# Quaternionic connections, induced holomorphic structures and a vanishing theorem

Liana David

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**Abstract:** We classify the holomorphic structures of the tangent vertical bundle $\Theta$ of the twistor fibration of a quaternionic manifold $(M, Q)$ of dimension $4n \geq 8$. In particular, we show that any self-dual quaternionic connection $D$ of $(M, Q)$ induces an holomorphic structure $\bar{\partial}^D$ on $\Theta$. We construct a Penrose transform which identifies solutions of the Penrose operator on $(M, Q)$ defined by $D$ with the space of $\bar{\partial}^D$-holomorphic purely imaginary sections of $\Theta$. As a consequence we show that, when $(M, Q)$ is compact and admits a compatible quaternionic-Kähler metric of negative scalar curvature, $\Theta$ admits no global non-trivial holomorphic sections with respect to any of its holomorphic structures induced by closed quaternionic connections of $(M, Q)$.

**Key words and phrases:** quaternionic manifolds, twistor spaces, holomorphic structures, Penrose transforms, Penrose operators.
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## 1 Introduction

An almost quaternionic structure on a manifold $M$ of dimension $4n \geq 8$ is a rank three subbundle $Q$ of $\mathrm{End}(TM)$ locally generated by three almost complex structures which satisfy the quaternionic relations. The bundle $Q$ has a natural Euclidian metric, with respect to which any such system of almost complex structures is orthonormal. $Q$ is called a quaternionic structure if it is preserved by a torsion-free linear connection on $M$, called a quaternionic connection. A quaternionic manifold is a manifold together with a fixed quaternionic structure.

One of the main techniques to study quaternionic manifolds is provided by twistor theory. The twistor space $Z$ of a quaternionic manifold $(M, Q)$ is the total space of the unit sphere bundle of $Q$, or the set of complex structures of tangent spaces of $M$ which belong to $Q$. It has a natural integrable almost complex structure which makes $Z$ a complex manifold. Of interest in this paper is the tangent vertical bundle $\Theta$ of the twistor fibration $\pi : Z \to M$, which is a hermitian complex line bundle over $Z$. We show that any quaternionic connection $D$ on $(M, Q)$ defines a hermitian connection $\nabla$ on $\Theta$, which is a Chern connection if and only if $D$ is self-dual, i.e. the curvature of connection on $\Lambda^{4n}(T^*M)$ induced by $D$ is $Q$-hermitian (see Proposition 1). When $D$ is a self-dual, the $(0,1)$-part $\bar{\partial}^D$ of $\nabla$ is a real holomorphic structure of $\Theta$, where by "real" we mean that the space of $\bar{\partial}^D$-holomorphic sections of $\Theta$ is invariant under the canonical anti-holomorphic involution of $Z$, defined as the antipodal map along the fibers of $\pi$ (lifted to $\Theta$). However, the complex line bundle $\Theta$ admits holomorphic structures which are not necessarily real. Our first main result is Theorem 3 of Section 3, and represents a classification of all holomorphic structures of $\Theta$, in terms of self-dual quaternionic connections on $(M, Q)$ and 1-forms on $M$ with $Q$-hermitian exterior derivative. This result is analogous to Theorem 1 of [4], which classifies the holomorphic structures of the tangent vertical bundle of the twistor fibration of a conformal self-dual 4-manifold, in terms of self-dual Weyl connections and Maxwell fields on the conformal 4-manifold. As shown in [4], the tangent vertical bundle of the twistor fibration of a conformal self-dual 4-manifold has a canonical class of equivalent holomorphic structures, defined by the Levi-Civita connections of the metrics in the conformal class. Corollary 4 of Section 3 represents a similar result in the quaternionic context. In the last two sections – Section 4 and Section 5 – we turn our attention to the holomorphic sections of $\Theta$, with respect to the holomorphic structures $\bar{\partial}^D$. More precisely, in Section 4 we construct a Penrose transform, which identifies the $\bar{\partial}^D$-holomorphic purely imaginary sections of $\Theta$ with the kernel of the Penrose operator $P^D$ of $(M, Q)$ defined by a self-dual quaternionic connection $D$ (see Proposition 5). In Section 5 we prove that the Penrose operator $P^D$ has no global non-trivial solutions when $(M, Q)$ is compact and admits a compatible quaternionic-Kähler metric $g$ of negative scalar curvature, and $D$ is related to the Levi-Civita connection of $g$ in a suitable way (see Theorem 9). This will readily imply that the holomorphic structures of $\Theta$ induced by closed quaternionic connections on $(M, Q)$ (i.e. quaternionic connections which induce flat connections on $\Lambda^{4n}(T^*M)$) admit no global non-trivial holomorphic sections, when $(M, Q)$ is compact and admits a compatible quaternionic-Kähler metric of negative scalar curvature (see Corollary 10). Similar results have been developed in [4], in the context of conformal self-dual 4-manifolds.

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# 2 Basic facts on quaternionic manifolds

In this preliminary section we recall some basic facts we shall need about quaternionic manifolds (quaternionic connections and twistor spaces of quaternionic manifolds). We follow the treatment of [2] and [3] (Chapter 14, Section $G$). All our quaternionic manifolds will be of dimension $4n \geq 8$. For a manifold $M$, $TM$, $T^*M$ and $\Omega^k(M)$ will denote the real tangent bundle of $M$, the real cotangent bundle of $M$ and the space of smooth real-valued $k$-forms on $M$, respectively. For a vector bundle $V \to M$, $\Omega^k(M, V)$ will denote the space of smooth $k$-forms on $M$ with values in $V$.

## 2.1 Quaternionic connections

Let $(M, Q)$ be a quaternionic manifold and $D$ a quaternionic connection on $(M, Q)$. Any other quaternionic connection $D'$ is related to $D$ by the formula $D'_X = D_X + S^\alpha_X$, where $\alpha \in \Omega^1(M)$ and

$$S^\alpha_X := \alpha(X)\mathrm{Id}_{TM} + \alpha \otimes X - \sum_{i=1}^{3} [\alpha(J_i X) J_i + (\alpha \circ J_i) \otimes J_i X], \quad X \in TM. \tag{1}$$

Here "$\mathrm{Id}_{TM}$"denotes the identity endomorphism of $TM$ and $\{J_1, J_2, J_3\}$ is an admissible basis of $Q$, i.e. a system of locally defined almost complex structures which satisfy the quaternionic relations and generate $Q$. The connections $D$ and $D'$ are **equivalent** if $\alpha$ is an exact 1-form. We say that $D$ is **closed** (respectively, **exact**) if it induces a flat connection on $\Lambda^{4n}(T^*M)$ (respectively, if there is a volume form on $M$ preserved by $D$). There always exist exact quaternionic connections on $(M, Q)$: using relation (1), one can check that $D'_X \underline{\mathrm{vol}} = D_X \underline{\mathrm{vol}} - 4(n+1)\alpha(X)\underline{\mathrm{vol}}$, where $\underline{\mathrm{vol}}$ is an arbitrary volume form on $M$; if $D_X \underline{\mathrm{vol}} = \omega(X)\underline{\mathrm{vol}}$ for a 1-form $\omega \in \Omega^1(M)$, then the quaternionic connection $D' := D + S^\alpha$ with $\alpha := \frac{1}{4(n+1)}\omega$ is exact, because $\underline{\mathrm{vol}}$ is $D'$-parallel). Equally easy can be shown that any two exact quaternionic connections are equivalent. The family of exact quaternionic connections forms the **canonical class of equivalent quaternionic connections of** $(M, Q)$. We shall meet a third class of connections, the so called **self-dual** quaternionic connections; a quaternionic connection is self-dual if the induced connection on the bundle $\Lambda^{4n}(T^*M)$ has $Q$-hermitian curvature, i.e. its curvature is compatible with any complex structure which belongs to $Q$.

