Stability and Extremal Metrics on Ruled Manifolds

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EXTREMAL KÄHLER METRICS

- (M, J): a compact complex manifold.
- $\Omega \in H^2(M, \mathbb{R})$: a Kähler class on M.
- Ω_J: the set of Kähler forms in Ω.

Calabi functional $\quad \mathcal{C}\colon \Omega_J \to \mathbb{R}$;

$$\mathcal{C}(\omega) = \int_M \mathsf{Scal}_\omega^2 \, \mathsf{vol}_\omega \,.$$

Scal_ω is the scalar curvature of the induced metric $g = g_{\omega,J}$ and vol_ω is the volume form.

Fact (Calabi): $\omega \in \Omega_J$ is a critical point of $\mathcal{C}$ iff $\mathsf{grad}_\omega \mathsf{Scal}_\omega$ is a Killing vector field.

Then ω is said to be **extremal** ($g_{J,\omega}$ is an extremal Kähler metric).

Call Ω an **extremal Kähler class** on (M, J) if $\exists \omega \in \Omega_J$ extremal.

BASIC QUESTIONS

1. Uniqueness? (Optimality)

2. Existence? (Usefulness)

SOME ANSWERS

1. Formally clear that extremal Kähler metrics in Ω are unique modulo automorphisms of (M, J). Now proven analytically by Chen–Tian (2005).

2. Conjectures of Donaldson, Tian and Yau:

$$
\begin{array}{ccc}
\Omega \text{ extremal} & \Longleftrightarrow & (M, J, \Omega) \text{ K-polystable} \\
\Updownarrow & & \Updownarrow \\
& \Omega \text{ has proper K-energy} &
\end{array}
$$

The precise notions of polystability and properness are not yet completely settled.

THE EXTREMAL VECTOR FIELD

$\mathfrak{h}_0(M,J)$: Lie algebra of holomorphic vector fields with zeros. $\mathsf{H}_0(M,J)$: corresponding connected group of automorphisms.

Fact (Calabi): for $\omega \in \Omega_J$ extremal, $G := \mathrm{Ham}(M,\omega) \cap \mathsf{H}_0(M,J)$ is a maximal compact subgroup of $\mathsf{H}_0(M,J)$.

Fix $G \subset \mathsf{H}_0(M,J)$ compact and let

$$\Omega_J^G = \{\omega \in \Omega_J : \omega \text{ is } G\text{-invariant}\}.$$

Note $\omega \in \Omega_J^G \Rightarrow G \subset \mathrm{Ham}(M,\omega)$. Let $\tilde{\mathfrak{g}}_\omega \subset C^\infty(M,\mathbb{R})$ be the subspace of hamiltonian generators for the G action ($\tilde{\mathfrak{g}}_\omega \cong \mathfrak{g} \oplus \mathbb{R}$).

$C^\infty(M,\mathbb{R}) = \tilde{\mathfrak{g}}_\omega \oplus \tilde{\mathfrak{g}}_\omega^\perp$ wrt. L_2 inner product

$$\mathrm{Scal}_\omega = s_\omega + s_\omega^\perp.$$

Fact (Futaki–Mabuchi): for G maximal, $\chi := \mathrm{grad}_\omega\, s_\omega \in \mathfrak{g}$ is independent of $\omega \in \Omega_J$.

ω extremal $\Leftrightarrow \mathrm{grad}_\omega \mathrm{Scal}_\omega = \chi \Leftrightarrow s_\omega^\perp = 0$

χ is called the **extremal vector field**.

RELATIVE (OR MODIFIED) K-ENERGY

Fact (Mabuchi, Guan, Simanca):

$$\sigma_\omega(dd^c f) = \int_M f s_\omega^\perp \,\mathrm{vol}_\omega$$

defines a closed 1-form on Ω_J^G.

Ω_J^G is contractible, so $\sigma = -d\mathcal{E}$, where $\mathcal{E}\colon \Omega_J^G \to \mathbb{R}$ is defined up to an additive constant. This is the relative K-energy.

Clearly: ω is extremal $\Leftrightarrow$ ω is critical for $\mathcal{E}$.

The **Mabuchi metric** $\mathbf{g}$ is the L_2 metric on Ω_J^G, i.e., $\mathbf{g}_\omega(dd^c f_1, dd^c f_2) = \int_M f_1 f_2 \,\mathrm{vol}_\omega$.

Fact: $\mathcal{E}$ is geodesically convex wrt. $\mathbf{g}$.

This underlies the Chen–Tian uniqueness result for extremal Kähler metrics. It also motivates the properness criterion. Chen–Tian show that extremal Kähler metrics minimize the relative K-energy.

THE SYMPLECTIC VIEWPOINT

• (M,ω): a compact symplectic manifold.

