0 and radius ϵ. But $S(0,\epsilon)$ is exactly $\delta[(0) \times I\!P^{n-1}]$ where δ is the residue in homology. Thus in the manifold $\tilde{\Omega}$, one has by the residue formula

$$\int_{S(0,\epsilon)} \phi = \int_{S(0,\epsilon)} p^*\phi = (2i\pi) \int_{I\!P^{n-1}} Res \quad p^*\phi =$$

$$= 2i\pi f(0) \int_{I\!P^{n-1}} \frac{\left(d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{n-1} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}\right)}{(1+\sum_{k=1}^{n-1}|\zeta_k|^2)^n}$$

This last integral is exactly the integral of the volume form of $I\!P^{n-1}$. Its value is

$$\frac{(2i\pi)^{n-1}}{(n-1)!}$$

Hence we have seen that the pullback of the Bochner-Martinelli form on $\tilde{\Omega}$ has a logarithmic singularity along the exceptional divisor, and that the Bochner-Martinelli formula is a residue formula in the blow-up manifold $\tilde{\Omega}$ with respect to the exceptional divisor.

The behaviour at the infinity of the Bochner-Martinelli form.

Next we study the behaviour at the infinity of the Bochner-Martinelli form. We consider the projective space $\mathbb{P}^n$ with complex coordinates $[x_0, \cdots, x_n]$ and we look at $\mathbb{C}^n$ as the complement of the hyperplane H defined by the equation $x_0 = 0$. Let P be a point of H where $x_1 \neq 0$. Then we can write

$$z_j = \frac{x_j}{x_0}, \quad j = 1, \cdots n$$

so that the local coordinates v_j around P are related to the coordinates z_j by the formulas

$$z_1 = \frac{1}{v_1}, \quad z_s = \frac{v_s}{v_1} \quad for \quad s = 2, \cdots, n$$

and H is defined by the equation $v_1 = 0$. Substituting the above relations in the formula (1.2) $(w = 0)$ we find that the Bochner-Martinelli form has a logarithmic singularity along H, whose residue is

$$-\frac{(d\bar{v}_2 \wedge \cdots \wedge d\bar{v}_n \wedge dv_2 \wedge \cdots \wedge dv_n)}{(1 + \sum_{k=2}^{n} |v_k|^2)^n}$$

2. The Leray residue theorem for a divisor with normal crossings.

In this section, X is a complex analytic manifold and $D = D_1 \cup \cdots \cup D_N$ is a *divisor with normal crossings*; that means that each D_i is a smooth hypersurface of X, and at each point $x \in X$, there are at most $n = \dim_{\mathbb{C}} X$ divisors D_j passing through x and which are transversal. In particular, given x, one can find complex analytic coordinates $(z_1, \cdots, z_n)$ in a neighborhood U of x, such that the local equation of $D \cap U$ in U is $z_1 \cdots z_s = 0$, s depending on x.

We define for any ordered multiindex $I = (i_1, \cdots, i_q) \subset (1, ..., N)$

$$D_I = D_{i_1} \cap \cdots \cap D_{i_q}$$

and

$$D^{[q]} = \amalg_{|I|=q} D_I, \quad D^{[0]} = X$$

where the symbol $\amalg$ denotes the disjoint union. Then the $D^{[q]}$ are manifolds (not connected in general).

The residue theorem.

Let $L_i \to X$ be the line bundle associated to D_i, and h_i a hermitian metrics on L_i. We denote by s_i a holomorphic section of L_i vanishing exactly on D_i. There exists a neighborhood of D_i in X diffeomorphic to a neighborhood of D_i (embedded as the zero section) in L_i. For $\epsilon_i > 0$ small enough we consider the tube around D_i:

$$T_{\epsilon_i} = \{x \in X : \|s_i(x)\| < \epsilon_i\}$$

where the length $\|s_i(x)\|$ is taken with respect to the metrics h_i.

Let us define

$$T_{\underline{\epsilon}} = \bigcup_i T_{\epsilon_i} \tag{2.1}$$

The boundary $\partial T_{\underline{\epsilon}}$ is a cycle of dimension $2n-1$ in $X \setminus D$.

Let ω be a form of type $(n, n-1)$ on a neighborhood of D, d-closed (hence $\overline{\partial}$-closed), having logarithmic singularities along D.

We define the residue of ω on D. Let x be a smooth point of D; it belongs to a unique component D_j of D. Let $\zeta_j = 0$ be the equation of D_j in a neighborhood U of x; on U we can write

$$\omega|_U = \frac{d\zeta_j}{\zeta_j} \wedge \psi + \theta$$

where ψ, θ are C^∞ on U. We put

$$Res\ \omega|_{U \cap D_j} = \psi|_{U \cap D_j}$$

and we get a well defined $(n-1, n-1)$ form $Res\ \omega$ on the disjoint union $\bigcup_j (D_j \setminus Sing(D))$ $= D \setminus Sing(D)$, having logarithmic singularities along each $D_j \cap Sing(D)$.

Lemma 2.1. *The form $Res\ \omega$ is integrable on D_j.*

Theorem 2.2. *Let X be a complex manifold of complex dimension $n \geq 2$, $D \subset X$ a divisor with normal crossings, ω a differential form of type $(n, n-1)$, with compact support, on X, having logarithmic singularities along D. Then*

$$\lim_{\epsilon \to 0} \int_{\partial T_{\underline{\epsilon}}} \omega = 2i\pi \int_D Res\ \omega$$

with $T_{\underline{\epsilon}} = \bigcup_i T_{\epsilon_i}$ as in (2.1).