A quaternionic curvature tensor of $(M, Q)$ is a curvature tensor $R$ of $M$ (i.e. a section of $\Lambda^2(T^*M) \otimes \mathrm{End}(TM)$ in the kernel of the Bianchi map) which takes values in the normalizer of $Q$, i.e. for any $X, Y \in TM$, $[R_{X,Y}, Q] \subset Q$. The space $\mathcal{R}(N(Q))$ of quaternionic curvature tensors decomposes into the direct sum $\mathcal{W} \oplus \mathcal{R}^{\mathrm{Bil}}$ where $\mathcal{W}$, called the space of quaternionic Weyl curvatures, is the kernel of the Ricci contraction Ricci : $\mathcal{R}(N(Q)) \to \mathrm{Bil}(TM)$, defined by $\mathrm{Ricci}(R)_{X,Y} := \mathrm{trace}\{Z \to R_{Z,X}Y\}$ and $\mathcal{R}^{\mathrm{Bil}}$ is isomorphic to the space $\mathrm{Bil}(TM)$ of bilinear forms on $TM$, by means of the isomorphism which associates to $\eta \in \mathrm{Bil}(TM)$ the quaternionic curvature

$$
\begin{aligned}
R^{\eta}_{X,Y} :=& (\eta(Y, X) - \eta(X, Y)) \,\mathrm{Id}_{TM} - \eta_X \otimes Y + \eta_Y \otimes X \\
&- \sum_{i=1}^{3} [(\eta(Y, J_i X) - \eta(X, J_i Y)) J_i + (\eta_Y \circ J_i) \otimes J_i X - (\eta_X \circ J_i) \otimes J_i Y],
\end{aligned}
$$

where $X, Y \in TM$ and $\eta_X := \eta(X, \cdot)$, $\eta_Y := \eta(Y, \cdot)$. Hence the curvature of any quaternionic connection $D$ decomposes as $R^D = W + R^{\eta}$, where $W \in \mathcal{W}$, called the quaternionic Weyl tensor, is an invariant of the quaternionic structure (i.e. is independent of the choice of quaternionic connection) and satisfies

$$[W_{X,Y}, A] = 0, \quad X, Y \in TM, \quad A \in Q.$$

With respect to an admissible basis $\{J_1, J_2, J_3\}$ of $Q$,

$$[R^D_{X,Y}, J_i] = \Omega_k(X, Y) J_j - \Omega_j(X, Y) J_k, \tag{2}$$

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$ and

$$\Omega_i(X, Y) := 2\left(\eta(X, J_i Y) - \eta(Y, J_i X)\right), \quad X, Y \in TM, \quad i \in \{1, 2, 3\}. \tag{3}$$

The bilinear form $\eta$ is related to the Ricci tensor $\mathrm{Ricci}(R^D)$ of $D$ in the following way (see [2], p. 223)

$$\eta = \frac{1}{4(n+1)} \mathrm{Ricci}(R^D)^{\mathrm{skew}} + \frac{1}{4n} \mathrm{Ricci}(R^D)^{\mathrm{sym}} - \frac{1}{2n(n+2)} P_h\left(\mathrm{Ricci}(R^D)^{\mathrm{sym}}\right), \tag{4}$$

where "sym" and "skew" denote the symmetric, respectively skew-symmetric parts of a bilinear form and $P_h$ is the projection

$$P_h(\eta) := \frac{1}{4}\left(\eta + \sum_{i=1}^{3} \eta(J_i\cdot, J_i\cdot)\right)$$

of $\mathrm{Bil}(TM)$ onto the space of $Q$-hermitian bilinear forms. We mention that in general, $\mathrm{Ricci}(R^D)$ is not symmetric. More precisely, $\mathrm{Ricci}(R^D)^{\mathrm{skew}}$ is half of the curvature of the connection induced by $D$ on the canonical bundle $\Lambda^{4n}(T^*M)$ (see [2], p. 222 and p. 224), so that $D$ has symmetric Ricci tensor (respectively, the skew part of the Ricci tensor of $D$ is $Q$-hermitian) if and only if $D$ is a closed (respectively, self-dual) quaternionic connection. Finally, we remark that if $D' = D + S^\alpha$, then the Ricci tensors of $D$ and $D'$ are related by the following formulas (see [2], p. 263)

$$\begin{aligned}\mathrm{Ricci}(R^{D'})^{\mathrm{sym}} &= \mathrm{Ricci}(R^D)^{\mathrm{sym}} + 4n\left(\alpha\otimes\alpha - \sum_{i=1}^{3}(\alpha\circ J_i)\otimes(\alpha\circ J_i) - (D\alpha)^{\mathrm{sym}}\right)\\ &\quad + 8P_h\left(\alpha\otimes\alpha - \sum_{i=1}^{3}(\alpha\circ J_i)\otimes(\alpha\circ J_i) - (D\alpha)^{\mathrm{sym}}\right)\\ \mathrm{Ricci}(R^{D'})^{\mathrm{skew}} &= \mathrm{Ricci}(R^D)^{\mathrm{skew}} - 4(n+1)d\alpha.\end{aligned}$$

## 2.2 Twistor theory of quaternionic manifolds

**The twistor space of a quaternionic manifold:** As mentioned in the Introduction, the twistor space $Z$ of $(M, Q)$, defined as the unit sphere bundle of $Q$, has a natural complex structure. In order to define it, we first consider a twistor line $Z_p$, i.e the fiber of the natural projection $\pi : Z \to M$ corresponding to a point $p \in M$. Let $\langle\cdot,\cdot\rangle$ be the natural Euclidian metric of the bundle $Q$. Then $T_J Z_p$ consists of all $J$-anti-linear endomorphisms of $T_pM$ which belong $Q_p$, or to the orthogonal complement $J^\perp$ of $J$ in $Q_p$, with respect to the metric $\langle\cdot,\cdot\rangle$ . Note that $Z_p$ is a Kähler manifold: it has a complex structure $\mathcal{J}$, defined as $\mathcal{J}(A) := J \circ A$, for any $A \in T_J Z_p$, and a compatible Riemannian metric, induced from the metric of $Q_p$, since $T_J Z_p \subset Q_p$. Now we are able to define the complex structure $\mathcal{J}$ of $Z$: chose a quaternionic connection $D$ of $(M, Q)$. Since it preserves $Q$ and $\langle\cdot,\cdot\rangle$, it induces a connection on the twistor bundle $\pi : Z \to M$, i.e. a decomposition of every tangent space $T_J Z$ into the vertical tangent space $T_J Z_p$ and horisontal space $\mathrm{Hor}_J$. On $\mathrm{Hor}_J$, identified with $T_pM$ by means of the differential $\pi_*$, $\mathcal{J}$ is equal to $J$. On $T_J Z_p$, $\mathcal{J}$ is defined as above. It can be shown that $\mathcal{J}$ so defined is independent of the choice of quaternionic connection and is integrable. The twistor space $Z$ becomes a complex manifold of dimension $2n+1$ and the twistor lines are complex projective lines of $Z$ with normal bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.

