HOLOMORPHIC CURVES IN COMPLEX SPACES

BARBARA DRINOVEC-DRNOVŠEK & FRANC FORSTNERIČ

Abstract. We study the existence of closed complex curves normalized by bordered Riemann surfaces in complex spaces. Our main result is that such curves abound in any non compact complex space admitting an exhaustion function whose Levi form has at least two positive eigenvalues at every point outside a compact set, and this condition is essential. We also construct a Stein neighborhood basis of any compact complex curve with $\mathcal{C}^2$ boundary in a complex space.

To Josip Globevnik for his 60th birthday

1. Introduction

Let X be an irreducible (reduced, paracompact) complex space of dimension $\dim X > 1$. For every topologically closed complex curve C in X we have a sequence of holomorphic maps

$$\{\mathbb{CP}^1, \mathbb{C}, \triangle\} \ni \widetilde{D} \to D \to C \hookrightarrow X$$

where $C \hookrightarrow X$ is the inclusion, $D \to C$ is a normalization of C by a Riemann surface D, and $\widetilde{D} \to D$ is a universal covering map (the uniformization map). Here $\triangle = \{z \in \mathbb{C} \colon |z| < 1\}$. Thus C is the image of a generically one to one proper holomorphic map $D \to X$; hence it is natural to ask which Riemann surfaces D admit any proper holomorphic maps to a given complex space, and how plentiful are they. This question has been investigated most intensively for compact complex curves $C \subset X$ which form a part of the *Douady space* and of the *cycle space* of X ([4], [7], [16]).

In this paper we obtain essentially optimal existence and approximation results in the case when D is a *finite bordered Riemann surface*, hence uniformized by the disc $\triangle$. This requires that X be non compact.

We begin by a brief survey of the known results. Every open Riemann surface properly immerses in $\mathbb{C}^2$ and embeds in $\mathbb{C}^3$ [6], [60]. Turning to more general spaces, we note that the Kobayashi hyperbolicity of X excludes curves uniformized by $\mathbb{C}$, but it imposes less restrictions on those uniformized by $\triangle$ [49], [50]. There are other, less tangible obstructions: Dor [15] found a bounded domain with non smooth boundary in $\mathbb{C}^n$ without any proper

Date: June 23, 2006.

2000 *Mathematics Subject Classification.* 32C25, 32F32, 32H02, 32H35, 14H55.

Key words and phrases. Complex spaces, holomorphic curves, holomorphic mappings.

Research supported by grants P1-0291 and J1-6173, Republic of Slovenia.

holomorphic images of $\triangle$; even in smoothly bounded (non pseudoconvex) domains in $\mathbb{C}^n$ the union of images of all proper analytic discs can omit a nonempty open subset [25].

On the positive side, every point in a Stein manifold X of dimension > 1 is contained in the image of a proper holomorphic map $\triangle \to X$ (Globevnik [33]; see also [14], [17], [18], [19], [25], [26], [27]). The same holds for discs in any connected complex manifold X which is q-complete for some $q < \dim X$ [19]. The first case of interest, inaccessible with the existing techniques, are Stein spaces with singularities.

Recall that a smooth function $\rho\colon X \to \mathbb{R}$ on a complex space X is said to be *q-convex* on an open subset $U \subset X$ (in the sense of Andreotti-Grauert [2], [36, def. 1.4, p. 263]) if there is a covering of U by open sets $V_j \subset U$, biholomorphic to closed analytic subsets of open sets $\Omega_j \subset \mathbb{C}^{n_j}$, such that for each j the restriction $\rho|_{V_j}$ admits an extension $\widetilde{\rho}_j\colon \Omega_j \to \mathbb{R}$ whose Levi form $i\partial\bar\partial\,\widetilde{\rho}_j$ has at most $q-1$ negative or zero eigenvalues at each point of Ω_j. The space X is *q-complete*, resp. *q-convex*, if it admits a smooth exhaustion function $\rho\colon X \to \mathbb{R}$ which is q-convex on X, resp. on $\{x \in X\colon \rho(x) > c\}$ for some $c \in \mathbb{R}$. A 1-complete complex space is just a Stein space, and a 1-convex space is a proper modification of a Stein space. We denote by X_{reg} (resp. X_{sing}) the set of regular (resp. singular) points of X.

A *bordered Riemann surface* is a compact one dimensional complex manifold $\bar D = D \cup bD$ with smooth boundary bD consisting of finitely many closed curves; its interior $D = \bar D \backslash bD$ is an open Riemann surface uniformized by $\triangle$.

We are now ready to state our first main result; it is proved in §6.

Theorem 1.1. *Let X be an irreducible complex space of $\dim X > 1$, and let $\rho\colon X \to \mathbb{R}$ be a smooth exhaustion function which is $(n-1)$-convex on $X_c = \{x \in X\colon \rho(x) > c\}$ for some $c \in \mathbb{R}$. Given a bordered Riemann surface D and a $\mathcal{C}^2$ map $f\colon \bar D \to X$ which is holomorphic in D and satisfies $f(D) \not\subset X_{sing}$ and $f(bD) \subset X_c$, there is a sequence of proper holomorphic maps $g_\nu\colon D \to X$ homotopic to $f|_D$ and converging to f uniformly on compacts in D as $\nu \to \infty$. Given an integer $k \in \mathbb{N}$ and finitely many points $\{z_j\} \subset D$, each g_ν can be chosen to have the same k-jet as f at each of the points z_j.*

The condition on ρ in Theorem 1.1 means that its Levi form has at least two positive eigenvalues at every point of $X_c = \{\rho > c\}$. One positive eigenvalue does not suffice in view of Dor's example of a domain in $\mathbb{C}^n$ without any proper analytic discs [15] and the fact that every domain in $\mathbb{C}^n$ is n-complete ([37], [63]). Necessity of the hypothesis $f(D) \not\subset X_{sing}$ is seen by Proposition 3 in [32] (based on an example of Kaliman and Zaidenberg [47]): An analytic disc contained in X_{sing} may be forced to remain there under analytic perturbations, and it need not be approximable by proper holomorphic maps $\triangle \to X$. The only possible (but inessential) improvement would be a reduction of the boundary regularity assumption on the initial map. If D is a planar domain bounded by finitely many Jordan curves and

X is a manifold, it suffices to assume that f is continuous on $\bar{D}$ by appealing to [8, Theorem 1.1.4] in order to approximate f by a more regular map.

If $f\colon \bar{D} \to X$ is generically injective then so is any proper holomorphic map $g_\nu\colon D \to X$ approximating f sufficiently closely; its image $g_\nu(D) = C_\nu \subset X$ is then a closed complex curve in X normalized by D. Assuming that $f(\bar{D}) \subset X_{reg}$ one can choose each g_ν to be an immersion, and even an embedding when $n \geq 3$. Each map g_ν will be a locally uniform limit in D of a sequence of $\mathcal{C}^2$ maps $f_j\colon \bar{D} \to X$ which are holomorphic in D and satisfy

$$\lim_{j\to\infty} \inf\{\rho \circ f_j(z)\colon z \in bD\} \to +\infty; \tag{1.1}$$

that is, their boundaries $f_j(bD)$ tend to infinity in X. Embedding $\bar{D}$ as a domain in an open Riemann surface S, we can choose each f_j to be holomorphic in open set $U_j \subset S$ containing $\bar{D}$.

Theorem 1.1 gives new information on *algebraic curves* in $(n-1)$-convex *quasi projective algebraic spaces* $X = Y\backslash Z$, where $Y, Z \subset \mathbb{CP}^N$ are closed complex (=algebraic) subvarieties in a complex projective space. We embed our bordered Riemann surface D as a domain with smooth real analytic boundary in its *double* $\hat{S}$, a compact Riemann surface obtained by gluing two copies of $\bar{D}$ along their boundaries [5, p. 581], [72, p. 217]. There is a meromorphic embedding $\hat{S} \hookrightarrow \mathbb{CP}^3$ with poles outside of $\bar{D}$; the subset $S \subset \hat{S}$ which is mapped to the affine part $\mathbb{C}^3 \subset \mathbb{CP}^3$ is a smooth affine algebraic curve, and $\bar{D}$ is holomorphically convex in S. A holomorphic map $f\colon U \to X$ from an open set $U \subset S$ to a quasi projective algebraic space X is said to be *Nash algebraic* (Nash [62]) if the graph

$$G_f = \{(z, f(z)) \in S \times X\colon z \in U\}$$

is contained in a one dimensional algebraic subvariety of $S \times X$.

Corollary 1.2. *Let X be an irreducible quasi projective algebraic space of $\dim X > 1$, and let D be a smoothly bounded Runge domain in an affine algebraic curve S. Assume that $\rho\colon X \to \mathbb{R}$ and $f\colon \bar{D} \to X$ satisfy the hypotheses of Theorem 1.1. Then there is a sequence of Nash algebraic maps $f_j\colon U_j \to X$ in open sets $U_j \supset \bar{D}$ satisfying (1.1) such that the sequence $f_j|_D$ converges to a proper holomorphic map $g\colon D \to X$.*

Of course we can choose g such that it also satisfies the additional properties in Theorem 1.1. Here is a geometric interpretation of Corollary 1.2. If $\Gamma_j \subset S \times X$ is an algebraic curve containing the graph of the Nash algebraic map $f_j\colon U_j \to X$ then its projection $C_j \subset X$ under the map $(z,x) \to x$ is an algebraic curve in X containing $f_j(U_j)$. As $j \to \infty$, the domains $f_j(D) \subset C_j$ converge to the closed transcendental curve $g(D) \subset X$ while their boundaries $f_j(bD)$ leave any compact subset of X.

Corollary 1.2 applies for example to $X = \mathbb{CP}^n\backslash A$ where A is a closed complex submanifold of dimension $d \in \{[\frac{n+1}{2}], \ldots, n-1\}$. Indeed, $\mathbb{CP}^n\backslash A$

is $(2(n-d)-1)$-complete [64], and hence is $(n-1)$-complete if $n \le 2d$. For further examples see [11].

Corollary 1.2 is obtained by approximating each of the holomorphic maps $f_j \colon U_j \to X$, obtained in the proof of Theorem 1.1, uniformly on $\bar D$ by a Nash algebraic map, appealing to a theorem of Demailly, Lempert and Shiffman [13, Theorem 1.1] and Lempert [53]. Their results give Nash algebraic approximations of any holomorphic map from an open Runge domain in an affine algebraic variety to a quasi projective algebraic space.

The following consequence of Theorem 1.1 was proved in [19] in the special case $X_{sing} = \emptyset$ and $D = \triangle$.