3. Logarithmic differential forms and the mixed Hodge structure on cohomology.

By a pair (of complex spaces) (X, Q) we mean the data of a complex space X and of a closed, nowhere dense complex subspace Q. Let $\rho : X \setminus Q \to X$ be the natural embedding.

If X is smooth and $Q = D$ is a divisor with normal crossings, a *logarithmic differential k-form* on X (with poles of order $\leq l$ along D) is a form ω on $X \setminus D$ which, in a sufficiently small neighborhood of any $x \in D$ can be written as

$$\omega = \sum_{|I| \leq l} \alpha_I \wedge \left(\frac{dz}{z} \right)^I \tag{3.1}$$

where $\left(\frac{dz}{z}\right)^I = \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}}$.

The differential $d\omega$ of a logarithmic form (with poles of order $\leq l$) is logarithmic (with poles of order $\leq l$). The above definition has a local nature: we can define a logarithmic form on $Y \setminus D$ for any open subset $Y \subset X$, hence the sheaf $\mathcal{E}^k_X < logD >$ of the logarithmic k-forms is well defined, and $\mathcal{E}^{\cdot}_X < logD >$ is a complex of fine sheaves on X.

The logarithmic forms on any open set $Y \subset X$ are particular differential forms on $Y \setminus D$, hence we have an inclusion

$$\mathcal{E}^{\cdot}_X < logD > \subset \rho_* \mathcal{E}^{\cdot}_{X \setminus D}$$

where $\rho : X \setminus D \hookrightarrow X$ is the natural inclusion map.

The following statements (Griffiths-Schmid) hold:

- every closed differential form on $X \setminus D$ is cohomologous to a logarithmic form;
- every logarithmic differential form on $X \setminus D$ which is exact, is the differential of a logarithmic form.

The main consequence of the above result is that the cohomology of $X \setminus D$ is the cohomology of the complex of global sections $\Gamma(X, \mathcal{E}^{\cdot}_X < logD >)$:

$$H^k(X, \mathcal{E}^{\cdot}_X < logD >) \simeq H^k(X, \rho_* \mathcal{E}^{\cdot}_{W \setminus D}) \simeq H^k(X \setminus D, \mathbb{C})$$

We introduce **the weight filtration** W on $\mathcal{E}^k_X < logD >$, just defining $W_l \mathcal{E}^k_X < logD >$ as the subsheaf of $\mathcal{E}^k_X < logD >$ of the forms having poles of order $\leq l$.

If X is smooth and Q is any closed subspace, a a differential k-form ω on $X \setminus Q$ is said *logarithmic along* Q if for some blowing-up

$$\begin{array}{ccc} D & \xrightarrow{i} & \tilde{X} \\ \downarrow & & \pi \downarrow \\ Q & \xrightarrow{j} & X \end{array}$$

such that D is a divisor with normal crossing, the pull-back $\pi^*\omega$ is logarithmic along D.

For a pair (X, Q), where X is possibly singular, we define a complex (in fact, a family of complexes) of fine sheaves $(\Lambda^{\cdot}_X < logQ >, d)$ on X with the following properties.

(I) The restriction $\Lambda^{\cdot}_{X\setminus Q} = \Lambda^{\cdot}_X < logQ > |_{X\setminus Q}$ of $\Lambda^{\cdot}_X < logQ >$ to $X \setminus Q$ is a resolution of the constant sheaf $\mathbb{C}$ on $X \setminus Q$, and the natural morphism of complexes

$$\Lambda^{\cdot}_X < logQ > \longrightarrow \rho_* \Lambda^{\cdot}_{X\setminus Q}$$

induces isomorphisms in cohomology:

$$H^k(X, \Lambda^{\cdot}_X < logQ >) = H^k(X, \rho_* \Lambda^{\cdot}_{X\setminus Q}) = H^k(X \setminus Q, \mathbb{C}) \tag{3.2}$$

in other words the cohomology of $X \setminus Q$ can be calculated as the cohomology of the complex of sections $(\Gamma(X, \Lambda^{\cdot}_X < logQ >), d)$ of $\Lambda^{\cdot}_X < logQ >$.

(II) For $k > 2dimX$, $\Lambda^{k}_X < logQ >= 0$.

The complex $\Lambda^{\cdot}_X < logQ >$ will be called a *logarithmic complex* for a pair (X, Q); we recall its construction.

Let (X, Q) be any pair, $E = Sing(X)$; let us consider a diagram of desingularization of X

$$\begin{array}{ccc} \tilde{E} & \stackrel{i}{\rightarrow} & \tilde{X} \\ q\downarrow & & \pi\downarrow \\ E & \stackrel{j}{\rightarrow} & X \end{array} \tag{3.3}$$

where $\tilde{X}$ is a smooth manifold, $\tilde{E} = \pi^{-1}(E)$, and π induces by restriction an isomorphism $\tilde{X} \setminus \tilde{E} \simeq X \setminus E$. Let

$$\tilde{Q} = \pi^{-1}(Q), \; M = E \cap Q, \; \tilde{M} = \tilde{E} \cap \tilde{Q}.$$

We suppose that $\tilde{Q}$ is a divisor with normal crossings.

