Algebraic deformations of compact Kähler manifolds

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This talk is concerned with questions arising from the problem of classification of compact complex manifolds.

Recall that the classification of compact Riemann surfaces is well-known. The Enriques-Kodaira classification of compact complex surfaces covers the two-dimensional case, and is considered complete modulo some questions concerning surfaces of class VII_0. In higher dimensions there is not a whole lot known in general.

At least in the algebraic category we do know quite a lot, thanks to the work of Mori and others. This is particularly true in dimension three, for which one sometimes hears the experts say that "*classification of algebraic 3-manifolds is essentially complete*".

On the other hand, if the algebraic condition is dropped, things became a great deal more complicated. To illustrate this, in 1990 Taubes proved that every smooth compact oriented 4-manifold admits metrics with anti-self-dual Weyl curvature after taking connected sums with enough copies of $\overline{\mathbb{P}}_2$. The twistor space of such a manifold is a complex 3-manifold fibred by $\mathbb{P}_1$'s. Taking connected sums with $\overline{\mathbb{P}}_2$ has no effect on the Donaldson polynomials, so in Taubes' words: "*Classification of compact complex 3-manifolds is at least as complicated as the classification of smooth 4-manifolds*".

In spite of this, in recent years there has been significant progress in the classification of compact complex manifolds using hard analytical and differential-geometric methods as opposed to algebraic methods. Such techniques generally apply to Kähler manifolds. (Note that the only twistor spaces which are Kähler are $\mathbb{P}_3$ and $\mathbb{F}_{1,2}$ by a result of Hitchin.)

Recall that a Kähler metric on a complex manifold corresponds to a positive $(1,1)$-form ω which is closed: $d\omega = 0$.

The de Rham class $[\omega]$ lies in $H^2(X,\mathbb{R}) \cap H^{1,1}(X)$. If this class lies in $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ then there is a holomorphic line bundle on X with hermitian connection which has $(2\pi/i)\,\omega$ as its curvature, so by the Kodaira Embedding Theorem, X is algebraic.

The same conclusion holds if $[\omega]$ lies in $H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$, just by multiplying by an appropriate integer. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it could be conjectured that any compact Kähler manifold is a small deformation of an algebraic manifold.

This is certainly true in dimension 1 since every Riemann surface is algebraic. In dimension 2, Kodaira proved in 1963 that every compact Kähler surface is a deformation of an algebraic surface. His proof relied heavily on the classification of surfaces and on his classification of elliptic surfaces.

In general, it is not necessarily possible to deform a compact complex manifold X. It is well known that the space of infinitesimal deformations of X is parameterised by $H^1(X,\Theta)$ (where Θ is the sheaf of germs of holomorphic sections of the holomorphic tangent bundle). Moreover, not every infinitesimal deformation can be realised by a genuine deformation.

A closely related question is then whether or not there are any rigid compact Kähler manifolds which are not algebraic, rigid meaning $H^1(X,\Theta) = 0$.

In 2002, Demailly-Eckl-Peternell proposed a nice construction for such a manifold by blowing up certain subvarieties in a projective bundle over a torus, but this didn't quite work—the only rigid examples which could be constructed by this method turned out to be algebraic.

In 2003 the question was resolved very much in the negative by Voisin, who proved that in every dimension greater than 3, there are compact Kähler manifolds which do not have the same homotopy type as an algebraic manifold. Using ideas of Deligne, she could even give simply connected examples.

The basic method of her construction involved blowing up certain subvarieties in products of tori and considering associated Hodge structures.

Blowing up subvarieties has the effect of making manifolds more rigid, and all of Voisin's examples are indeed rigid. This leaves open possibility that every compact Kähler manifold is nevertheless birational to one which is a deformation of algebraic.

However, in 2004 Voisin showed that in all even dimensions above 7, there are compact Kähler manifolds which are not birational to anything which has same homotopy type as an algebraic manifold.

In spite of Voisin's results, there are still some very interesting features associated with the problem which are worth considering, and it is primarily these with which this talk is concerned.