Corollary 1.3. *Let X be an irreducible $(n-1)$-complete complex space of dimension $n > 1$, and let D be a bordered Riemann surface. Given a $\mathcal{C}^2$ map $f \colon \bar D \to X$ which is holomorphic in D and satisfies $f(D) \not\subset X_{sing}$, a positive integer $k \in \mathbb{N}$ and finitely many points $\{z_j\} \subset D$, there is a sequence of proper holomorphic maps $g_\nu \colon D \to X$ converging to $f|_D$ uniformly on compacts in D such that each g_ν has the same k-jets as f at each of the points z_j. This holds in particular if X is a Stein space.*

Let X be a complex manifold. The *Kobayashi-Royden pseudonorm* of a tangent vector $v \in T_xX$ is given by

$$\kappa_X(v) = \inf\left\{\lambda > 0 \colon \exists f \colon \triangle \to X \text{ holomorphic, } f(0) = x,\ f'(0) = \lambda^{-1}v\right\}.$$

The same quantity is obtained by using only maps which are holomorphic in small neighborhoods of $\bar\triangle$ in $\mathbb{C}$. Corollary 1.3 implies:

Corollary 1.4. *If X is an $(n-1)$-complete complex manifold of dimension $n > 1$ then its infinitesimal Kobayashi-Royden pseudometric κ_X is computable in terms of* proper *holomorphic discs $f \colon \triangle \to X$.*

On a quasi projective algebraic manifold X, the pseudometric κ_X and its integrated form, the Kobayashi pseudodistance, are also computable by algebraic curves [13, Corollary 1.2].

It is natural to inquire which homotopy classes of maps $D \to X$ from a bordered Riemann surface admit a proper holomorphic representative. Hyperbolicity properties of X ([49], [50], [20]) impose serious obstructions on the existence of a holomorphic map in the given homotopy class. The following opposite property is important in the Oka-Grauert theory:

A complex manifold X is said to enjoy the *m-dimensional convex approximation property* (CAP_m) if every holomorphic map $U \to X$ from an open set $U \subset \mathbb{C}^m$ can be approximated uniformly on any compact convex set $K \subset U$ by entire maps $\mathbb{C}^m \to X$ [24].

Corollary 1.5. *Let X be an $(n-1)$-complete complex manifold of dimension $n > 1$. If X satisfies CAP_{n+1} then for every continuous map $f \colon D \to X$ from a bordered Riemann surface D there exists a proper holomorphic map $g \colon D \to X$ homotopic to f. If f is holomorphic on a neighborhood of a* *compact subset $K \subset D$ then g can be chosen to approximate f as close as desired on K. This holds in particular if $X = \mathbb{CP}^n \backslash A$ where $n \geq 4$ and $A \subset \mathbb{CP}^n$ is a closed complex submanifold of dimension $d \in \{[\frac{n+1}{2}], \ldots, n-2\}$.*

Proof. We may assume that $\bar{D} = \{z \in S\colon v(z) \leq 0\}$ where S is an open Riemann surface and $v\colon S \to \mathbb{R}$ is a smooth function with $dv \neq 0$ on $bD = \{v = 0\}$. Choose numbers $c_0 < 0 < c_1$ close to 0 such that v has no critical values on $[c_0, c_1]$. Let $D_j = \{z \in S\colon v(z) < c_j\}$ for $j = 0, 1$. We may assume $K \subset D_0$. There is a homotopy of smooth maps $\tau_t\colon D_1 \to D_1$ ($t \in [0,1]$) such that τ_0 is the identity on D_1, $\tau_1(D_1) = D_0$, and for all $t \in [0,1]$ we have $\tau_t(D) \subset D$ and τ_t equals the identity map near K. Set $\widetilde{f} = f \circ \tau_1\colon D_1 \to X$. Note that $\widetilde{f}|_D$ is homotopic to f via the homotopy $f \circ \tau_t|_D$ ($t \in [0,1]$).

By the main result of [24] the CAP_{n+1} property of X implies the existence of a holomorphic map $f_1\colon D_1 \to X$ homotopic to $\widetilde{f}\colon D_1 \to X$. Then $f_1|_D$ is homotopic to $\widetilde{f}|_D$ and hence to f. Theorem 1.1, applied to the map $f_1|_{\bar{D}}\colon \bar{D} \to X$, furnishes a proper holomorphic map $g\colon D \to X$ homotopic to $f_1|_D$, and hence to f. In addition, f_1 and g can be chosen to approximate f uniformly on K.

The last statement follows from the already mentioned fact that $\mathbb{CP}^n \backslash A$ is $(n-1)$-complete if A is as in the statement of the corollary (see [64]), and it enjoys CAP_m for all $m \in \mathbb{N}$ provided that $\dim A \leq n-2$ [24]. $\square$

By [24] the property $\mathrm{CAP} = \cap_{m=1}^{\infty} \mathrm{CAP}_m$ of a complex manifold X is equivalent to the classical *Oka property* concerning the existence and the homotopy classification of holomorphic maps from Stein manifolds to X. Examples in [38] and [24] show that Corollary 1.5 fails in general if X does not enjoy CAP, and the most one can expect is to find a proper map $D \to X$ in the given homotopy class which is holomorphic with respect to some choice of the complex structure on the given smooth 2-surface D. This indeed follows by combining Theorem 1.1 with the main result of [31].

Corollary 1.6. *Let X be a $(n-1)$-complete complex manifold of dimension $n > 1$ and let $\bar{D}$ be a compact, connected, oriented real surface with boundary. For every continuous map $f\colon D \to X$ there exist a complex structure J on D and a proper J-holomorphic map $g\colon D \to X$ which is homotopic to f.*

Another result of independent interest is Theorem 2.1 to the effect that a compact complex curve with $\mathcal{C}^2$ boundary in a complex space admits a basis of open Stein neighborhoods. The following special case is proved in §2.

Theorem 1.7. *Let X be an n-dimensional complex manifold. If D is a relatively compact, smoothly bounded domain in an open Riemann surface S and $f\colon \bar{D} \hookrightarrow X$ is a $\mathcal{C}^2$ embedding which is holomorphic in D then $f(\bar{D})$ has a basis of open Stein neighborhoods in X which are biholomorphic to domains in $S \times \mathbb{C}^{n-1}$. In particular, if D is a smoothly bounded planar* *domain then* $f(\bar{D})$ *has a basis of open Stein neighborhoods in* X *which are biholomorphic to domains in* $\mathbb{C}^n$.

Royden showed in [69] that for any holomorphically embedded polydisc $f\colon \triangle^k \hookrightarrow X$ in a complex manifold X and for any $r < 1$ the smaller polydisc $f(r\triangle^k) \subset X$ admits open neighborhoods in X biholomorphic to $\triangle^n$ with $n = \dim X$. We have the analogous result for closed analytic discs, showing that they have no appreciation whatsoever of their surroundings.

Corollary 1.8. *Let* X *be an* n*-dimensional complex manifold. For every* $\mathcal{C}^2$ *embedding* $f\colon \bar{\triangle} \hookrightarrow X$ *which is holomorphic in* $\triangle$ *the image* $f(\bar{\triangle})$ *has a basis of open neighborhoods in* X *which are biholomorphic to* $\triangle^n$.

These and related result are used to obtain new holomorphic approximation theorems (Corollary 2.7 and Theorem 5.1).

Outline of proof of Theorem 1.1. Theorem 1.1 is proved in §6 after developing the necessary tools in §2–§5.

We begin by perturbing the initial map $f\colon \bar{D} \to X$ to a new map for which $f(bD) \subset X_{reg}$ (Theorem 5.1). The rest of the construction is done in such a way that the image of bD remains in the regular part of X. A proper holomorphic map $g\colon D \to X$ is obtained as a limit $g = \lim_{j\to\infty} f_j|_D$ of a sequence of $\mathcal{C}^2$ maps $f_j\colon \bar{D} \to X$ which are holomorphic in D such that the boundaries $f_j(bD)$ converge to infinity.

Our local method of lifting the boundary $f(bD)$ is similar to the one used (in the special case $D = \triangle$) in earlier papers on the subject ([14], [17], [18], [25], [26], [33]). Since the Levi form $\mathcal{L}_\rho$ is assumed to have at least two positive eigenvalues at every point of $f(bD)$, we get at least one positive eigenvalue in a direction tangential to the level set of ρ at each point $f(z)$, $z \in bD$; this gives a small analytic disc in X, tangential to the level set of ρ at $f(z)$, along which ρ increases quadratically. By solving a certain Riemann-Hilbert boundary value problem we obtain a local holomorphic map whose boundary values on the relevant part of bD are close to the boundaries of these discs, and hence $\rho \circ f$ has increased there. (One positive eigenvalue of $\mathcal{L}_\rho$ does not suffice since the corresponding eigenvector may be transverse to the level set of ρ and cannot be used in the construction.)

To globalize the construction we develop a new method of patching holomorphic maps by improving a technique from the recent work of the second author on localization of the Oka principle [24]. We embed a given map $f\colon \bar{D} \to X$ into a *spray of maps*, i.e., a family of maps $f_t\colon \bar{D} \to X$ depending holomorphically on the parameter t in a Euclidean space and satisfying a certain submersivity property (dominability) outside of an exceptional subvariety. The local modification method explained above gives a new spray near a part of the boundary bD; by insuring that the two sprays are sufficiently close to each other on the intersection of their domains $\overline{D_0 \cap D_1}$, we patch them into a new spray over $\overline{D_0 \cup D_1}$ (Proposition 4.3). This is accomplished by finding a fiberwise biholomorphic transition map between them and decomposing it into a pair of maps over $\bar{D}_0$ resp. $\bar{D}_1$ which are used to correct the two sprays so as to make them agree over $\overline{D_0 \cap D_1}$.

The most difficult step — a decomposition of the transition map (Theorem 3.2) — is achieved by a rapidly convergent iteration. (It cannot be obtained by the implicit function theorem due to inherent shrinking of the domain in the fiber direction.) Unlike in [24, Lemma 2.1], the base domains don't shrink in our present construction — this is not allowed since all action in the construction of proper maps takes place at the boundary. Theorem 3.2 generalizes the classical Cartan's lemma to non linear maps, with $\mathcal{C}^r$ estimates up to the boundary.

Our method of gluing sprays is also useful in proving holomorphic approximation theorems; see e.g. Theorem 5.1.

One of the difficult problems in earlier papers has been to avoid running into a critical point of the given exhaustion function $\rho\colon X \to \mathbb{R}$. For Stein manifolds this problem was solved by Globevnik [33]. Here we use an alternative method from [22]; we cross each critical level by using a different function constructed especially for this purpose.

We hope that the methods developed in this paper will be applicable in other problems involving geometric control of the range of a holomorphic map. Most of the new technical tools are obtained in the context of smoothly bounded strongly pseudoconvex domains in Stein manifolds since the proofs are not essentially more difficult than for open Riemann surfaces.