By induction on $dim(X)$ we can find complexes $\Lambda^{\cdot}_E < logM >$ and $\Lambda^{\cdot}_{\tilde{E}} < log\tilde{M} >$, corresponding to the pairs (E, M) and $(\tilde{E}, \tilde{M})$, a pullback

$$\phi : \Lambda^{\cdot}_E < logM > \longrightarrow \Lambda^{\cdot}_{\tilde{E}} < log\tilde{M} > \tag{3.4}$$

a pullback

$$\psi : \mathcal{E}^{\cdot}_{\tilde{X}} < log\tilde{Q} > \longrightarrow \Lambda^{\cdot}_{\tilde{E}} < log\tilde{M} >$$

so that we define the complex

$$\Lambda^{k}_X < logQ >= \pi_* \mathcal{E}^{k}_{\tilde{X}} < log\tilde{Q} > \oplus j_* \Lambda^{k}_E < logM > \oplus (j \circ q)_* \Lambda^{k-1}_{\tilde{E}} < log\tilde{M} > \tag{3.5}$$

whose differential is by definition

$$d(\omega, \sigma, \theta) = (d\omega, d\sigma, d\theta + (-1)^k(\psi(\omega) - \phi(\sigma)). \tag{3.6}$$

Note that $\Lambda^k_X < logQ >$ is a fine sheaf defined on all of X.
From the construction of $\Lambda^{\cdot}_X < logQ >$ it follows that there is a uniquely determined family $((X_a, Q_a), h_a)_{a\in A}$ of pairs (X_a, Q_a), where X_a is a smooth manifold and Q_a is (either empty or) a divisor with normal crossings in X_a, and proper maps of pairs $h_a : (X_a, Q_a) \to (X, Q)$ such that

$$\Lambda^k_X < logQ >= \bigoplus_{a\in A} (h_a)_* \mathcal{E}^{k-q(a)}_{X_a} < logQ_a > \tag{3.7}$$

where $q(a) = q_X(a)$ is a nonnegative integer, which depends only on $a \in A$ and not on k. The family $(X_a, Q_a)_{a\in A}$ will be called the *hypercovering* of (X, Q) associated to the complex $\Lambda^{\cdot}_X < logQ >$, and $q_X(a)$ will be the **rank** of (X_a, Q_a).

Remark.

1) *In the situation of the diagram (3.3) and of the complex (3.5), we notice that* $(\tilde{X}, \tilde{Q})$ *is a pair* (X_a, Q_a) *of the hypercovering, with* $q(a) = 0$.
2) *Notice also that* $dimX_a \leq dimX$*, and equality holds if and only if* $X_a = \tilde{X}$.

The weight filtration W and the Hodge filtration F.

If $\Lambda^{\cdot}_X < logQ >$ is a logarithmic complex, we can rewrite the equation (3.5) defining the complex as

$$\Lambda^k_X < logQ >= \mathcal{E}^k_{\tilde{X}} < log\tilde{Q} > \oplus \Lambda^k_E < logM > \oplus \Lambda^{k-1}_{\tilde{E}} < log\tilde{M} > \tag{3.8}$$

where we have skipped the symbols of direct images of sheaves. The *weight filtration* W on the complex $(\Lambda^{\cdot}_X < logQ >, d)$ is defined by the formula

$$W_m\Lambda^k_X < logQ >=$$

$$= W_m\mathcal{E}^k_{\tilde{X}} < log\tilde{Q} > \oplus W_m\Lambda^k_E < logM > \oplus\ W_{m+1}\Lambda^{k-1}_{\tilde{E}} < log\tilde{M} > \tag{3.9}$$

In (3.9) $W_m\Lambda^k_E < logM >$ and $W_{m+1}\Lambda^{k-1}_{\tilde{E}} < log\tilde{M} >$ are defined by recursion on the dimension of the spaces, and $W_m\mathcal{E}^{\cdot}_{\tilde{X}} < log\tilde{Q} >$ is the filtration by the order of the poles.
$(\Lambda^{\cdot}_X < logQ >, d)$ is a filtered complex for W_m:

$$d(\ W_m\Lambda^k_X < logQ >)\ \subset W_m\Lambda^{k+1}_X < logQ > \tag{3.10}$$

As well, the *Hodge filtration* F on the complex $(\Lambda^{\cdot}_X < logQ >, d)$ is defined by the formula

$$F^p\Lambda^k_X < logQ >=$$

$$= F^p\mathcal{E}^k_{\tilde{X}} < log\tilde{Q} > \oplus F^p\Lambda^k_E < logM > \oplus\ F^p\Lambda^{k-1}_{\tilde{E}} < log\tilde{M} > \tag{3.11}$$

where $F^p\Lambda^{k}_{E} < logM >$ and $F^p\Lambda^{k-1}_{\tilde{E}} < log\tilde{M} >$ are defined by recursion on the dimension of the spaces, and $F^p\mathcal{E}^{\cdot}_{\tilde{X}} < log\tilde{Q} >$ is the usual Hodge filtration.

By the isomorphism (3.2) the filtrations W and F induce a weight and a Hodge filtrations on the cohomology spaces $H^k(X \setminus Q, \mathbb{C})$, which we denote by the same symbols.