In general, a Kähler structure on a smooth manifold consists of a Riemannian metric together with a compatible symplectic form which is covariantly constant with respect to the Levi-Civita connection.

To deform a compact Kähler manifold, consider a smooth manifold and smooth family of Riemannian metrics g_t with compatible symplectic forms ω_t such that $\nabla_t \omega_t = 0$, where ∇_t is the Levi-Civita connection associated to g_t. Taking the derivative at $t = 0$ gives an equation of the form $\nabla_0 \dot{\omega}_0 + \mathcal{D}_0 \dot{g}_0 = 0$, where $\mathcal{D}_0$ is a certain first order linear operator.

If for each closed 2-form w on X there is some symmetric 2-tensor u on X such that $\nabla_0 w + \mathcal{D}_0 u = 0$, we can hope to be able to find arbitrarily small deformations of X which are algebraic.

After changing to the complex coordinates of the base Kähler manifold X, this equation is somewhat easier to understand and a cohomological interpretation becomes apparent. Namely, we expect to be able to find arbitrarily small deformations of X which are algebraic if the natural map (induced by contraction with ω) $H^1(X,\Theta) \to H^2(X,\mathcal{O})$ is surjective.

[Note that the entire problem is cohomological in nature: A theorem of Moser states that two nearby symplectic forms which are cohomologous are related by a self-diffeomorphism of the manifold.]

However, surjectivity of this map is not in general sufficient to deduce the desired result: infinitesimal deformations need to be "integrated", but there may be obstructions to doing this.

But, if $H^2(X, \Theta) = 0$, there are no formal obstructions to integrating the infinitesimal deformations, and the Kuranishi space $\mathcal{K}$ of deformations of X is smooth. Then an appropriate application of the holomorphic Implicit Function Theorem does imply the desired result.

In 2003, I proved that if X is a compact Kähler surface for which $H^1(X, \Theta) \to H^2(X, \mathcal{O})$ is not surjective, X must already be algebraic. As a result, in the case of surfaces X with $H^2(X, \Theta) = 0$, a new proof of Kodaira's result is obtained, one which does not rely on the classification of surfaces.

The result above on deforming compact Kähler manifolds into algebraic manifolds can be generalised somewhat. To apply the holomorphic Implicit Function Theorem, it is only necessary that $\mathcal{K}$ be smooth. This is implied by $H^2(X, \Theta) = 0$, but is not equivalent—for example, tori. All that is required is that the infinitesimal deformations be unobstructed.

Extending this idea a little further, if we have a smooth holomorphic family $\mathcal{F} \to B$ of deformations of X such that the composition of Kodaira-Spencer map $TB_0 \to H^1(X, \Theta)$ with $H^1(X, \Theta) \to H^2(X, \mathcal{O})$ is surjective, then in the same way as before it is possible to conclude that in this family there are arbitrarily small deformations of X which are algebraic.

In case that X is a compact Kähler surface with a non-constant holomorphic map $X \to S$ Riemann surface, one can in fact find a family of infinitesimal deformations of X which are all unobstructed: the deformations themselves can be written down explicitly.

Some careful analysis then shows that for this family, the composition above is surjective.

For a compact Kähler surface which does not have any non-constant map to Riemann surface, it is not hard to show that all infinitesimal deformations are unobstructed.

(There are two approaches to showing this: Such surfaces must be blowups of tori or $K3$ surfaces, but it is not necessary to classify them: Some recent results of Clemens show that that the obstruction classes to deforming a compact Kähler manifold annihilate ambient cohomology: $H^2(X, \Theta) \otimes H^p(X, \Omega^q) \to H^{p+2}(X, \Omega^{q-1})$ is 0 on obstruction classes. These imply that all infinitesimal deformations of X are unobstructed.)

In conclusion, we have a complete and new proof of Kodaira's result that every compact Kähler surface is a deformation of an algebraic surface, a proof which does not require any reference to classification.

There is another approach to the problem of deforming compact Kähler surfaces which did not succeed (for me) but which provokes some interesting questions.