**The tangent vertical bundle $\Theta$:** The tangent vertical bundle $\Theta$ of the twistor projection $\pi : Z \to M$ is a complex line bundle over the complex manifold $Z$, with complex structure of the fibers defined by the complex structure of the twistor lines. Moreover, it has a canonical hermitian metric $h(X,Y) := \frac{1}{2}(\langle X,Y\rangle - i\langle \mathcal{J}X,Y\rangle)$, for any $X,Y \in \Theta_J = T_J Z_p \subset Q_p$. Due to this, there is an isomorphism between Chern connections of $\Theta$ (i.e. hermitian connections with $\mathcal{J}$-invariant curvature) and holomorphic structures of $\Theta$, i.e. operators

$$\bar{\partial} : \Gamma(\Theta) \to \Omega^{0,1}(Z,\Theta)$$

which satisfy the Liebniz rule

$$\bar{\partial}(fs) = f\bar{\partial}(s) + \bar{\partial}(f)s, \quad f \in C^\infty(Z,\mathbb{C}), \quad s \in \Gamma(\Theta)$$

and whose natural extension to the complex $\Omega^{0,*}(Z,\Theta)$ satisfies $\bar{\partial}^2 = 0$. The isomorphism associates to a Chern connection $\nabla$ its $(0,1)$-part

$$\bar{\partial}_U s := \frac{1}{2}\left(\nabla_U s + \mathcal{J}\nabla_{\mathcal{J}U} s\right), \quad U \in TZ, \quad s \in \Gamma(\Theta).$$

Hence the study of holomorphic structures of $\Theta$ reduces to the study of Chern connections.

**Distinguished sections of $\Theta$:** Note that any section $A \in \Gamma(Q)$ defines a section $\tilde{A}$ of $\Theta$, by the formula:

$$\tilde{A}(J) = \Pi_J(A) := \frac{1}{2}\left(A + J \circ A \circ J\right) = A - \langle A, J\rangle J, \quad J \in Z,$$

where the bundle homomorphism $\Pi : \pi^*Q \to \Theta$ is the orthogonal projection onto $\Theta \subset \pi^*Q$ with respect to the metric of $\pi^*Q$ induced by the natural Euclidian metric $\langle \cdot, \cdot \rangle$ of $Q$. Such sections of $\Theta$ will be called **distinguished**. The differential $\sigma_* : TZ \to TZ$ of the antipodal map $\sigma : Z \to Z$, $\sigma(J) = -J$ induces an involution on the space of smooth sections of $\Theta$, which associates to a section $s$ the section $\bar{s}$ defined as follows: for any $J \in Z$, $\bar{s}_J := \sigma_*\left(s_{\sigma(J)}\right)$. If $s := \tilde{A}$ is distinguished, then $\bar{s} = -s$. This is why the distinguished sections are also called **purely imaginary**. Moreover, $\mathcal{J}s$ is **real**, i.e. $\overline{\mathcal{J}s} = \mathcal{J}s$. The distinguished sections of $\Theta$ will play a fundamental role in our treatment.

# 3 Holomorphic structures on $\Theta$

In this section, we adapt the arguments used in [4] (Sections II2, II4, II5) to the quaternionic context. To keep our text short, we refer to [4] whenever the analogy is straightforward.

Consider a quaternionic connection $D$ on $(M, Q)$. Then $\pi^*D$ is a connection on the pull-back bundle $\pi^*Q$ and $\nabla := \Pi \circ \pi^*D$ is a connection on $\Theta$. Since $D$ preserves $\langle \cdot, \cdot \rangle$, the connection $\nabla$ preserves the Euclidian metric of $\Theta$. Like in [4], one shows that $\nabla$ is $\mathbb{C}$-linear, i.e. that $\nabla \mathcal{J} = 0$, where $\mathcal{J}$ denotes the complex structure of the fibers of $\Theta$.

**Proposition 1.** *The connection $\nabla$ is a Chern connection if and only if $D$ is self-dual.*

*Proof.* For any distinguished section $\tilde{A}$ of $\Theta$ and $U \in T_J Z$, with $\pi_* U = X \in T_p M$,

$$\nabla_U \tilde{A} = \widetilde{D_X A} - \langle J, A \rangle v^{\bar{D}}(U). \tag{5}$$

Here $v^{\bar{D}}(U) \in T_J Z_p = \Theta_J$ denotes the vertical part of $U$ with respect to the connection $\bar{D}$ induced by $D$ on the twistor bundle $\pi : Z \to M$. From relation (5) we obtain, like in [4] (see p. 585 and Appendix A), the following expression of the curvature $R^\nabla$ :

$$\begin{aligned} R^\nabla_{\tilde{X},\tilde{Y}} A &= \Pi_J \left( [R^D_{X,Y}, A] \right) \\ R^\nabla_{B,C} A &= -\Omega_p(B, C) \mathcal{J}(A) \\ R^\nabla_{\tilde{X},B} A &= 0, \end{aligned}$$

where $\tilde{X}, \tilde{Y} \in T_J Z$ are $\bar{D}$-horisontal lifts of $X, Y \in T_p M$, $B, C \in T_J Z_p$, $A \in \Theta_J$, $\Pi_J : Q_p \to J^\perp$ is the orthogonal projection and $\Omega_p$ is the Kähler form of the twistor line $Z_p$, which is obviously $\mathcal{J}$-invariant. Hence $\nabla$ is a Chern connection if and only if the horisontal part of $R^\nabla$ is $\mathcal{J}$-invariant, i.e. for every $J \in Z$ and $A \in Q$ with $A \perp J$,

$$\Pi_J \left( [R^D_{JX \wedge JY - X \wedge Y}, A] \right) = 0. \tag{6}$$

In order to study condition (6), we take an admissible basis $\{J_1, J_2, J_3\}$ of $Q$ with $J = J_1$, so that $A = \lambda_2 J_2 + \lambda_3 J_3$ for some $\lambda_2, \lambda_3 \in \mathbb{R}$. Then $\Pi_J$ becomes the projection onto the subspace generated by $J_2$ and $J_3$. Recall now that $R^D = W + R^\eta$, for some $\eta \in \mathrm{Bil}(TM)$ and that the quaternionic Weyl tensor $W$ commutes with the endomorphisms of $Q$. Using relations (2) and (3), we easily obtain:

$$\begin{aligned} \Pi_{J_1} \left( [R^D_{J_1 X \wedge J_1 Y - X \wedge Y}, A] \right) &= \left( \Omega_1(J_1 X, J_1 Y) - \Omega_1(X, Y) \right) J_1 A \\ &= -4 \left( \eta^{\mathrm{skew}}(J_1 X, Y) + \eta^{\mathrm{skew}}(X, J_1 Y) \right) J_1 A. \end{aligned}$$

From relation (4) we deduce that $\mathrm{Ricci}(R^D)^{\mathrm{skew}}$ is $Q$-hermitian. It follows that $D$ is a self-dual quaternionic connection.