The spectral sequence associated to the weight filtration.

For the spectral sequences (associated to a filtration) we use notations, which are different from those which usually appear in the literature. In our notation $E_r^{m,k}$, m is the degree of the filtration and k is the degree of the complex (the degree of differential forms in the case of the De Rham complex). In particular

$$d_r : E_r^{m,k} \to E_r^{m-r,k+1}$$

If one is willing to work with the classical indices $E_r'^{p,q}$ can use the following dictionary:

$$E_r^{m,k} = E_r'^{-p,p+q}$$

$$E_r'^{p,q} = E_r^{-m,k+m}$$

The mixed Hodge structure.

Let us suppose that X is a compact complex space bimeromorphic to a Kähler manifold.

We consider the spectral sequence $E_r^{m,k}$ attached to the weight filtration of the complex $\Gamma(X, \Lambda^{\cdot}_X < logQ >)$.

The following (highly non trivial) results hold.

1) the first terms are

$$E_1^{m,k} = E_1^{m,k}(X) = \bigoplus_a E_1^{m+q(a),k-q(a)}(X_a) \tag{3.12}$$

where

$$E_1^{r,s}(X_a) = H^{s-r}({Q_a}^{[r]}, \mathbb{C}) \tag{3.13}$$

(recall: ${Q_a}^{[0]} = X_a$);

2) the spectral sequence degenerates at the level 2: $d_r = 0$, hence $E_r^{m,k} = E_2^{m,k}$, for $r \geq 2$;

3) the second terms $E_2^{m,k}$ carry a pure Hodge structure, and they are isomorphic to the graded quotients $\frac{W_m H^k(X \setminus Q, \mathbb{C})}{W_{m-1} H^k(X \setminus Q, \mathbb{C})}$ of the cohomology $H^k(X \setminus Q, \mathbb{C})$ with respect to the weight filtration;

4) the Hodge filtration on $E_2^{m,k}$ coincides with the filtration induced in cohomology, by means of residues, by the Hodge filtration of the complex $\Lambda^{\cdot}_X < logQ >$.

4. The general Bochner-Martinelli formula.

Definition 4.1. *Let U be a complex space, $S \subset U$ a closed subspace. Let α be a differential form on $U \setminus (Sing(U) \cup S)$. We say that α is logarithmic along S if there exists a proper modification $\pi : \tilde{U} \to U$ with the following properties*

1) $\tilde{U}$ is non singular;

2) π induces an isomorphism $h : \tilde{U} \setminus \pi^{-1}(Sing(U) \cup S) \to U \setminus (Sing(U) \cup S)$;

3) $\tilde{S} = \pi^{-1}(S)$ is a divisor with normal crossings in $\tilde{U}$;

4) $\pi^*\alpha$ extends to a differential form on $\tilde{U} \setminus \tilde{S}$, logarithmic along $\tilde{S}$.

If S is a point $P \in U$, we will say that α *is logarithmic at* P.

Throughout the present talk, U will be a complex space of complex dimension $n \geq 2$, subject to the following assumption: *there exist a compact complex space X bimeromorphic to a Kähler manifold, and a closed subspace $T \subset X$, such that $X \backslash T = U$.* An affine, or a quasi projective variety U satisfies the above property (X is a projective compactification of U).

Let $p : Y \to X$ be a desingularization of X (so that $p^{-1}(U)$ is a desingularization of U).

Let P be a point of U. Let $u : \tilde{X} \to Y$ be the blowing-up of Y along a suitable subspace of the fiber $p^{-1}(P)$ such that $D = (p \circ u)^{-1}(P)$ is a divisor with normal crossings in $\tilde{X}$. We denote $\pi : \tilde{X} \to X$ the composition $p \circ u$. Let $\tilde{U} = \pi^{-1}(U)$. Then $\tilde{U} \setminus D$ is a desingularization of $U \setminus \{P\}$. Replacing Y by a suitable blowing-up along a subspace of $p^{-1}(T)$ (not affecting $p^{-1}(U)$) we can assume that $H = Y \setminus p^{-1}(U) = \tilde{X} \setminus \tilde{U}$ is also a divisor with normal crossings.

The cohomology groups $H^k(U \setminus \{P\}, \mathbb{C})$ carry a mixed Hodge structure [D]. We are interested in the case $k = 2n-1$. We follow the contructions in [AG], part II, chapters 2 and 4 to describe the mixed Hodge structure.

Let

$$Q = T \cup \{P\}$$

Let us fix a complex $\Lambda^{\cdot}_X < logQ >$ corresponding to the above desingularization $\tilde{X}$ of X, and let $((X_a, Q_a))$ be the associated hypercovering. Let $n_a = dimX_a$. Let us notice that

$$\tilde{Q} = D \cup H$$

hence $(\tilde{X}, \tilde{Q})$ is a pair of the hypercovering. The weight filtration W_m of the complex $\Gamma(X, \Lambda^{\cdot}_X < logQ >)$ gives rise to a spectral sequence whose first term, by (3.12) and (3.13), is

$$E_1^{m,k} = E_1^{m,k}(X) = \bigoplus_a E_1^{m+q(a),k-q(a)}(X_a) \tag{4.1}$$

where

$$E_1^{m+q(a),k-q(a)}(X_a) = H^{k-m-2q(a)}({Q_a}^{[m+q(a)]}, \mathbb{C}) \tag{4.2}$$

