Canonical metrics on blow ups

[redacted]

- Given $(M, [\omega])$ Kähler orbifold, find $\omega_\varphi = \omega + i\partial\bar\partial\varphi$

  "best" representative in $[\omega]$.

Calabi's proposal: Critical points of

$$\mathcal{C}(\varphi) = \int_M (\mathrm{Scal}_\varphi)^2 \frac{\omega_\varphi^n}{n!}$$

Euler-Lagrange: $\bar\partial\left(J\nabla\mathrm{Scal} + i\nabla\mathrm{Scal}\right) = 0$

Fundamental properties:

1. Extend Einstein, Kcsc...
2. Reach maximal symmetry (Isom_0 max cpt in $\mathrm{Aut}_0(M)$).
3. Uniqueness up to automorphisms
4. Conjecturally equivalent to "relative" K-stability (Tian-Szekelyhidi)

Existence results (very quickly!)

For KE: 1. $c_1(M) < 0$ or $\equiv 0$ — Aubin–Yau

2. Complex surfaces: $Bl_{p_1 \cdots p_k} \mathbb{P}^2$, $k \in [3,8]$ in general position. — Tian

3. Toric Manifolds: KE iff Futaki invariant vanishes — Wang–Zhu

4. Intersection of 2 quadrics: — A. Ghigi–Pirola

5. Fano Fermat's hypersurfaces — Nadel–Tian

6. $\{ x_0^d + \dots + x_{k-1}^d + f(x_k, \dots, x_{n+1}) = 0 \} \subset \mathbb{P}^n$
$k > n+2-d$ — A. Ghigi–Pirola

4-6 obtained checking Tian's analytic stability.

For extremal & Kcsc not having reverse arrows (stable $\to \exists$ metric)

we are forced to produce metrics directly.

Existence results (II)

- For Kcsc: 1. $\mathbb{P}(E)$ $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ vector bdle, X Kcsc Hong
  $\mathbb{P}(E)$ with discrete autom.

  2. $X^2 \longrightarrow C$ with fibres of genus ≥ 2 Fine

  3. (X^2, ω) s.t. $\int_X (\mathrm{Scal}_\omega)^1 \frac{\omega^n}{n!} \geq 0$, $c_1(X) \neq 0$

  Then $Bl\, X^2$ at suffic. many points has

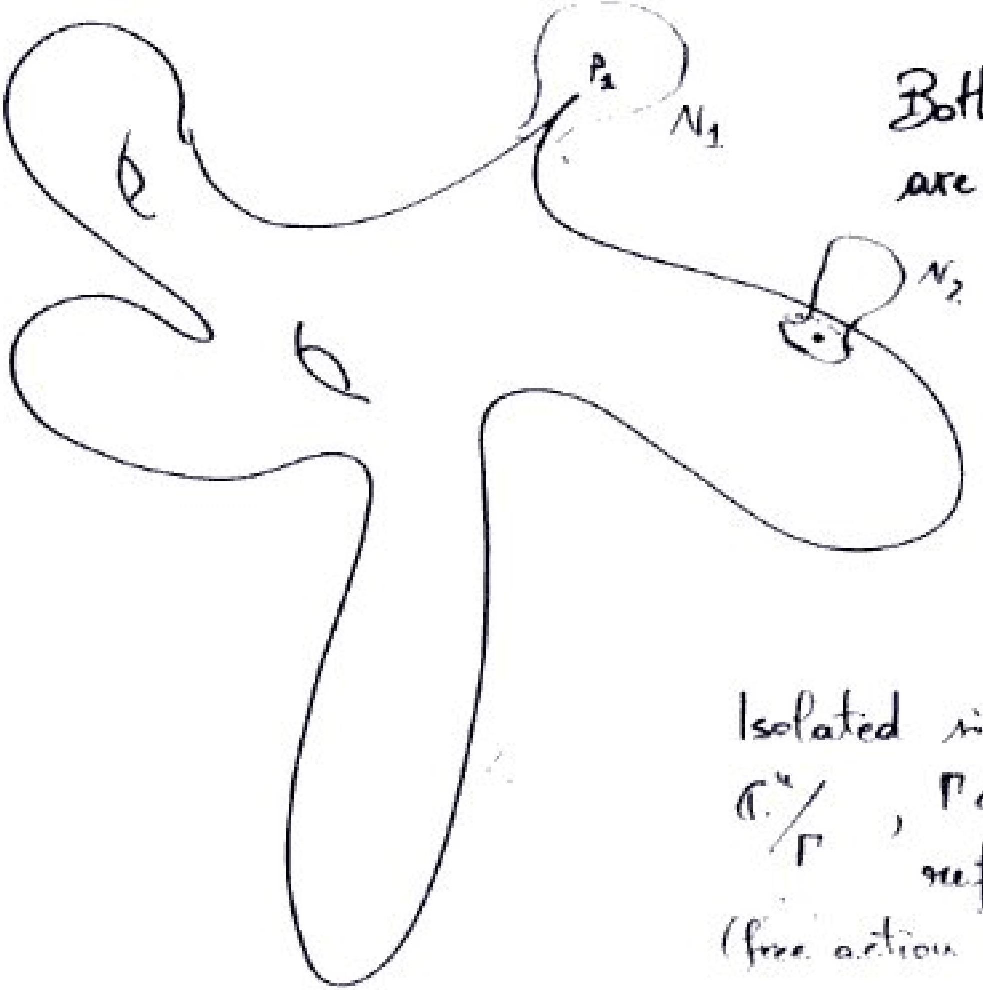
  Kcsc $\equiv 0$ metrics Kim-LeBrun-Pontecorvo
  Rollin-Singer (10 pts $\mathbb{CP}^2$)