$\square$

Self-dual quaternionic connections exist on any quaternionic manifold. If $D$ is a self-dual quaternionic connection, then any other self-dual quaternionic connection is of the form $D' = D + S^\alpha$, with $d\alpha$ $Q$-hermitian. The way the Chern connections of $\Theta$ determined by two self-dual quaternionic connections of $(M, Q)$ are related is described in the following proposition.

**Proposition 2.** *Let $D$, $D' = D + S^\alpha$ be two self-dual quaternionic connections. Then the Chern connections $\nabla$ and $\nabla'$ induced by $D$ and $D'$ are related as follows:*

$$\nabla' = \nabla + 2\mathcal{J}\,(\pi^*\alpha) \otimes \mathcal{J}.$$

*Proof.* Fix an arbitrary $J \in Z$ and an admissible basis $\{J_1, J_2, J_3\}$ of $Q$ with $J = J_1$. Any $A \in \Theta_J$ is of the form $\lambda_2 J_2 + \lambda_3 J_3$, for some $\lambda_2, \lambda_3 \in \mathbb{R}$. For every $U \in T_J Z$ with $\pi_* U = X$,

$$\begin{aligned}(\nabla' - \nabla)_U(A) = \Pi_{J_1}[D'_X - D_X, A] &= 2\alpha(J_1 X)(-\lambda_2 J_3 + \lambda_3 J_2)\\ &= -2\alpha(J_1 X) J_1(\lambda_2 J_2 + \lambda_3 J_3)\\ &= 2(\mathcal{J}\pi^*\alpha)(U) J_1 A.\end{aligned}$$

$\square$

**Remark:** Recall that two holomorphic structures on a complex line bundle $V \to N$ over a complex manifold $(N, J)$ are equivalent, if they are conjugated by an element of the gauge group $C^\infty(N, \mathbb{C}^*)$. Suppose now that $V$ has a hermitian structure. Let $\bar\partial^1$ and $\bar\partial^2$ be two holomorphic structures of $V$ and $\nabla^1$ and $\nabla^2$ the Chern connections, with $(0,1)$-parts $\bar\partial^1$ and $\bar\partial^2$ respectively. Then $\bar\partial^1$ and $\bar\partial^2$ are equivalent if and only if

$$\nabla^2 = \nabla^1 + (d^J \log\rho - d\theta) \otimes \mathcal{J}, \tag{7}$$

where $\rho$ is a positive smooth function, $\theta$ is a smooth function with values in $S^1$ and $\mathcal{J}$ is the complex structure of the fibers of $V$. The connections $\nabla^1$ and $\nabla^2$ are equivalent as hermitian connections if $d^J \log\rho = 0$.

The main result of this Section is the following classification theorem:

**Theorem 3.** *Any holomorphic structure of $\Theta$ is equivalent with an holomorphic structure $\bar\partial^{D,\beta} := \bar\partial^D + \tilde\beta$, where $\bar\partial^D$ is the $(0,1)$-part of the Chern connection of $\Theta$ induced by a self-dual quaternionic connection $D$ of $(M, Q)$,*

*$\beta \in \Omega^1(M)$ has $Q$-hermitian exterior differential and $\tilde{\beta} \in \Omega^{0,1}(Z, \mathrm{End}_{\mathbb{C}}(\Theta))$ is defined as follows: for any $U \in T_J Z$ with $\pi_* U = X$ and $s \in \Gamma(\Theta)$,*

$$\tilde{\beta}_U(s) := \frac{1}{2}\left(\beta(X)\mathcal{J}s - \beta(JX)s\right).$$

*Moreover, two holomorphic structures $\bar{\partial}^{D,\beta}$ and $\bar{\partial}^{D',\beta'}$ are equivalent if and only if $D$ and $D'$ are equivalent as quaternionic connections of $(M, Q)$ and $d^{\beta} := d + i\beta$ and $d^{\beta'} := d + i\beta'$ are equivalent as hermitian connections of the hermitian trivial line bundle $M \times \mathbb{C}$.*

*Proof.* The proof follows the same steps as the proof of Theorem 1 of [4] (note that, our Proposition 1 corresponds to Proposition 2 of [4] and our Proposition 2 corresponds to Lemma 4 of [4]). Due to this, we content ourselves to explain why $\bar{\partial}^{D,\beta}$ is an holomorphic structure. Since $d\beta$ is $Q$-hermitian, the pull-back connection $d + i\pi^*\beta$ is a Chern connection on the hermitian trivial line bundle $Z \times \mathbb{C}$. Let $\nabla$ be the Chern connection of $\Theta$ induced by a self-dual quaternionic connection $D$, as in Proposition 1. The tensor product connection $\nabla^{\beta} := \nabla \otimes (d + i\pi^*\beta) = \nabla + \pi^*\beta \otimes \mathcal{J}$ on $\Theta = \Theta \otimes_{\mathbb{C}} \mathbb{C}$ is also a Chern connection on $\Theta$. It can be readily checked that its $(0,1)$-part is precisely $\bar{\partial}^{D,\beta}$. In particular, $\bar{\partial}^{D,\beta}$ is an holomorphic structure of $\Theta$.

$\square$

Recall now that any two exact quaternionic connections are equivalent. The following Corollary is a consequence of Theorem 3.

**Corollary 4.** *The tangent vertical bundle of the twistor fibration of a quaternionic manifold $(M, Q)$ has a canonical class of equivalent holomorphic structures, determined by the exact quaternionic connections of $(M, Q)$.*

# 4 A Penrose transform

In this section we shall use the $E - H$ formalism developed in [5]. We begin with a brief review of some basic facts we shall need about the representation theory of the group $Sp(1)$. Let $H \cong \mathbb{C}^2$ be an abstract 2-dimensional complex vector space on which $Sp(1) \cong SU(2)$ acts, leaving invariant a complex symplectic form $\omega$ and a compatible quaternionic structure, i.e. a $\mathbb{C}$-anti-linear map $q : H \to H$, which satisfies $q^2 = -\mathrm{Id}_H$, $\omega(qv, qw) = \overline{\omega(v, w)}$ and $\omega(v, qv) > 0$, for any $v, w \in H$. The 2-form $\omega$ together with $q$ define an invariant hermitian positive definite metric $\langle \cdot, \cdot \rangle := \omega(\cdot, q\cdot)$ on $H$. By means of the identification $H \ni h \to \omega(h, \cdot) \in H$ between $H$ and its dual $H^*$, $S^2(H) \subset H \otimes H \cong H^* \otimes H \subset \mathrm{End}(H)$ acts on $H$ and its real part (with respect to the real structure induced by $q$) is isomorphic to the Lie algebra $sp(1) \subset \mathrm{End}(H)$ of imaginary quaternions. We also need to recall that $H \otimes S^2(H)$ has two $Sp(1)$-irreducible components: $S^3(H)$, which is the kernel of the map $F : H \otimes S^2(H) \to H$ defined by

$$F(h, h_1 h_2 + h_2 h_1) = \omega(h, h_1)h_2 + \omega(h, h_2)h_1, \quad h_1, h_2, h \in H, \tag{8}$$

and $H$, isomorphic to the hermitian orthogonal of $S^3(H)$ in $H \otimes S^2(H)$ with respect to the hermitian metric of $H \otimes S^2(H)$ induced by the hermitian metric $\langle \cdot, \cdot \rangle$ of $H$ (to simplify notations, we sometimes omit the tensor product signs, so that $h_1 h_2 + h_2 h_1$ denotes $h_1 \otimes h_2 + h_2 \otimes h_1$).