which vanishes unless

$$k - m - 2q(a) \leq 2n_a - 2m - 2q(a)$$

that is

$$k \leq 2n_a - m \tag{4.3}$$

We want to compute the term $E_2^{1,2n-1}$ of the spectral sequence, hence we are interested in $E_1^{0,2n}$, $E_1^{1,2n-1}$ and $E_1^{2,2n-2}$. For $(m,k) = (0,2n), (1,2n-1), (2,2n-2)$, from (4.3) we obtain $n_a = n$. There is only one X_a with $n_a = n$, that is $X_a = \tilde{X}$. The corresponding $q(a)$ is zero.
It follows

$$E_1^{0,2n} = H^{2n}(\tilde{X}, \mathbb{C})$$

$$E_1^{1,2n-1} = \bigoplus_{l=1}^{p} H^{2n-2}(H_l, \mathbb{C}) \oplus \bigoplus_{ls=1}^{N} H^{2n-2}(D_s, \mathbb{C})$$

where $H = H_1 \cup \cdots \cup H_p$ and $D = D_1 \cup \cdots \cup D_N$ are the decompositions of the divisors H and D into irreducible components, and

$$E_1^{2,2n-2} = H^{2n-4}(H^{[2]}, \mathbb{C}) \oplus H^{2n-4}(D^{[2]}, \mathbb{C})$$

It follows that $E_1^{0,2n}$, $E_1^{1,2n-1}$ and $E_1^{2,2n-2}$ carry a pure Hodge structure of weight $2n$, $2n-2$ and $2n-4$ respectively, admitting only elements of type (n,n), $(n-1,n-1)$, $(n-2,n-2)$ respectively.
The differentials

$$d_1^{1,2n-1} : E_1^{1,2n-1} \to E_1^{0,2n}, \quad d_1^{2,2n} : E_1^{2,2n-2} \to E_1^{1,2n-1}$$

are sums with coefficients ± 1 of Gysin maps, which are morphism of pure Hodge structures (suitably shifted).
The term

$$E_2^{1,2n-1} = \frac{Ker\ d_1^{1,2n-1}}{Im\ d_1^{0,2n}}$$

carries a pure Hodge structure of weight $2n-2$ whose elements are of pure type $(n-1,n-1)$. Since the Hodge filtration on $E_2^{1,2n-1}$ is induced, by means of residues, by the Hodge filtration on $\Lambda_X^{\cdot} < logQ >$, and the spectral sequence degenerates at E_2, we obtain the following

Theorem 4.2. *Let W_m be the weight filtration on the cohomology $H^{2n-1}(U \setminus \{P\}, \mathbb{C})$, and denote $BM(U \setminus \{P\}) = \frac{W_1 H^{2n-1}(U \setminus \{P\}, \mathbb{C})}{W_0 H^{2n-1}(U \setminus \{P\}, \mathbb{C})}$, which is isomorphic to $E_2^{1,2n-1}$. For any element $\alpha \in BM(U \setminus \{P\})$ there exists a differential form ω on $U \setminus \{P\}$, logarithmic at P*

and along T (in sense of definition 4.1), of type $(n, n-1)$, d-closed (hence $\overline{\partial}$-closed), whose class modulo W_0 is α. Such a form will be called a Bochner-Martinelli form on $U \setminus \{P\}$.

By construction the form ω is in fact a differential form on $\tilde{X} \setminus (D \cup H)$, logarithmic along $D \cup H$, and has the property that the residue $Res\ \omega$ is a closed form of type $(n-1, n-1)$ on $D_1 \sqcup \cdots \sqcup D_N \sqcup H_1 \sqcup \cdots \sqcup H_p$ which detects an element of $Ker\ d_1^{1,2n-1}$.

A Bochner-Martinelli form ω on $U \setminus \{P\}$ induces an ordinary differential form of type $(n, n-1)$ on $U \setminus \{P\} \setminus Sing(U)$ which we still denote ω.

Theorem 4.3. *Let ω be a Bochner-Martinelli form on $U \setminus \{P\}$, $\Omega \subset U$ a relatively compact domain containing the point P, such that $\bar{\Omega}$ is subanalytic, and no component of $\partial\Omega$ is contained in $Sing(U)$. Let f be a holomorphic function on Ω, continuous on $\bar{\Omega}$. Then the integral of $f\omega$ converges on $\partial\Omega \setminus Sing(U)$ and the following equality holds:*

$$2i\pi f(P) \int_D Res\ \omega = \int_{\partial\Omega \setminus Sing(U)} f\omega \tag{4.4}$$

Proof. Since $\bar{\Omega}$ is subanalytic, the closure of $\pi^{-1}\left(\bar{\Omega} \setminus Sing(U)\right)$ in $\tilde{U}$ is subanalytic. Hence one can define the strict transforms $\tilde{\Omega}$ and $\bar{\tilde{\Omega}}$, through π, of Ω and $\bar{\Omega}$ respectively. Because ω is logarithmic along D, by theorem 2.2

$$\lim_{\epsilon \to 0} \int_{\partial T_{\underline{\epsilon}}} (f \circ \pi)\ \omega = 2i\pi \int_D Res\ \left((f \circ \pi)\ \omega\right)$$

where $T_{\underline{\epsilon}} = \bigcup_i T_{\epsilon_i}$ is a neighborhood of D defined as in (2.1). Because $(f \circ \pi)$ is constant on H, $Res\ \left((f \circ \pi)\ \omega\right) = f(P) Res\ \omega$; because $f\omega$ is $\overline{\partial}$-closed, hence d-closed, on $\tilde{U} \setminus D$,

$$\int_{\partial T_{\underline{\epsilon}}} (f \circ \pi)\ \omega = \int_{\partial\tilde{\Omega}} (f \circ \pi)\ \omega$$

