

Geometry of currents, intersection theory and dynamics of horizontal-like maps

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Abstract

We introduce a geometry on the cone of positive closed currents of bidegree (p, p) and apply it to define the intersection of such currents. We also construct and study the Green currents and the equilibrium measure for horizontal-like mappings. The Green currents satisfy some extremality properties. The equilibrium measure is invariant, mixing and has maximal entropy. It is equal to the intersection of the Green currents associated to the horizontal-like map and to its inverse.

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1 Introduction

In this paper we develop the theory of positive closed currents of any degree in order to continue our exploration of dynamical systems in several variables, with emphasis on systems not defined by rational maps.

In [6], we developed the theory of polynomial-like maps in higher dimension. Recall that a polynomial-like map is a holomorphic map $f : U \rightarrow V$, with $U \Subset V \Subset \mathbb{C}^k$, and that f is proper of topological degree $d_t > 1$. In some sense, such a map is expanding, but it has critical points in general.

Here, we consider horizontal-like maps in any dimension. Basically, a horizontal-like map is a holomorphic map defined on a domain in \mathbb{C}^k , which is “expanding” in p directions and “contracting” in $k - p$ directions. The expansion and contraction are of global nature, but the map is, in general, not uniformly hyperbolic in the dynamical sense [20]. The precise definition is given in Section 4.

This situation has been already studied by Dujardin for $k = 2$ with emphasis on biholomorphic maps [11]. The study was developed in dimension 2 by Dujardin and the authors to deal with the random iteration of meromorphic horizontal-like

maps, in order to study rates of escape to infinity for polynomial mappings in \mathbb{C}^2 [5]. It turns out that, as for polynomial-like maps, the building blocs for a large class of polynomial maps are horizontal-like maps. We should observe that to treat the case of \mathbb{C}^2 with the methods of the present paper one should deal with horizontal-like maps in \mathbb{C}^4 or \mathbb{C}^8 and that we obtain new results even in the \mathbb{C}^2 case (see Theorem 6.4 and [4]). The main technical problem is to deal with currents of higher bidegree.

One of the difficulties is that the potentials of currents of higher bidegree are not functions. Hence, the techniques used in the case of dimension 2 do not work for general horizontal-like maps. It seems that considering the potentials is not the best way to prove properties of currents of higher bidegree. We propose here another approach to deal directly with the cone of positive closed currents that we consider as a space of infinite dimension with some plurisubharmonic (p.s.h.) structure.

We introduce in Section 2 the notion of *structural varieties* in this cone which allows us to use the complex structure of \mathbb{C}^k . Structural varieties connect currents in this cone. So, we will study singular currents using their smooth approximation in structural discs. For example, a structural disc of currents of bidimension (p, p) is the collection of slices of a positive closed current of bidimension $(p + 1, p + 1)$. The family is not always continuous in term of slices, but when it acts on forms Φ such that $dd^c\Phi \geq 0$ we get p.s.h. functions on the space parametrizing slices. To prove the convergence of a sequence of currents we embed it in some sequence of structural discs passing through a common smooth current. We then use systematically the convergence properties of the sequence of p.s.h. functions produced by the action on a test form Φ with $dd^c\Phi \geq 0$. An analog Λ_Φ of the Abel-Radon transform is also introduced. It plays the role of *p.s.h. functions* on the space of currents.

In Section 3, we use the structural discs in order to define the wedge product $T \wedge S$ where T is a vertical positive closed current and S is a horizontal one, of the right bidegrees, such that the supports intersect on a compact set. Let φ be a p.s.h. function on a small neighbourhood W of $\text{supp}(T) \cap \text{supp}(S)$. We define

$$\langle T \wedge S, \varphi \rangle := \limsup_{\substack{T' \rightarrow T \\ S' \rightarrow S}} \langle T' \wedge S', \varphi \rangle \quad (1.1)$$

where T' are smooth vertical currents approaching T and S' are smooth horizontal currents approaching S with $\text{supp}(T') \cap \text{supp}(S') \subset W$. We use structural discs in order to show that the right hand side of (1.1) depends linearly on T , S and φ . This wedge product has interesting continuity properties.

We believe that the notion of structural discs will be useful in other situations. It is a notion of deformation of a positive closed current into another one in the same ‘‘homology’’ class. This can be also useful in the context of compact manifolds.

We apply the above theory of currents to study horizontal-like maps in \mathbb{C}^k , $k \geq 2$. A horizontal-like map has a (main) dynamical degree d ; this allows us to define an operator $\mathcal{L}_v := \frac{1}{d}f^*$ (resp. $\mathcal{L}_h := \frac{1}{d}f_*$) on vertical (resp. horizontal) currents. One of our main results is the following (Theorem 5.1). Let f_n be a sequence of invertible horizontal-like maps and let R_n be a sequence of normalized vertical positive closed forms. If R_n are uniformly bounded, then $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n)$ converge to a normalized vertical current T_+ which is independent of (R_n) . If the R_n 's are continuous, the convergence is stronger than weak convergence (see Remark 5.3). We use structural discs in the proof in order to deduce the convergence of currents from the convergence of subharmonic functions on structural discs.

When all the f_n 's are equal to f , we obtain a Green current satisfying $f^*(T_+) = dT_+$ (Corollary 6.1). We are then able to produce in this case a mixing invariant measure μ (Theorem 7.1). This is done by going to the product space and applying our formalism to the horizontal-like map $F : (x_1, x_2) \mapsto (f(x_1), f^{-1}(x_2))$ of dynamical degree d^2 . More precisely, if R (resp. S) is a normalized smooth vertical (resp. horizontal) positive closed form then the equilibrium measure is constructed as $\mu := \lim d^{-2n}(f^n)^*R \wedge (f^n)_*S$. Formally, if Δ is the diagonal of the product space, we obtain μ as the limit of

$$d^{-2n}((f^n)^*R \otimes (f^n)_*S) \wedge [\Delta] = d^{-2n}F^{n*}(R \otimes S) \wedge [\Delta].$$

This reduces the problem to the study of strong convergence of the vertical currents $d^{-2n}F^{n*}(R \otimes S)$ (see Remark 5.3). We finally show that $\mu = T_+ \wedge T_-$ (Theorem 7.10). Here, T_- is the Green current associated to f^{-1} .

Our proof of the mixing of the equilibrium measure uses also a new idea different from the approach in Bedford-Smillie [2] for Hénon maps or in [22] for regular polynomial automorphisms. The method is to use the maps of type $(x_1, x_2) \mapsto (f(x_1), f^{-1}(x_2))$ in order to reduce the problem to a linear one.

Using classical arguments [16, 25, 24, 2, 6], we show that μ has maximal entropy $\log d$ (Theorem 8.1).

2 Geometry of currents

In this Section we study the geometry of the cones of positive closed currents which are supported in vertical or horizontal subsets of a domain $D = M \times N$. We define structural discs, p.s.h. functions and the Kobayashi pseudo-distance on these cones. We refer to [13, 21, 3, 18] for the basics on the theory of currents. For the reader's convenience, we recall some properties, that we use in this article, of the slicing operation in the complex setting.

- **Slicing theory.** Let X, V be two complex manifolds of dimension $k + l$ and l respectively. Let $\Pi_V : X \rightarrow V$ be a surjective holomorphic map and \mathcal{R} be a

current on X of degree $2k + 2l - m$ and of dimension m with $m \geq 2l$. Assume that \mathcal{R} , $\partial\mathcal{R}$ and $\bar{\partial}\mathcal{R}$ are of order 0. One can define *the slice* $\langle \mathcal{R}, \Pi_V, \theta \rangle$ for almost every $\theta \in V$. This is a current of dimension $m - 2l$ on $\Pi_V^{-1}(\theta)$. One can of course consider it as a current on X . When \mathcal{R} is of bidimension (n, n) , $\langle \mathcal{R}, \Pi_V, \theta \rangle$ are of bidimension $(n - l, n - l)$. The slicing commutes with the operations ∂ and $\bar{\partial}$. In particular, if \mathcal{R} is closed then $\langle \mathcal{R}, \Pi_V, \theta \rangle$ is also closed.

Slicing is the generalization of restriction of forms to level sets of holomorphic maps. When \mathcal{R} is a continuous form, $\langle \mathcal{R}, \Pi_V, \theta \rangle$ is simply the restriction of \mathcal{R} to $\Pi_V^{-1}(\theta)$. When \mathcal{R} is the current of integration on an analytic subset Y of X , $\langle \mathcal{R}, \Pi_V, \theta \rangle$ is the current of integration on the analytic set $Y \cap \Pi_V^{-1}(\theta)$ for θ generic. If φ is a continuous form on X then $\langle \mathcal{R} \wedge \varphi, \Pi_V, \theta \rangle = \langle \mathcal{R}, \Pi_V, \theta \rangle \wedge \varphi$.

Let y denote the coordinates in a chart of V and λ_V the standard volume form. Let $\psi(y)$ be a positive smooth function with compact support such that $\int \psi \lambda_V = 1$. Define $\psi_\epsilon(y) := \epsilon^{-2l} \psi(\epsilon^{-1}y)$ and $\psi_{\theta, \epsilon}(y) := \psi_\epsilon(y - \theta)$ (the measures $\psi_{\theta, \epsilon} \lambda_V$ approximate the Dirac mass at θ). Then, for every smooth test form Ψ of the right degree with compact support in X one has

$$\langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi) = \lim_{\epsilon \rightarrow 0} \langle \mathcal{R} \wedge \Pi_V^*(\psi_{\theta, \epsilon} \lambda_V), \Psi \rangle$$

when $\langle \mathcal{R}, \Pi_V, \theta \rangle$ exists. This property holds for all choice of the function ψ and for Ψ such that Π_V is proper on $\text{supp}(\Psi) \cap \text{supp}(\mathcal{R})$. Conversely, when the previous limit exists and is independent of ψ , it defines $\langle \mathcal{R}, \Pi_V, \theta \rangle$ and one says that $\langle \mathcal{R}, \Pi_V, \theta \rangle$ is *well defined*. We have the following formula for every continuous form Ω of maximal degree with compact support in V :

$$\int_V \langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi) \Omega(\theta) = \langle \mathcal{R} \wedge \Pi_V^*(\Omega), \Psi \rangle. \quad (2.1)$$

We will show that in the situation we consider, slices are always well defined.

• **Vertical and horizontal currents.** Let $M \subset \mathbb{C}^p$ and $N \subset \mathbb{C}^{k-p}$ be two bounded convex open sets (see Remark 2.6). Consider the domain $D := M \times N$ in \mathbb{C}^k . We call *vertical* (resp. *horizontal*) *boundary* of D the set $\partial_v D := \partial M \times N$ (resp. $\partial_h D := M \times \partial N$). A subset E of D is called *vertical* (resp. *horizontal*) if \bar{E} does not intersect $\bar{\partial}_v D$ (resp. $\bar{\partial}_h D$). Let π_1 and π_2 denote the canonical projections of D on M and N . Then, E is vertical (resp. horizontal) if and only if $\pi_1(E) \Subset M$ (resp. $\pi_2(E) \Subset N$). A current on D is *vertical* (resp. *horizontal*) if its support is vertical (resp. horizontal).

Let $\mathcal{C}_v(D)$ (resp. $\mathcal{C}_h(D)$) denote the cone of **positive closed** vertical (resp. horizontal) currents of bidegree (p, p) (resp. $(k - p, k - p)$) on D . Consider a current R in $\mathcal{C}_v(D)$. Since π_2 is proper on $\text{supp}(R)$, $(\pi_2)_*(R)$ is a positive closed current of bidegree $(0, 0)$ on N . Hence, $(\pi_2)_*(R)$ is given by a constant function c on N . Formula (2.1) implies that the mass of the slice measure $\langle R, \pi_2, w \rangle$ is independent of w and is equal to c . We will show in Theorem 2.1 that in

this situation, the slice measure is defined for every $w \in N$ (see also Theorem 3.1). We say that c is the *slice mass* of R and we denote it by $\|R\|_v$. For every smooth probability measure Ω with compact support in N , we have $\|R\|_v := \langle R, (\pi_2)^*(\Omega) \rangle$. When $\|R\|_v = 1$ we say that R is *normalized*. Let $\mathcal{C}_v^1(D)$ denote the set of such currents. The slice mass $\|\cdot\|_h$ and the convex $\mathcal{C}_h^1(D)$ for horizontal currents are similarly defined.

• **Structural varieties and p.s.h. functions.** In order to use the complex structure of D , we introduce the notion of structural varieties in $\mathcal{C}_v^1(D)$. Let V be a connected complex manifold. Let \mathcal{R} be a positive closed current of bidegree (p, p) in $V \times D$. Let $\Pi_V : V \times D \rightarrow V$, $\Pi_D : V \times D \rightarrow D$, $\Pi_M : V \times D \rightarrow M$ and $\Pi_N : V \times D \rightarrow N$ be the canonical projections. We assume that for every compact set $K \subset V$ the projection of $\text{supp}(\mathcal{R}) \cap \Pi_V^{-1}(K)$ on M is relatively compact in M . In particular $\text{supp}(\mathcal{R}) \cap \Pi_V^{-1}(\theta)$ is a vertical set of $\{\theta\} \times D$ for every $\theta \in V$.