- For extremal: 1. $\mathbb{P}(\theta \oplus L)$ $\begin{array}{c} \theta \oplus L \\ \downarrow \\ \Sigma_g \end{array}$, $g \geq 2$, $\deg L > 0$
  Tønnesen

  2. $Bl_p \mathbb{P}^2$ Calabi
  + generalizations Hwang-Singer
  Apostolov-Calderbank-Gauduchon
  Tønnesen...

How to construct new ones out of old ones?

Desingularizing and/or Blowing up points



Both procedures are "connected sums"...

Isolated singular points $\mathbb{C}^n/\Gamma$, $\Gamma \subset U(n)$, no reflections

(free action on $\mathbb{C}^n \setminus \{0\}$)

Given (M,ω) extremal orbifold, does there exist a family, $0<\varepsilon<\varepsilon_0$, $(M_\varepsilon, \omega_\varepsilon)$ s.t. "$M_\varepsilon \to M$", $\omega_\varepsilon \to \omega$ away from the exceptional divisors, and $(M_\varepsilon, \omega_\varepsilon)$ extremal

?

Pretend the solution exists and guess how the bubble should look like:

1. N_j must be ALE (asympt. to $\mathbb{C}^n/\Gamma$ metrically!!), Kähler and of zero scalar curvature

2. (only for extremal not Kcsc) $Scal(\omega_\varepsilon) \to Scal(\omega)$ $\Rightarrow$ the holom. vect. field $J\nabla S + i\nabla S$ must lift

For Bl_p this is asking $J\nabla S(p) = 0$, for desingularizing this could create a problem.

For Bl_p: N_j is forced to be $\begin{array}{c} \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^{n-1} \end{array}$

This satisfies (1) thanks to the existence of a famous K metric (Burns-Calabi) η which is ALE of order $\boxed{2n-2}$ at infinity. (Ricci-flat are of order $4n-2$)

This gives a local obstruction:

Local Obstruction: We need to be able to "prepare" M to receive the bubble, i.e. on $M \setminus \{p\}$ find φ s.t. $\omega + i\partial\bar\partial\varphi$ is K, still "canonical" and blows up at 0 of the same order of η.

This is achieved (for $2n-2$) by looking at Green's function of the linearization of the scalar curvature map.

$\omega + it\partial\bar\partial\varphi \quad Scal(\omega + i\partial\bar\partial t\varphi) = Scal(\omega) + t\,\mathbb{L}_\omega(\varphi) + \Theta(t^2)$

For the desingularising problem this is a genuine obstruction ~~in~~ for general $\Gamma \subset \mathcal{U}(n)$.