• $G \subset \mathsf{Ham}(M,\omega)$: a compact subgroup.

• $\mathsf{Diff}_0^G(M)$: connected normalizer of G in $\mathsf{Diff}(M)$ (modulo G).

Let $\mathcal{J}^G$ be an orbit of $\mathsf{Diff}_0^G(M)$ on the space of G-invariant complex structures on M, and let $\mathcal{J}_\omega^G$ be the set of ω-compatible $J \in \mathcal{J}^G$.

Fact: for any $\omega \in \Omega_J^G$, there is an isomorphism $\mathcal{J}_\omega^G/\mathsf{Sp}_0^G(M,\omega) \cong \Omega_J^G/\mathsf{H}_0^G(M,J)$.

Let $\tilde{\mathfrak{g}} \subset C^\infty(M,\mathbb{R})$ be the subspace of hamiltonian generators for the G action.

$$C^\infty(M,\mathbb{R}) = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^\perp \quad \text{wrt. } L_2 \text{ inner product}$$
$$\mathsf{Scal}_J = s_J + s_J^\perp.$$

Fact (Apostolov): $s_J \in \tilde{\mathfrak{g}}$ is independent of J and so defines an potential for a symplectic version of the extremal vector field.

SCALAR CURVATURE AND STABILITY

Note $\mu_J(\mathrm{grad}_\omega f) = \int_M f s_J^\perp \mathrm{vol}_\omega$ defines a map $\mu\colon \mathcal{J}_\omega^G \to \mathfrak{ham}^G(M,\omega)^*$

Fact (Donaldson, Fujiki): $\mathcal{J}_\omega^G$ is a (∞-diml.) Kähler manifold with an isometric action of $\mathrm{Ham}^G(M,\omega)$ with momentum map μ.

$\mu^{-1}(0) = \{J : s_J^\perp = 0\}$ is then the space of extremal Kähler metrics in $\mathcal{J}_\omega^G$.

Geometric Invariant Theory then motivates the idea that there is a stability condition for the existence of extremal Kähler metrics.

SYMPLECTIC K-ENERGY

Suppose $\mathcal{E}$ is $\mathsf{H}_0^G(M,J)$-invariant on Ω_J^G. Then it defines a symplectic version $\widehat{\mathcal{E}}$ of K-energy.

Note that $T_J\mathcal{J}_\omega^G = \{\mathcal{L}_Z J : Z = Z_1 + JZ_2\}$ where $Z_1 \in \mathfrak{sp}^G(M,\omega)$, $Z_2 \in \mathfrak{ham}^G(M,\omega)$. Then

$$\widehat{\sigma}_J(\mathcal{L}_Z J) = \int_M f_2 s_J^\perp \,\mathrm{vol}_\omega,$$

where $Z_2 = \mathrm{grad}_\omega f_2$, is a closed 1-form and $\widehat{\sigma} = d\widehat{\mathcal{E}}$.

Fact (Gauduchon): $\widehat{\mathcal{E}}$ is a $\mathsf{Ham}(M,\omega)$-invariant Kähler potential along integral manifolds of $\{\mathcal{L}_Z J : Z_1, Z_2 \in \mathfrak{ham}^G(M,\omega)\}$. Hence it is strongly plurisubharmonic.

Observation: geodesics in Ω_J^G correspond to integral curves of the vector field $J \mapsto \mathcal{L}_{JZ_2} J$ on $\mathcal{J}_\omega^G$, where $Z_2 \in \mathfrak{ham}^G(M,\omega)$.

Extremal Kähler metrics are critical for $\widehat{\mathcal{E}}$ and this provides another way to see their formal uniqueness.

K-STABILITY

Suppose $\Omega = 2\pi c_1(L)$, so (M, Ω) is Hodge, and there is a lift of the G-action to L. It is convenient to take $G \subset \mathrm{H}_0(M, J)$ to be a maximal torus instead of a maximal compact.

A **test configuration** $\mathcal{T}$ for (M, L, G) is:

• a polarized complex variety (or scheme) $(X, \mathcal{L})$ with an action of G;

• a G-equivariant $\mathbb{C}^\times$ action α on $(X, \mathcal{L})$;

• a G-invariant and $\mathbb{C}^\times$-equivariant flat morphism $p \colon X \to \mathbb{C}$

such that $(X_t, \mathcal{L}_t) \cong (M, L)$ for some $t \neq 0$.

(Here $X_t = p^{-1}(t)$ and $\mathcal{L}_t = \mathcal{L}|_{X_t}$.)

α induces a $\mathbb{C}^\times$ action on $(X_0, \mathcal{L}_0)$ which is called the **central fibre** of $(X, \mathcal{L})$.

This action has a weight called the relative (or modified) **Futaki invariant** $\mathcal{F}_\Omega(\mathcal{T})$ of $\mathcal{T}$.