It is clear that $\partial\tilde{\Omega}$ and $\partial\Omega \setminus Sing(U)$ differ by a set of measure zero, so that

$$\int_{\partial\tilde{\Omega}} (f \circ \pi)\ \omega = \int_{\partial\Omega \setminus Sing(U)} f\omega$$

which implies (4.4).

Remark. *The space $BM(U \setminus \{P\})$ is intrinsic, because the mixed Hodge structure on $H^{2n-1}(U \setminus \{P\}, \mathbb{C})$ is unique. In particular, it does not depend on the choice of a desingularization of X.*

Let us remark also that a closed $(2n-1)$-form whose class is in $BM(U \setminus \{P\})$ is a Bochner-Martinelli form if and only if it has type $(n, n-1)$; otherwise it is not necessarily $\overline{\partial}$-closed.

The following theorem gives more information about the Bochner-Martinelli forms.

Theorem 4.4. *Under the assumptions of theorem 4.3, let ω_1 and ω_2 be two Bochner-Martinelli forms on $U \setminus \{P\}$. Then*

i) ω_1 and ω_2 are cohomologous for d as logarithmic forms if and only if they are cohomologous for d as forms on $\tilde{X} \setminus \tilde{Q}$; as well, they are cohomologous for $\overline{\partial}$ as logarithmic forms if and only if they are cohomologous for $\overline{\partial}$ as forms on $\tilde{X} \setminus \tilde{Q}$.

ii) ω_1 and ω_2 are cohomologous for d as logarithmic forms if and only if they are cohomologous for $\overline{\partial}$ as logarithmic forms.

iii) ω_1 and ω_2 have same class in $H^{2n-1}(U \setminus \{P\}, \mathbb{C})$ if and only if they are cohomologous for d on $\tilde{X} \setminus \tilde{Q}$.

iv) Let $\Omega^k_{\tilde{X}} < log\tilde{Q} >$ be the sheaf of holomorphic k-forms on $\tilde{X} \setminus \tilde{Q}$ which are logarithmic along $\tilde{Q}$. There is a natural surjective morphism

$$H^{n-1}(\tilde{X}, \Omega^n_{\tilde{X}} < log\tilde{Q} >) \to BM(U \setminus \{P\}) \tag{4.5}$$

inducing an isomorphism

$$\frac{W_1 H^{n-1}(\tilde{X}, \Omega^n_{\tilde{X}} < log\tilde{Q} >)}{W_0 H^{n-1}(\tilde{X}, \Omega^n_{\tilde{X}} < log\tilde{Q} >)} \simeq BM(U \setminus \{P\}) \tag{4.6}$$

5. Dependence on the point: the Bochner-Martinelli Kernels.

Let us keep the notations and the assumptions of the previous section. Let $V \subset U$ be an open set of U. Let $Z = V \times X$, the diagram of desingularization of X:

$$\begin{array}{ccc} \tilde{E} & \to & Y \\ \downarrow & & p\downarrow \\ E & \to & X \end{array}$$

We can suppose that $H = p^{-1}(T)$ is a divisor with normal crossings in Y. In $Z' = p^{-1}(V) \times Y$ let $R' = \{(P, Q) \in Z : P = Q\}$. Then R' is a closed subspace of Z', contained in $p^{-1}(V) \times p^{-1}(V)$, isomorphic to $p^{-1}(V)$. Let $\pi' : \tilde{Z} \to Z'$ be the blowing-up of Z' along R', $\pi : \tilde{Z} \to Z$ the composite mapping, $\tilde{R} = \pi^{-1}(R')$. Finally, let $S = R' \cup (p^{-1}(V) \times T)$, $\tilde{S} = \tilde{R} \cup \pi^{-1}(p^{-1}(V) \times H)$.

Let V_1 be the open set of smooth points of V. For a point P of $V_!$ let

$$X_P = \{P\} \times X, \quad \tilde{X}_P = \pi^{-1}(\{P\} \times X)$$

$$D_P = \tilde{X}_P \cap \tilde{R}$$

$$T_P = \{P\} \times T, \quad H_P = \{P\} \times H$$

$$Q_P = \{(P,P)\} \cup T_P, \quad \tilde{Q}_P = D_P \cup H_P$$

It is easy to see that $\tilde{X}_P$ is the blowing-up of $Y_P = \{P\} \times Y$ at the point (P,P), whose exceptional divisor is D_P. The pair (X_P, Q_P) gives rise to a complex $\Lambda^{\cdot}_{X_P} < log\ Q_P >$ which describes the cohomology of $U \setminus \{P\}$, and there is a natural restriction mapping

$$\Lambda^{\cdot}_Z < logS > \rightarrow \Lambda^{\cdot}_{X_P} < log\ Q_P >$$

which is compatible with the respective differentials and weight filtrations.