Coming back to geometry, the quaternionic structure of $(M, Q)$ determines a $G = GL(n, \mathbb{H})Sp(1)$ structure $F_0$, i.e. a $G$-subbundle of the principal frame bundle of $M$, consisting of all frames $f : T_pM \to \mathbb{H}^n$ which convert the standard basis of imaginary quaternions, acting by multiplication on $\mathbb{H}^n$ on the right, onto an admissible basis of $Q_p$, acting naturally on $T_pM$. Any representation of $\tilde{G} = GL(n, \mathbb{H}) \times Sp(1)$ determines a locally defined bundle over $M$, which is globally defined when the representation descends to $G$ (in which case the bundle is associated to the principal $G$-bundle $F_0$). Real representations and real vector bundles over $M$ will be automatically complexified. There are two locally defined complex vector bundles $\mathbf{E}$ and $\mathbf{H}$ on $M$, which are associated to the standard representations of $GL(n, \mathbb{H})$ and $Sp(1)$ on $E \cong \mathbb{C}^{2n}$ and $H \cong \mathbb{C}^2$ respectively, extended trivially to $GL(n, \mathbb{H}) \times Sp(1)$. The $Sp(1)$-invariant structures of $H$ induce similar structures on the bundle $\mathbf{H}$, which will be denoted with the same symbols (e.g. $\omega$ will denote the symplectic form of $H$ as well as the induced symplectic form on the bundle $\mathbf{H}$; in particular, we shall identify $\mathbf{H}$ with its dual $\mathbf{H}^*$ by means of the isomorphism $\mathbf{H} \ni h \to \omega(h, \cdot) \in \mathbf{H}^*$; similarly, $\langle \cdot, \cdot \rangle$ will denote the hermitian inner product of $H$ and the induced hermitian metric on the bundle $\mathbf{H}$). Some of the natural bundles over $M$ are isomorphic with tensor products and direct sums of $\mathbf{H}$ and $\mathbf{E}$. For example, $TM$ is isomorphic with $\mathbf{E} \otimes \mathbf{H}$, $T^*M$ with $\mathbf{E}^* \otimes \mathbf{H}$, $Q$ with $S^2(\mathbf{H})$ and the product $T^*M \otimes Q$ decomposes as

$$T^*M \otimes Q \cong \mathbf{E}^* \otimes \mathbf{H} \otimes S^2(\mathbf{H}) \cong \mathbf{E}^* \otimes S^3(\mathbf{H}) \oplus \mathbf{E}^* \otimes \mathbf{H}, \tag{9}$$

since $H \otimes S^2(H) \cong S^3(H) \oplus H$. The **Penrose operator** $P^D : \Gamma(Q) \to \Gamma(\mathbf{E}^* \otimes S^3(\mathbf{H}))$ defined by a quaternionic connection $D$ of $(M, Q)$ is the composition of $D : \Gamma(Q) \to \Gamma(T^*M \otimes Q)$ with the projection onto the first component of the decomposition (9).

**Proposition 5.** *Let $D$ be a self-dual quaternionic connection on $(M, Q)$ and $A \in \Gamma(Q)$. Then the distinguished section $\tilde{A}$ of $\Theta$ is $\bar{\partial}^D$-holomorphic if and only if $A$ is a solution of the Penrose operator $P^D$.*

*Proof.* The section $\tilde{A}$ is $\bar{\partial}^D$-holomorphic if and only if it satisfies

$$\nabla_{\mathcal{J}U}(\tilde{A}) = \mathcal{J}\nabla_U(\tilde{A}), \quad \forall U \in T_J Z, \quad \forall J \in Z, \tag{10}$$

where $\nabla$ is the Chern connection of $\Theta$ induced by $D$. Using relation (5), it can be seen that (10) is equivalent with

$$D_{JX}A - \langle D_{JX}A, J\rangle J - JD_X A - \langle D_X A, J\rangle \mathrm{Id}_{TM} = 0, \quad \forall X \in TM, \quad \forall J \in Z.$$

For every unit $j \in sp(1) \subset \mathrm{End}(E \otimes H)$ (acting trivially on $E$), define

$$T_j : E^* \otimes H^* \otimes S^2(H) \to E^* \otimes H^* \otimes S^2(H)$$

in the following way: for any $\gamma \in E^* \otimes H^* \otimes S^2(H)$ and $v \in E \otimes H$,

$$T_j(\gamma)(v) := \gamma(jv) - \langle \gamma(jv), j\rangle j - j \circ \gamma(v) - \langle \gamma(v), j\rangle \mathrm{Id}_{E\otimes H}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the hermitian inner product of $H \otimes H$ induced by the $Sp(1)$-invariant hermitian inner product $\langle \cdot, \cdot \rangle$ of $H$, i.e.

$$\langle h_1 h_2, h_3 h_4 \rangle = \frac{1}{2}\langle h_1, h_3\rangle \langle h_2, h_4\rangle, \quad \forall h_i \in H,$$

so that the restriction of $\langle \cdot, \cdot \rangle$ to $sp(1) \subset S^2(H)$ induces the natural Euclidian metric of the bundle $Q$; recall that $Q$ is associated to the adjoint representation of $Sp(1)$ on its Lie algebra $sp(1)$, extended trivially to $\tilde{G}$. The group $Sp(1)$ acts naturally on $H^* \otimes S^2(H) \subset H^* \otimes \mathrm{End}(H)$, by

$$(A \cdot \alpha)(h) = A \circ \alpha(A^{-1}h) \circ A^{-1}, \quad A \in Sp(1), \quad \alpha \in H^* \otimes S^2(H), \quad h \in H.$$