Theorem 2.1. *For every $\theta \in V$ the slice $\langle \mathcal{R}, \Pi_V, \theta \rangle$ exists and is a vertical positive closed current on $\{\theta\} \times D$. Moreover its slice mass is independent of θ . If Ψ is a real continuous $(k-p, k-p)$ -form on $V \times D$ such that $\text{dd}^c \Psi \geq 0$ and $\Pi_N(\text{supp}(\Psi)) \Subset N$ then $\langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi)$ defines a p.s.h. function on V . If $\text{dd}^c \Psi = 0$ then $\langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi)$ is pluriharmonic.*

Proof. The problem is local, so we can assume that V is a ball. Consider the current $\mathcal{R}' := \mathcal{R} \wedge \Psi$ of bidegree (k, k) on $V \times D$. It satisfies $\text{dd}^c \mathcal{R}' \geq 0$. Observe that for every $\theta \in V$, $\text{supp}(\mathcal{R}') \cap \Pi_V^{-1}(\theta)$ is compact in $\{\theta\} \times D$ and Π_V is proper on the support of \mathcal{R}' . Then $(\Pi_V)_*(\mathcal{R}')$ is well defined. It is a current of bidegree $(0, 0)$ on V which satisfies $\text{dd}^c(\Pi_V)_*(\mathcal{R}') \geq 0$. Therefore, it is defined by a p.s.h. function φ . It follows that if ψ , $\psi_{\theta, \epsilon}$ and λ_V are as above then $\int \varphi \psi_{\theta, \epsilon} \lambda_V$ converges to $\varphi(\theta)$.

The last integral is equal to $\langle \mathcal{R} \wedge \Pi_V^*(\psi_{\theta, \epsilon} \lambda_V), \Psi \rangle$. Hence $\langle \mathcal{R} \wedge \Pi_V^*(\psi_{\theta, \epsilon} \lambda_V), \Psi \rangle$ converge to $\varphi(\theta)$ which does not depend on ψ . This holds also for every smooth test form Ψ' with compact support in $V \times D$. Indeed, we have the following lemma.

Lemma 2.2. *Smooth $(k-p, k-p)$ -forms with compact support in $V \times D$ belong to the space generated by the forms Ψ satisfying the hypotheses of Theorem 2.1, i.e. $\Pi_N(\text{supp}(\Psi)) \Subset N$ and $\text{dd}^c \Psi \geq 0$.*

Proof. Let Ψ' be a real smooth $(k-p, k-p)$ -form with compact support in $V \times D$. Let Ω be a positive form of maximal degree on N , with compact support and strictly positive on $\Pi_N(\text{supp}(\Psi'))$. If ρ is a smooth strictly p.s.h. function on $V \times D$ then $\Psi_0 := \rho \Pi_D^*(\pi_2^*(\Omega))$ is a smooth form satisfying the hypothesis of Theorem 2.1. If $\pi_{2, \epsilon}$ is a small perturbation of π_2 then $\Psi_\epsilon := \rho \Pi_D^*(\pi_{2, \epsilon}^*(\Omega))$ satisfies the same properties. Taking a linear combination of such forms we obtain a form Ψ such that $\text{dd}^c \Psi$ is strictly positive on $\text{supp}(\Psi')$. Then we can write $\Psi' = (A\Psi + \Psi') - A\Psi$ with $A > 0$ large enough. The forms $A\Psi + \Psi'$ and $A\Psi$

satisfy the hypotheses of Theorem 2.1, in particular, we have $\text{dd}^c(A\Psi + \Psi') \geq 0$ and $\text{dd}^c(A\Psi) \geq 0$. \square

Hence $\langle \mathcal{R}, \Pi_V, \theta \rangle$ is well defined and $\langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi) = \varphi(\theta)$ is a p.s.h. function on θ . When $\text{dd}^c\Psi = 0$ the function $\langle \mathcal{R}, \Pi_V, \theta \rangle(-\Psi)$ is also p.s.h. Hence $\langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi)$ is pluriharmonic.

Let Ω be as above. Consider $\Psi := \Pi_D^*(\pi_2^*(\Omega))$. In this case, since Ψ is closed, φ is also closed. It follows that φ is a constant function. By definition $\varphi(\theta) = \langle \mathcal{R}, \Pi_V, \theta \rangle(\Psi)$ is equal to the slice mass of $\langle \mathcal{R}, \Pi_V, \theta \rangle$. Therefore, the slice mass is independent of θ . \square

Remark 2.3. One can identify $R_\theta = \langle \mathcal{R}, \Pi_V, \theta \rangle$ with a current in $\mathcal{C}_v(D)$. Theorem 2.1 implies that the family (R_θ) is continuous for the *plurifine topology* on V , i.e. the coarsest topology for which p.s.h. functions are continuous. Let Φ be a real horizontal current of bidegree $(k-p, k-p)$, of finite mass on D such that $\text{dd}^c\Phi \geq 0$. If \mathcal{R} or Φ is a continuous form then $\langle \mathcal{R}, \Pi_V, \theta \rangle(\Phi)$ defines a p.s.h. function on V . Indeed, we can apply Theorem 2.1 to $\Psi := \Pi_D^*(\Phi)$.

Definition 2.4. Theorem 2.1 allows us to define a map $\tau : V \rightarrow \mathcal{C}_v(D)$

$$\tau(\theta) := R_\theta = \langle \mathcal{R}, \Pi_V, \theta \rangle.$$

If we multiply \mathcal{R} by a suitable constant, all the values of τ are normalized. We say that τ defines a *structural variety* in $\mathcal{C}_v^1(D)$.

A function $\Lambda : \mathcal{C}_v^1(D) \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *p.s.h.* if it is not identically equal to $-\infty$ and for every structural variety $\tau : V \rightarrow \mathcal{C}_v^1(D)$ the function $\Lambda \circ \tau$ is either p.s.h. or identically $-\infty$ on V . If Λ and $-\Lambda$ are p.s.h. we say that Λ is *pluriharmonic*.

Let Φ be a real continuous horizontal $(k-p, k-p)$ -form on D . Define the linear map $\Lambda_\Phi : \mathcal{C}_v(D) \rightarrow \mathbb{R}$ by $\Lambda_\Phi(R) := \langle R, \Phi \rangle$. Such an operator is a version of the Abel-Radon transform in complex analysis. Observe that real smooth $(k-p, k-p)$ -forms with compact support in D belong to the space generated by the smooth horizontal forms $\Phi \geq 0$ with $\text{dd}^c\Phi \geq 0$ (see Lemma 2.2). Hence, the maps Λ_Φ with $\Phi \geq 0$ and $\text{dd}^c\Phi \geq 0$, separate currents in $\mathcal{C}_v(D)$. Theorem 2.1 and Remark 2.3 show that Λ_Φ is p.s.h. on $\mathcal{C}_v^1(D)$ when $\text{dd}^c\Phi \geq 0$.

We can summarize our construction of the function $\theta \mapsto \langle \tau(\theta), \Phi \rangle$ by the following diagram:

$$V \xrightarrow{\tau} \mathcal{C}_v^1(D) \xrightarrow{\Lambda_\Phi} \mathbb{R} \tag{2.2}$$

• **Some structural discs.** Given a current $R \in \mathcal{C}_v^1(D)$, we construct some special structural discs passing through R , that we will use in the next sections. For these discs, the map τ is continuous with respect to the weak topology on

currents. In order to construct the current \mathcal{R} , we consider the images of R under holomorphic families of maps.

Let $M' \Subset M$ and $N'' \Subset N$ be open sets. Define $D' := M' \times N$ and $D'' := M \times N''$. In order to simplify the notations, assume that 0 belongs to $(M \setminus \overline{M'}) \times (N \setminus \overline{N''})$. Fix a domain $D^* = M^* \times N^* \Subset D$ with $M \setminus M^*$ and $N \setminus N^*$ small enough, $M' \Subset M^*$, $N' \Subset N^*$. Choose a small simply connected neighbourhood V of $[0, 1]$ in \mathbb{C} . Finally, choose a small open neighbourhood $U \Subset D^*$ of 0 in $\mathbb{C}^p \times \mathbb{C}^{k-p}$ and a smooth positive function ρ with support in U such that $\int \rho(a, b) \lambda(a, b) = 1$. Here, λ denotes the standard volume form on \mathbb{C}^k .

For $\theta \in V$ and $(a, b) \in U$, define the affine map $h_{a,b,\theta} : \mathbb{C}^p \times \mathbb{C}^{k-p} \rightarrow \mathbb{C}^p \times \mathbb{C}^{k-p}$ by

$$h_{a,b,\theta}(z, w) := (\theta z + (1 - \theta)a, w + (\theta - 1)b).$$

These maps are small perturbations of the map $(z, w) \mapsto (\theta z, w)$. When $\theta = 1$ we obtain the identity map and when $\theta = 0$ we obtain an affine map onto the subspace $\{z = a\}$. Let R be a current in $\mathcal{C}_v^1(D')$. We will show that the currents $R_{a,b,\theta} := (h_{a,b,\theta})_*(R)$ define a structural disc in $\mathcal{C}_v^1(D^*)$, i.e. they are slices of a current $\mathcal{R}_{a,b}$ in $V \times D^*$.

Observe that $R_{a,b,\theta}$ is well defined, since $h_{a,b,\theta} : \text{supp}(R) \cap h_{a,b,\theta}^{-1}(D^*) \rightarrow D^*$ is proper. This last property follows from the fact that M is convex and $h_{a,b,\theta}$ is close to the map $(z, w) \mapsto (\theta z, w)$. Moreover, $R_{a,b,\theta}$ is well defined on some open set D_θ which converges to D when $\theta \rightarrow 1$. The dependence of currents $R_{0,0,\theta}$ on θ has been used by Dujardin in order to study Hénon-like maps [11] (see also [5]).

Define the meromorphic map $H_{a,b} : V \times D^* \rightarrow \mathbb{C}^p \times N$ by

$$H_{a,b}(\theta, z, w) := h_{a,b,\theta}^{-1}(z, w) = \left(\frac{z + (\theta - 1)a}{\theta}, w - (\theta - 1)b \right).$$

The current $\mathcal{R}_{a,b} := H_{a,b}^*(R)$, which is of bidimension $(k - p + 1, k - p + 1)$, is well defined out of the pole set $\{\theta = 0\}$ of $H_{a,b}$. Since $\text{supp}(\mathcal{R}_{a,b}) \subset H_{a,b}^{-1}(\text{supp}(R))$, then when θ approaches 0 , $\text{supp}(\mathcal{R}_{a,b})$ clusters only on the set $\{z = a\}$. So, this current is well defined out of $\{\theta = 0\} \cap \{z = a\}$. The dimension of $\{\theta = 0\} \cap \{z = a\}$, which is equal to $k - p$, is smaller than the dimension of $\mathcal{R}_{a,b}$. Hence, one can extend $\mathcal{R}_{a,b}$ across $\{\theta = 0\} \cap \{z = a\}$ with no mass on this set [17].

Since M is convex and since $h_{a,b,\theta}$ is close to the map $(z, w) \mapsto (\theta z, w)$, $\text{supp}(\mathcal{R}_{a,b}) \cap \Pi_V^{-1}(\theta)$, which is isomorphic to $\text{supp}(R_{a,b,\theta})$, is a vertical set of $\{\theta\} \times D^*$ for every $\theta \in V$. Hence, the slice currents $\langle \mathcal{R}_{a,b}, \Pi_V, \theta \rangle$ define a structural disc in $\mathcal{C}_v^1(D^*)$. By Theorem 2.1, these slices exist for every $\theta \in V$ and are equal to $R_{a,b,\theta}$ (we identify $\{\theta\} \times D$ with D). The currents $R_{a,b,\theta}$ depend continuously on θ for the weak topology on currents. This is clear for $\theta \neq 0$, and as we have seen, the limit at $\theta = 0$ is $[z = a]$ (see also Lemmas 2.5 and 2.7 below). Recall that $[z = a]$ denotes the current of integration on the analytic set $\{z = a\}$.

We have $R_{a,b,1} = R$ and $R_{a,b,0} = [z = a]$. Hence, $R_{a,b,0}$ is independent of R . In other words, when R varies we obtain a family of structural discs passing through

the same point $[z = a]$ in $\mathcal{C}_v^1(D^*)$.

We introduce a smoothing. Define

$$\mathcal{R} := \int \mathcal{R}_{a,b} \rho(a, b) \lambda(a, b).$$

The current \mathcal{R} satisfies the hypothesis of Theorem 2.1 for D^* . Hence, the slice currents $R_\theta := \langle \mathcal{R}, \Pi, \theta \rangle$ define a structural disc in $\mathcal{C}_v^1(D^*)$. These slices are well defined for every $\theta \in V$ and

$$R_\theta = \int R_{a,b,\theta} \rho(a, b) \lambda(a, b). \quad (2.3)$$

Observe that R_θ depends continuously on θ for the weak topology. We have $R_1 = R$ and

$$R_0 = \int [z = a] \rho(a, b) \lambda(a, b) = \pi_1^*(\pi_1)_*(\rho\lambda).$$

The last current is independent of R . When R varies, we obtain again a family of structural discs which pass through the same point $\pi_1^*(\pi_1)_*(\rho\lambda)$ in $\mathcal{C}_v^1(D^*)$.

In the following two lemmas, we study the continuity of R_θ near 0 and near 1. We will use these facts in our convergence theorems. Lemma 2.5 shows that every current in $\mathcal{C}_v^1(D')$ can be joined to a fixed vertical current R_0 through smooth ones.

Lemma 2.5. *Let $R \in \mathcal{C}_v^1(D')$. Then, for $\theta \in V \setminus \{1\}$, R_θ is a smooth form on D^* . For $\theta \in V \setminus \{0, 1\}$, R_θ depends continuously on (R, θ) for the \mathcal{C}^∞ topology. Moreover, there exist $r > 0$ and $c > 0$ independent of R such that if $|\theta| \leq r$*

$$\|R_\theta - R_0\|_{\mathcal{L}^\infty(D^*)} \leq c|\theta|$$

where the \mathcal{L}^∞ norm on forms is the sum of \mathcal{L}^∞ norms of coefficients.