2. Stein neighborhoods of smoothly bounded complex curves

Let $(X, \mathcal{O}_X)$ be a complex space. We denote by $\mathcal{O}(X)$ the algebra of all holomorphic functions on X, endowed with the compact-open topology. A compact subset K of X is said to be $\mathcal{O}(X)$*-convex* if for any point $p \in X\backslash K$ there exists $f \in \mathcal{O}(X)$ with $|f(p)| > \sup_K |f|$. If X is Stein and K is contained in a closed complex subvariety X' of X then K is $\mathcal{O}(X')$-convex if and only if it is $\mathcal{O}(X)$-convex. (For Stein spaces we refer to [39] and [46].)

We will say that a compact set A in a complex space X is a *complex curve with $\mathcal{C}^r$ boundary bA in X* if

- (i) $A\backslash bA$ is a closed, purely one dimensional complex subvariety of $X\backslash bA$ without compact irreducible components, and
- (ii) every point $p \in bA$ has an open neighborhood $V \subset X$ and a biholomorphic map $\phi\colon V \to V' \subset \Omega \subset \mathbb{C}^N$ onto a closed complex subvariety V' in an open subset $\Omega \subset\subset \mathbb{C}^N$ such that $\phi(A \cap V)$ is a one dimensional complex submanifold of Ω with $\mathcal{C}^r$ boundary $\phi(bA \cap V)$.

Note that bA consists of finitely many closed Jordan curves and has no isolated points, but it may contain some singular points of X.

Theorem 2.1. *Let A be a compact complex curve with $\mathcal{C}^2$ boundary in a complex space X. Let K be a compact $\mathcal{O}(\Omega)$-convex set in a Stein open set $\Omega \subset X$. If $bA \cap K = \emptyset$ and $A \cap K$ is $\mathcal{O}(A)$-convex then $A \cup K$ has a fundamental basis of open Stein neighborhoods ω in X.*

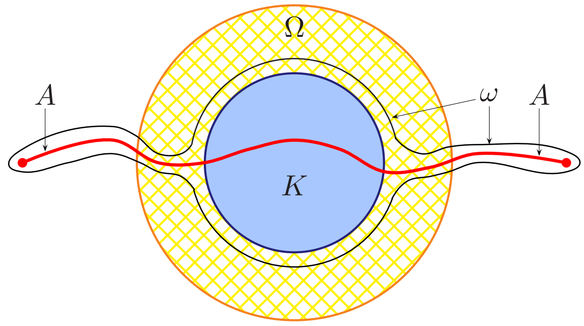

FIGURE 1. Theorem 2.1

Theorem 2.1 is the main result of this section (but see also Theorem 2.6). For $X = \mathbb{C}^n$ this follows from results of Wermer [74] and Stolzenberg [73]. We shall only use the special case with $K = \emptyset$, but the proof of the general case is not essentially more difficult and we include it for future applications. The necessity of $\mathcal{O}(A)$-convexity of $K \cap A$ is seen by taking $X = \mathbb{C}^2$, $A = \{(z, 0) \colon |z| \le 3\}$, and $K = \{(z, w) \colon 1 \le |z| \le 2,\ |w| \le 1\}$: Every Stein neighborhood of $A \cup K$ contains the bidisc $\{(z, w) \colon |z| \le 2,\ |w| \le 1\}$.

In this connection we mention a result of Siu [71] to the effect that a closed Stein subspace (withouth boundary) of any complex space admits an open Stein neighborhood. Extensions to the q-convex case and simplifications of the proof were given by Colţoiu [10] and Demailly [12]. These results do not apply directly to subvarieties with boundaries.

Proof. We shall adapt the proof of Theorem 2.1 in [23]. (It is based on the proof of Siu's theorem [71] given in [12].) We begin by preliminary results. We have $bA = \cup_{j=1}^m C_j$ where each C_j is a closed Jordan curve of class $\mathcal{C}^2$ (a diffeomorphic image of the circle $T = \{z \in \mathbb{C} \colon |z| = 1\}$).

Lemma 2.2. *There are a Stein open neighborhood $U_j \subset X$ of C_j, with $\overline{U}_j \cap K = \emptyset$, and a holomorphic embedding $Z = (z, w) \colon U_j \to \mathbb{C}^{1+n_j}$ for some $n_j \in \mathbb{N}$ such that $Z(U_j)$ is a closed complex subvariety of the set*

$$U'_j = \{(z, w) \in \mathbb{C}^{1+n_j} : 1 - r_j < |z| < 1 + r_j,\ |w_1| < 1, \ldots,\ |w_{n_j}| < 1\}$$

for some $0 < r_j < 1$, and

$$Z(A \cap U_j) = \{(z, w) \in U'_j \colon z \in \Gamma_j,\ w = g_j(z)\}$$

where

$$\Gamma_j = \{z = re^{i\theta} \in \mathbb{C} \colon 1 - r_j < r \le h_j(\theta)\},$$

h_j is a $\mathcal{C}^2$ function close to 1 (in particular, $|h_j(\theta)-1| < r_j$ for every $\theta \in \mathbb{R}$), and $g_j = (g_{j,1},\dots,g_{j,n_j})\colon \Gamma_j \to \triangle^{n_j}$ is a $\mathcal{C}^2$ map which is holomorphic in the interior of Γ_j.

Proof. Note that C_j is a totally real submanifold of class $\mathcal{C}^2$ in X and hence it admits a basis of open Stein neighboorhoods in X. Therefore a Stein neighborhood $U_j \subset\subset X$ of C_j embeds holomorphically to a Euclidean space $\mathbb{C}^{1+n_j}$. Denote by $C'_j \subset \mathbb{C}^{1+n_j}$ (resp. by A') the image of C_j (resp. of $A \cap U_j$) under this embedding. We identify the circle T with $T \times \{0\}^{n_j} \subset \mathbb{C}^{1+n_j}$.

The complexified tangent bundle to C'_j, and the complex normal bundle to C'_j in $\mathbb{C}^{1+n_j}$, are trivial (since every complex vector bundle over a circle is trivial). Using standard techniques for totally real submanifolds (see e.g. [29]) we find a $\mathcal{C}^2$ diffeomorphism Φ_j from a tube around C'_j in $\mathbb{C}^{1+n_j}$ onto a tube around the circle T such that $\Phi_j(C'_j) = T$, and such that $\bar\partial\,\Phi_j$ and its total first derivative $D^1(\bar\partial\,\Phi_j)$ vanish on C'_j.

By Theorems 1.1 and 1.2 in [29] we can approximate Φ_j in a tube around C'_j by a biholomorphic map Φ'_j which maps C'_j very close to T and which spreads a collar around C'_j in A' as a graph over an annular domain in the first cordinate axis. Composing the initial embedding $U_j \hookrightarrow \mathbb{C}^{1+n_j}$ with Φ'_j we obtain (after shrinking U_j around C_j) the situation in the lemma. □

Using the notation in Lemma 2.2 we set

$$\Lambda_j = \{x \in U_j \colon z(x) \in \Gamma_j\} \subset X, \tag{2.1}$$

$$\phi_j(x) = w(x) - g_j(z(x)) \in \mathbb{C}^{n_j}, \quad x \in \Lambda_j. \tag{2.2}$$

We can extend $|\phi_j|^2$ to a $\mathcal{C}^2$ function on U_j which is positive on $U_j\backslash\Gamma_j$. Choose additional open sets $U_{m+1},\dots,U_N$ in X whose closures do not intersect any of the sets $U_j\backslash\Lambda_j$ for $j = 1,\dots,m$ such that $A \cup K \subset \cup_{j=1}^N U_j$. By choosing these sets sufficiently small we also get for each $j \in \{m+1,\dots,N\}$ a holomorphic map $\phi_j \colon U_j \to \mathbb{C}^{n_j}$ whose components generate the ideal sheaf of A at every point of U_j. If $U_j \cap A = \emptyset$ for some j, we take $n_j = 1$ and $\phi_j(x) = 1$. Choose slightly smaller open sets $V_j \subset\subset U_j$ ($j = 1,\dots,N$) such that $A \cup K \subset \cup_{j=1}^N V_j$. Choose an open set $V \subset X$ with $A \cup K \subset V \subset\subset \cup_{j=1}^N V_j$ and let

$$\Lambda = \bigcup_{j=1}^m (\overline{V} \cap \Lambda_j) \cup \bigcup_{j=m+1}^N (\overline{V} \cap V_j). \tag{2.3}$$

Lemma 2.3. *There is a family of $\mathcal{C}^2$ functions $v_\delta \colon V \to \mathbb{R}$ ($\delta \in (0,1]$) and a constant $M > -\infty$ such that $i\partial\bar\partial\, v_\delta \geq M$ on Λ for all $\delta \in (0,1)$, and such that $v_0(x) = \lim_{\delta\to 0} v_\delta(x)$ is of class $\mathcal{C}^2$ on $V\backslash A$ and satisfies $v_0|_A = -\infty$.*

Proof. We adapt the proof of Lemma 5 in [12]. Let rmax denote a regularized maximum (p. 286 in [12]); this function is increasing and convex in all variables (hence it preserves plurisubharmonicity), and it can be chosen as close as desired to the usual maximum. On every set V_j we choose a smooth function $\tau_j\colon V_j \to \mathbb{R}$ which tends to $-\infty$ at bV_j. For each $\delta \in [0,1]$ we set

$$v_{\delta,j}(x) = \log(\delta + |\phi_j(x)|^2) + \tau_j(x), \quad x \in V_j,$$

and $v_\delta(x) = \mathrm{rmax}(\ldots, v_{\delta,j}(x), \ldots)$, where the regularized maximum is taken over all indices $j \in \{1,\ldots,N\}$ for which $x \in V_j$. As $\delta \to 0$, v_δ decreases to v_0 and $\{v_0 = -\infty\} = A$. Since the generators ϕ_j and ϕ_k for the ideal sheaf of A can be expressed in terms of one another on $U_j \cap U_k$, the quotient $|\phi_j|/|\phi_k|$ is bounded on $\overline{V}_j \cap \overline{V}_k$, and hence $(\delta + |\phi_j|^2)/(\delta + |\phi_k|^2)$ is bounded on $\overline{V}_j \cap \overline{V}_k$ uniformly with respect to $\delta \in [0,1]$. Since τ_j tends to $-\infty$ along bV_j, none of the values $v_{\delta,j}(x)$ for x sufficiently near bV_j contributes to the value of $v_\delta(x)$ since the other functions take over in rmax, and this property is uniform with respect to $\delta \in [0,1]$. Since $\log(\delta + |\phi_j(x)|^2)$ is plurisubharmonic on Λ_j if $j \in \{1,\ldots,m\}$, resp. on U_j if $j \in \{m+1,\ldots,N\}$, we have $i\partial\bar\partial\, v_{\delta,j} = i\partial\bar\partial\, \tau_j$ on the respective sets. The above argument therefore gives a uniform lower bound for $i\partial\bar\partial\, v_\delta$ on the compact set Λ (2.3). However, we cannot control the Levi forms of v_δ from below on the sets $V_j \backslash \Lambda_j$ for $j \in \{1,\ldots,m\}$ since ϕ_j fails to be holomorphic there. $\square$

Lemma 2.4. *Let $U \subset X$ be an open set containing $A \cup K$. There exists a neighborhood W of $A \cup K$ with $\overline{W} \subset U$ and a $\mathcal{C}^2$ function $\rho\colon X \to \mathbb{R}$ which is strongly plurisubharmonic on $\overline{W}$ such that $\rho < 0$ on K and $\rho > 0$ on bW.*

Proof. Since $A \cap K$ is $\mathcal{O}(A)$-convex, there exists a compact neighborhood $K' \subset U \cap \Omega$ of K such that the set $K' \cap A \subset A \backslash bA$ is also $\mathcal{O}(A)$-convex. Since K is $\mathcal{O}(\Omega)$-convex, there is a smooth strongly plurisubharmonic function $\rho_0\colon \Omega \to \mathbb{R}$ such that $\rho_0 < 0$ on K and $\rho_0 > 1$ on $\Omega \backslash K'$ [46, Theorem 5.1.5, p. 117]. Set $\Omega_c = \{x \in \Omega\colon \rho_0(x) < c\}$. Fixing a number c with $0 < c < 1/2$ we have $K \subset \Omega_c \subset \Omega_{2c} \subset K'$.