We can extend this action to an action of $\tilde{G}$ on $E^* \otimes H^* \otimes S^2(H)$, with $GL(n, \mathbb{H})$ acting naturally on $E^*$. This extended action preserves the $\mathbb{C}$-linear condition

$$T_j(\gamma) = 0, \quad \forall j \in sp(1), \quad j^2 = -\mathrm{Id}_H \tag{11}$$

on $E^* \otimes H^* \otimes S^2(H)$, since

$$T_j(A \cdot \gamma)(v) = A \circ T_{j'}(\gamma)(A^{-1}v) \circ A^{-1}, \quad v \in E \otimes H,$$

where $j' := A^{-1} \circ j \circ A$. Since $E^* \otimes H$ and $E^* \otimes S^3(H)$ are irreducible components of $E^* \otimes H \otimes S^2(H)$ and there are distinguished sections of $\Theta$ which are not $\bar{\partial}^D$-holomorphic, from Shur's lemma it is enough to check that any element of $E^* \otimes H$ satisfies (11). This can be done in the following way: let $e^* h \in E^* \otimes H$ be decomposable. Without loss of generality, we can take $\langle h, h\rangle = 1$. Define $\tilde{h} := q(h)$. Then the basis $\{h, \tilde{h}\}$ is unitary with respect $\langle\cdot,\cdot\rangle$ and $\omega(h, \tilde{h}) = 1$. As an element of $E^* \otimes H \otimes S^2(H) \subset E^* \otimes H \otimes \mathrm{End}(E \otimes H)$, $e^*h$ has the following form

$$\gamma(v) = \Big(2(e^*\tilde{h})(v)hh - (e^*h)(v)(\tilde{h}h + h\tilde{h})\Big)\,\mathrm{Id}_E, \quad v \in E \otimes H. \tag{12}$$

The identity endomorphism of $H$ is $\mathrm{Id}_H = h\tilde{h} - \tilde{h}h$ and a basis of unit imaginary quaternions can be chosen to be

$$\begin{aligned} j_1 &:= -i(h\tilde{h} + \tilde{h}h)\mathrm{Id}_E \\ j_2 &:= -(hh + \tilde{h}\tilde{h})\mathrm{Id}_E \\ j_3 &:= i(\tilde{h}\tilde{h} - hh)\mathrm{Id}_E. \end{aligned}$$

Consider now $j = a_1j_1 + a_2j_2 + a_3j_3 \in sp(1)$ an arbitrary unit imaginary quaternion (so that the $a_i$'s are real and $a_1^2 + a_2^2 + a_3^2 = 1$). Then, for every $v \in E \otimes H$,

$$\begin{aligned} \gamma(jv) &= 2hh\left(a_1 i(e^*\tilde{h})(v) + (a_2 + ia_3)(e^*h)(v)\right)\mathrm{Id}_E \\ &\quad + (h\tilde{h} + \tilde{h}h)\left(a_1 i(e^*h)(v) + (a_2 - ia_3)(e^*\tilde{h})(v)\right)\mathrm{Id}_E; \\ \langle\gamma(jv), j\rangle &= (-a_2 + ia_3)\left(a_1 i(e^*\tilde{h})(v) + (a_2 + ia_3)(e^*h)(v)\right) \\ &\quad + a_1\left(-a_1(e^*h)(v) + (a_3 + ia_2)(e^*\tilde{h})(v)\right); \\ j \circ \gamma(v) &= (e^*h)(v)\left((a_2 + ia_3)hh + (-a_2 + ia_3)\tilde{h}\tilde{h} + ia_1(h\tilde{h} - \tilde{h}h)\right)\mathrm{Id}_E \\ &\quad + 2(e^*\tilde{h})(v)\left(ia_1 hh + (a_2 - ia_3)h\tilde{h}\right)\mathrm{Id}_E; \\ \langle\gamma(v), j\rangle &= (e^*\tilde{h})(v)(-a_2 + ia_3) - (e^*h)(v)ia_1. \end{aligned}$$

From these relations it is straightforward to check that $T_j(\gamma) = 0$.

$\square$

# 5 A vanishing theorem

In this section we consider a quaternionic manifold $(M, Q)$ which admits a compatible quaternionic-Kähler metric $g$, i.e. the Levi-Civita connection $D^g$ of $g$ is a quaternionic connection of $(M, Q)$ and the endomorphisms of $Q$ are skew-symmetric with respect to $g$. Let $g^* : T^*M \to TM$ be the isomorphism defined by $g$.

Borrowing the terminology of [6], we define a conformal weight operator

$$B : T^*M \otimes Q \to T^*M \otimes Q$$

by the following formula:

$$B(\alpha \otimes A)(X) := [S^{\alpha}_X, A], \quad X \in TM, \quad \alpha \in T^*M, \quad A \in Q, \tag{13}$$

where $S^{\alpha}$ was defined in (1). (A conformal weight operator has been defined in [4], for vector bundles on conformal manifolds, associated to the principal bundle of conformal frames, and in [6] for vector bundles associated to the reduced frame bundle of a Riemannian manifold with special holonomy). The following lemma is a straightforward calculation:

**Lemma 6.** *Let $\{J_1, J_2, J_3\}$ be an admissible basis of $Q$. Then for every $X \in TM$, $\alpha \in T^*M$ and $A \in Q$,*

$$B(\alpha \otimes A)(X) = \alpha\left([J_1, A](X)\right) J_1 + \alpha\left([J_2, A](X)\right) J_2 + \alpha\left([J_3, A](X)\right) J_3.$$

**Proposition 7.** *Consider the decomposition (9) of $T^*M \otimes Q$. The conformal weight operator $B$ acts as $-2 \cdot \mathrm{Id}_{\mathbf{E}^* \otimes S^3(\mathbf{H})}$ on $\mathbf{E}^* \otimes S^3(\mathbf{H})$ and as $4 \cdot \mathrm{Id}_{\mathbf{E}^* \otimes \mathbf{H}}$ on $\mathbf{E}^* \otimes \mathbf{H}$.*

*Proof.* Let $e^*h \in \Gamma(\mathbf{E}^* \otimes \mathbf{H})$ be a decomposable local section, with $\langle h, h \rangle = 1$. Define $\tilde{h} := q(h)$. As a section of $T^*M \otimes Q$, $e^*h = \alpha \otimes A + \tilde{\alpha} \otimes \tilde{A}$, where

$$\begin{aligned} A &:= 2hh\mathrm{Id}_{\mathbf{E}} \in \Gamma(Q), \\ \tilde{A} &:= -(\tilde{h}h + h\tilde{h})\mathrm{Id}_{\mathbf{E}} \in \Gamma(Q) \\ \alpha &:= e^*\tilde{h} \in \Omega^1(M) \\ \tilde{\alpha} &:= e^*h \in \Omega^1(M). \end{aligned}$$

As in the proof of Proposition 5, we chose the admissible basis of $Q$:

$$\begin{aligned} J_1 &:= -i(h\tilde{h} + \tilde{h}h)\mathrm{Id}_{\mathbf{E}} \\ J_2 &:= -(hh + \tilde{h}\tilde{h})\mathrm{Id}_{\mathbf{E}} \\ J_3 &:= i(\tilde{h}\tilde{h} - hh)\mathrm{Id}_{\mathbf{E}}. \end{aligned}$$

It is easy to check the equalities:

$$\begin{aligned} [J_1, A] &= -2i[h\tilde{h} + \tilde{h}h, hh]\mathrm{Id}_{\mathbf{E}} = 4ihh\mathrm{Id}_{\mathbf{E}} \\ [J_2, A] &= -2[hh + \tilde{h}\tilde{h}, hh]\mathrm{Id}_{\mathbf{E}} = 2(h\tilde{h} + \tilde{h}h)\mathrm{Id}_{\mathbf{E}} \\ [J_3, A] &= 2i[\tilde{h}\tilde{h} - hh, hh]\mathrm{Id}_{\mathbf{E}} = -2i(h\tilde{h} + \tilde{h}h)\mathrm{Id}_{\mathbf{E}}. \end{aligned}$$