Proof. The smoothness of R_θ for $\theta \neq 1$, and their dependence of (R, θ) are checked using a classical change of variables in (2.3) as follows. Let \tilde{R} , $\tilde{R}_{a,b,\theta}$ and \tilde{R}_θ be the coefficients of $dz_I \wedge d\bar{z}_J \wedge dw_K \wedge d\bar{w}_L$ in R , $R_{a,b,\theta}$ and R_θ respectively, for some multi-indices I, J, K, L . Let

$$(A, B) := h_{a,b,\theta}^{-1}(z, w) = \left(\frac{z + (\theta - 1)a}{\theta}, w - (\theta - 1)b \right)$$

be the new variables. Since $R_{a,b,\theta} = (h_{a,b,\theta}^{-1})^* R$ we have

$$\tilde{R}_{a,b,\theta}(z, w) = \theta^{-|I|\bar{\theta}^{-|J|}} \tilde{R}(A, B)$$

and from (2.3)

$$\tilde{R}_\theta(z, w) = \theta^{-|I|\bar{\theta}^{-|J|}} \int_{A,B} \tilde{R}(A, B) \cdot (\rho\lambda) \left(\frac{\theta A - z}{\theta - 1}, \frac{w - B}{\theta - 1} \right).$$

The smoothness of R_θ for $\theta \in V \setminus \{0, 1\}$ and the dependence of (R, θ) are clear.

Let Φ be a smooth $(k-p, k-p)$ -form with compact support in D^* . By duality, the inequality that we have to prove is equivalent to

$$|\langle R_\theta - R_0, \Phi \rangle| \leq c|\theta| \|\Phi\|_{\mathcal{L}^1}.$$

From (2.3), we get

$$\begin{aligned} \langle R_\theta, \Phi \rangle &= \int \langle R_{a,b,\theta}, \Phi \rangle \rho(a,b) \lambda(a,b) = \int \langle R, h_{a,b,\theta}^*(\Phi) \rangle \rho(a,b) \lambda(a,b) \\ &= \left\langle R, \int h_{a,b,\theta}^*(\Phi) \rho(a,b) \lambda(a,b) \right\rangle =: \langle R, \Phi_\theta \rangle. \end{aligned}$$

This also holds for $\theta = 0$ by continuity. The forms Φ_θ are obtained by convolution. They are smooth and uniformly bounded by $c\|\Phi\|_{\mathcal{L}^1}$. Using the change of variables $(a,b) \mapsto (A,B) := h_{a,b,\theta}(z,w)$, we get

$$\|\Phi_\theta - \Phi_0\|_{\mathcal{L}^\infty} \leq c|\theta| \|\Phi\|_{\mathcal{L}^1}$$

for θ small. Lemma 2.5 follows. \square

Remark 2.6. When M is not convex and $V' \subset V$ is a small neighbourhood of 1, then $(R_\theta)_{\theta \in V'}$ defines also a structural disc in $\mathcal{C}_v^1(D'')$. The first part of Lemma 2.5 holds in this case.

Lemma 2.7. *Let $R \in \mathcal{C}_v^1(D')$ be a continuous form. Let $m(R, \epsilon)$ denote the modulus of continuity of R . Then, there exist $r > 0$, $c > 0$, $A > 0$ independent of R such that for $|\theta - 1| \leq r$*

$$\|R_\theta - R\|_{\mathcal{L}^\infty(D^*)} \leq c(\|R\|_{\mathcal{L}^\infty(D)}|\theta - 1| + m(R, A|\theta - 1|)).$$

Proof. Let W denote the disc $\{|\theta - 1| \leq r\}$ with $r > 0$ small enough, so we are away of $\{\theta = 0\}$. Then, there exists $A > 0$ such that $\|h_{a,b,\theta}^{-1}(z,w) - (z,w)\|_{\mathcal{C}^1} \leq A|\theta - 1|$ when $(z,w,a,b,\theta) \in D \times U \times W$. Hence, there exists $c > 0$ such that for $(a,b) \in U$

$$\|R_{a,b,\theta} - R\|_{\mathcal{L}^\infty(D^*)} \leq c(\|R\|_{\mathcal{L}^\infty(D)}|\theta - 1| + m(R, A|\theta - 1|)).$$

We can also prove this inequality using the description of $\tilde{R}_{a,b,\theta}$ as in Lemma 2.5. Finally, we obtain the desired inequality by integration using (2.3). \square

• **Kobayashi pseudo-distance.** Let $\mathcal{C}_v^1(\overline{D})$ be the set of currents in $\mathcal{C}_v^1(D)$ which can be extended to a current in $\mathcal{C}_v^1(M \times N')$ for some neighbourhood N' of \overline{N} . We introduce the Kobayashi pseudo-distance ρ_v on $\mathcal{C}_v^1(\overline{D})$. Let R and S be two currents in $\mathcal{C}_v^1(\overline{D})$. Let Δ be the unit disc and ρ_0 denote the hyperbolic distance on Δ . Consider chains of continuous structural discs $\tau_i : \Delta \rightarrow \mathcal{C}_v^1(\overline{D})$

which connect R and S . More precisely, suppose $\theta_i, \theta'_i \in \Delta$ such that $\tau_1(\theta_1) = R$, $\tau_i(\theta'_i) = \tau_{i+1}(\theta_{i+1})$ and $\tau_n(\theta'_n) = S$. Define

$$\rho_v(R, S) := \inf \sum_{i=1}^n \rho_0(\theta_i, \theta'_i) \quad (2.4)$$

where the infimum is taken over all the n, τ_i, θ_i and θ'_i . We have seen that R and S can be connected by a chain of two continuous structural discs. Hence $\rho_v(R, S)$ is finite. It is easy to check that ρ_v satisfies the triangle inequality.

Proposition 2.8. *The pseudo-distance ρ_v is not a distance. If real continuous horizontal dd^c -closed forms on D separate R and S then $\rho_v(R, S) > 0$.*

Proof. We have to construct two different currents R and S such that $\rho_v(R, S) = 0$. We can replace N by a ball containing N and M by a polydisc contained in M . So, we can assume that N is the unit ball and M is the unit polydisc. It is sufficient to consider the case where $p = 1$ and M is the unit disc Δ . We obtain the general case by taking the product of M and D by Δ^{p-1} .

Let ν_r be the Lebesgue measure on the circle $\{|z| = r\}$ normalized by $\|\nu_r\| = 1$. Consider $R := \pi_1^*(\nu_0) = [z = 0]$ and $S := \pi_1^*(\nu_{1/2})$. Let \mathcal{R} be the positive closed current of bidegree $(1, 1)$ on $\Delta \times (\Delta \times \mathbb{C}^{k-1})$ given by $\mathcal{R} := \text{dd}^c \mathcal{U}$ where $\mathcal{U}(\theta, z, w) := \max\{\log |z|, \frac{1}{A} \log |\theta|\}$ and $A > 1$. This current has support in $\{|z|^A = |\theta|\}$. Hence, if Π_Δ is the projection on the first factor Δ , the slices $R_\theta := \langle \mathcal{R}, \Pi_\Delta, \theta \rangle = \text{dd}^c \mathcal{U}(\theta, \cdot, \cdot)$ define a continuous structural disc in $\mathcal{E}_v^1(\overline{D})$. Moreover, we have $R_\theta = \pi_1^*(\nu_r)$ where $r^A = |\theta|$. In particular, we have $R_0 = R$ and $R_\theta = S$ for $\theta := 2^{-A}$. It follows that $\rho_v(R, S) \leq \rho_0(0, \theta)$. When $A \rightarrow \infty$, we have $\theta \rightarrow 0$ and then $\rho_0(0, \theta) \rightarrow 0$. Therefore, $\rho_v(R, S) = 0$.

Now assume that R, S satisfy the hypothesis of Proposition 2.8 and consider structural discs τ_i as above. Let Φ be a real continuous horizontal form such that $\text{dd}^c \Phi = 0$ and $\langle R, \Phi \rangle \neq \langle S, \Phi \rangle$. Using a regularization we can assume that Φ is smooth and is defined on a neighbourhood of D . Hence there exists a smooth current Φ' in $\mathcal{E}_h(D)$ such that $-\Phi' \leq \Phi \leq \Phi'$. Using coordinate changes on Δ , one can also assume that $\theta_i = 0$. If $\rho_0(0, \theta'_i) > 1$ then the right hand side of (2.4) is larger than 1. We have only to consider the case where $\rho_0(0, \theta'_i) \leq 1$ for every i .

Define $\psi_i := \Lambda_\Phi \circ \tau_i$. Theorem 2.1 implies that these functions are harmonic. Lemma 3.2 below implies that they are uniformly bounded by $\pm \|\Phi'\|_h$. Hence by Harnack's inequality $|\psi_i(\theta'_i) - \psi_i(0)| \leq c \rho_0(0, \theta'_i)$, $c > 0$. On the other hand, we have $\psi_1(0) = \langle R, \Phi \rangle$, $\psi_i(\theta'_i) = \psi_{i+1}(0)$ and $\psi_n(\theta'_n) = \langle S, \Phi \rangle$. We then deduce that the right hand side of (2.4) is bounded from below by $c^{-1} |\psi_1(0) - \psi_n(\theta'_n)| = c^{-1} |\langle R, \Phi \rangle - \langle S, \Phi \rangle|$. Hence $\rho_v(R, S) > 0$. \square

Proposition 2.9. *The space $\mathcal{E}_v^1(\overline{D})$ is hyperbolic in the sense of Brody. More precisely, there exists no non-constant structural line $\tau : \mathbb{C} \rightarrow \mathcal{E}_v^1(\overline{D})$.*

Proof. Consider a horizontal positive test forms Φ such that $\text{dd}^c\Phi \geq 0$ and assume $\Phi \leq \Phi'$ with Φ' a smooth form in $\mathcal{C}_h^1(D)$. Then, $\Lambda_\Phi \circ \tau$ is constant since, by Theorem 2.1 and Lemma 3.2 below, it is a subharmonic function on \mathbb{C} , bounded from above by $\|\Phi'\|_h$. Proposition 2.9 follows. \square

• **Case of bidegree (1,1).** Assume that $p = 1$. We will construct an example of non-continuous structural discs in $\mathcal{C}_v^1(D)$. Let \mathcal{R} be a positive closed current of bidegree (1,1) on $\Delta \times D$ satisfying the hypotheses of Theorem 2.1. We can write $\mathcal{R} = \text{dd}^c\mathcal{U}$ where \mathcal{U} is a p.s.h. function on $\Delta \times D$ which is pluriharmonic near $\Delta \times \partial_v D$. The slice $R_\theta := \langle \mathcal{R}, \Pi_\Delta, \theta \rangle$ is equal to $\text{dd}^c U_\theta$ where $U_\theta := \mathcal{U}|_{\{\theta\} \times D}$. The geometry of the support of \mathcal{R} insures that U_θ is not identically equal to $-\infty$. Hence slice currents exist for every θ .

Let v be a bounded subharmonic function on Δ . In order to simplify the notation assume that 0 belongs to D . Consider the case where

$$\mathcal{U}(\theta, z, w) = \max\{v(\theta) - A, \log|z|\}.$$

The constant A is chosen large enough so that $\mathcal{U} = \log|z|$ near $\Delta \times \partial_v D$. Then \mathcal{R} vanishes near $\Delta \times \partial_v D$. One easily check that $R_\theta = \pi_1^*(\nu_r)$ where $r := \exp(v(\theta) - A)$. Hence (R_θ) is continuous with respect to θ , if and only if v is continuous.

The following proposition gives the converse of Theorem 2.1 in the bidegree (1,1) case.

Proposition 2.10. *Let R_θ be a family of currents of bidegree (1,1) in $\mathcal{C}_v^1(\{\theta\} \times D)$, $\theta \in \Delta$. Assume that the projection of $\cup \text{supp}(R_\theta)$ on M is relatively compact in M . Assume also that for every real continuous $(k-1, k-1)$ -form Ψ on $\Delta \times D$ such that $\text{dd}^c\Psi \geq 0$ and $\Pi_N(\text{supp}(\Psi)) \Subset N$, the function $\theta \mapsto \langle R_\theta, \Psi \rangle$ is subharmonic on Δ . Then $\theta \mapsto R_\theta$ defines a structural disc in $\mathcal{C}_v^1(D)$.*

Proof. We want to construct a potential \mathcal{U} of a current \mathcal{R} with given slices R_θ . We will obtain \mathcal{U} as a decreasing limit of some p.s.h. functions $\mathcal{U}_{\epsilon, \delta}$.

Let λ denote the canonical volume form on \mathbb{C}^{k-1} and ψ be a positive radial function with compact support in \mathbb{C}^{k-1} such that $\int \psi \lambda = 1$. Define continuous functions $\psi_\epsilon(w) := \epsilon^{2-2k} \psi(\epsilon^{-1}w)$, $\epsilon > 0$, and $\log_\delta|z| := \max\{\log|z|, \log\delta\}$, $\delta > 0$. Define also

$$\Phi_{z_0, w_0}^{\epsilon, \delta}(z, w) := \log_\delta|z - z_0| \psi_\epsilon(w - w_0) \lambda(w - w_0)$$

which is a regularization of the current $\log|z - z_0| \cdot [w = w_0]$, and

$$\mathcal{U}_{\epsilon, \delta}(\theta_0, z_0, w_0) := \langle R_{\theta_0}, \Phi_{z_0, w_0}^{\epsilon, \delta} \rangle.$$

Here we identify R_{θ_0} to a current on D .

We first prove that for every domain $N^* \Subset N$, the function $\mathcal{U}_{\epsilon, \delta}$ is p.s.h. on $\Delta \times \mathbb{C} \times N^*$ for ϵ small enough. Assume that $z_0 = g(\theta_0)$ and $w_0 = h(\theta_0)$ where (g, h) is a holomorphic map from Δ to $\mathbb{C} \times N^*$. It is enough to prove that

$\mathcal{U}_{\epsilon,\delta}(\theta_0, g(\theta_0), h(\theta_0))$ is a subharmonic function with respect to θ_0 . This follows from the hypothesis. Indeed, in this case $\Phi_{z_0, w_0}^{\epsilon,\delta}(z, w)$ is equal to a continuous form $\Psi^{\epsilon,\delta}(\theta_0, z, w)$ which satisfies $\text{dd}^c \Psi^{\epsilon,\delta} \geq 0$ on $\Delta \times D$ and if ϵ is small enough $\Pi_N(\text{supp}(\Psi^{\epsilon,\delta}))$ is compact in N .