If X is a torus or a $K3$ surface, by virtue of Yau's solution of the Calabi conjecture, in each Kähler class there is a metric with zero Ricci curvature with respect to which a holomorphic 2-form is covariantly constant. Using such a 2-form and Kähler form, one can explicitly write down isometric deformations of the complex structure.

If (X, ω) is any compact Kähler surface and $\kappa \in H^0(X, \Omega^2)$ satisfies $\int_X \omega^2 = \int_X \kappa \wedge \bar{\kappa}$, one can try to solve the equation $(\omega + i\bar{\partial}\partial u)^2 = \kappa \wedge \bar{\kappa}$ for u. At the end of Yau's paper, he solved equations of this form (i.e., with degenerate right-hand-side), and showed that a solution always exists which is smooth on $X \setminus \kappa^{-1}(0)$.

If $\omega + \epsilon\kappa + \epsilon\bar{\kappa}$ defines an integral $(1,1)$-class for some such complex structure, is there a version of the Kodaira embedding theorem which work with this degenerate complex structure so as to be able to realise an explicit deformation of X which is algebraic?

Another question which arises in the context of the general deformation theory is to try to understand what is happening in the case of obstructed deformations.

For general compact complex X, given $\theta \in H^1(X, \Theta)$ there is a first obstruction $[\theta, \theta] \in H^2(X, \Theta)$ to finding a genuine family of deformations realising θ as its derivative. If this vanishes, there is another obstruction in $H^2(X, \Theta)$, etc. If all obstructions vanish, it is possible to show that there is a genuine deformation of X, (but proving convergence of the formal series is hard).

If (X, ω) is a compact Kähler manifold, the form ω defines an element of $H^1(X, \Omega^1)$ and thus an extension $0 \to \mathcal{O} \to \mathcal{S} \to \Theta \to 0$.

Using Young tableaux, there is a fine resolution

$$
\begin{aligned}
0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \xrightarrow{\bar{\partial}_0\bar{\partial}_0} \ydiagram{2}(\mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}_0} \ydiagram{2,1}(\mathcal{E}^{0,1}) \\
\xrightarrow{\bar{\partial}_0} \ydiagram{2,1,1}(\mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}_0} \ydiagram{2,1,1,1}(\mathcal{E}^{0,1}) \longrightarrow \cdots .
\end{aligned}
$$

Here $\bar{\partial}_0$ is the $(0,1)$ component of the covariant derivative induced by the Levi-Civita connection. Since the curvature of this connection is a $(1,1)$-form, it cannot appear in the sequence above, so the sequence is a complex.

[Note: The Young tableaux $\ydiagram{2,1}(\Lambda^{0,1})$ for example can be regarded as the kernel of $\wedge$: $\Lambda^{0,2} \otimes \Lambda^{0,1} \to \Lambda^{0,3}$, with analogous identities for other numbers of vertical boxes.]

(Aside: The Ricci scalar is a section of $\mathcal{S}$ if and only if the metric is extremal in the sense of Calabi.)

If $w \in \Lambda^{0,2}(X)$ and $\bar{\partial}w = 0$, $\bar{\partial}_0 w$ defines element of $H^2(X, \mathcal{S})$. This class vanishes for every w if and only if $H^1(X, \Theta) \to H^2(X, \mathcal{O})$ is surjective.

Sequences of this kind are of considerable interest in the area of parabolic geometry, known as subcomplexes of *curved Bernstein-Gelfand-Gelfand sequences*.

There is some evidence to suggest that if ω_t is a smooth family of symplectic forms ($t \in \mathbb{R}$)

and g_t is a family of metrics with $\nabla_t \omega_t = O(t^{m+1})$, the obstruction to finding $\tilde{g}_t$ with $\tilde{g}_t = g_t$ to order m and with $\tilde{\nabla}_t \omega_t = O(t^{m+2})$ lies in $H^2(X, \mathcal{S})$.

The case $m = 1$ is the linear case, which is straightforward. The case $m = 2$ can be verified directly, albeit with some difficulty. The higher order cases have not been verified completely: a better formalism is required for the algebra.

References

This list is intended to cover most of the main references mentioned either implicitly or explicitly in the talk. There are quite likely several important omissions for which I offer my apologies.

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