Since the restricted function $\rho_0|_{A\cap\Omega}$ is strongly subharmonic and the set $K' \cap A$ is $\mathcal{O}(A)$-convex, a standard argument [23, p. 737] gives another smooth function $\widetilde\rho_0\colon X \to \mathbb{R}$ which agrees with ρ_0 in a neighborhood of K' in X such that $\widetilde\rho_0|_A$ is strongly subharmonic, $\widetilde\rho_0 > c$ on $A \backslash \overline{\Omega}_c$, $\widetilde\rho_0 > 2c$ on $A \backslash \overline{\Omega}_{2c}$, and $\widetilde\rho_0|_{bA} = c_0 \geq 1$ is constant.

Choose a strongly increasing convex function $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $h(t) \geq t$ for all $t \in \mathbb{R}$, $h(t) = t$ for $t \leq c$, and $h(t) > t + 1$ for $t \geq 2c$. The function

$$\rho_1 = h \circ \widetilde\rho_0\colon X \to \mathbb{R} \tag{2.4}$$

is then strongly plurisubharmonic on K' and along A, and it satisfies

(i) $\rho_1 = \widetilde\rho_0 = \rho_0$ on $\overline{\Omega}_c$,
(ii) $\rho_1 \geq \widetilde\rho_0 > c$ on $A \backslash \overline{\Omega}_c$,
(iii) $\rho_1 > \widetilde\rho_0 + 1$ on $A \backslash \overline{\Omega}_{2c}$, and
(iv) $\rho_1|_{bA} = c_1 > 2$.

To complete the proof of Lemma 2.4 we shall need the following result; compare with [12, Theorem 4].

Lemma 2.5. *Let A be a compact complex curve with $\mathcal{C}^2$ boundary in a complex space X. For every function $\rho_1\colon X \to \mathbb{R}$ of class $\mathcal{C}^2$ such that $\rho_1|_A$ is strongly subharmonic there exists a $\mathcal{C}^2$ function $\rho_2\colon X \to \mathbb{R}$ which is strongly plurisubharmonic in a neighborhood of A and satisfies $\rho_2|_A = \rho_1|_A$.*

Proof. Let $\{U_j\colon j = 1, \ldots, N\}$ be the open covering of A chosen at the beginning of the proof of Theorem 2.1. (For the present purpose we delete those sets which do not intersect A.) For each index $j \in \{1, \ldots, m\}$ let $Z = (z, w)\colon U_j \to U'_j \subset \mathbb{C}^{1+n_j}$, Γ_j, Λ_j and ϕ_j be as above. Denote by $\psi'_j\colon \Gamma_j \times \mathbb{C}^{n_j} \to \mathbb{R}$ the unique function which is independent of the variable $w \in \mathbb{C}^{n_j}$ and satisfies $\rho_1 = \psi'_j \circ Z$ on $A \cap U_j$. We extend ψ'_j to a $\mathcal{C}^2$ function $\psi'_j\colon U'_j \to \mathbb{R}$ which is independent of the w variable and set

$$\psi_j = \psi'_j \circ Z\colon U_j \to \mathbb{R}. \tag{2.5}$$

Then $\psi_j|_{A\cap U_j} = \rho_1$, and there is an open set $\widetilde{\Gamma}_j \subset \{1 - r_j < |z| < 1 + r_j\}$, with $\Gamma_j \subset \widetilde{\Gamma}_j$, such that ψ_j is subharmonic on the open set

$$\widetilde{U}_j = \{x \in U_j\colon z(x) \in \widetilde{\Gamma}_j\} \subset X. \tag{2.6}$$

By choosing the remaining sets U_j for $j \in \{m+1, \ldots, N\}$ sufficiently small we also get a holomorphic map $\phi_j\colon U_j \to \mathbb{C}^{n_j}$ whose components generate the ideal sheaf of A at every point of U_j, and a strongly plurisubharmonic function $\psi_j\colon U_j \to \mathbb{R}$ extending $\rho_1|_{A\cap U_j}$.

Choose a smooth partition of unity $\{\theta_j\}$ on a neighborhood of A in X with $\operatorname{supp}\theta_j \subset U_j$ for $j = 1, \ldots, N$. Fix an $\epsilon > 0$ and set

$$\rho_2(x) = \sum_{j=1}^{N} \theta_j(x) \left(\psi_j(x) + \epsilon^3 \log\bigl(1 + \epsilon^{-4}|\phi_j(x)|^2\bigr)\right).$$

For $x \in A$ we have $\rho_2(x) = \sum_j \theta_j(x)\psi_j(x) = \rho_1(x)$. One can easily verify that ρ_2 is strongly plurisubharmonic in a neighborhood of A in X provided that $\epsilon > 0$ is chosen sufficiently small. Indeed, as $\epsilon \to 0$, the function $\epsilon^3 \log\bigl(1 + \epsilon^{-4}|\phi_j(x)|^2\bigr)$ is of size $O(\epsilon^3)$, its first derivative are of size $O(\epsilon)$, and its Levi form at points of $A_{reg} \cap U_j$ in the direction normal to A is of size comparable to ϵ^{-1}, which implies that the Levi form of ρ_2 is positive definite at each point of A provided that ϵ is chosen sufficiently small. (See the proof of Theorem 4 in [12] for the details.) $\square$

With ρ_1 given by (2.4), and ρ_2 furnished by Lemma 2.5, we set

$$\rho = \operatorname{rmax}\{\widetilde{\rho}_0, \rho_2 - 1\}.$$

It is easily verified that ρ is strongly plurisubharmonic on a compact neighborhood $\overline{W} \subset U$ of the set $A \cup \overline{\Omega}_c$, $\rho = \widetilde{\rho}_0 = \rho_0$ on $\overline{\Omega}_c$ (hence $\rho < 0$ on K),

$\rho = \rho_2 - 1 > \widetilde{\rho}_0$ in a neighborhood of $A \backslash \Omega_{2c}$, and $\rho|_{bA}$ has a constant value $C > 1$. After shrinking W around $A \cup \overline{\Omega}_c$ we also have $\rho > 0$ on bW. $\square$

Completion of the proof of Theorem 2.1. We shall use the notation established at the beginning of the proof: $U_j \subset X$ is an open Stein neighborhood of a boundary curve $C_j \subset bA$, Λ_j and $\phi_j \colon U_j \to \mathbb{C}^{n_j}$ are defined by (2.1) resp. by (2.2), and $\psi_j \colon U_j \to \mathbb{R}$ is defined by (2.5).

Let V be an open set containg $A \cup K$, and let $v_\delta \colon V \to \mathbb{R}$ ($\delta \in [0,1]$) be a family of functions furnished by Lemma 2.3. Let Λ denote the corresponding set (2.3) on which $i\partial\bar\partial\, v_\delta$ is bounded from below uniformly with respect to $\delta \in (0,1]$. As δ decreases to 0, the functions v_δ decrease monotonically to a function v_0 satisfying $\{v_0 = -\infty\} = A$. By subtracting a constant we may assume that $v_\delta \le v_1 < 0$ on K for every $\delta \in [0,1]$.

Given an open set $U \subset X$ containing $A \cup K$, we must find a Stein neighborhod $\omega \subset U$ of $A \cup K$. We may assume that $\overline{U} \subset V$. Let ρ be a function furnished by Lemma 2.4; thus ρ is strongly plurisubharmonic on the closure $\overline{W} \subset U$ of an open set $W \supset A \cup K$, $\rho|_K < 0$, and $\rho|_{bW} > 0$. Let

$$\rho_{\epsilon,\delta} = \rho + \epsilon\, v_\delta \colon \overline{W} \to \mathbb{R}.$$

Choose $\epsilon > 0$ sufficiently small such that $\rho_{\epsilon,0} > 0$ on bW; hence $\rho_{\epsilon,\delta} \ge \rho_{\epsilon,0} > 0$ on bW for every $\delta \in [0,1]$. Decreasing $\epsilon > 0$ if necessary we may assume that $\rho_{\epsilon,\delta}$ is strongly plurisubharmonic on $\Lambda \cap \overline{W}$ for every $\delta \in (0,1]$ (since the positive Levi form of ρ will compensate the small negative part of the Levi form of ϵv_δ). Fix an ϵ with these properties. Now choose a sufficiently small $\delta > 0$ such that $\rho_{\epsilon,\delta} < 0$ on A (this is possible since v_δ decreases to v_0 which equals $-\infty$ on A). Note that $\rho_{\epsilon,\delta} < 0$ on K since both ρ and v_δ are negative on K. By continuity $\rho_{\epsilon,\delta}$ remains strongly plurisubharmonic also on the set $\overline{W} \cap \widetilde{U}_j$ for every $j = 1, \ldots, m$, where $\widetilde{U}_j \subset U_j$ is an open set of the form (2.6).