Let $\pi_1 : \tilde{Z} \rightarrow V$ be the composition of the blowing-up $\pi' : \tilde{Z} \rightarrow Z' = p^{-1}(V) \times Y$ with the projection $p^{-1}(V) \times Y \rightarrow V$.

Theorem 5.1. *Let $U = X \setminus T$ be a complex space, such that X is a compact complex space bimeromorphic to a Kähler manifold, and T is a closed subspace of X; let $V \subset U$ be a Stein open subset of U, and P' a smooth point of V. Let $R = \{(P,Q) \in V \times U : P = Q\}$. For every Bochner-Martinelli form $\omega \in BM(U \setminus \{P'\})$ on $U \setminus \{P'\}$ there exists a form $\omega(P,Q)$ of type $(n, n-1)$ on $(V \times U) \setminus R$, $\overline{\partial}$-closed, logarithmic along R and $V \times T$ (in sense of definition 4.1), which induces ω and for each $P \in V$ a induces Bochner-Martinelli form on $U \setminus \{P\}$.*

In other terms, the above theorem states, under the above assumptions, that Bochner-Martinelli forms at smooth points admit $\overline{\partial}$-closed, logarithmic kernels. We do not know if it is possible in general to find kernels of pure type $(n, n-1)$ which are also d-closed.

Corollary 5.2. *Let U be a smooth affine variety, and $\Delta \subset U \times U$ be the diagonal. For every Bochner-Martinelli form $\omega \in BM(U \setminus \{P'\})$ on $U \setminus \{P'\}$ there exists a form $\omega(P,Q)$ of type $(n, n-1)$ on $(U \times U) \setminus \Delta$, $\overline{\partial}$-closed, logarithmic along Δ and at infinity (in sense of definition 4.1), which induces ω and for each $P \in U$ induces a Bochner-Martinelli form on $U \setminus \{P\}$.*

The Bochner -Martinelli form (1.2), as a form on $\mathbb{C}^n \times \mathbb{C}^n$, *is not logarithmic along the diagonal* $\Delta = \{(w,z) : z = w\}$. In order to fulfill the conclusions of corollary 5.2 it must be replaced by the form

$$\tilde{\omega}(w,z) = \frac{\tilde{\omega}'\left(\overline{z-w}, d(\overline{z-w})\right) \wedge d(z_1 - w_1) \wedge \cdots \wedge d(z_n - w_n)}{\|z-w\|^{2n}} \tag{5.1}$$

with

$$\tilde{\omega}'\left(\overline{z-w}, d(\overline{z-w})\right) = \sum_{k=1}^{n} (-1)^k (\overline{z_k - w_k}) d(\overline{z_1 - w_1}) \wedge \cdots \wedge d(\widehat{\overline{z_k - w_k}}) \wedge \cdots \wedge d(\overline{z_n - w_n})$$

The form $\tilde{\omega}(w,z)$ is $\overline{\partial}$-closed and d-closed on $(\mathbb{C}^n \times \mathbb{C}^n) \setminus \Delta$, and is logaritmic along Δ (that is, its pullback to the blowing-up of $\mathbb{C}^n \times \mathbb{C}^n$ along Δ is logarithmic along the exceptional divisor).

Corollary 5.3. *Let $\omega(P,Q)$ be a Bochner-Martinelli kernel as in theorem 5.1. The integral $\int_{D_P} Res\ \omega(P,Q)\, dQ$ appearing in formula (4.4) is locally constant with respect to $P \in V$.*

6. Integral formulas for differential forms.

Theorem 6.1. *Let U be a complex manifold of complex dimension n, $V \subset U$ a connected open subset, $Z = V \times U$, $R = \{(P,Q) \in V \times U : P = Q\}$, $\pi : \tilde{Z} \to Z$ the blowing-up of Z along R, $\tilde{R} = \pi^{-1}(R)$. Let $\omega(w,z)$ be a differential form of type $(n, n-1)$ on $(V \times U) \setminus R$, $\overline{\partial}$-closed, logarithmic along R, (that is, its pullback to $\tilde{Z}$ is logarithmic along $\tilde{R}$). Let $\Omega \subset U$ be a relatively compact domain with piecewise C^1-boundary, $\phi = \phi(z)$ a differential form of type (p,q) defined in a neighborhood of Ω. Then*

(i) The integral

$$C = \int_{\tilde{R}\cap\{w=const\}} Res\ \omega(w,z)$$

is constant with respect to $w \in V$.

(ii) One has the equality

$$(-1)^{p+q} 2i\pi C\phi = \int_{z\in\partial\Omega} \phi(z) \wedge \omega(w,z) - \int_{z\in\Omega} \overline{\partial}\phi(z) \wedge \omega(w,z) +$$

$$+\overline{\partial} \int_{z\in\Omega} \phi(z) \wedge \omega(w,z) \quad (on \quad V) \tag{6.1}$$

We define, for a form η and $A = \Omega$ or $\partial\Omega$:

$$(B_A \eta)(w) = \int_{z\in A} \eta(z) \wedge \omega(w,z)$$

so that (6.1) can be written

$$(-1)^{p+q} 2i\pi C\phi = B_{\partial\Omega}\phi - B_\Omega(\overline{\partial}\phi) + \overline{\partial}(B_\Omega \phi) \quad (on \quad V) \tag{6.2}$$