Global Obstruction:

We first need to guess where $J\nabla S_\varepsilon$ could be.

We know it must be close to $J\nabla S$ but could have

changed using hamiltonian real-holom. vect. fields
(those for which $X - iJX$ is hol.)
lifting to the blow up. Call this space $\mathfrak{h}$.

An affirmative solution implies $\exists\, u$ & $X' \in \mathfrak{h}$ s.t.

$$\mathbb{L}_\omega u + ham_\omega(X') \text{ is a distribution concentrated at the } p_j.$$

In the simplest case (that we assume)

$$(*) \qquad \mathbb{L}_\omega u + ham_\omega(X') = \sum a_j \delta_{p_j}, \quad a_j > 0$$

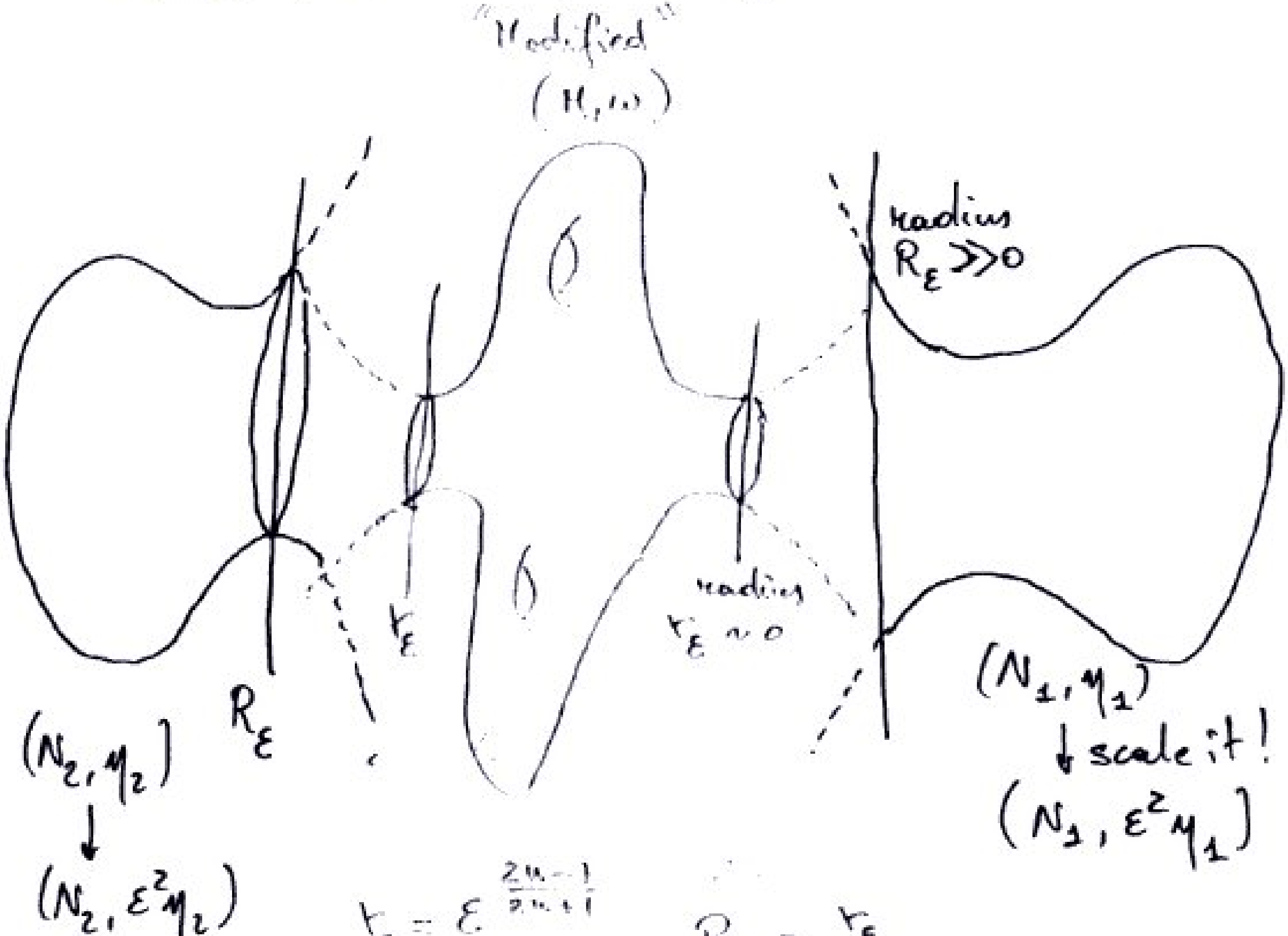
Luckily $\mathbb{L}_\omega$ is self-adjoint elliptic
(Δ^2_ω + horrible lower order terms!)