The function $\psi_j \colon \widetilde{U}_j \to \mathbb{R}$ (2.5) is plurisubharmonic on the open set $\widetilde{U}_j$ (2.6) which contains Λ_j, ψ_j has a constant value c_1 on the curve $C_j \subset bA$, and $\{\psi_j \le c_1\} = \Lambda_j \supset A \cap U_j$. Let $\chi \colon \mathbb{R} \to \mathbb{R}_+$ be a smooth increasing convex function with $\chi(t) = 0$ for $t \le c_1$ and $\chi(t) > 0$ for $t > c_1$. The plurisubharmonic function $\chi \circ \psi_j \colon \widetilde{U}_j \to \mathbb{R}$ then vanishes on Λ_j and is positive on $\widetilde{U}_j \backslash \Lambda_j$. Extending it by zero along A we obtain a plurisubharmonic function $\psi \colon V \to \mathbb{R}_+$ which vanishes of $\overline{W} \cap \Lambda$ and is positive on each of the sets $\widetilde{U}_j \backslash \Lambda_j$ (where it agrees with $\chi \circ \psi_j$). By choosing χ to grow sufficiently fast on $\{t > c_1\}$ we may insure that the sublevel set

$$\omega = \{x \in W \colon \psi(x) + \rho_{\epsilon,\delta}(x) < 0\} \subset\subset W$$

(which contains $A \cup K$) is contained in the set on which $\rho_{\epsilon,\delta}$ is strongly plurisubharmonic. The purpose of adding ψ is to round off the sublevel set sufficiently close to bA where it exists from $\Lambda \cap \overline{W}$, thereby insuring that ω remains in the region where the defining function $\psi + \rho_{\epsilon,\delta}$ is strongly plurisubharmonic. Narasimhan's theorem [61] now implies that ω is a Stein domain. This completes the proof of Theorem 2.1. $\square$

The restriction to one dimensional subvarieties $A \subset X$ was essential only in the proof of Lemma 2.2. For higher dimensional subvarieties we have the following partial result.

Theorem 2.6. *Let $h\colon X \to S$ be a holomorphic map of a complex space X to a complex manifold S, and let $D \subset\subset S$ be a strongly pseudoconvex Stein domain in S. Let $f\colon \bar D \to X$ be a $\mathcal{C}^2$ section of h (i.e., $h(f(z)) = z$ for $z \in \bar D$) which is holomorphic in D. If $f(bD) \subset X_{reg}$ and h is a submersion near $f(bD)$ then $A = f(\bar D)$ has a basis of open Stein neighborhoods in X.*

Proof. The only necessary change in the proof is in the construction of the sets Λ_j (2.1) and the functions ϕ_j (2.2) which describe the subvariety $A \subset X$ in a neighborhood of its boundary. When $\dim A = 1$, we could choose ϕ_j globally around the respective boundary curve $C_j \subset bA$ due to the existence of a Stein neighborhood of C_j. When $\dim A > 1$, this is no longer possible and hence this step must be localized as follows.

Fix a point $p \in bD$ and let $q = f(p) \in bA \subset X_{reg}$. Since h is a submersion near q, there are local holomorphic coordinates $x = (z, w)$ in an open neighborhood $U \subset X$ of q, and there is an open neighborhood $U' \subset S$ of the point $p = h(q)$ such that $h(x) = h(z, w) = z \in U'$ for $x \in U$, and $f(z) = (z, g(z))$ for $z \in U' \cap \bar D$. We take $\Lambda = \{x = (z, w) \in U : z \in U' \cap \bar D\}$ and $\phi(x) = \phi(z, w) = w - g(z)$. Covering bA by finitely many such neighborhoods, the rest of the proof of Theorem 2.1 applies mutatis mutandis. $\square$

Corollary 2.7. *Let S and X be complex manifolds, and let $D \subset\subset S$ be a strongly pseudoconvex Stein domain with boundary of class $\mathcal{C}^\ell$. If $2 \le r \le \ell$ then every $\mathcal{C}^r$ map $f\colon \bar D \to X$ which is holomorphic in D is a $\mathcal{C}^r(\bar D)$ limit of a sequence of maps $f_j\colon U_j \to X$ which are holomorphic in open neighborhoods of $\bar D$ in S.*

For maps from Riemann surfaces a stronger result is proved in §5 below.

Proof. When $S = \mathbb{C}^n$, $X = \mathbb{C}^N$ and $r = 0$, this classical result follows from the Henkin-Ramírez integral kernel representation of functions in $\mathcal{A}(D)$ (Henkin [40], Ramírez [65], Kerzman [48], Lieb [54], Henkin and Leiterer [42, p. 87]). Another approach which works for $0 \le r \le \ell$ is via the solution to the $\bar\partial$-equation with $\mathcal{C}^r$ estimates (see the paper of Range and Siu [67], Lieb and Range [56], Michel and Perotti [59], and [55, Theorem 3.43, VIII/3]).

Assume now that X is a complex manifold and $2 \le r \le \ell$. By Theorem 2.6 the graph $G_f = \{(z, f(z)) : z \in \bar D\}$ admits an open Stein neighborhood Ω in $S \times X$. Choose a proper holomorphic embedding $\psi\colon \Omega \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\pi\colon W \to \psi(\Omega)$ from an open neighborhood $W \subset \mathbb{C}^N$ of $\psi(\Omega)$ onto $\psi(\Omega)$. Choose a neighborhood $U \subset S$ of $\bar D$ and a sequence of holomorphic maps $g_j\colon U \to \mathbb{C}^N$ such that the sequence $g_j|_{\bar D}$ converges in

$\mathcal{C}^r(\bar{D})$ to the map $z \to \psi(z, f(z))$ as $j \to +\infty$. Denote by $pr_X : S \times X \to X$ the projection $(z, x) \to x$. Let $U_j = \{z \in U : g_j(z) \in W\}$. The sequence $f_j = pr_X \circ \psi^{-1} \circ \pi \circ g_j : U_j \to X$ then satisfies Corollary 2.7. □

Proof of Theorem 1.7 and Corollary 1.8. Let $D \subset\subset S$ be a smoothly bounded domain in an open Riemann surface S, and let $f : \bar{D} \hookrightarrow X$ be a $\mathcal{C}^2$ embedding which is holomorphic in D. By Theorem 2.1 the set $f(\bar{D}) \subset X$ has an open Stein neighborhood $\Omega \subset X$. Choose a proper holomorphic embedding $\psi : \Omega \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\pi : W \to \psi(\Omega) =: \Sigma$ from an open neighborhood $W \subset \mathbb{C}^N$ of Σ onto Σ. The embedding $\psi \circ f : \bar{D} \hookrightarrow \Sigma$ extends to a map $F : S \to \Sigma$ which is smooth on $S \backslash \bar{D}$. Then $\bar{\partial} F$ and its total first derivative $D^1(\bar{\partial} F)$ vanish on $\bar{D}$.

Set $A = F(\bar{D}) \subset \Sigma$. Let $\nu = T\Sigma|_A / TA$ denote the complex normal bundle of the embedding $F = \psi \circ f : D \hookrightarrow \Sigma$; this bundle is holomorphic over $\mathrm{Int} A = F(D)$ and is continuous (even of class $\mathcal{C}^1$) up to the boundary. An application of Theorem B for vector bundles which are holomorphic in the interior and continuous up to the boundary ([45], [52], [67]) gives a direct sum splitting $T\Sigma|_A = TA \oplus \nu$ which is holomorphic over $\mathrm{Int}\, A$ and continuous up to the boundary. (It suffices to follow the proof for vector bundles over open Stein manifolds, see e.g. [39, p. 256].)

Since A is a bordered Riemann surface, every complex vector bundle on A is topologically trivial, and hence also holomorphically trivial in the sense that it is isomorphic to the product bundle $A \times \mathbb{C}^{n-1}$ ($n = \dim X = \dim \Sigma$) by a continuous complex vector bundle isomorphism which is holomorphic over the interior of A [44, Theorem 2], [51]. Hence there exist continuous vector fields $v_1, \ldots, v_{n-1}$ tangent to $\nu \subset T\Sigma|_A$ which are holomorphic in the interior of A and generate ν at every point of A. Considering these fields as maps $A \to T\mathbb{C}^N = \mathbb{C}^N \times \mathbb{C}^N$ we can approximate them uniformly on A by vector fields (still denoted $v_1, \ldots, v_{n-1}$) which are holomorphic in a neighborhood of A in Σ and tangent to Σ. (The last condition can be fulfilled by composing them with the differential of the retraction $\pi : W \to \Omega$.) If the approximations are sufficiently close on A then the new vector fields are also linearly independent at each point of A and transverse to TA. The flow θ_j^t of v_j is defined and holomorphic for sufficiently small values of $t \in \mathbb{C}$ beginning at any point near A. The map

$$\widetilde{F}(z, t_1, \ldots, t_{n-1}) = \theta_1^{t_1} \circ \cdots \circ \theta_{n-1}^{t_{n-1}} \circ F(z)$$

is a diffeomorphism from an open neighborhood of $\bar{D} \times \{0\}^{n-1}$ in $S \times \mathbb{C}^{n-1}$ onto an open neighborhood of $A = F(\bar{D})$ in $\Sigma \subset \mathbb{C}^N$. $\widetilde{F}$ is holomorphic in the variables $t = (t_1, \ldots, t_{n-1})$ and satisfies $\frac{\partial \widetilde{F}}{\partial \bar{z}}(z, t) = 0$ for $z \in \bar{D}$.

Choose a $\mathcal{C}^2$ strongly subharmonic function $\rho : S \to \mathbb{R}$ such that $D = \{z \in S : \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for every $z \in bD = \{\rho = 0\}$. For $\epsilon \geq 0$ (small and variable) and $M > 0$ (large and fixed) the set

$$O_\epsilon = \{(z, t) \in S \times \mathbb{C}^{n-1} : \rho(z) + M|t|^2 < 0\}$$

is strongly pseudoconvex with $\mathcal{C}^2$ boundary and contained in the domain of $\widetilde{F}$ (the latter condition is achieved by choosing $M > 0$ sufficiently large). Note that $\bar{D} \times \{0\}^{n-1} \subset O_\epsilon$ for $\epsilon > 0$, and $||\bar{\partial}\widetilde{F}||_{L^\infty(O_\epsilon)} = o(\epsilon)$ as $\epsilon \to 0$.

There are constants $C > 0$ and $\epsilon_0 > 0$ such that the equation $\bar{\partial}U = \bar{\partial}\widetilde{F}$ has a solution $U = U_\epsilon \in \mathcal{C}^2(O_\epsilon)$ satisfying a uniform estimate

$$||U_\epsilon||_{L^\infty(O_\epsilon)} \le C||\bar{\partial}\widetilde{F}||_{L^\infty(O_\epsilon)} = o(\epsilon) \tag{2.7}$$

for $0 < \epsilon \le \epsilon_0$ (see [41], [55], [67] and the discussion in §3 below). The map

$$G_\epsilon = \pi \circ (\widetilde{F} - U_\epsilon)\colon O_\epsilon \to \Sigma \subset \mathbb{C}^N$$

is then holomorphic, and it is homotopic to $\widetilde{F}|_{O_\epsilon}$ through the homotopy $G_{\epsilon,s} = \pi \circ (\widetilde{F} - sU_\epsilon) \in \Sigma$ ($s \in [0,1]$) satisfying $||G_{\epsilon,s} - \widetilde{F}||_{L^\infty(O_\epsilon)} = o(\epsilon)$ as $\epsilon \to 0$, uniformly in $s \in [0,1]$. Choosing $\epsilon > 0$ sufficiently small we conclude that $G_{\epsilon,s}(z,t) \in \Sigma\backslash\widetilde{F}(\bar{O}_0)$ for each $(z,t) \in bO_{\epsilon/2}$ and $s \in [0,1]$. It follows that for each point $x \in \widetilde{F}(\bar{O}_0)$ the number of solutions $(z,t) \in O_{\epsilon/2}$ of the equation $G_{\epsilon,s}(z,t) = x$, counted with algebraic multiplicities, does not depend on $s \in [0,1]$, and hence it equals one (its value at $s = 0$). Taking $s = 1$ we see that the set $G_\epsilon(O_{\epsilon/2})$ contains $\widetilde{F}(\bar{O}_0) \supset A$.