Now let ϵ decrease to 0. Observe that $(\pi_2)_*(\log_\delta |z - z_0| \cdot R_{\theta_0})$ is defined by a p.s.h. function $\varphi_{\theta_0}^\delta$ on N and

$$\mathcal{U}_{\epsilon,\delta}(\theta_0, z_0, w_0) = \int \varphi_{\theta_0}^\delta(w) \psi_\epsilon(w - w_0) \lambda(w - w_0).$$

Since ψ is radial and $\varphi_{\theta_0}^\delta$ is p.s.h., the submean inequality implies that $\mathcal{U}_{\epsilon,\delta}$ decreases to a p.s.h. function \mathcal{U}_δ on $\Delta \times \mathbb{C} \times N$. The definition of $\Phi_{z_0, w_0}^{\epsilon,\delta}$ and slicing theory imply that

$$\mathcal{U}_\delta(\theta_0, z_0, w_0) = \langle \langle R_{\theta_0}, \pi_2, w_0 \rangle, \log_\delta |z - z_0| \rangle.$$

Recall that $\langle R_\theta, \pi_2, w_0 \rangle$ is a probability measure. When δ decreases to 0, \mathcal{U}_δ decreases to the p.s.h. function

$$\mathcal{U}(\theta_0, z_0, w_0) := \langle \langle R_{\theta_0}, \pi_2, w_0 \rangle, \log |z - z_0| \rangle.$$

The last formula says that for every fixed θ_0 , $\mathcal{U}(\theta_0, \cdot, \cdot)$ defines a potential of R_{θ_0} . In particular, the restriction of \mathcal{U} to $\{\theta_0\} \times \mathbb{C} \times N$ is pluriharmonic outside the support of R_{θ_0} . Recall that the projection of $\cup \text{supp}(R_\theta)$ on M is relatively compact in M . On the other hand, for $|z_0|$ large enough, $\log_\delta |z - z_0|$ is pluriharmonic for $z \in M$. Then, it is easy to check that $\mathcal{U}_{\epsilon,\delta}$ and \mathcal{U} are pluriharmonic for $|z_0|$ large enough. Now, by Hartogs extension theorem, \mathcal{U} is pluriharmonic near $\Delta \times \partial_v D$ and then $\mathcal{R} := \text{dd}^c \mathcal{U}$ vanishes near $\Delta \times \partial_v D$. It follows that the slices of \mathcal{R} , which are equal to R_θ , define a structural disc in $\mathcal{C}_v^1(D)$. \square

3 Intersection of currents

In this section, we define the intersection (wedge product) $R \wedge S$ of a vertical positive closed current $R \in \mathcal{C}_v(D)$ and a horizontal positive closed current $S \in \mathcal{C}_h(D)$. When one of these currents, for example R , has bidegree $(1, 1)$, using a regularization, the reader can verify that our definition coincides with the classical definition $R \wedge S := \text{dd}^c(uS)$ where u is a potential of R . The current uS is well defined since, by Oka's inequality [15, Prop. 3.1], u is integrable with respect to the trace measure of S . This case is very simple since the mass of uS on a compact set can be estimated using Stokes' theorem and the geometry of the supports of R and S .

Theorem 3.1. *Let R be a current in $\mathcal{C}_v(D)$ and S be a current in $\mathcal{C}_h(D)$. Then $R \wedge S$ is defined such that for every p.s.h. function φ on D*

$$\langle R \wedge S, \varphi \rangle = \limsup_{\substack{R' \rightarrow R \\ S' \rightarrow S}} \langle R' \wedge S', \varphi \rangle$$

where $R' \in \mathcal{C}_v(D)$ and $S' \in \mathcal{C}_v(D)$ are smooth with supports converging in the Hausdorff sense to those of R and S . The value of $\langle R \wedge S, \varphi \rangle$ depends linearly on R , S and φ . The wedge product $R \wedge S$ is a positive measure of mass $\|R\|_v \|S\|_h$ and supported in $\text{supp}(R) \cap \text{supp}(S)$.

In the previous theorem, the convexity of D is not necessary and we can take R' and S' such that $\text{supp}(R') \cap \text{supp}(S')$ is contained in a fixed neighbourhood $W \Subset D$ of $\text{supp}(R) \cap \text{supp}(S)$ (see Propositions 3.4 and 3.5).

Choose M' and N'' such that $R \in \mathcal{C}_v(D')$ and $S \in \mathcal{C}_h(D'')$. We can assume that R and S are normalized. We will construct explicitly the probability measure $R \wedge S$. We first prove the following lemma.

Lemma 3.2. *Assume that R is a continuous form. Then, $R \wedge S$ is a probability measure.*

Proof. By regularization of currents, we can assume that S is smooth. Let \mathcal{R} be the structural disc associated to R which was constructed in Section 2. The current $\mathcal{R}' := \mathcal{R} \wedge \Pi_D^*(S)$ is positive closed and of bidimension $(1, 1)$. Moreover, the restriction of Π_V to $\text{supp}(\mathcal{R}')$ is proper. Hence, $(\Pi_V)_*(\mathcal{R}')$, which is positive closed and of bidimension $(1, 1)$, is defined by a constant function c on V . It follows that $\|R_\theta \wedge S\| = c$ for almost every θ . By Lemmas 2.5 and 2.7, R_θ depends continuously on θ . Then $\|R_\theta \wedge S\| = c$ for every θ . From the explicit form of R_0 , we get $c = \|R_0 \wedge S\| = \|S\|_h = 1$. For $\theta = 1$, we get $\|R \wedge S\| = c = 1$. \square

Proposition 3.3. *Let K be a compact subset of M . Then, the set of currents $R \in \mathcal{C}_v^1(D)$ with support in $K \times N$, is compact for the weak topology on currents.*

Proof. Let L be a compact subset of D . Let $S \in \mathcal{C}_h^1(D)$ be a normalized smooth form, strictly positive on L (see lemma 2.2). Lemma 3.2 implies that $\langle R, S \rangle = 1$. Hence, the mass of R on L is bounded from above by a constant independent of R . The proposition follows. \square

Consider a function φ continuous and p.s.h. in a neighbourhood W' of $\text{supp}(R) \cap \text{supp}(S)$ in D . Let W be another neighbourhood of $\text{supp}(R) \cap \text{supp}(S)$ such that $W \Subset W'$. Consider smooth forms $R_n \in \mathcal{C}_v^1(D')$ and $S_n \in \mathcal{C}_h^1(D'')$ such that $R_n \rightarrow R$, $S_n \rightarrow S$, $\text{supp}(R_n) \cap \text{supp}(S_n) \subset W$ and $\langle R_n \wedge S_n, \varphi \rangle$ converge to a constant m_φ . Assume that m_φ is the maximal constant that we can obtain in this way. It follows from Lemma 3.2 that m_φ is finite.

Let R_θ , $\theta \in V$, be the currents of the structural disc in $\mathcal{C}_v^1(D')$ associated to R that we constructed in Section 2. Recall that $R_1 = R$. We construct in the same way the horizontal currents $S_{\theta'}$, $\theta' \in V$, with $S_1 = S$. They define a structural disc in $\mathcal{C}_h^1(D'')$. Observe that when $\theta, \theta' \rightarrow 1$, we have $\text{supp}(R_\theta) \rightarrow \text{supp}(R)$ and $\text{supp}(S_{\theta'}) \rightarrow \text{supp}(S)$. In particular, $\text{supp}(R_\theta) \cap \text{supp}(S_{\theta'}) \subset W$ when θ and θ' are close to 1.

Proposition 3.4. *We have*

$$m_\varphi = \limsup_{\theta \rightarrow 1} \langle R_\theta \wedge S, \varphi \rangle = \limsup_{\theta \rightarrow 1} \langle R \wedge S_\theta, \varphi \rangle = \limsup_{\theta, \theta' \rightarrow 1} \langle R_\theta \wedge S_{\theta'}, \varphi \rangle.$$

Moreover, m_φ does not depend on W' , W , and depends linearly on φ , R , S .

Proof. Define $\psi(\theta, \theta') := \langle R_\theta \wedge S_{\theta'}, \varphi \rangle$. By Lemma 2.5 and Remark 2.6, there exists a small neighbourhood U of $(1, 1)$ in V^2 such that ψ is defined and continuous on $U \setminus (1, 1)$. Lemma 3.2 shows that ψ is bounded. We first show that ψ is p.s.h. on $U' := \{(\theta, \theta') \in U, \theta \neq 1, \theta' \neq 1\}$. This allows us to extend ψ to a p.s.h. function on U with

$$\psi(1, 1) := \limsup_{\theta, \theta' \rightarrow 1} \psi(\theta, \theta') = \limsup_{\theta, \theta' \rightarrow 1} \langle R_\theta \wedge S_{\theta'}, \varphi \rangle.$$

Let \mathcal{R} and \mathcal{S} be currents as in Section 2 whose slices are R_θ and $S_{\theta'}$. These currents are smooth for $\theta \neq 1$ and $\theta' \neq 1$. It follows that the form

$$\tilde{\mathcal{R}}(\theta, \theta', z, w) := \varphi(z, w) \mathcal{R}(\theta, z, w) \wedge \mathcal{S}(\theta', z, w)$$

is continuous on $U' \times D$. We also have $\text{dd}^c \tilde{\mathcal{R}} \geq 0$ and the projection of $\text{supp}(\tilde{\mathcal{R}})$ on U' is proper. As in Theorem 2.1, we obtain ψ as the push-forward of $\tilde{\mathcal{R}}$ on U' . Hence ψ is p.s.h.

Define $m'_\varphi := \psi(1, 1)$. We first prove that $m'_\varphi = m_\varphi$. This implies that m_φ depends linearly on φ , R and S since ψ depends linearly on φ , R and S . The current R_θ is a priori not defined on D but it is a vertical current on a domain D_θ with $D_\theta \rightarrow D$ when $\theta \rightarrow 1$. The current $S_{\theta'}$ satisfies the same properties. Hence, by definition of m_φ , we have $m'_\varphi \leq m_\varphi$.

We define the structural discs $(R_{n,\theta})$ and $(S_{n,\theta})$ associated to R_n and S_n as in Section 2 with $R_{n,1} = R_n$ and $S_{n,1} = S_n$. Recall that $R_{n,\theta}$ and $S_{n,\theta}$ are smooth currents when $\theta \neq 1$. By Lemma 2.5, the bounded sequence of continuous p.s.h. functions $\psi_n(\theta, \theta') := \langle R_{n,\theta} \wedge S_{n,\theta'}, \varphi \rangle$ converges to ψ on $U \setminus (1, 1)$. It follows that $\psi_n \rightarrow \psi$ in $\mathcal{L}_{loc}^1(U)$. By Hartogs lemma,

$$m'_\varphi = \psi(1, 1) \geq \limsup_{n \rightarrow \infty} \psi_n(1, 1) = \limsup_{n \rightarrow \infty} \langle R_n \wedge S_n, \varphi \rangle = m_\varphi.$$

Hence $m'_\varphi = m_\varphi$.

Since p.s.h. functions on U are decreasing limits of smooth p.s.h. functions, their restrictions to $V \times \{1\}$ are subharmonic functions. It follows that

$$\limsup_{\theta \rightarrow 1} \langle R_\theta \wedge S, \varphi \rangle = \limsup_{\theta \rightarrow 1} \psi(\theta, 1) = \psi(1, 1) = m_\varphi.$$

We prove in the same way that $\limsup \langle R \wedge S_\theta, \varphi \rangle = m_\varphi$. □

End of the proof of Theorem 3.1. For functions φ continuous p.s.h. on a neighbourhood of $\text{supp}(R) \cap \text{supp}(S)$, define

$$\langle R \wedge S, \varphi \rangle := m_\varphi.$$

Since smooth functions on neighbourhoods of $\text{supp}(R) \cap \text{supp}(S)$ can be written as differences of continuous p.s.h. functions, we can extend the definition to smooth functions.

Proposition 3.4 shows that the current $R \wedge S$ is supported in $\text{supp}(R) \cap \text{supp}(S)$. It is clear that the definition does not depend on coordinate systems of M, N . If $\varphi \leq \varphi'$ we have $m_\varphi \leq m_{\varphi'}$. Then $R \wedge S$ is a positive measure. When $\varphi = 1$, Lemma 3.2 implies that $m_\varphi = 1$. Hence $R \wedge S$ is a probability measure. \square

Proposition 3.5. *Let R, S, φ, W' and W be as above. Let $R_n \in \mathcal{C}_v(D')$ and $S_n \in \mathcal{C}_h(D'')$ such that $R_n \rightarrow R, S_n \rightarrow S$ and $\text{supp}(R_n) \cap \text{supp}(S_n) \subset W$. Then*

$$\limsup_{n \rightarrow \infty} \langle R_n \wedge S_n, \varphi \rangle \leq \langle R \wedge S, \varphi \rangle.$$

The measures $R_n \wedge S_n$ converge to $R \wedge S$ if and only if $\langle R_n \wedge S_n, \varphi \rangle \rightarrow \langle R \wedge S, \varphi \rangle$ for one function φ strictly p.s.h. on W' . In particular, there exists $(\theta_n) \subset V \setminus \{1\}$ converging to 1 such that

$$R_{\theta_n} \wedge S \rightarrow R \wedge S, \quad R \wedge S_{\theta_n} \rightarrow R \wedge S \quad \text{and} \quad R_{\theta_n} \wedge S_{\theta_n} \rightarrow R \wedge S.$$

More generally, if $(\theta, \theta') \rightarrow (1, 1)$ in the plurifine topology, then $R_\theta \wedge S_{\theta'} \rightarrow R \wedge S$.