(M, L) is (relatively) **K-polystable** if $\mathcal{F}_\Omega(\mathcal{T}) \geq 0$ for all $\mathcal{T}$, with equality if $X = M \times \mathbb{C}$ and α is induced by a $\mathbb{C}^\times$ action on M.

BUNDLE CONSTRUCTIONS

Build Kähler metrics on bundles $M \to S$ for:

- S a compact Kähler $2d$-manifold (e.g., $\Sigma_{\mathbf{g}}$),
- T an ℓ-torus (e.g., S^1),
- P a principal T bundle over S (e.g., U($\mathcal{L}$)),
- V a compact Kähler manifold with an isometric hamiltonian T-action (e.g., $\mathbb{C}P^1$),

such that M is (covered by) $P \times_T V$, a compact complex $2m$-manifold with $m = d + \dim_{\mathbb{C}} V \geq d + \ell$ (e.g., $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_{\mathbf{g}}$).

Simplifying assumptions:

S is (covered by) $\prod_j (S_j, \omega_j)$ such that $2\pi c_1(P)$ pulls back to $\sum_j [\omega_j] \otimes b_j$ for $b_j \in \mathfrak{t}$;

(V, T) is essentially toric, i.e., its blow-up along the fixed point sets of circle subgroups of T is (covered by) $\tilde{P} \times_T \tilde{V} \to \tilde{S}$, with $\tilde{V}$ toric, and $\tilde{S}$ a product of projective spaces.

Say M has order ℓ. ($P(\mathcal{O} \oplus \mathcal{L})$ has order 1).

EXAMPLES

$M = P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_1 \times \cdots \times \Sigma_d$ (order 1).

$M = P(\mathcal{O} \oplus \mathcal{L}) \to S_1 \times \cdots \times S_N$ with $\mathcal{L} = \otimes_j \mathcal{L}_j$ and $\mathcal{L}_j$ a power of an ample line bundle on S_j (order 1).

$M = P(E) \to S$ with E a projectively-flat hermitian vector bundle (order 0).

$M = P(E_0 \oplus E_1 \oplus \cdots \oplus E_\ell) \to S_1 \times \cdots \times S_N$ where E_j are projectively-flat hermitian vector bundles and $c_1(E_i)/\mathrm{rk}(E_i) - c_1(E_j)/\mathrm{rk}(E_j)$ is a linear combination of the Kähler forms on the S_k (order ℓ).

$M = V$ toric (order $\dim_{\mathbb{C}} V$).

$M = P \times_T V \to S$ with V toric and S as before (order $\dim_{\mathbb{C}} V$).

M of order ℓ admit Kähler metrics of the form

$$\begin{aligned} g &= \textstyle\sum_j (\langle b_j, z\rangle + c_j) g_j \\ &\qquad + \langle dz, \Theta^{-1}(z), dz\rangle + \langle \alpha, \Theta(z), \alpha\rangle, \\ \omega &= \textstyle\sum_j (\langle b_j, z\rangle + c_j)\omega_j + \langle dz \wedge \theta\rangle, \\ d\alpha &= \textstyle\sum_j b_j \omega_j. \end{aligned}$$

($z \in C^\infty(M, \mathfrak{t}^*)$ is the momentum map of the T-action, and $\Theta(z) \in S^2\mathfrak{t}^*$ the matrix of inner products of the generators, while θ is a connection 1-form for P and $c_j \in \mathbb{R}$.)

In particular, if $\ell = 1$, with $c_j = b_j/x_j$, then rescaling ω_j by b_j gives

$$\begin{aligned} g &= \sum_j \frac{1 + x_j z}{x_j} g_j + \frac{dz^2}{\Theta(z)} + \Theta(z)\alpha^2, \\ \omega &= \sum_j \frac{1 + x_j z}{x_j} \omega_j + dz \wedge \theta, \\ d\alpha &= \sum_j \omega_j. \end{aligned}$$

If the image of z is $[-1, 1]$ and $0 < |x_j| \leq 1$, this generalizes the form of metric on ruled surfaces as presented by Christina.

Note the symplectic viewpoint.

EXTREMAL KÄHLER METRICS OF ORDER ONE

Suppose M has order one. Then for any admissible Kähler class $\Omega = \Omega(x)$ (i.e., containing a metric of the previous form) $\exists!$ polynomial $F_\Omega(z)$ (the extremal polynomial) s.t. TFAE

- Ω is extremal;

- g (as before), with $\Theta(z) = F_\Omega(z)/P_\Omega(z)$ and $P_\Omega(z) = \prod_j (1 + x_j z)^{d_j}$, is extremal;

- $F_\Omega(z) > 0$ for $z \in (-1, 1)$.

This completely solves the existence problem for a large class of ruled manifolds. How does it relate to stability and properness?