From (2.7) and the interior elliptic regularity estimates [29, Lemma 3.2] we also see that $||dU_\epsilon||_{L^\infty(O_{\epsilon/2})} = o(1)$ as $\epsilon \to 0$, and hence G_ϵ is an injective immersion on $D_{\epsilon/2}$ for every sufficiently small $\epsilon > 0$ (since it is a $\mathcal{C}^1$-small perturbation of such a map $\widetilde{F}$). For such values of ϵ the set $U_\epsilon = \psi^{-1}(G_\epsilon(O_{\epsilon/2})) \subset X$ is an open Stein neighborhood of the set $f(\bar{D})$ which is biholomorphic (via $\psi^{-1} \circ G_\epsilon$) to $O_{\epsilon/2} \subset S \times \mathbb{C}^{n-1}$.

Since X can be replaced by an arbitrary open neighborhood of $f(\bar{D})$ in the above construction, this concludes the proof of Theorem 1.7. The same proof gives Corollary 1.8.

3. A Cartan type lemma with estimates up to the boundary

In this section we prove one of our main tools, Theorem 3.2.

Definition 3.1. A pair of relatively compact open subsets $D_0, D_1 \Subset S$ in a complex manifold S is said to be a *Cartan pair* of class $\mathcal{C}^\ell$ ($\ell \ge 2$) if

(i) the sets D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are Stein domains with strongly pseudoconvex boundaries of class $\mathbb{C}^\ell$, and
(ii) $\overline{D_0\backslash D_1} \cap \overline{D_1\backslash D_0} = \emptyset$ (the separation property).

Replacing S by a suitably chosen neighborhood of $\overline{D_0 \cup D_1}$ we can assume that S is a Stein manifold.

Let P be a bounded open set in $\mathbb{C}^n$. We shall denote the variable in S by z and the variable in $\mathbb{C}^n$ by $t = (t_1, \ldots, t_n)$. For each pair of integers $r, s \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ we denote by $\mathcal{C}^{r,s}(\bar{D} \times P)$ the space of all functions

$f\colon \bar D \times P \to \mathbb{C}$ with bounded partial derivatives up to order r in the z variable and up to order s in the t variable, endowed with the norm

$$||f||_{\mathcal{C}^{r,s}(D\times P)} = \sup\{|D^\mu_z D^\nu_t f(z,t)|\colon z\in \bar D,\ t\in P,\ |\mu|\le r,\ |\nu|\le s\} < +\infty.$$

Here D^ν_t denotes the partial derivative of order $\nu \in \mathbb{Z}^{2n}$ with respect to the real and imaginary parts of the components t_j of $t\in\mathbb{C}^n$. The same definition applies to D^μ_z when $S=\mathbb{C}^m$; in general we cover $\bar D$ by a finite system of local holomorphic charts $U_j \Subset V_j \subset S$, with biholomorphic maps $\phi_j\colon V_j \to V'_j \subset \mathbb{C}^m$, and take at each point $z\in\bar D$ the maximum of the above norms calculated in the ϕ_j-coordinates with respect to those charts (V_j,ϕ_j) for which $z\in U_j$. Alternatively, we can measure the z-derivatives with respect to a smooth Hermitian metric on S; the two choices yield equivalent norms on $\mathcal{C}^{r,s}(\bar D\times P)$. Set

$$\mathcal{A}^{r,s}(D\times P) = \mathcal{O}(D\times P)\cap \mathcal{C}^{r,s}(\bar D\times P), \quad r,s\in\mathbb{Z}_+.$$

For $t=(t_1,\ldots,t_n)\in\mathbb{C}^n$ we write $|t| = (\sum |t_j|^2)^{1/2}$. For a map $f = (f_1,\ldots,f_n)\colon \bar D\times P\to\mathbb{C}^n$ with components $f_j\in\mathcal{C}^{r,s}(\bar D\times P)$ we set

$$||f||_{\mathcal{C}^{r,s}(D\times P)} = \left(\sum_{j=1}^n ||f_j||^2_{\mathcal{C}^{r,s}(D\times P)}\right)^{1/2}.$$

Let $\mathbb{B}(t;\delta)\subset\mathbb{C}^n$ denote the ball of radius $\delta>0$ centered at $t\in\mathbb{C}^n$. For any subset $P\subset\mathbb{C}^n$ and $\delta>0$ we set

$$P_{-\delta} = \{t\in P\colon \mathbb{B}(t;\delta)\subset P\}.$$

Theorem 3.2. (Generalized Cartan's lemma) *Let (D_0,D_1) be a Cartan pair of class $\mathcal{C}^\ell$ ($\ell\ge 2$) and let P be a bounded open set in $\mathbb{C}^n$. Set $D = D_0\cup D_1$ and $D_{0,1}=D_0\cap D_1$. Given $\delta^*>0$ and $r\in\{0,1,\ldots,\ell\}$ there exist numbers $\epsilon^*>0$ and $M_{r,s}\ge 1$ ($s=0,1,2,\ldots$) satisfying the following. For every map $\gamma\colon \bar D_{0,1}\times P\to\mathbb{C}^n$ of class $\mathcal{A}^{r,0}(D_{0,1}\times P)^n$ satisfying*

$$\gamma(z,t) = t + c(z,t), \qquad ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)} < \epsilon^*$$

there exist maps $\alpha\colon \bar D_0\times P_{-\delta^*}\to\mathbb{C}^n$, $\beta\colon \bar D_1\times P_{-\delta^*}\to\mathbb{C}^n$ of the form

$$\alpha(z,t) = t + a(z,t), \qquad \beta(z,t) = t+b(z,t),$$

with $a\in\mathcal{A}^{r,s}(D_0\times P_{-\delta^*})^n$ and $b\in\mathcal{A}^{r,s}(D_1\times P_{-\delta^*})^n$ for all $s\in\mathbb{Z}_+$, which are fiberwise injective holomorphic and satisfy

$$\gamma(z,\alpha(z,t)) = \beta(z,t), \qquad z\in\bar D_{0,1},\ t\in P_{-\delta^*} \tag{3.1}$$

and also the estimates

$$\begin{aligned} ||a||_{\mathcal{C}^{r,s}(D_0\times P_{-\delta^*})} &\le M_{r,s}\cdot ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)},\\ ||b||_{\mathcal{C}^{r,s}(D_1\times P_{-\delta^*})} &\le M_{r,s}\cdot ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)}.\end{aligned}$$

If $\gamma(z,t)=t+c(z,t)$ is tangent to the map $\gamma_0(z,t)=t$ to order $m\in\mathbb{N}$ at $t=0$ (i.e., the function $c(\cdot\,,t)$ vanishes to order m at $t=0$) then α and β can be chosen to satisfy the same property.

Remark 3.3. The relation (3.1) is equivalent to

$$\gamma_z = \beta_z \circ \alpha_z^{-1}, \qquad z \in \bar{D}_{0,1}.$$

The classical *Cartan's lemma* [39, p. 199, Theorem 7] corresponds to the special case when $\alpha_z = \alpha(z, \cdot\,)$, β_z and γ_z are linear automorphism of $\mathbb{C}^n$ depending holomorphically on the point z in the respective base domain. A version of Cartan's lemma without shrinking the base domains was proved in Douady's thesis (see e.g. [16]), and for matrix valued functions of class $\mathcal{A}^\infty$ by A. Sebbar [70, Theorem 1.4]. Berndtsson and Rosay proved a splitting lemma over the disc $\triangle$ for bounded holomorphic maps into $GL_n(\mathbb{C})$ [3]. A key difference between all these results and Theorem 3.2 is that we do not restrict ourselves to fiberwise linear maps.

A result similar to Theorem 3.2, but less precise as it requires shrinking of the base domains, is Lemma 2.1 in [24] which follows from Theorem 4.1 in [22]. That lemma does not suffice for the application in this paper where it is essential that no shrinking be allowed in the base domain. On the other hand, shrinking is necessary (and admissible) in the fiber variable since we are taking compositions of maps.

Theorem 3.2 will be proved by a rapidly convergent iteration similar to the one in the proof of Theorem 4.1 in [22], but with estimates of derivatives. At an inductive step we split the map $c(z,t) = \gamma(z,t) - t$ into a difference $c = b - a$ where the maps $a\colon \bar{D}_0 \times P \to \mathbb{C}^n$ and $b\colon \bar{D}_1 \times P \to \mathbb{C}^n$ are of class $\mathcal{A}^{r,0}$, with estimates of their $\mathcal{C}^{r,0}$ norms in terms of the $\mathcal{C}^{r,0}$ norm of c (Lemma 3.4). Set

$$\alpha_z(t) = \alpha(z,t) := t + a(z,t), \qquad \beta_z(t) = \beta(z,t) := t + b(z,t).$$

We then show that for $z \in \bar{D}_{0,1}$ and t in a smaller set $P_{-\delta} \subset \mathbb{C}^n$, with ϵ sufficiently small compared to δ, there exists a map $\widetilde{\gamma}\colon \bar{D}_{0,1} \times P_{-\delta} \to \mathbb{C}^n$ of the form $\widetilde{\gamma}(z,t) = t + \widetilde{c}(z,t)$ satisfying

$$\gamma_z \circ \alpha_z = \beta_z \circ \widetilde{\gamma}_z, \qquad z \in \bar{D}_{0,1}$$

and a quadratic estimate

$$\widetilde{\epsilon} = ||\widetilde{c}||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} \leq const\cdot \frac{||c||^2_{\mathcal{C}^{r,0}(D_{0,1}\times P)}}{\delta}$$

(Lemma 3.5). If $\epsilon = ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)}$ is sufficiently small compared to δ then $\widetilde{\epsilon}$ is much smaller than ϵ. Choosing a sequence of δ's with the sum $\frac{\delta^*}{2}$ and assuming that the initial map c is sufficiently small, the sequences of compositions of the maps α_z (resp. β_z), obtained in the individual steps, converge on $P_{-\delta^*/2}$ to limit maps α (resp. β) satisfying $\gamma_z \circ \alpha_z = \beta_z$ for $z \in \bar{D}_{0,1}$. After an additional shrinking of the fiber for $\frac{\delta^*}{2}$ we obtain injective holomorphic maps on $P_{-\delta^*}$ satisfying the estimates in Theorem 3.2.