Using Lemma 6, we get that

$$B(\alpha \otimes A)(X) = -4(e^*h)(X)(h\tilde{h} + \tilde{h}h)\mathrm{Id}_{\mathbf{E}} + 4(e^*\tilde{h})(X)hh\mathrm{Id}_{\mathbf{E}}, \quad X \in TM.$$

A similar calculation shows that $B(\tilde{\alpha} \otimes \tilde{A})(X) = 4(e^*\tilde{h})(X)hh\mathrm{Id}_{\mathbf{E}}$, which readily implies that

$$B(\alpha \otimes A + \tilde{\alpha} \otimes \tilde{A}) = 4(\alpha \otimes A + \tilde{\alpha} \otimes \tilde{A}),$$

i.e. that $\mathbf{E}^* \otimes \mathbf{H}$ is the eigenspace of $B$ corresponding to the eigenvalue 4. In order to show that $\mathbf{E}^* \otimes S^3(\mathbf{H})$ is the eigenspace of $B$ corresponding to the eigenvalue 2, we notice that $\mathbf{E}^* \otimes S^3(\mathbf{H}) \subset T^*M \otimes Q$ is locally generated by sections $\gamma_i^{e^*}$ (for $e^* \in \Gamma(\mathbf{E}^*)$ and $i \in \{1, \cdots 4\}$) defined as follows: for every $X \in TM$,

$$\begin{aligned}
\gamma_1^{e^*}(X) &:= \Big((e^*h)(X)(h\tilde{h} + \tilde{h}h) + (e^*\tilde{h})(X)hh\Big)\,\mathrm{Id}_{\mathbf{E}} \\
\gamma_2^{e^*}(X) &:= \Big((e^*\tilde{h})(X)(h\tilde{h} + \tilde{h}h) + (e^*h)(X)\tilde{h}\tilde{h}\Big)\,\mathrm{Id}_{\mathbf{E}} \\
\gamma_3^{e^*}(X) &:= (e^*h)(X)hh\mathrm{Id}_{\mathbf{E}} \\
\gamma_4^{e^*}(X) &:= (e^*\tilde{h})(X)\tilde{h}\tilde{h}\mathrm{Id}_{\mathbf{E}}.
\end{aligned}$$

As before, one checks that $B(\gamma_i^{e^*}) = -2\gamma_i^{e^*}$, for every $e^* \in \Gamma(\mathbf{E}^*)$ and $i \in \{1, \cdots, 4\}$. The conclusion follows.

□

**Proposition 8.** *Let $D = D^g + S^\alpha$ be a quaternionic connection on a quaternionic-Kähler manifold $(M, Q, g)$, with $\alpha \in \Omega^1(M)$ a co-closed 1-form. Let $A \in \Gamma(Q)$ be a solution of the Penrose operator $P^D$. Then*

$$\langle \mathrm{trace}_g(D^2A), A\rangle = -2|A|^2\left(\frac{1}{4(n+2)}\mathrm{Scal}^g - 2|\alpha|^2\right).$$

*Proof.* It is straightforward to check that $D \circ B = \tilde{B} \circ D$, where the connection $D$ (on the right and left hand side of the equality) acts on $T^*M \otimes Q$ and $\tilde{B} := \mathrm{Id}_{T^*M} \otimes B$ is an automorphism of $T^*M \otimes T^*M \otimes Q$. Define

$$\mathrm{trace}_g(\tilde{B}) : T^*M \otimes T^*M \otimes Q \to Q$$

as follows: for any $A \in Q$, $\alpha, \beta \in T^*M$,

$$\mathrm{trace}_g(\tilde{B})(\alpha \otimes \beta \otimes A) := \sum_{i=1}^{4n} \tilde{B}(\alpha \otimes \beta \otimes A)(e_i, e_i) = B(\beta \otimes A)(g^*\alpha),$$

where $\{e_1, \cdots, e_{4n}\}$ is an arbitrary $g$-orthonormal basis of $TM$. Writing $A = a_1J_1 + a_2J_2 + a_3J_3$ in terms of an admissible basis $\{J_1, J_2, J_3\}$ of $Q$, we readily obtain, from Lemma 6, that

$$\begin{aligned}\mathrm{trace}_g(\tilde{B})(D^2A) &= \sum_{i<j} \left( g\left( [J_1, R^D_{e_i,e_j}(A)]e_i, e_j \right) J_1 + g\left( [J_2, R^D_{e_i,e_j}(A)]e_i, e_j \right) J_2 \right) \\ &\quad + \sum_{i<j} g\left( [J_3, R^D_{e_i,e_j}(A)]e_i, e_j \right) J_3,\end{aligned}$$

where $R^D_{e_i,e_j}(A) = [R^D_{e_i,e_j}, A]$ is the commutator of the endomorphisms $R^D_{e_1,e_j}$ and $A$ of $TM$. In particular,

$$\begin{aligned}\langle \mathrm{trace}_g(\tilde{B})(D^2A), A\rangle &= \sum_{i<j} g\left( [A, R^D_{e_i,e_j}(A)]e_i, e_j \right) \\ &= \sum_{i<j} \sum_{k,p} a_k a_p g\left( [J_k, R^D_{e_i,e_j}(J_p)]e_i, e_j \right).\end{aligned}$$

Using relations (2) and (3) we readily get that, for every $k \in \{1, 2, 3\}$,

$$\sum_{i<j} g\left( [J_k, R^D_{e_i,e_j}(J_k)]e_i, e_j \right) = -8\mathrm{trace}_g(\eta)$$

and for every $k \neq p$,

$$\sum_{i<j} g\left( [J_k, R^D_{e_i,e_j}(J_p)]e_i + [J_p, R^D_{e_i,e_j}(J_k)]e_i, e_j \right) = 0$$

from where we conclude that

$$\langle \mathrm{trace}_g(\tilde{B})(D^2A), A\rangle = -8|A|^2\mathrm{trace}_g(\eta). \tag{14}$$

Recall now that $\eta$ is related to the Ricci curvature $\mathrm{Ricci}(R^D)$ of $D$ as in relation (4). According to Section 2, we can express $\mathrm{Ricci}(R^D)$ in terms of $\alpha$ and the Ricci tensor of $g$, so that, taking traces and using the fact that $\alpha$ is co-closed, we easily obtain the following relation:

$$\langle \mathrm{trace}_g(\tilde{B})(D^2A), A\rangle = -8|A|^2\mathrm{trace}_g(\eta) = -8|A|^2 \left( \frac{1}{4(n+2)}\mathrm{Scal}^g - 2|\alpha|^2 \right). \tag{15}$$

On the other hand, from Proposition 7 and the very definition of the Penrose operator,

$$\tilde{B}(D^2A) = D \circ B(DA) = 4D^2A - 6D(P^DA).$$

In particular, if $A \in \Gamma(Q)$ is a solution of the Penrose operator, then $P^D A = 0$ and

$$\langle \mathrm{trace}_g(D^2 A), A\rangle = \frac{1}{4}\langle \mathrm{trace}_g(\tilde{B})(D^2 A), A\rangle = -2|A|^2\left(\frac{1}{4(n+2)}\mathrm{Scal}^g - 2|\alpha|^2\right). \tag{16}$$

□

We now restrict to the situation when $(M, Q, g)$ is a compact quaternionic-Kähler manifold of negative scalar curvature. An important class of such manifolds can be constructed in the following way [1]: take a non-compact symmetric quaternionic-Kähler manifold $\mathrm{M} = \mathrm{G}/\mathrm{K}$, which is dual to a Wolf space. The non-compact simple Lie group G has a torsion free cocompact discrete subgroup $\Gamma$. Then the double quotient $M/\Gamma$ is a compact quaternionic-Kähler manifold of negative scalar curvature.