Proof. The first inequality follows from the definition of m_φ . Now assume that φ is strictly p.s.h. on W' . Let ϕ be a real smooth function with support in W' . If $A > 0$ is large enough then $\varphi^\pm := A\varphi \pm \phi$ are p.s.h. on W' . Then $\limsup \langle R_n \wedge S_n, \varphi^\pm \rangle \leq \langle R \wedge S, \varphi^\pm \rangle$. When $\langle R_n \wedge S_n, \varphi \rangle \rightarrow \langle R \wedge S, \varphi \rangle$, we deduce easily that $\langle R_n \wedge S_n, \phi \rangle \rightarrow \langle R \wedge S, \phi \rangle$. It follows that $R_n \wedge S_n \rightarrow R \wedge S$.

The functions $\psi(\cdot, 1), \psi(1, \cdot)$ and $\psi(\cdot, \cdot)$ associated to φ are subharmonic or p.s.h. Then there exists $(\theta_n) \rightarrow 1$ such that $\psi(\theta_n, 1), \psi(1, \theta_n)$ and $\psi(\theta_n, \theta_n)$ converge to $\psi(1, 1)$. Hence $\langle R_{\theta_n} \wedge S, \varphi \rangle = \psi(\theta_n, 1)$ converge to $\langle R \wedge S, \varphi \rangle = \psi(1, 1)$. It follows that $R_{\theta_n} \wedge S \rightarrow R \wedge S$. Other convergences are obtained in the same way. If $(\theta, \theta') \rightarrow (1, 1)$ in the plurifine topology (i.e. the coarsest topology which makes p.s.h. functions continuous), we get $R_\theta \wedge S_{\theta'} \rightarrow R \wedge S$. \square

Remarks 3.6. a. Proposition 3.5 and Lemma 2.7 imply that when R or S is continuous, our definition of $R \wedge S$ coincides with the usual one.

b. When φ is a uniform limit of continuous functions p.s.h. on neighbourhoods of $\text{supp}(R) \cap \text{supp}(S)$, we can apply Proposition 3.5 and get $\limsup \langle R_n \wedge S_n, \varphi \rangle \leq \langle R \wedge S, \varphi \rangle$. Hence, if there exists a compact set $K \subset D$ containing $\text{supp}(R) \cap \text{supp}(S)$ such that continuous p.s.h. functions on neighbourhoods of K are dense in $\mathcal{C}^0(K)$, then $R_n \wedge S_n \rightarrow R \wedge S$ provided that $R_n \rightarrow R, S_n \rightarrow S$

and $\text{supp}(R_n) \cap \text{supp}(S_n) \rightarrow K$. In particular, this holds when K is totally disconnected. In the last case, continuous functions on K can be approximated by functions locally constant in neighbourhoods of K .

Let λ_ϵ denote the Lebesgue measure on the disc of center 1 and of radius ϵ normalized by $\|\lambda_\epsilon\| = 1$. Since the function ψ in Proposition 3.4 is p.s.h. we have

$$\begin{aligned} \psi(1, 1) &= \lim_{\epsilon \rightarrow 0} \int \psi(\theta, 1) d\lambda_\epsilon(\theta) = \lim_{\epsilon \rightarrow 0} \int \psi(1, \theta) d\lambda_\epsilon(\theta) \\ &= \lim_{\epsilon \rightarrow 0} \int \psi(\theta, \theta') d\lambda_\epsilon(\theta) d\lambda_\epsilon(\theta'). \end{aligned}$$

We define the vertical and horizontal currents $R^{(\epsilon)}$ and $S^{(\epsilon)}$ by

$$R^{(\epsilon)} := \int R_\theta d\lambda_\epsilon(\theta) \quad \text{and} \quad S^{(\epsilon)} := \int S_\theta d\lambda_\epsilon(\theta)$$

and deduce from the previous relations that

$$\psi(1, 1) = \lim_{\epsilon \rightarrow 0} \langle R^{(\epsilon)} \wedge S, \varphi \rangle = \lim_{\epsilon \rightarrow 0} \langle R \wedge S^{(\epsilon)}, \varphi \rangle = \lim_{\epsilon \rightarrow 0} \langle R^{(\epsilon)} \wedge S^{(\epsilon)}, \varphi \rangle.$$

This and Proposition 3.5 imply the following result which can be considered as a “less abstract” definition of $R \wedge S$.

Proposition 3.7. *Let $R, S, R^{(\epsilon)}$ and $S^{(\epsilon)}$ be as above. Then*

$$R \wedge S = \lim_{\epsilon \rightarrow 0} R^{(\epsilon)} \wedge S = \lim_{\epsilon \rightarrow 0} R \wedge S^{(\epsilon)} = \lim_{\epsilon \rightarrow 0} R^{(\epsilon)} \wedge S^{(\epsilon)}.$$

Remarks 3.8. It follows from the definition of $R \wedge S$ and from Proposition 3.7, that for φ p.s.h. on W'

$$\langle R \wedge S, \varphi \rangle = \limsup \langle R' \wedge S', \varphi \rangle$$

where the limit is taken over all currents, not necessarily smooth, $R' \rightarrow R$ and $S' \rightarrow S$ with $\text{supp}(R') \cap \text{supp}(S') \subset W$.

Let (R'_θ) (resp. $(S'_{\theta'})$) be an *arbitrary* structural variety in $\mathcal{C}_v^1(D)$ (rep. in $\mathcal{C}_h^1(D)$). Let φ be a bounded p.s.h. function on D . Then one can prove as in Theorem 2.1 and Proposition 3.4 that the function $\lambda(\theta, \theta') := \langle R'_\theta \wedge S'_{\theta'}, \varphi \rangle$ is p.s.h. and $(\theta, \theta') \mapsto R'_\theta \wedge S'_{\theta'}$ is continuous for the plurifine topology.

4 Horizontal-like maps

In general, a horizontal-like map f on D is not defined on the whole domain D but only on a vertical subset $f^{-1}(D)$ of D . It takes values in a horizontal subset $f(D)$ of D . We define these maps using their graphs as follows (see [11, 5]). Let pr_1 and pr_2 be the canonical projections of $D \times D$ on its factors. From now on we assume always that D is convex.

Definition 4.1. A *horizontal-like map* f on D is a holomorphic map with graph Γ such that

1. Γ is an irreducible submanifold of $D \times D$.
2. $\text{pr}_{1\Gamma}$ is injective; $\text{pr}_{2\Gamma}$ has finite fibers.
3. $\bar{\Gamma}$ does not intersect $\overline{\partial_v D} \times \bar{D}$ nor $\bar{D} \times \overline{\partial_h D}$.

The map f is defined on $f^{-1}(D) := \text{pr}_1(\Gamma)$ and its image is equal to $f(D) := \text{pr}_2(\Gamma)$ (if we assume only that $\pi_{1\Gamma}$ has finite fibers, we obtain a horizontal-like correspondence). Observe that there exist open sets $M' \Subset M$ and $N'' \Subset N$ such that $f^{-1}(D) \subset D' := M' \times N$ and $f(D) \subset D'' := M \times N''$. We have $\Gamma \subset D' \times D''$. This property characterizes horizontal-like maps. Since Γ is a submanifold of $D \times D$, when x converges to $\partial f^{-1}(D) \cap D$, $f(x)$ converges to $\partial_v D$. When y converges to $\partial f(D) \cap D$, $f^{-1}(y)$ converges to $\partial_h D$. So, the vertical (resp. horizontal) part of $\partial f^{-1}(D)$ is sent to the vertical (resp. horizontal) part of $\partial f(D)$. If g is another horizontal-like map on D , $f \circ g$ is also a horizontal-like map. When $p = k$, we obtain the polynomial-like maps studied in [6].

If $\text{pr}_{2\Gamma}$ is injective, we say that f is *invertible*. In this case, up to a coordinate change, $f^{-1} : \text{pr}_2(\Gamma) \rightarrow \text{pr}_1(\Gamma)$ is a horizontal-like map. When $k = 2$ and $p = 1$, we obtain the Hénon-like maps which are studied in [11, 5]. In order to simplify the paper, we consider only invertible horizontal-like maps. The results in Sections 4, 5 and 6 hold for non-invertible maps, but for the construction of T_+ , we need to define inverse images of positive closed currents by open holomorphic maps, see also [9].

The operator $f_* = (\text{pr}_{2\Gamma})_* \circ (\text{pr}_{1\Gamma})^*$ acts continuously on horizontal currents. If S is a horizontal current (form), so is $f_*(S)$. The operator $f^* = (\text{pr}_{1\Gamma})_* \circ (\text{pr}_{2\Gamma})^*$ acts continuously on vertical currents. If R is a vertical current (form), so is $f^*(R)$. We have the following proposition for positive closed currents.

Proposition 4.2. *The operator $f_* : \mathcal{C}_h(D') \rightarrow \mathcal{C}_h(D'')$ is well defined and continuous. Moreover, there exists an integer $d \geq 1$ such that $\|f_*(S)\|_h = d\|S\|_h$ for every $S \in \mathcal{C}_h(D')$. The operator $f^* : \mathcal{C}_v(D'') \rightarrow \mathcal{C}_v(D')$ is well defined and continuous. If R belongs to $\mathcal{C}_v(D'')$, we have $\|f^*(R)\|_v = d\|R\|_v$.*

Proof. Using Definition 4.1, one can check that f^* and f_* are well defined.

Let R be a current in $\mathcal{C}_v^1(D'')$. We want to compute the slice mass of $f^*(R)$. We can assume that R is smooth. Let $S = [w = b]$ be the current of integration on the subspace $\{w = b\}$ with $b \in N$. Since S is normalized, we have

$$\|f^*(R)\|_v = \langle f^*(R), S \rangle = \langle R, f_*(S) \rangle.$$

The current $f_*(S)$ is defined by a horizontal analytic subset of D'' . Hence, it is a ramified covering of degree d over M . We have $\|f_*(S)\|_h = d$. Theorem 3.1 implies that $\langle R, f_*(S) \rangle = d$. Hence, $\|f^*(R)\|_v = d$.

If S is an arbitrary current in $\mathcal{C}_h^1(D')$, then Theorem 3.1 implies that

$$\|f_*(S)\|_h = \langle f_*(S), R \rangle = \langle S, f^*(R) \rangle = d.$$

□

The integer d in Proposition 4.2 is called the (*main*) *dynamical degree* of f . Define $\mathcal{L}_v := \frac{1}{d}f^*$ and $\mathcal{L}_h := \frac{1}{d}f_*$. Using Cesàro means, one can easily construct a current $T_+ \in \mathcal{C}_v^1(D)$ such that $\mathcal{L}_v(T_+) = T_+$. A priori such T_+ is not unique. Our aim is to construct such a current T_+ with a good convergence theorem and some extremality properties. This allows us to construct an interesting invariant measure. The following diagram is one of the main objects we consider:

$$V \xrightarrow{\tau} \mathcal{C}_v^1(D^*) \xrightarrow{\Lambda_\Phi} \mathbb{R} \quad (4.1)$$

Example 4.3. Let f be a polynomial automorphism of \mathbb{C}^k . Denote also by f its meromorphic extension to \mathbb{P}^k . Let (z_1, \dots, z_k) be the coordinates of \mathbb{C}^k and $[z_1 : \dots : z_k]$ be homogeneous coordinates of the hyperplane at infinity L . Assume that the indeterminacy set I_+ of f is the subspace $\{z_1 = \dots = z_p = 0\}$ of L and the indeterminacy set I_- of f^{-1} is the subspace $\{z_{p+1} = \dots = z_k = 0\}$ of L . This map is regular in the sense of [22]; that is $I_+ \cap I_- = \emptyset$ (see also [10]).

If M and N are the balls of center 0 and of radius r in \mathbb{C}^p and \mathbb{C}^{k-p} , then f^{n_0} defines a horizontal-like map in $D = M \times N$ when r and n_0 are big enough. This follows from the description of Julia sets of f and f^{-1} in [22].

Observe that every small perturbation of f^{n_0} on D is still horizontal-like. One can construct such a map which admits both attractive and repelling fixed points [11]. The map is not conjugated to a polynomial automorphism since polynomial automorphisms have constant jacobian and hence cannot have such fixed points.

Example 4.4. Let f_i be horizontal-like maps on $D_i = M_i \times N_i$. Define $D = D_1 \times D_2$ and the product map $f(x_1, x_2) := (f_1(x_1), f_2(x_2))$. Up to a coordinate change, we can identify D to $M \times N$, with $M = M_1 \times M_2$ and $N = N_1 \times N_2$. Then, one can check easily that f defines a horizontal-like map on D .

When $M_1 = N_2$ and $N_1 = M_2$, let Δ denote the diagonal of D . Then, Δ is not a horizontal set but $f(\Delta)$ is horizontal.

We will see in Section 7 that this simple example can be used to linearize some problems.

5 Random iteration

Let (f_n) be a sequence of invertible horizontal-like maps on D of dynamical degrees d_n . Define $\mathcal{L}_{v,n} := \frac{1}{d_n}f_n^*$ and $\mathcal{L}_{h,n} := \frac{1}{d_n}(f_n)_*$. Assume there exist

open sets $M' \Subset M$ and $N'' \Subset N$ such that $f_n^{-1}(D) \subset D' := M' \times N$ and $f_n(D) \subset D'' := M \times N''$ for every n . Define the *filled Julia set* associated to (f_n) as

$$\mathcal{K}_+ := \bigcap_{n \geq 1} f_1^{-1} \circ \dots \circ f_n^{-1}(D) = \bigcap_{n \geq 1} f_1^{-1} \circ \dots \circ f_n^{-1}(\overline{D}).$$

This is a vertical closed subset of D' .

Theorem 5.1. *Let $(R_n) \subset \mathcal{C}_v^1(D')$ be a uniformly bounded family of forms. Then, the sequence $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n)$ converges weakly to a current $T_+ \in \mathcal{C}_v^1(D')$ supported in $\partial \mathcal{K}_+$. Moreover, T_+ is independent of (R_n) .*

We say that T_+ is the *Green current* associated to the sequence (f_n) . We say that (R_n) is *uniformly bounded* if the coefficients of R_n are uniformly bounded. Observe that $\mathcal{L}_{v,n}$ is “distance decreasing” for the Kobayashi pseudo-distance on $\mathcal{C}_v^1(\overline{D}'')$. However, the fact that it is not a distance makes the convergence questions more delicate. We first prove the following proposition.