We begin by recalling the relevant results on the solvability of the $\bar\partial$-equation. Let D be a relatively compact strongly pseudoconvex domain with boundary of class $\mathcal{C}^\ell$ ($\ell \geq 2$) in a Stein manifold S. Let $\mathcal{C}^r_{0,1}(\bar D)$ denote the space of $(0,1)$-forms with $\mathcal{C}^r$ coefficients on $\bar D$, and $\mathcal{Z}^r_{0,1}(\bar D) = \{f \in \mathcal{C}^r_{0,1}(\bar D) \colon \bar\partial f = 0\}$. According to Range and Siu [67] and Lieb and Range [56, Theorem 1] (see also [59, Theorem 1']) there exists a linear operator $T \colon \mathcal{C}^0_{0,1}(D) \to \mathcal{C}^0(D)$ satisfying the following properties:

(i) If $f \in \mathcal{C}^0_{0,1}(\bar D) \cap \mathcal{C}^1_{0,1}(D)$ and $\bar\partial f = 0$ then $\bar\partial(Tf) = f$.
(ii) If $f \in \mathcal{C}^0_{0,1}(\bar D) \cap \mathcal{C}^r_{0,1}(D)$ ($1 \leq r \leq \ell$) then for each $l = 0, 1, \ldots, r$

$$||Tf||_{\mathcal{C}^{l,1/2}(\bar D)} \leq C_l ||f||_{\mathcal{C}^l_{0,1}(\bar D)}. \tag{3.2}$$

The results in [56] are stated only for the case $bD \in \mathcal{C}^\infty$, but a more careful analysis shows that one only needs $\mathcal{C}^\ell$ boundary in order to get estimates up to order ℓ; this is implicitly contained in the paper by Michel and Perotti [59] (the special case of domains without corners). The case of domains in Stein manifolds easily reduces to the Euclidean case by standard techniques (holomorphic embeddings and retractions). Lieb and Range showed that for strongly pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$ the estimates (3.2) also hold for the Kohn solution operator $T = \bar\partial^* N$ ([57], [58, Corollary 2]). Here $\bar\partial^*$ is the formal adjoint of $\bar\partial$ on $(0,1)$-forms (under a suitable choice of a Hermitean metric on S) and N is the corresponding Neumann operator on $(0,1)$-forms on D (the inverse of the complex Laplacian $\Box = \bar\partial\,\bar\partial^* + \bar\partial^*\bar\partial$ acting on $(0,1)$-forms). (See also [55, Theorem 3.43, VIII/3]; for Sobolev estimates see [9, Theorem 5.2.6, p. 103].)

Lemma 3.4. *Let $D = D_0 \cup D_1 \Subset S$, $D_{0,1} = D_0 \cap D_1$ and $P \subset \mathbb{C}^n$ be as in Theorem 3.2. For every $r \in \{0, 1, \ldots, \ell\}$ there are a constant $C_r \geq 1$, independent of P, and linear operators*

$$A \colon \mathcal{A}^{r,0}(D_{0,1} \times P)^n \to \mathcal{A}^{r,0}(D_0 \times P)^n, \quad B \colon \mathcal{A}^{r,0}(D_{0,1} \times P)^n \to \mathcal{A}^{r,0}(D_1 \times P)^n$$

satisfying

$$c = Bc|_{\bar D_{0,1} \times P} - Ac|_{\bar D_{0,1} \times P}, \quad c \in \mathcal{A}^{r,0}(D_{0,1} \times P)^n,$$

and the estimates

$$\begin{aligned} ||Ac||_{\mathcal{C}^{r,0}(D_0 \times P)} &\leq C_r \cdot ||c||_{\mathcal{C}^{r,0}(D_{0,1} \times P)}, \\ ||Bc||_{\mathcal{C}^{r,0}(D_1 \times P)} &\leq C_r \cdot ||c||_{\mathcal{C}^{r,0}(D_{0,1} \times P)}. \end{aligned}$$

If c vanishes to order $m \in \mathbb{N}$ at $t = 0$ then so do Ac and Bc.

Proof. The separation condition (ii) in the definition of a Cartan pair implies that there exists a smooth function χ on S with values in $[0,1]$ such that $\chi = 0$ in an open neighborhood of $\overline{D_0 \backslash D_1}$ and $\chi = 1$ in an open neighborhood of $\overline{D_1 \backslash D_0}$. Note that $\chi(z)c(z,t)$ extends to a function in $\mathcal{C}^{r,0}(\bar D_0 \times P)$ which vanishes on $\overline{D_0 \backslash D_1} \times P$, and $\big(\chi(z) - 1\big)c(z,t)$ extends to a function in $\mathcal{C}^{r,0}(\bar D_1 \times P)$ which vanishes on $\overline{D_1 \backslash D_0} \times P$. Furthermore, $\bar\partial(\chi c) = \bar\partial((\chi - 1)c) = c\bar\partial\chi$ is a $(0,1)$-form on $\bar D$ with $\mathcal{C}^r$ coefficients and with support in $\bar D_{0,1} \times P$, depending holomorphically on $t \in P$.

Let T denote a solution operator to the $\bar\partial$ equation satisfying (3.2). For any $c \in \mathcal{A}^{r,0}(D_{0,1} \times P)$ and $t \in P$ we set

$$\begin{aligned}(Ac)(z,t) &= \big(\chi(z)-1\big)c(z,t) - T\big(c(\cdot\,,t)\bar\partial\chi\big)(z), \quad z \in \bar D_0.\\ (Bc)(z,t) &= \chi(z)c(z,t) - T\big(c(\cdot\,,t)\bar\partial\chi\big)(z), \qquad\quad z \in \bar D_1.\end{aligned}$$

Then $Ac - Bc = c$ on $\bar D_{0,1} \times P$, $\bar\partial_z(Ac) = 0$, and $\bar\partial_z(Bc) = 0$ on their respective domains. The bounded linear operator T commutes with the derivative $\bar\partial_t$ on the parameter t. Since $\bar\partial_t\,\big(c(z,t)\bar\partial\chi(z)\big) = 0$, we get $\bar\partial_t(Ac) = 0$ and $\bar\partial_t(Bc) = 0$. The estimates follow from boundedness of T (3.2). $\square$

Lemma 3.5. *Let $D = D_0 \cup D_1 \Subset S$, $D_{0,1} = D_0 \cap D_1$ and $P \subset \mathbb{C}^n$ be as in Theorem 3.2. Given $c \in \mathcal{A}^{r,0}(D_{0,1} \times P)^n$, let $a = Ac$ and $b = Bc$ be as in Lemma 3.4. Let $\alpha\colon \bar D_0 \times P \to \mathbb{C}^n$, $\beta\colon \bar D_1 \times P \to \mathbb{C}^n$ and $\gamma\colon \bar D_{0,1} \times P \to \mathbb{C}^n$ be given by*

$$\alpha(z,t) = t + a(z,t), \quad \beta(z,t) = t + b(z,t), \quad \gamma(z,t) = t + c(z,t).$$

Let $C_r \ge 1$ be the constant in Lemma 3.4. There is a constant $K_r > 0$ with the following property. If $4\sqrt{n}C_r||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)} < \delta$ then there is a map $\widetilde\gamma\colon \bar D_{0,1} \times P_{-\delta} \to \mathbb{C}^n$ of the form $\widetilde\gamma(z,t) = t + \widetilde c(z,t)$, with $\widetilde c \in \mathcal{A}^{r,0}(D_{0,1} \times P_{-\delta})^n$, satisfying

$$\gamma_z \circ \alpha_z = \beta_z \circ \widetilde\gamma_z, \qquad z \in \bar D_{0,1}$$

and the estimate

$$||\widetilde c||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le K_r \cdot \frac{||c||^2_{\mathcal{C}^{r,0}(D_{0,1}\times P)}}{\delta}.$$

If the functions a, b and c vanish to order $m \in \mathbb{N}$ at $t = 0$ then so does $\widetilde c$.

Proof. We begin by estimating the composition $\gamma_z \circ \alpha_z$. Since the same estimate will be used below for other compositions as well, we formulate the result as an independent lemma.

Lemma 3.6. *Let D be a domain with $\mathcal{C}^1$ boundary in a complex manifold S, let P be an open set in $\mathbb{C}^n$, and let $0 < \delta < 1$. Given maps $\alpha_j(z,t) = t + a_j(z,t)$ $(j = 0,1)$ with $a_0 \in \mathcal{A}^{r,0}(D \times P)^n$, $a_1 \in \mathcal{A}^{r,0}(D \times P_{-\delta})^n$, and $||a_1||_{\mathcal{C}^{r,0}(D\times P_{-\delta})} < \frac{\delta}{2}$ we have for all $(z,t) \in \bar D \times P_{-\delta}$*

$$\alpha_0\big(z, \alpha_1(z,t)\big) = t + a_0(z,t) + a_1(z,t) + e(z,t)$$

where

$$||e||_{\mathcal{C}^{r,0}(D\times P_{-\delta})} \le \frac{L_r}{\delta} \cdot ||a_0||_{\mathcal{C}^{r,0}(D\times P)} \cdot ||a_1||_{\mathcal{C}^{r,0}(D\times P_{-\delta})}$$

for some constant $L_r > 0$ depending only on r and n.

Proof. We have

$$\begin{aligned}\alpha_0(z,\alpha_1(z,t)) &= \alpha_1(z,t) + a_0(z,\alpha_1(z,t))\\ &= t + a_1(z,t) + a_0(z, t + a_1(z,t))\\ &= t + a_0(z,t) + a_1(z,t) + e(z,t)\end{aligned}$$

where the error term equals

$$e(z,t) = a_0(z, t + a_1(z,t)) - a_0(z,t).$$

Fix a point $(z,t) \in \bar{D} \times P_{-\delta}$. Since $|a_1(z,t)| < \frac{\delta}{2}$, the line segment $\lambda \subset \mathbb{C}^n$ with the endpoints t and $\alpha_1(z,t) = t + a_1(z,t)$ is contained in $P_{-\delta/2}$. Using the Cauchy estimates for the partial derivative $\partial_t a_0$ we obtain

$$\begin{aligned} |e(z,t)| &= \Big| \int_0^1 (\partial_t a_0)(z, t + \tau a_1(z,t)) \cdot a_1(z,t)\, d\tau \Big| \\ &\leq \sup_{t' \in \lambda} ||\partial_t a_0(z,t')|| \cdot |a_1(z,t)| \\ &\leq \frac{2\sqrt{n}}{\delta} \cdot ||a_0||_{\mathcal{C}^{0,0}(D\times P)} \cdot ||a_1||_{\mathcal{C}^{0,0}(D\times P_{-\delta})} \end{aligned}$$

which is the required estimate for $r = 0$. We proceed to estimate the partial differential:

$$\begin{aligned} \partial_z e(z,t) &= (\partial_z a_0)(z, t + a_1(z,t)) - (\partial_z a_0)(z,t) + \\ &\qquad + (\partial_t a_0)(z, t + a_1(z,t)) \cdot (\partial_z a_1)(z,t). \end{aligned}$$

The difference in the first line equals

$$\int_0^1 \partial_t (\partial_z a_0)(z, t + \tau a_1(z,t)) \cdot a_1(z,t)\, d\tau$$

which can be estimated exactly as above (using the Cauchy estimates for $\partial_t \partial_z a_0$) by

$$\frac{const}{\delta} \cdot ||a_0||_{\mathcal{C}^{1,0}(D\times P)} \cdot ||a_1||_{\mathcal{C}^{0,0}(D\times P_{-\delta})}.$$

Applying the Cauchy estimate for $\partial_t a_0$ we estimate the remaining term in the expression for $e(z,t)$ by

$$\frac{const}{\delta} \cdot ||a_0||_{\mathcal{C}^{0,0}(D\times P)} \cdot ||a_1||_{\mathcal{C}^{1,0}(D\times P_{-\delta})}.$$

This proves the estimate in Lemma 3.6 for $r = 1$.