**Theorem 9.** *Let $(M, Q, g)$ be a compact quaternionic-Kähler manifold with negative scalar curvature. Let $D$ be a quaternionic connection on $(M, Q, g)$, such that $D = D^g + S^\alpha$, with $\alpha \in \Omega^1(M)$ a co-closed 1-form. Then the Penrose operator $P^D$ has no non-trivial global solutions.*

*Proof.* Choose an admissible basis $\{J_1, J_2, J_3\}$ of $Q$ and recall the formula

$$D_X A = D^g_X A + B(\alpha \otimes A)(X) = D^g_X A + \sum_{j=1}^{3} \alpha\left([J_j, A](X)\right) J_j, \quad X \in TM \tag{17}$$

which relates $D$ and $D^g$ when they act on the bundle $Q$. Using relation (17) it is straightforward to check that

$$\begin{aligned}\langle (D^2 A)_{X,X}, A\rangle &= \langle (D^g)^2(A)_{X,X}, A\rangle + \alpha\left([A, D^g_X A](X)\right) - 2\alpha(X)\langle D^g_X A, A\rangle \\ &+ \sum_{j=1}^{3}[X\left(\alpha([J_j, A](X))\right)\langle J_j, A\rangle + \alpha\left([J_j, A](X)\right)\langle D_X J_j, A\rangle] \\ &+ 2\sum_{j=1}^{3}\alpha(J_j X)\langle D^g_{J_j X} A, A\rangle.\end{aligned}$$

Choosing a $g$-orthonormal basis $\{e_1, \cdots, e_{4n}\}$ of $TM$ and letting $X := e_i$ in the previous relation, we get

$$\begin{aligned}\langle \mathrm{trace}_g(D^2 A), A\rangle &= \langle \mathrm{trace}_g(D^g)^2 A, A\rangle + 4\langle D^g_{g^*\alpha} A, A\rangle + 2\sum_{j=1}^{4n}\alpha\left([A, D^g_{e_j} A]e_j\right) \\ &- \sum_{i,j=1}^{4n}\alpha\left([J_j, A]e_i\right)^2.\end{aligned}$$

On the other hand, again from relation (17), we deduce that

$$
\begin{aligned}
\langle DA, DA\rangle &:= \sum_{i=1}^{4n} \langle D_{e_i}A, D_{e_i}A\rangle = \sum_{i=1}^{4n}\left(\langle D^g_{e_i}A, D^g_{e_i}A\rangle + 2\alpha\left([D^g_{e_i}A, A]e_i\right)\right)\\
&\quad + \sum_{i,j=1}^{4n}\alpha\left([J_j, A]e_i\right)^2\\
&= \langle D^gA, D^gA\rangle + 2\sum_{i=1}^{4n}\alpha\left([D^g_{e_i}A, A]e_i\right) + \sum_{i,j=1}^{4n}\alpha\left([J_j, A]e_i\right)^2.
\end{aligned}
$$

Combining the above relations, we get

$$
\begin{aligned}
\langle \mathrm{trace}_g(D^2A), A\rangle =& \langle \mathrm{trace}_g(D^g)^2(A), A\rangle + 4\langle D^g_{g^*\alpha}A, A\rangle\\
&+ \langle D^gA, D^gA\rangle - \langle DA, DA\rangle.
\end{aligned}
$$

Suppose now that $P^DA = 0$. Using Proposition 8, this relation becomes

$$
\begin{aligned}
&\langle DA, DA\rangle - \langle D^gA, D^gA\rangle - 4\langle D^g_{g^*\alpha}A, A\rangle - \langle \mathrm{trace}_g(D^g)^2A, A\rangle\\
&= 2|A|^2\left(\frac{1}{4(n+2)}\mathrm{Scal}^g - 2|\alpha|^2\right).
\end{aligned}
$$

Integrating over $M$ and using $\int_M \langle D^g_{g^*\alpha}A, A\rangle \mathrm{vol}_g = 0$, the 1-form $\alpha$ being co-closed, we get

$$
\int_M \langle DA, DA\rangle \mathrm{vol}_g + 4\int_M |A|^2|\alpha|^2\mathrm{vol}_g - \frac{1}{2(n+2)}\mathrm{Scal}^g\int_M |A|^2\mathrm{vol}_g = 0.
$$

Since $\mathrm{Scal}^g < 0$, $A$ must be necessarily identically zero.

□

**Corollary 10.** *Let $D$ be a closed quaternionic connection on a compact quaternionic-Kähler manifold $(M, Q, g)$ of negative scalar curvature. There is no global non-trivial $\bar\partial^D$-holomorphic section of $\Theta$.*

*Proof.* Let us consider an arbitrary $\bar\partial^D$-holomorphic section $s$ of $\Theta$. As in [4], we can prove that $s = \tilde A + \mathcal{J}\tilde B$, for two sections $A, B \in \Gamma(Q)$. It can be checked that $\bar s = -\tilde A + \mathcal{J}\tilde B$ is also $\bar\partial^D$-holomorphic, from where we deduce that both $\tilde A$ and $\tilde B$ are $\bar\partial^D$-holomorphic. Therefore, to prove our claim it is enough to show that there are no global non-trivial $\bar\partial^D$-holomorphic distinguished sections of $\Theta$. The Levi-Civita connection $D^g$ is exact, and hence $D = D^g + S^\alpha$, for some closed 1-form $\alpha \in \Omega^1(M)$. From Theorem 3, the holomorphic structure $\bar\partial^D$ depends (up to isomorphism) only on the cohomology class of $\alpha$. Hence, without loss of generality, we can take $\alpha$ to be harmonic. We conclude from the Penrose transform developed in Proposition 5 and from Theorem 9. □

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**Author's addresses:** *Present:* Centro di Ricerca Matematica "Ennio de Giorgi", Scuola Normale Superiore, Piazza dei Cavalieri nr. 3, Pisa, Italia; e-mail: l.david@sns.it; *Permanent:* Institute of Mathematics of the Romanian Academy "Simion Stoilow", Calea Grivitei nr. 21, Sector 1, Bucharest, Romania; e-mail: liana.david@imar.ro