Proposition 5.2. *Let Φ be a real continuous horizontal $(k-p, k-p)$ -form with $\text{dd}^c \Phi \geq 0$. There exists a constant M_Φ such that if R_n are currents in $\mathcal{C}_v^1(D)$, then $\limsup \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n), \Phi \rangle \leq M_\Phi$; if R_n are as in Theorem 5.1, then $\lim \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n), \Phi \rangle = M_\Phi$.*

Proof. By regularization, we can assume that R_n are smooth. Observe that by Theorem 3.1 if Φ is positive and closed then $\langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n), \Phi \rangle = \|\Phi\|_h$. So in this case the convergence is clear. If we add to Φ a form in $\mathcal{C}_h(D)$, we can assume that Φ is positive on D' . We can also assume that Φ is smaller than a smooth form in $\mathcal{C}_h^1(D')$. It follows from Proposition 4.2 that each form $\mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi)$ is positive and bounded from above by a current in $\mathcal{C}_h^1(D'')$, depending on n .

Let (\tilde{R}'_{i_n}) be a sequence of continuous forms in $\mathcal{C}_v^1(D'')$ with (i_n) a sequence of integers, $i_n > n$, such that $\langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,i_n}(\tilde{R}'_{i_n}), \Phi \rangle$ converge to a real number M_Φ . We choose (i_n) and (\tilde{R}'_{i_n}) so that M_Φ is the maximal value that we can obtain in this way. Since $\mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi)$ are bounded by normalized currents, Theorem 3.1 implies that M_Φ is finite. Hence, M_Φ satisfies the inequality in Proposition 5.2.

Define $\tilde{R}_n := \mathcal{L}_{v,n+1} \dots \mathcal{L}_{v,i_n}(\tilde{R}'_{i_n})$. We have $\tilde{R}_n \in \mathcal{C}_v^1(D')$ and $\langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(\tilde{R}_n), \Phi \rangle \rightarrow M_\Phi$. We will use the structural discs $(\tilde{R}_{n,\theta})$ of $\mathcal{C}_v^1(D'')$ constructed in Section 2 (see also (2.2) and (4.1)) associated to \tilde{R}_n in order to prove that the convergence holds when R_n is replaced by $\tilde{R}_{n,0}$.

Theorem 2.1 allows us to define continuous subharmonic functions on V by

$$\varphi_n(\theta) := \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(\tilde{R}_{n,\theta}), \Phi \rangle = \langle \tilde{R}_{n,\theta}, \mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi) \rangle.$$

Since $\varphi_n(1)$ tends to the maximal value M_Φ , Hartogs lemma [19] and the maximum principle imply that $\varphi_n \rightarrow M_\Phi$ in $\mathcal{L}_{loc}^1(V)$.

On the other hand, since each $\mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi)$ is bounded by a current in $\mathcal{C}_h^1(D)$, Lemma 2.5 implies that $|\varphi_n(\theta) - \varphi_n(0)| \leq c|\theta|$ for $|\theta| \leq r$. Hence, $\varphi_n(0)$ converge to M_Φ . Since $\tilde{R}_0 := \tilde{R}_{n,0}$ is independent of n , we obtain that $\langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(\tilde{R}_0), \Phi \rangle \rightarrow M_\Phi$.

Now assume that R_n satisfy the hypothesis of Theorem 5.1. If we replace M' by a bigger domain, we can assume that there exists an open set $M'' \Subset M'$ such that $f_n^{-1}(D) \subset M'' \times N$ and $\text{supp}(R_n) \subset M'' \times N$. Then, we can find a continuous form $R \in \mathcal{C}_v^1(D')$ and $c > 0$ such that $R_n \leq cR$ for every n .

Define the currents R_θ associated to R as in Section 2 and

$$\psi_n(\theta) := \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_\theta), \Phi \rangle = \langle R_\theta, \mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi) \rangle.$$

Recall that $R_0 = \tilde{R}_0$. Since $\psi_n(0) = \varphi_n(0) \rightarrow M_\Phi$ and $\limsup \psi_n \leq M_\Phi$, we have $\psi_n \rightarrow M_\Phi$ in $\mathcal{L}_{loc}^1(V)$. On the other hand, since $\mathcal{L}_{h,n} \dots \mathcal{L}_{h,1}(\Phi)$ are bounded by currents in $\mathcal{C}_h^1(D'')$, Lemma 2.7 implies that

$$\lim_{\theta \rightarrow 1} \left(\sup_{n \geq 1} |\psi_n(\theta) - \psi_n(1)| \right) = 0.$$

It follows that $\psi_n(1) \rightarrow M_\Phi$. We obtain that $\langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R), \Phi \rangle \rightarrow M_\Phi$.

We turn to the general case. Since R_n and $cR - R_n$ belong to $\mathcal{C}_v(D')$, by definition of M_Φ , we have

$$\limsup \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n), \Phi \rangle \leq M_\Phi \tag{5.1}$$

and since $cR - R_n$ have slice mass $c - 1$

$$\limsup \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(cR - R_n), \Phi \rangle \leq (c - 1)M_\Phi. \tag{5.2}$$

We consider the sum of (5.1) and (5.2) and deduce that these inequalities are in fact equalities. It follows that $\lim \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n), \Phi \rangle = M_\Phi$. \square

Remark 5.3. Proposition 5.2 still holds when R_n are continuous forms and Φ is a non-smooth horizontal current such that $\text{dd}^c \Phi \geq 0$ and $-\Psi \leq \Phi \leq \Psi$ for some current $\Psi \in \mathcal{C}_h(D)$. If R_n are continuous then $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n)$ are continuous and they act on currents of order 0, like Φ . All the arguments in the above proof make sense. In this case, the convergence in Theorem 5.1 is stronger than the usual weak convergence.

Proof of Theorem 5.1. Since the maps Λ_Φ separate the currents in $\mathcal{C}_v(D)$, Proposition 5.2 implies that $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n)$ converge to a current T_+ in $\mathcal{C}_v^1(D)$ which is defined by $\langle T_+, \Phi \rangle := M_\Phi$. This current is independent of (R_n) .

Now, we prove that T_+ is supported in $\partial \mathcal{K}_+$. It is clear that $\text{supp}(T_+) \subset \mathcal{K}_+$. If $U \Subset \mathcal{K}_+$ is an open set, then $f_n \circ \dots \circ f_1(U) \subset f_{n+1}^{-1}(D) \subset M'' \times N$ for some $M'' \Subset M$ and for every n . It follows that if $\text{supp}(R_n) \subset (M' \setminus \overline{M''}) \times N$ we get

$\text{supp}(T_+) \cap U = \emptyset$, since T_+ is independent of R_n . \square

The following corollary is a direct consequence of Proposition 5.2. It gives an extremality property of T_+ .

Corollary 5.4. *Let $(R_n) \subset \mathcal{C}_v^1(D)$. Let Φ be a real continuous horizontal $(k-p, k-p)$ -form such that $\text{dd}^c \Phi \geq 0$. Then, every limit value R of the sequence of currents $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n)$ satisfies*

$$\langle R, \Phi \rangle \leq \langle T_+, \Phi \rangle.$$

If $\text{dd}^c \Phi = 0$, then $\langle R, \Phi \rangle = \langle T_+, \Phi \rangle$.

Corollary 5.5. *Let (n_i) be an increasing sequence of integers and R_{n_i}, R'_{n_i} be currents in $\mathcal{C}_v^1(D)$. Assume that $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(R_{n_i})$ converge to T_+ and that $R'_{n_i} \leq cR_{n_i}$ with $c > 0$ independent of n_i . Then, $\mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(R'_{n_i})$ converge also to T_+ .*

Proof. Let Φ be as above. Proposition 5.2 implies that

$$\limsup \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(R'_{n_i}), \Phi \rangle \leq \langle T_+, \Phi \rangle. \quad (5.3)$$

On the other hand, the currents $cR_{n_i} - R'_{n_i}$ belong to $\mathcal{C}_v(D)$ and have slice mass $c - 1$. Hence

$$\limsup \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(cR_{n_i} - R'_{n_i}), \Phi \rangle \leq (c - 1) \langle T_+, \Phi \rangle. \quad (5.4)$$

By hypothesis,

$$\lim \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(R_{n_i}), \Phi \rangle = \langle T_+, \Phi \rangle.$$

We consider the sum of (5.3) and (5.4) and deduce that

$$\lim \langle \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n_i}(R'_{n_i}), \Phi \rangle = \langle T_+, \Phi \rangle.$$

The corollary follows. \square

The following proposition allows us to check that $\lim \mathcal{L}_{v,1} \dots \mathcal{L}_{v,n}(R_n) = T_+$ with only one test form.

Proposition 5.6. *Let R_n, R and Φ be as in Corollary 5.4. Assume that $\text{dd}^c \Phi$ is strictly positive on an open set V . If $\langle R, \Phi \rangle = \langle T_+, \Phi \rangle$, then $R = T_+$ on V .*

Proof. Let Ψ be a real test form with compact support in V . Let $A > 0$ be a constant such that $\text{dd}^c(A\Phi \pm \Psi) \geq 0$. Corollary 5.4 implies that $\langle R, A\Phi \pm \Psi \rangle \leq \langle T_+, A\Phi \pm \Psi \rangle$. We deduce that $\langle R, \Psi \rangle = \langle T_+, \Psi \rangle$ if $\langle R, \Phi \rangle = \langle T_+, \Phi \rangle$. Therefore, $R = T_+$ on V . \square

Corollary 5.7. *Let (n_i) be an increasing sequence of integers. Then, there exist a subsequence (m_i) and a pluripolar set $\mathcal{E}_+ \subset M$ such that, for every $a \in M \setminus \mathcal{E}_+$, we have*

$$\mathcal{L}_{v,1} \cdots \mathcal{L}_{v,m_i}[z = a] \rightarrow T_+$$

where $[z = a]$ is the current of integration on the vertical analytic set $\{a\} \times N$.

Proof. Let Φ and V be as above. Consider the locally uniformly bounded p.s.h. functions $\varphi_{n_i}(a) := \langle \mathcal{L}_{v,1} \cdots \mathcal{L}_{v,n_i}[z = a], \Phi \rangle$ (see Section 2). By extracting a subsequence, we can assume that φ_{n_i} converge in $\mathcal{L}_{loc}^1(M)$ to a p.s.h. function φ . Proposition 5.2 implies $\varphi \leq M_\Phi$.

Let ν be a smooth probability measure with compact support in M' . Consider the current $R := \pi_1^*(\nu)$ in $\mathcal{C}_v^1(D')$. Since R is smooth, Proposition 5.2 implies

$$\int \varphi_{n_i} d\nu = \langle \mathcal{L}_{v,1} \cdots \mathcal{L}_{v,n_i}(R), \Phi \rangle \rightarrow M_\Phi.$$

It follows that $\varphi = M_\Phi$. Hence, there exists a subsequence $(m_i) \subset (n_i)$ and a pluripolar set $\mathcal{E}_+(\Phi) \subset M$ such that $\varphi_{m_i} \rightarrow M_\Phi$ pointwise on $M \setminus \mathcal{E}_+(\Phi)$ [6, Proposition 3.9.4]. Proposition 5.6 implies that $\mathcal{L}_{v,1} \cdots \mathcal{L}_{v,m_i}[z = a] \rightarrow T_+$ on V for $a \notin \mathcal{E}_+(\Phi)$.

Consider a sequence of (Φ_n, V_n) such that $\cup_n V_n = D$. Extracting subsequences of (m_i) gives $\mathcal{L}_{v,1} \cdots \mathcal{L}_{v,m_i}[z = a] \rightarrow T_+$ on D for $a \notin \mathcal{E}_+ := \cup_n \mathcal{E}_+(\Phi_n)$. \square

Remark 5.8. Corollary 5.7 implies that T_+ can be approximated by currents of integration on vertical manifolds with control of support. When $p = 1$, this holds for every current in $\mathcal{C}_v(D)$ [12]. The problem is still open for general currents of higher bidegree.

6 Green currents

In the rest of the paper, we study the dynamics of an invertible horizontal-like map. The following result is a direct consequence of Theorem 5.1.

Corollary 6.1. *Let f be an invertible horizontal-like map on D of dynamical degree $d \geq 1$. Let $\mathcal{K}_+ := \cap_{n \geq 1} f^{-n}(D)$ be the filled Julia set of f . Let $(R_n) \subset \mathcal{C}_v^1(D')$ be a uniformly bounded family of forms. Then, $d^{-n} f^{n*}(R_n)$ converge weakly to a current $T_+ \in \mathcal{C}_v^1(D')$ supported in $\partial \mathcal{K}_+$. Moreover, T_+ does not depend on (R_n) and satisfies $f^*(T_+) = dT_+$.*

We call T_+ the *Green current* of f . Corollary 5.7 shows that T_+ is a limit value of $(d^{-n} f^{n*}[z = a])$ for $a \in M$ generic. We construct in the same way the Green current $T_- \in \mathcal{C}_h^1(D'')$ for f^{-1} . This current is supported in the boundary of $\mathcal{K}_- := \cap_{n \geq 1} f^n(D)$ and satisfies $f_*(T_-) = dT_-$. Now, we give some properties of the Green currents.

Let (R_n) be an arbitrary sequence of currents in $\mathcal{C}_v^1(D)$ and Φ be a smooth real horizontal test form such that $\text{dd}^c\Phi \geq 0$. Corollary 5.4 implies that every limit value R of $(d^{-n}f^{n*}(R_n))$ satisfies $\langle R, \Phi \rangle \leq \langle T_+, \Phi \rangle$. Proposition 5.6 implies that if $\langle R, \Phi \rangle = \langle T_+, \Phi \rangle$, then $R = T_+$ in the open set where $\text{dd}^c\Phi$ is strictly positive. We deduce from this the following corollary.