hence $(*)$ is equivalent to $\boxed{\sum a_j \, ham_\omega(X'')(p_j) = 0 \ \forall\, X'' \ \underline{\text{not}} \text{ lifting}}$ to the blow up

R. Thomas: $\Box \longleftrightarrow$ asymptotic Chow stability

for Kcsc. (X'' is free to move in the whole space of

The local & global obstruction tell us that we are in the following situation:



$r_\varepsilon = \varepsilon^{\frac{2n-1}{2n+1}}$ $\qquad$ $R_\varepsilon = \frac{r_\varepsilon}{\varepsilon}$

Need to match at all orders for $0 < \varepsilon < \varepsilon_0$.

i.e. Need to solve Cauchy data problem for truncated manifolds.

No FURTHER OBSTRUCTIONS !

The relationship between $\square$ and GIT-stability of configurations of points in under scrutiny (J. Stoppa $\to$ Kcsc, A. Della Vedova $\to$ extremal)

Special case:

1) Aut(M) discrete (more generally no hol. v. f with zeros hence automatically kcsc $\mathbb{L}_\omega u = \sum a_j \delta_{p_j}$

Classical: $\ker \mathbb{L}_\omega \leftrightarrow$ Hol. v. f. with zeros
$f \rightsquigarrow J\nabla f + i \nabla f$

2. Kcsc, Aut(M) free: We need $\sum a_j \delta_{p_j} \perp \ker \mathbb{L}_\omega$

OUTPUT FOR Kcsc

1. Aut(M) discrete $\to$ blow up any smooth or singular point $((Bl_0 \mathbb{C}^n)/\Gamma)$

2. Aut(M) discrete $\to$ $\Gamma_{sing} \subset SU(2), SU(3)$
then we can keep Kcsc on "minimal" resolution.
More generally if a Kähler crepant resolution exists using Ricci-flat models

3. Every surface of general type has Kcsc < 0 metric (look at pluricanonical image of the minimal model)

5. It is delicate (& fun!) to decide if we can blow up few points, but for many:

<u>Main Thm</u>: Given (M, ω) Kcsc $\exists\, m(\omega)$ s.t.

$$Bl_{p_1 \cdots p_k} M \text{ has Kcsc (of the same sign as } \omega)$$

for $k > m(\omega)$. The pts can move in an open set of M^k

<u>Note</u>: $m(\omega) \neq \dim Aut(M) + 1$. it depends on the potentials and not on the zero set of h.v.f.!

<u>Problem</u>: Algebraic versions of these results? In fact, base Kcsc could (should) be unnecessar

<u>Comment</u>: For Kcsc the a_js satisfy a very special equation $\square$ (e.g. for $\mathbb{P}^2$, $k=3$, $a_1 = a_2 = a_3$).
For extremal metrics we have (often) great freedom

<u>Thm</u> (A-Pacard-Singer) (M, ω) toric extremal smooth then $Bl_{p_1 \cdots p_k} M$ has extremal in

$$\pi^*[\omega] - \varepsilon^2 (\Sigma a_j [E_j]) \quad \forall a_j > 0 \;\&$$

$$\forall \{p_1 \cdots p_k\} \subset \text{Fix}\{\text{torus action}\}$$

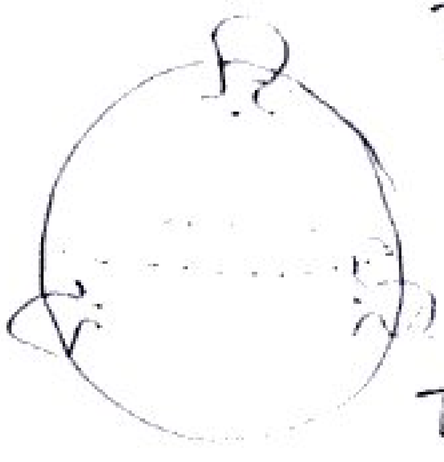
$Bl_p \mathbb{P}^2 \to$ Calabi, $Bl_{p_1 p_2} \mathbb{P}^2$, $Bl_{p_1 p_2 p_3} \mathbb{P}^2$.