Proof. (i) is a consequence of corollary 5.3.
(ii) Let $v(w)$ be a differential form of type $(n-p, n-q)$ with compact support on V. As in [HL] (theorem 1.11.1) we must prove the identity

$$(-1)^{p+q} 2i\pi C \int_{w\in V} \phi(w) \wedge v(w) = \int_{(w,z)\in V\times\partial\Omega} \phi(z) \wedge \omega(w,z) \wedge v(w)$$

$$-\int_{(w,z)\in V\times\Omega}\overline{\partial}\phi(z)\wedge\omega(w,z)\wedge v(w)-(-1)^{p+q-1}\int_{(w,z)\in V\times\Omega}\phi(z)\wedge\omega(w,z)\wedge\overline{\partial}v(w)$$

The form $\phi(z)\wedge\omega(w,z)\wedge v(w)$ has type $(2n,2n-1)$, hence

$$d(\phi(z)\wedge\omega(w,z)\wedge v(w))=\overline{\partial}(\phi(z)\wedge\omega(w,z)\wedge v(w))$$

$$=\overline{\partial}\phi(z)\wedge\omega(w,z)\wedge v(w)+(-1)^{p+q-1}\phi(z)\wedge\omega(w,z)\wedge\overline{\partial}v(w)$$

It follows from Stokes formula

$$\int_{(w,z)\in(V\times\Omega)\setminus T_\epsilon}\overline{\partial}\phi(z)\wedge\omega(w,z)\wedge v(w)+(-1)^{p+q-1}\int_{(w,z)\in(V\times\Omega)\setminus T_\epsilon}\phi)(z)\wedge\omega(w,z)\wedge\overline{\partial}v(w)$$

$$=\int_{(w,z)\in V\times\partial\Omega}\phi(z)\wedge\omega(w,z)\wedge v(w)-\int_{(w,z)\in\partial T_\epsilon}\phi(z)\wedge\omega(w,z)\wedge v(w)$$

where T_ϵ is a small tubular neighborhood of R in $V\times U$. T_ϵ is also a tubular neighborhood of $\tilde{R}$ in $\tilde{Z}$. Thus the last integral can be computed on $\tilde{Z}$ using the residue theorem:

$$\lim_{\epsilon\to 0}\int_{(w,z)\in\partial T_\epsilon}\phi(z)\wedge\omega(w,z)\wedge v(w)=2i\pi\int_{\tilde{R}}Res\ (\pi_2^*\phi\wedge\pi^*\omega\wedge\pi_1^*v)$$

where $\pi:\tilde{Z}\to V\times U$ is the blowing-up along R and $\pi_1:\tilde{Z}\to V$, $\pi_2:\tilde{Z}\to U$ are the projections. On $\tilde{R}$ one has $\pi_2^*\phi=\pi_1^*\phi$ so that the integral becomes

$$(-1)^{p+q}2i\pi\int_{\tilde{R}}\pi_1^*(\phi\wedge v)\wedge Res\ \pi^*\omega=$$

$$(-1)^{p+q}2i\pi\int_{w\in V}\phi(w)\wedge v(w)\int_{\tilde{R}\cap\{w=const\}}Res\ \omega(w,z)=(-1)^{p+q}2i\pi C\int_{w\in V}\phi(w)\wedge v(w)$$

Taking the limit for $\epsilon\to 0$ in the above formulas we get (6.1).

Theorem 6.2. *(Bochner-Martinelli formula for differential forms). Let U be a normal complex space of complex dimension n; let $V\subset U$ be a connected open subset of U, $Z=V\times U$, $R=\{(P,Q)\in V\times U: P=Q\}$, $\pi:\tilde{Z}\to Z$ the blowing-up of Z along R, $\tilde{R}=\pi^{-1}(R)$, and $\omega(w,z)$ a Bochner-Martinelli kernel on $(V\times U)\setminus R$ ($\overline{\partial}$-closed, logarithmic along R). Let $\Omega\subset U$ be a relatively compact domain such that $\bar{\Omega}$ is subanalytic, and no component of $\partial\Omega$ is contained in $Sing(U)$, $\phi=\phi(w)$ a differential form in sense of Grauert, of type (p,q), defined in a neighborhood of Ω. Then*

(i) The integral

$$C=\int_{\tilde{R}\cap\{w=const\}}Res\ \omega(w,z)$$

is constant with respect to $w\in V$.

(ii) The integrals

$$B_{\partial\Omega\setminus Sing(U)}\phi, B_{\Omega\setminus Sing(U)}(\overline{\partial}\phi), (B_{\Omega\setminus Sing(U)}\phi)$$

converge on $V \setminus Sing(U)$ and we have the equality

$$(-1)^{p+q}2i\pi C\phi|_{V\setminus Sing(U)} = B_{\partial\Omega\setminus Sing(U)}\phi - B_{\Omega\setminus Sing(U)}(\overline{\partial}\phi) + \overline{\partial}(B_{\Omega\setminus Sing(U)}\phi) \quad (6.3)$$

The form $\omega(w, z)$ lives by construction on a desingularization $\tilde{X}$ of X, and the form ϕ extends to a form on $\tilde{X}$. Hence the proof of the theorem is an easy consequence of theorem 6.1 (on $\tilde{X}$) and its proof.

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