Corollary 6.2. *Let T be a current in $\mathcal{C}_v^1(D)$ and Φ be a real horizontal continuous form. Assume that $\text{dd}^c\Phi \geq 0$ on D and $\text{dd}^c\Phi > 0$ on a neighbourhood W of $\mathcal{K}_+ \cap \mathcal{K}_-$. Then, $d^{-n}f^{n*}(T) \rightarrow T_+$ if and only if $\langle d^{-n}f^{n*}(T), \Phi \rangle \rightarrow \langle T_+, \Phi \rangle$.*

Proof. Assume that $\langle d^{-n}f^{n*}(T), \Phi \rangle \rightarrow \langle T_+, \Phi \rangle$. Hence, every limit value R of $(d^{-n}f^{n*}(T))$ is equal to T_+ on W . For every $m \geq 0$, there exists a limit value R' of $(d^{-n+m}(f^{n-m})^*(T))$ such that $R = d^{-m}f^{m*}(R')$. We also have $R' = T_+$ on W . This implies $R = T_+$ on $f^{-m}(W)$. It follows that $R = T_+$ on $\cup_{m \geq 0} f^{-m}(W)$ which is a neighbourhood of \mathcal{K}_+ . Since both the currents R and T_+ are supported in \mathcal{K}_+ , we have $R = T_+$. \square

The following result is a direct consequence of Corollary 5.5.

Corollary 6.3. *Let T be a current in $\mathcal{C}_v^1(D)$. Assume there exist $c > 0$, an increasing sequence (n_i) and currents $T_{n_i} \in \mathcal{C}_v^1(D)$ such that $T_{n_i} \leq cT_+$ and $T = d^{-n_i}(f^{n_i})^*(T_{n_i})$. Then $T = T_+$. In particular, T_+ is extremal in the cone of currents $T \in \mathcal{C}_v^1(D)$ satisfying $f^*(T) = dT$.*

Theorem 6.4. *Let R be a real continuous vertical form of bidegree (p, p) not necessarily closed. Then, $d^{-n}(f^n)^*(R)$ converge to cT_+ where $c := \langle R, T_- \rangle$.*

Proof. We can write R as a difference of positive forms (scale D if necessary). Hence, we can assume that R is positive and that $R \leq R'$ for a suitable continuous form $R' \in \mathcal{C}_v(D)$. We can extract from $d^{-n}(f^n)^*(R)$ convergent subsequences. Corollary 6.1 implies that every limit value is bounded by $\|R'\|_v T_+$.

Let (n_i) and T such that $\lim d^{-n_i}(f^{n_i})^*(R) = T$. We have $T \leq \|R'\|_v T_+$. Moreover, for every $m \geq 0$, we have $T = d^{-m}(f^m)^*(T')$ where T' is a limit value of $(d^{-n_i+m}(f^{n_i-m})^*(R))$.

Let $\Phi \in \mathcal{C}_h^1(D)$ be a continuous form. We have

$$\langle T, \Phi \rangle = \lim \langle d^{-n_i}(f^{n_i})^*(R), \Phi \rangle = \lim \langle R, d^{-n_i}(f^{n_i})_*(\Phi) \rangle = \langle R, T_- \rangle = c.$$

It follows that if T were closed, it has slice mass c (this also holds for T'). Hence, Corollary 6.3 implies that it is sufficient to prove that T is closed. We first prove that it is dd^c -closed.

Lemma 6.5. *Let T be a real vertical current of bidegree (p, p) and of finite mass. Consider smooth forms $\Phi \in \mathcal{C}_h^1(D)$. Assume that $\langle T, \Phi \rangle$ does not depend on Φ . Then T is dd^c -closed.*

Proof. Consider a real smooth $(k-p-1, k-p-1)$ -form α with compact support in D . Let Φ be a smooth form in $\mathcal{C}_h(D)$ strictly positive in a neighbourhood of $\text{supp}(\alpha)$. Write $\text{dd}^c\alpha = (A\Phi + \text{dd}^c\alpha) - A\Phi$. When A is big enough, both $A\Phi + \text{dd}^c\alpha$ and $A\Phi$ are positive closed and have the same slice mass. By hypothesis, $\langle T, A\Phi + \text{dd}^c\alpha \rangle = \langle T, A\Phi \rangle$. Hence, $\langle T, \text{dd}^c\alpha \rangle = 0$ and T is dd^c -closed. \square

Consider the product map $F(x_1, x_2) = (f(x_1), f(x_2))$ on D^2 as in Example 4.4. The same arguments applied to F and to $R \otimes R$ imply that $T \otimes T$ is dd^c -closed. It follows that T is closed. For this end, it suffices to compute $\text{dd}^c(T \otimes T)$. \square

7 Equilibrium measure

The main result of this section is the following theorem.

Theorem 7.1. *Let f be an invertible horizontal-like map of dynamical degree d on D . Let $(R_n) \subset \mathcal{C}_v^1(D')$ and $(S_n) \subset \mathcal{C}_h^1(D'')$ be uniformly bounded sequences of continuous forms. Then, $d^{-2n}(f^n)^*(R_n) \wedge (f^n)_*(S_n)$ converge weakly to an invariant probability measure μ which does not depend on (R_n) and (S_n) . Moreover, μ is mixing and is supported on the boundary of the compact set $\mathcal{K} := \bigcap_{n \in \mathbb{Z}} f^n(D)$.*

We say that μ is the *equilibrium measure* of f . We will see that the convergence part of Theorem 7.1 is a consequence of Proposition 5.2 and Remark 5.3 (see also Proposition 7.8 and Corollary 7.9).

Let M_i and N_i be copies of M and N . Consider the domain

$$D^2 = D \times D = (M_1 \times N_1) \times (M_2 \times N_2) \subset \mathbb{C}^{2k}$$

and the product map (see Example 4.4)

$$F(z_1, w_1, z_2, w_2) := (f(z_1, w_1), f^{-1}(z_2, w_2)).$$

Using the coordinate change $(z_1, w_1, z_2, w_2) \mapsto (z_1, w_2, z_2, w_1)$, write

$$F(z_1, w_2, z_2, w_1) = (f_M(z_1, w_1), f_N^{-1}(z_2, w_2), f_M^{-1}(z_2, w_2), f_N(z_1, w_1))$$

where $f = (f_M, f_N)$ and $f^{-1} = (f_M^{-1}, f_N^{-1})$.

Recall that, the coordinate change $(z, w) \mapsto (w, z)$ makes f^{-1} a horizontal-like map. One can check that F is an invertible horizontal-like map of dynamical degree d^2 on $D^2 \simeq (M_1 \times N_2) \times (M_2 \times N_1)$. The diagonal

$$\Delta := \{z_1 = z_2, w_1 = w_2\}$$

is not a horizontal set but $F(\Delta)$ is horizontal. If $\tilde{\varphi}$ is a (positive) p.s.h. function on Δ , then $\tilde{\varphi}[\Delta]$ is a (positive) current such that $\text{dd}^c(\tilde{\varphi}[\Delta]) \geq 0$. Hence, we can apply Proposition 5.2 and Remark 5.3.

Proposition 7.2. *Let φ be a continuous p.s.h. function on D . There exists a constant M_φ such that if $(R_m) \subset \mathcal{C}_v^1(D)$ and $(S_n) \subset \mathcal{C}_h^1(D)$, then*

$$\limsup_{m,n \rightarrow \infty} \langle d^{-m-n}(f^m)^* R_m \wedge (f^n)_* S_n, \varphi \rangle \leq M_\varphi.$$

If R_n and S_n are as in Theorem 7.1, we have

$$\lim_{n \rightarrow \infty} \langle d^{-2n}(f^n)^* R_n \wedge (f^n)_* S_n, \varphi \rangle = M_\varphi.$$

Proof. By Proposition 3.7, we can assume that R_n and S_n are smooth forms. We can also assume that $m \geq n$ and $n \rightarrow \infty$. Write $d^{-m}(f^m)^* R_m = d^{-n}(f^n)^* R_{m,n}$ with $R_{m,n} := d^{-m+n}(f^{m-n})^* R_m$. This allows us to suppose that $m = n$.

Define the currents T_n in $\mathcal{C}_v^1(D^2)$ by $T_n := R_n \otimes S_n$ and $\tilde{\varphi}(z_1, w_2, z_2, w_1) := \varphi(z_1, w_1)$. Then

$$\langle (f^n)^* R_n \wedge (f^n)_* S_n, \varphi \rangle = \langle F^{n*}(T_n), \tilde{\varphi}[\Delta] \rangle.$$

The current $\Phi := \tilde{\varphi}[\Delta]$ is not horizontal, but $F_*(\Phi)$ is horizontal. Hence, Proposition 7.2 is a consequence of Proposition 5.2 and Remark 5.3 applied to F . \square

We can now define the positive measure μ by

$$\langle \mu, \varphi \rangle := M_\varphi.$$

Consider smooth forms $R \in \mathcal{C}_v^1(D')$ with support in $D' \setminus \mathcal{K}_+$ and $S \in \mathcal{C}_h^1(D'')$ with support in $D'' \setminus \mathcal{K}_-$. We have $\mu = \lim_{n \rightarrow \infty} d^{-2n}(f^n)^* R \wedge (f^n)_* S$. Hence, μ is supported in the boundary of $\mathcal{K} = \mathcal{K}_+ \cap \mathcal{K}_-$. Theorem 3.1 shows that μ is a probability measure.

We also have

$$\begin{aligned} f^*(\mu) &= \lim_{n \rightarrow \infty} d^{-2n} f^* ((f^n)^* R \wedge (f^n)_* S) = \lim_{n \rightarrow \infty} d^{-2n} (f^{n+1})^* R \wedge (f^{n-1})_* S \\ &= \lim_{n \rightarrow \infty} d^{-2n+2} (f^{n-1})^* (d^{-2} f^{2*} R) \wedge (f^{n-1})_* S = \mu. \end{aligned}$$

Hence, μ is invariant.

The following corollary gives us an extremality property of μ :

Corollary 7.3. *Let $(R_m) \subset \mathcal{C}_v^1(D)$ and $(S_n) \subset \mathcal{C}_h^1(D)$. Let ν be a limit value of $d^{-m-n}(f^m)^* R_m \wedge (f^n)_* S_n$ when $\min(m, n) \rightarrow \infty$. Then*

$$\langle \nu, \varphi \rangle \leq \langle \mu, \varphi \rangle \quad \text{for } \varphi \text{ p.s.h. on } D.$$

If φ is pluriharmonic, then $\langle \nu, \varphi \rangle = \langle \mu, \varphi \rangle$.

Proof. We can assume that φ is continuous since we can approximate it by a decreasing sequence of continuous p.s.h. functions. Proposition 7.2 implies that $\langle \nu, \varphi \rangle \leq \langle \mu, \varphi \rangle$. When φ is pluriharmonic, this inequality holds for $-\varphi$. Hence $\langle \nu, \varphi \rangle = \langle \mu, \varphi \rangle$. \square

The proof of the following results are left to the reader (see Corollaries 5.5, 5.7, 6.2 and Proposition 5.6).

Corollary 7.4. *Let R_n, R'_n in $\mathcal{C}_v^1(D)$ and S_n, S'_n in $\mathcal{C}_h^1(D)$ and $c > 0$ such that $R'_n \leq cR_n, S'_n \leq cS_n$ for every n . Let (m_i) and (n_i) be increasing sequences of integers. If*

$$d^{-m_i-n_i}(f^{m_i})^*R_{m_i} \wedge (f^{n_i})_*S_{n_i} \rightarrow \mu,$$

then

$$d^{-m_i-n_i}(f^{m_i})^*R'_{m_i} \wedge (f^{n_i})_*S'_{n_i} \rightarrow \mu.$$

Proposition 7.5. *Let R_m, S_n, m_i, n_i be as in Corollary 7.4. Let φ be a function strictly p.s.h. on D . Then,*

$$d^{-m_i-n_i}(f^{m_i})^*R_{m_i} \wedge (f^{n_i})_*S_{n_i} \rightarrow \mu$$

if and only if

$$\langle d^{-m_i-n_i}(f^{m_i})^*R_{m_i} \wedge (f^{n_i})_*S_{n_i}, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle.$$

Corollary 7.6. *Let (n_i) be an increasing sequence of integers. Then, there exist a subsequence (m_i) and a pluripolar set $\mathcal{E} \subset D$ such that, for every $(a, b) \in D \setminus \mathcal{E}$, we have*

$$d^{-2m_i}(f^{m_i})^*[z = a] \wedge (f^{m_i})_*[w = b] \rightarrow \mu$$

where (z, w) are the coordinates of $\mathbb{C}^p \times \mathbb{C}^{k-p}$.