Note: We always produce metrics in

$\pi^*[\omega] - \varepsilon^2 \left(\Sigma a_j [E_j] \right)$, E_j exceptional divisors.

so if $\varepsilon^2 a_j$ rational $\to$ polarized mfds.

4. If Aut(M) free could be difficult to calculate $ham_\omega(X)$. Doable for toric varieties, e.g. $\mathbb{P}^2$ with FS "round" metric. We must blow up at least 3 pts



The condition asks for a "barycenter" to stay in the origin (which is necessary too Futaki $\equiv 0$).

This seems to suggest that the pts must be in special position, but this is not true if we move the round metric with Aut($\mathbb{P}^2$).

Same for 4 pts which has Aut discrete so we can use it as base for further blow ups. We can also use as base Tian's KE metrics and iterate...

Moral: Wealth of K classes in the K cone represented by Kcsc metrics.

Output for extremal

(Arezzo - Pacard - Singer)

<u>Theorem</u> (simplified version) (M, J, ω, g) extremal

$K \subset \mathrm{Isom}(M,g)$ cpt s.t. $K \ni J\nabla \mathrm{scal}$ & $\mathfrak{h} \subseteq \mathfrak{k}$.

<u>Then</u> given any $\{p_1, \dots, p_n\} \subset \mathrm{Fix}\, K_0$ and weights $a_j > 0$,

$Bl_{p_1 \dots p_n} M$ has extremal metric in

$$\pi^*[\omega] - \varepsilon^2 (a_1 PDE_1 + \dots + a_n PDE_n)$$

$\forall\, \varepsilon < \varepsilon_0$

Extremal Results for $Bl_{p_1 \ldots p_n}\mathbb{P}^2$:

1. On $\underline{Bl_p \mathbb{P}^2}$ the whole Kähler cone is spanned by extremal metrics (Calabi).

   With our construction we can get $\pi^*[\omega_{FS}] - \varepsilon^2 PD[E_1]$.

2. On $Bl_{p,q}\mathbb{P}^2$ we can get the following K classes:

   i) $\pi^*[\omega_{FS}] - \left(a_1 PD[E_1] + \varepsilon^2 a_2 PD[E_2]\right)$, $a_1 < 1$

   ii) $\pi^*[\omega_{FS}] - \dfrac{a_1 - \varepsilon^2}{a_1 + a_2 - \varepsilon^2} PD[E_1] - \dfrac{a_2 - \varepsilon^2}{a_1 + a_2 - \varepsilon^2} PD[E_2]$

   For i) use 1. For ii) use $Bl_q(\mathbb{P}^1 \times \mathbb{P}^1)$ and translate.

3. On $Bl_{p_1, p_2, p_3}\mathbb{P}^2$ not on a line

   i) Add $\varepsilon^2 a_3 PD[E_3]$ above i).

   ii) Add $\varepsilon^4 a_3 PD[E_3]$ to above ii)

   iii) Apply Cremona Transform to i) and get K classes with exceptional volumes close to $\frac{1}{2}$

   iv) exceptional volumes $\sim \frac{1}{3}$ (canonical class) Siu-Tian-Yau + LeBrun-Simanca.

4. On $Bl_{p_1 p_2 p_3}\mathbb{P}^2$ on a line. (b > 2a impossible)

   i) $\pi^*[\omega_{FS}] - \left(a_1 E_1 + \varepsilon^2 a_2 E_2 + \varepsilon^4 a_3 E_3\right)$

   ii) $\pi^*[\omega_{FS}] - \varepsilon^2 \left(a\, PD E_1 + a\, PD E_2 + b\, PD E_3\right)$ $b < a$.