AMAZING FACT: for $z \in (-1, 1) \cap \mathbb{Q}$ $\exists$ a test configuration $\mathcal{T}(z)$ for (M, Ω, T) such that $\mathcal{F}_\Omega(\mathcal{T}(z)) = F_\Omega(z)$.

So $F_\Omega(z) > 0$ for $z \in (-1, 1) \cap \mathbb{Q}$ is a stability condition!

AN EXAMPLE

What if $F_\Omega(z) > 0$ for $z \in (-1,1) \cap \mathbb{Q}$ but $F_\Omega(z) = 0$ for some $z \in (-1,1) \setminus \mathbb{Q}$?

This can happen.

Let $M = P(\mathcal{O} \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)) \to \Sigma_1 \times \Sigma_2 \times \Sigma_3$.

Then the freedom in the genera of Σ_j and degrees of $\mathcal{L}_j$ can be used to obtain $F_\Omega(z) = (1-z^2)(z^2+rz-1)^2$ for any $r \in \mathbb{Q}^+$. z^2+rz-1 has an irrational root in $(-1,1)$ for r in a nonempty open subset of $\mathbb{Q}$.

How to handle this problem?

1. Allow analytic test configurations.

2. Require a uniform bound on $\mathcal{F}_\Omega(\mathcal{T})$:

$$\mathcal{F}_\Omega(\mathcal{T}) \geq \lambda ||\mathcal{T}||$$

$\to$ notion of uniform K-polystability (Székelyhidi).

UNIFORM K-STABILITY

Let $||\pi(\mathcal{T})||$ be the L_2 norm of the generator of the $\mathbb{C}^\times$ action on the central fibre $(X_0, \mathcal{L}_0)$ projected orthogonally to $\mathfrak{g}$.

Defn (M, Ω, G) is L_2-uniformly K-polystable if $\exists \lambda > 0$ s.t. $\forall$ test configurations $\mathcal{T}$,

$$\mathcal{F}_\Omega(\mathcal{T}) \geq \lambda ||\pi(\mathcal{T})||$$

A similar definition can be made for a wide range of semi-norms on test configurations as long as the semi-norm vanishes when the test configuration is a product $M \times \mathbb{C}$.

In his work on toric surfaces, Donaldson uses a boundary integral over the momentum polygon to bound the Futaki invariant below.

TORIC KÄHLER MANIFOLDS AND BUNDLES

Let $M = P \times_T V \to S$ be a bundle of toric Kähler manifolds. The image of z is a compact convex polytope Δ in $\mathfrak{t}^*$, generalizing $[-1,1]$ in the order one case.

Assume M is toric for simplicity ($S = \{\mathrm{pt}\}$).

Let $\mathcal{C} = \{f \colon \Delta \to \mathbb{R} \text{ convex}\}$.

Let $\mathcal{S}$ be the space of "symplectic potentials": a subspace of strictly convex functions such that $\mathcal{S}/\{\text{affine linear functions on } \Delta\} \cong \mathcal{J}^T_\omega/\mathrm{Ham}^T(M,\omega) \cong \Omega^T_J/\mathrm{H}^T_0(M,\omega)$.

Then (Donaldson) as a function on $\mathcal{S}$:

$$\mathcal{E}(u) = -\int_M \log\det\mathrm{Hess}(u)\,d\mu + F_\Omega(u)$$

where $F_\Omega(u) \colon \mathcal{C} \to \mathbb{R}$ is linear.

The "amazing fact" generalizes: for any PL $f \in \mathcal{C}$ there is a test configuration $\mathcal{T}(f)$ with Futaki invariant $\mathcal{F}_\Omega(\mathcal{T}(f)) = F_\Omega(f)$.

TWO OBSERVATIONS OF DONALDSON, ENHANCED.

Theorem For any $\lambda > 0$ TFAE:

1. (Uniform K-stability)

$$F_\Omega(f) \geq \lambda ||\pi(f)|| \quad \forall \text{ PL } f \in \mathcal{C}.$$

2. (Proper K-energy) For $0 \leq \delta < \lambda$ $\exists C_\delta$ s.t.

$$\mathcal{E}(u) \geq \delta ||\pi(u)|| + C_\delta \quad \forall u \in \mathcal{S}$$

Idea of proof By approximation, can suppose f, u smooth. Also without loss, $\pi(f) = f$ and $\pi(u) = u$.

(i) $\Rightarrow$ (ii) Compare $\mathcal{E}$ to

$$\mathcal{E}_a(u) := -\int_M \log \det \text{Hess}(u) d\mu + F_a(u)$$

where $F_a(u)$ (linear) is chosen so that $\mathcal{E}_a$ is bounded below.

(ii) $\Rightarrow$ (i) Consider $\mathcal{E}(u + kf)$ and let $k \to \infty$.