To complete the proof of Theorem 7.1, we have only to check that μ is mixing. That is

$$\lim_{m \rightarrow \infty} \langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle = \langle \mu, \phi \rangle \langle \mu, \psi \rangle \quad (7.1)$$

for every functions ϕ and ψ smooth in a neighbourhood of \overline{D} . Define a function φ on D^2 by

$$\varphi(z_1, w_2, z_2, w_1) := \phi(z_1, w_1)\psi(z_2, w_2).$$

Lemma 7.7. *Assume that φ is p.s.h. Then*

$$\limsup_{m \rightarrow \infty} \langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle \leq \langle \mu, \phi \rangle \langle \mu, \psi \rangle.$$

Proof. Let $R \in \mathcal{C}_v^1(D')$ and $S \in \mathcal{C}_h^1(D'')$ be smooth forms. Define $T := R \otimes S$ and $T' = S \otimes R$. We have

$$\begin{aligned}
\langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle &= \lim_{n \rightarrow \infty} \langle d^{-2n}(F^n)^*T, (\varphi \circ F^m)[\Delta] \rangle \\
&= \lim_{n \rightarrow \infty} \langle d^{-2n}(F^n)^*((F^{n-m})^*T\varphi), [\Delta] \rangle \\
&= \lim_{n \rightarrow \infty} \langle d^{-2n}(F^{n-m})^*T\varphi, (F^m)_*[\Delta] \rangle \\
&= \lim_{n \rightarrow \infty} \langle d^{-2n}(F^{n-m})^*T \wedge (F^m)_*[\Delta], \varphi \rangle.
\end{aligned}$$

Applying Proposition 7.2 to F gives

$$\begin{aligned}
&\limsup_{m \rightarrow \infty} \langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle \\
&\leq \lim_{m \rightarrow \infty} \langle d^{-4m}(F^m)^*T \wedge (F^m)_*T', \varphi \rangle \\
&= \lim_{m \rightarrow \infty} \langle d^{-2m}(f^m)^*R \wedge (f^m)_*S, \phi \rangle \langle d^{-2m}(f^m)^*R \wedge (f^m)_*S, \psi \rangle \\
&= \langle \mu, \phi \rangle \langle \mu, \psi \rangle.
\end{aligned}$$

□

End of the proof of Theorem 7.1. Since ϕ and ψ can be written as differences of smooth strictly p.s.h. functions, in order to prove (7.1), it is sufficient to consider ϕ and ψ smooth strictly p.s.h. in a neighbourhood of \overline{D} . Let $A > 0$ be a large constant. Then, $(\phi(z_1, w_1) + A)(\psi(z_2, w_2) + A)$ is p.s.h. Lemma 7.7 implies that

$$\limsup_{m \rightarrow \infty} \langle \mu, (\phi \circ f^m + A)(\psi \circ f^{-m} + A) \rangle \leq \langle \mu, \phi + A \rangle \langle \mu, \psi + A \rangle.$$

Since μ is invariant, we have $\langle \mu, \phi \circ f^m \rangle = \langle \mu, \phi \rangle$ and $\langle \mu, \psi \circ f^{-m} \rangle = \langle \mu, \psi \rangle$. We deduce from the last inequality that

$$\limsup_{m \rightarrow \infty} \langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle \leq \langle \mu, \phi \rangle \langle \mu, \psi \rangle. \quad (7.2)$$

On the other hand, the function $(\phi(z_1, w_1) - A)(-\psi(z_2, w_2) + A)$ is also p.s.h. in a neighbourhood of \overline{D} . In the same way, we obtain

$$\limsup_{m \rightarrow \infty} -\langle \mu, (\phi \circ f^m)(\psi \circ f^{-m}) \rangle \leq -\langle \mu, \phi \rangle \langle \mu, \psi \rangle. \quad (7.3)$$

The inequalities (7.2) and (7.3) imply (7.1). Hence, μ is mixing. □

The following proposition generalizes the convergence in Theorem 7.1.

Proposition 7.8. *Let $(R_m) \subset \mathcal{C}_v^1(D')$ and $(S_n) \subset \mathcal{C}_h^1(D'')$ be uniformly bounded sequences of continuous forms. Then, $d^{-m-n}(f^m)^*R_m \wedge (f^n)_*S_n$ converges weakly to μ when $\min(m, n) \rightarrow \infty$.*

Proof. It is sufficient to consider the case where $m \leq n$ and $m \rightarrow \infty$. If we replace M', N'' by bigger domains, we can assume that there exist $c > 0$ and continuous forms $R \in \mathcal{C}_v^1(D')$ and $S \in \mathcal{C}_h^1(D'')$ such that $R_m \leq cR$ and $S_n \leq cS$ for every m and n . By Corollary 7.4, it is sufficient to prove that $d^{-m-n}(f^m)^*R \wedge (f^n)_*S \rightarrow \mu$. We will use the same idea as in Theorem 5.1.

Let φ be a continuous function strictly p.s.h. on D with $0 \leq \varphi \leq 1$. By Proposition 7.5, we only need to check that $\langle d^{-m-n}(f^m)^*R \wedge (f^n)_*S, \varphi \rangle \rightarrow M_\varphi$. Write

$$\begin{aligned} \langle d^{-m-n}(f^m)^*R \wedge (f^n)_*S, \varphi \rangle &= \langle R, d^{-m-n}(\varphi \circ f^{-m})(f^{m+n})_*S \rangle \\ &=: \langle R, \Psi_{m,n} \rangle. \end{aligned}$$

Observe that each $\Psi_{m,n}$ is positive, bounded by a current in $\mathcal{C}_h^1(D'')$ and verifies $\text{dd}^c \Psi_{m,n} \geq 0$. If R_θ is defined as in Section 2, then $\phi_{m,n}(\theta) := \langle R_\theta, \Psi_{m,n} \rangle$ define a uniformly bounded family of subharmonic functions on $\theta \in V$. Since $R = R_1$, we want to prove that $\phi_{m,n}(1) \rightarrow M_\varphi$. By Proposition 7.2

$$\limsup_{m,n \rightarrow \infty} \phi_{m,n}(\theta) \leq M_\varphi$$

and by Lemma 2.7

$$\limsup_{\theta \rightarrow 1} \limsup_{m,n} |\phi_{m,n}(\theta) - \phi_{m,n}(1)| = 0.$$

Hence, it is sufficient to prove that $\phi_{m,n}$ converge to M_φ in $\mathcal{L}_{loc}^1(V)$. By maximum principle, we only have to check that $\phi_{m,n}(0) = \langle R_0, \Psi_{m,n} \rangle \rightarrow M_\varphi$.

Consider a smooth form $R' \in \mathcal{C}_v^1(D')$ and define $R'_{m,n} := d^{m-n}(f^{n-m})^*R'$. Theorem 7.1 implies that $d^{-m-n}(f^m)^*R'_{m,n} \wedge (f^n)_*(S) \rightarrow \mu$. Let $R'_{m,n,\theta}$ be the currents of the structural discs associated to $R'_{m,n}$ constructed in Section 2. Then, $\phi'_{m,n}(\theta) := \langle R'_{m,n,\theta}, \Psi_{m,n} \rangle$ define a uniformly bounded family of subharmonic functions on $\theta \in V$. We also have $\limsup \phi'_{m,n}(\theta) \leq M_\varphi$ and $\lim \phi'_{m,n}(1) = M_\varphi$ since $R'_{m,n,1} = R'_{m,n}$. By maximum principle, $\phi'_{m,n} \rightarrow M_\varphi$ in $\mathcal{L}_{loc}^1(V)$. Lemma 2.5 implies that

$$\limsup_{\theta \rightarrow 0} \limsup_{m,n} |\phi'_{m,n}(\theta) - \phi'_{m,n}(0)| = 0.$$

Hence, $\langle R'_{m,n,0}, \Psi_{m,n} \rangle = \phi'_{m,n}(0) \rightarrow M_\varphi$. We have seen in Section 2 that $R_0 = R'_{m,n,0}$. It follows that $\langle R_0, \Psi_{m,n} \rangle \rightarrow M_\varphi$. \square

Corollary 7.9. *Let $S \in \mathcal{C}_h^1(D)$ be a continuous form. Then $d^{-n}T_+ \wedge (f^n)_*S$ converge weakly to μ .*

Proof. Let φ be a continuous strictly p.s.h. function on D . Let $R \in \mathcal{C}_v^1(D)$ be a smooth form. Corollary 6.1 implies that $T_+ = \lim d^{-n}f^{n*}(R)$. Hence, there exists $m > n$ such that

$$|\langle d^{-m-n}(f^m)^*R \wedge (f^n)_*S, \varphi \rangle - \langle d^{-n}T_+ \wedge (f^n)_*S, \varphi \rangle| \leq 1/n.$$

By Proposition 7.8, this implies that $\lim \langle d^{-n}T_+ \wedge (f^n)_*S, \varphi \rangle = \langle \mu, \varphi \rangle$. Proposition 7.5 implies that $\lim d^{-n}T_+ \wedge (f^n)_*S = \mu$. \square

We now show that the equilibrium measure is equal to the wedge product of the Green currents.

Theorem 7.10. *Let f be an invertible horizontal-like map and μ, T_+, T_- be as above. Then*

$$\mu = T_+ \wedge T_-.$$

Proof. Let φ be a continuous p.s.h. function on D . Let $R \in \mathcal{C}_v^1(D)$ and $S \in \mathcal{C}_h^1(D)$ be smooth forms. Corollary 6.1 and Theorem 7.1 implies that $d^{-n}(f^n)^*R \rightarrow T_+$, $d^{-n}(f^n)_*S \rightarrow T_-$ and $d^{-2n}(f^n)^*R \wedge (f^n)_*S \rightarrow \mu$. It follows from Theorem 3.1 that $\langle \mu, \varphi \rangle \leq \langle T_+ \wedge T_-, \varphi \rangle$.

On the other hand, we have $f^*T_+ = dT_+$ and $f_*T_- = dT_-$. Hence Proposition 7.2 imply that

$$\langle T_+ \wedge T_-, \varphi \rangle = \lim \langle d^{-2n}(f^n)^*T_+ \wedge (f^n)_*T_-, \varphi \rangle \leq \langle \mu, \varphi \rangle.$$

Theorem 7.10 follows. \square

8 Entropy

We will show that the topological entropy $h_t(f|_{\mathcal{K}})$ of the restriction of f to the invariant compact set \mathcal{K} is equal to $\log d$. From the variational principle [24], it follows that the entropy of μ is bounded from above by $\log d$. We will show that this measure has entropy $h(\mu) = \log d$. This also implies that $h_t(f|_{\text{supp}(\mu)}) = \log d$.

Theorem 8.1. *Let f, d, \mathcal{K}, μ be as above. Then, the topological entropy of $f|_{\mathcal{K}}$ is equal to $\log d$ and μ is an invariant measure of maximal entropy $\log d$.*

We have to prove that $h_t(f|_{\mathcal{K}}) \leq \log d$ and $h(\mu) \geq \log d$. Using Yomdin's results [25], Bedford-Smillie proved the second inequality for Hénon maps [2] (see also Smillie [23]). We only need the following lemma applied to a closed form S strictly positive in a neighbourhood of \mathcal{K}_- in order to adapt their proof and get $h(\mu) \geq \log d$.

Lemma 8.2. *Let $S \in \mathcal{C}_h^1(D'')$ be a smooth form. Then, there exist an increasing sequence (n_i) of positive integers and a point $a \in M'$ such that*

$$\frac{1}{n_i} \sum_{j=0}^{n_i-1} d^{-n_i}(f^j)^*[z = a] \wedge (f^{n_i-j})_*S \rightarrow \mu.$$

Proof. Let φ be a smooth strictly p.s.h. function on D . Define a sequence of p.s.h. functions, for $a \in M$ (see Theorem 2.1):

$$\phi_n(a) := \left\langle \frac{1}{n} \sum_{j=0}^{n-1} d^{-n}(f^j)^*[z = a] \wedge (f^{n-j})_*S, \varphi \right\rangle.$$

Let ν be a smooth probability measure on M' . Consider the smooth form $R := \pi_1^*(\nu)$ in $\mathcal{C}_v^1(D')$. Proposition 7.8 implies that

$$\frac{1}{n} \sum_{j=0}^{n-1} d^{-n} (f^j)^* R \wedge (f^{n-j})_* S \rightarrow \mu.$$

Hence, $\int \phi_n(a) d\nu(a) \rightarrow M_\varphi$. On the other hand, Proposition 7.2 implies that

$$\limsup_{n \rightarrow \infty} \sup_{a \in M'} \phi_n(a) \leq M_\varphi.$$

It follows that there exist (n_i) and $a \in M'$ such that $\lim \phi_{n_i}(a) = M_\varphi$. As in Propositions 5.6 and 7.5, we prove that (n_i) and a satisfy the lemma. \square

Now, we prove the first inequality $h_t(f|_{\mathcal{X}}) \leq \log d$. Analogous inequalities have been proved in [16, 6, 7, 8]. We use here some arguments in Gromov [16] and in [6].

Let $\Gamma_{[n]}$ be the graph of the map $x \mapsto (f(x), \dots, f^{n-1}(x))$. This is the set of points $(x, f(x), \dots, f^{n-1}(x))$. We use the canonical euclidian metric on D^n . Let $D_* := M' \times N''$. We have $\mathcal{X} \subset D_* \Subset D$. Define

$$\text{lov}(f) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{volume}(\Gamma_{[n]} \cap D_*).$$

Following Gromov [16, 6], we have $h_t(f|_{\mathcal{X}}) \leq \text{lov}(f)$. We will show that $\text{lov}(f) \leq \log d$; then $h_t(f|_{\mathcal{X}}) = \text{lov}(f) = \log d$ since $h_t(f|_{\mathcal{X}}) \geq h(\mu) \geq \log d$.

Let Π denote the projection of $D^n = (M \times N)^n$ on the product $M \times N$ of the last factor M and the first factor N . Let Π_1 (resp. Π_2) denote the projections of D^n on the product M^{n-1} (resp. N^{n-1}) of the other factors M (resp. N). Observe that $\Pi : \Gamma_{[n]} \rightarrow M \times N$ is proper and defines a ramified covering of degree d^{n-1} over $M \times N$. Indeed, for a generic point $(a, b) \in M \times N$ the fiber $\Pi^{-1}(a, b) \cap \Gamma_{[n]}$ contains a number of points equal to the number of points in $\{z = a\} \cap f^{n-1}\{w = b\}$, i.e. equal to d^{n-1} (see Proposition 4.2). Moreover, we have $\Gamma_{[n]} \subset \Pi_1^{-1}(M^{n-1})$ and $\Gamma_{[n]} \subset \Pi_2^{-1}(N^{n-1})$. Now, it is sufficient to apply the following lemma (see [6, lemme 3.3.3] for the proof).

Lemma 8.3. *Let Γ be an analytic subset of dimension k of $D \times M^m \times N^m$ such that $\Gamma \subset D \times M^m \times N^m$. We assume that Γ is a ramified covering over D of degree d_Γ . Then, there exist $c > 0$, $s > 0$ independent of Γ and of m such that*

$$\text{volume}(\Gamma \cap D_* \times M^m \times N^m) \leq cm^s d_\Gamma.$$

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