We proceed in a similar way to estimate the higher order derivatives of e. In the expression for $\partial_z^k e(z,t)$ we shall have a main term

$$(\partial_z^k a_0)(z, t + a_1(z,t)) - (\partial_z^k a_0)(z,t) = \int_0^1 \partial_t (\partial_z^k a_0)(z, t + \tau a_1(z,t)) \cdot a_1(z,t)\, d\tau$$

which is estimated by $const \cdot \delta^{-1} ||a_0||_{\mathcal{C}^{k,0}(D\times P)} \cdot ||a_1||_{\mathcal{C}^{0,0}(D\times P_{-\delta})}$. The remaining terms in $e(z,t)$ are products of partial derivatives of order $\leq k$ of a_0 (with respect to both z and t variables) with partial derivatives of a_1 of order $\leq k$ with respect to the z variable. Each t-derivative of a_0 can be removed by using the Cauchy estimates, contributing another δ in the denominator. The chain rule shows that each term containing l derivatives of a_0 on the t variable is multiplied by l factors involving a_1 and its z-derivatives; this gives an estimate $const\cdot \delta^{-l}\, ||a_0||_{\mathcal{C}^{k,0}(D\times P)}\cdot ||a_1||^l_{\mathcal{C}^{k,0}(D\times P_{-\delta})}$. Since we have assumed $||a_1||_{\mathcal{C}^{r,0}(D\times P)} < \frac{\delta}{2}$, this is less than

$$\frac{const}{\delta}\cdot ||a_0||_{\mathcal{C}^{k,0}(D\times P)}\cdot ||a_1||_{\mathcal{C}^{k,0}(D\times P_{-\delta})}$$

and the lemma is proved. □

Let now α, β and γ be as in Lemma 3.5. Set $\epsilon = ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)}$; then $||a||_{\mathcal{C}^{r,0}(D_0\times P)} \le C_r\epsilon$ and $||b||_{\mathcal{C}^{r,0}(D_1\times P)} \le C_r\epsilon$ by Lemma 3.4. Since we have assumed $4\sqrt{n}C_r\epsilon < \delta$, Lemma 3.6 with $\alpha_0 = \gamma$ and $\alpha_1 = \alpha$ gives for $z \in \bar{D}_{0,1}$ and $t \in P_{-\delta}$:

$$\gamma(z, \alpha(z,t)) = t + c(z,t) + a(z,t) + e(z,t) = \beta(z,t) + e(z,t) \in P_{-\delta/2}$$

where

$$||e||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le \frac{L_r}{\delta}\cdot ||c||_{\mathcal{C}^{r,0}(D_{0,1}\times P)}\cdot ||a||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le \frac{L_rC_r\epsilon^2}{\delta}.$$

It remains to find a map $\widetilde{\gamma}(z,t) = t + \widetilde{c}(z,t)$ on $\bar{D}_{0,1}\times P_{-\delta}$ satisfying

$$\beta(z,t) + e(z,t) = \beta(z, t+\widetilde{c}(z,t)) = t + \widetilde{c}(z,t) + b(z, t+\widetilde{c}(z,t))$$

and an estimate

$$||\widetilde{c}||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} \le const\cdot \epsilon^2\delta^{-1}.$$

For the existence of $\widetilde{\gamma}$ it suffices to see that the map β_z is injective on $P_{-\delta/4}$ and $\beta_z(P_{-\delta/4}) \supset P_{-\delta/2}$ for every $z \in \bar{D}_{0,1}$; since $\gamma_z\circ\alpha_z \in P_{-\delta/2}$, we can then take $\widetilde{\gamma}_z = \beta_z^{-1}\circ\gamma_z\circ\alpha_z$. To see the injectivity of β_z note that for $t, t' \in P_{-\delta/4}$, $t \ne t'$, we have

$$|\beta_z(t) - \beta_z(t')| \ge |t-t'| - |b_z(t) - b_z(t')| \ge |t-t'|\left(1 - \frac{4\sqrt{n}C_0\epsilon}{\delta}\right) > 0.$$

(We applied the Cauchy estimate to $\partial_t b_z$.) The inclusion $P_{-\delta/2} \subset \beta_z(P_{-\delta/4})$ follows from the estimate $||b||_{\mathcal{C}^{r,0}(D_1\times P)} \le C_r\epsilon \le \frac{\delta}{4\sqrt{n}}$ by Rouché's theorem.

In order to estimate $\widetilde{c}$ we rewrite its defining equation in the form

$$\begin{aligned}\widetilde{c}(z,t) &= b(z,t) - b(z, t+\widetilde{c}(z,t)) + e(z,t)\\ &= -\int_0^1 (\partial_t b)\big(z, t+\tau\widetilde{c}(z,t)\big)\cdot \widetilde{c}(z,t)\, d\tau + e(z,t).\end{aligned}$$

Since the path of integration lies in $P_{-\delta/2}$, the Cauchy estimates for $\partial_t b$ give

$$|\widetilde{c}(z,t)| \le \frac{2\sqrt{n}C_0\epsilon}{\delta}\cdot |\widetilde{c}(z,t)| + |e(z,t)| \le \frac{1}{2}\,|\widetilde{c}(z,t)| + |e(z,t)|$$

and hence $|\widetilde{c}(z,t)| \le 2|e(z,t)| \le const\cdot \epsilon^2\delta^{-1}$. We proceed inductively to estimate the derivatives $\partial_z^k\widetilde{c}$ for $k \le r$ by differentiating the implicit equation for $\widetilde{c}$. The top order differential $|\partial_z^k\widetilde{c}|$ appearing on the right hand side is multiplied by a constant < 1 arising from an estimate on b (just as was done above); subsuming this term by the left hand side we obtain the estimates of $|\partial_z^k\widetilde{c}|$ for all $k \le r$. Although we obtain a term δ^r in the denominator, we can cancel $r-1$ powers of δ by appropriate terms of size $O(\epsilon)$ just as we did at the end of proof of Lemma 3.6 to get $||\widetilde{c}||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{-\delta})} = O(\epsilon^2\delta^{-1})$. $\square$

Proof of Theorem 3.2. We shall write $(\gamma\alpha)(z,t) = \gamma(z,\alpha(z,t))$, and similarly for the fiberwise composition of several maps. Let

$$\gamma(z,t) = \gamma_0(z,t) = t + c_0(z,t), \quad \epsilon_0 = ||c_0||_{\mathcal{C}^{r,0}(D_{0,1}\times P)}$$

and $\delta^* > 0$ be as in Theorem 3.2. We first describe the inductive procedure and subsequently show convergence provided that $\epsilon_0 > 0$ is sufficiently small. Let $P_0 = P$ and $P_* = P_{-\delta^*/2}$. For every $k \in \mathbb{Z}_+$ set

$$\delta_k = 2^{-k-2}\delta^*, \quad P_{k+1} = (P_k)_{-\delta_k}.$$

Then $\sum_{k=0}^{\infty}\delta_k = \frac{\delta^*}{2}$ and $\cap_{k=0}^{\infty} P_k = \bar{P}_*$. Let $C_r \geq 1$, $K_r \geq 1$ and $L_r \geq 1$ be the constants in Lemmas 3.4, 3.5 and 3.6, respectively. We shall inductively construct sequences of maps

$$\begin{aligned} \alpha_k(z,t) &= t + a_k(z,t), \quad a_k \in \mathcal{A}^{r,0}(D_0\times P_k)^n \\ \beta_k(z,t) &= t + b_k(z,t), \quad b_k \in \mathcal{A}^{r,0}(D_1\times P_k)^n \\ \gamma_k(z,t) &= t + c_k(z,t), \quad c_k \in \mathcal{A}^{r,0}(D_{0,1}\times P_k)^n \end{aligned}$$

such that, setting $\epsilon_k = ||c_k||_{\mathcal{C}^{r,0}(D_{0,1}\times P_k)}$, the following hold for all $k\in\mathbb{Z}_+$:

- (1_k) $||a_k||_{\mathcal{C}^{r,0}(D_0\times P_k)} \leq C_r\epsilon_k$, $\quad ||b_k||_{\mathcal{C}^{r,0}(D_1\times P_k)} \leq C_r\epsilon_k$.
- (2_k) $4\sqrt{n}C_r\epsilon_k < \delta_k = 2^{-k-2}\delta^*$.
- (3_k) $\gamma_k\alpha_k = \beta_k\gamma_{k+1}$ on $\bar{D}_{0,1}\times P_{k+1}$.
- (4_k) $\epsilon_{k+1} = ||c_{k+1}||_{\mathcal{C}^{r,0}(D_{0,1}\times P_{k+1})} \leq K_r\epsilon_k^2\delta_k^{-1} = (4K_r\delta^{*-1})2^k\epsilon_k^2$.

These conditions imply for every $k\in\mathbb{Z}_+$

$$\gamma_0(\alpha_0\alpha_1\cdots\alpha_k) = (\beta_0\beta_1\cdots\beta_k)\gamma_{k+1} \quad \text{on } \bar{D}_{0,1}\times P_{k+1}. \tag{3.3}$$

Assuming that $\epsilon_0 = ||c_0||_{\mathcal{C}^{r,0}(D_{0,1}\times P)} > 0$ is sufficiently small we shall prove that, as $k\to+\infty$, the sequence of maps

$$\widetilde{\alpha}_k = \alpha_0\alpha_1\cdots\alpha_k \colon \bar{D}_0\times P_k \to \mathbb{C}^n \tag{3.4}$$

converges to a map $\alpha\colon \bar{D}_0\times P_*\to\mathbb{C}^n$, the sequence

$$\widetilde{\beta}_k = \beta_0\beta_1\cdots\beta_k \colon \bar{D}_1\times P_k\to\mathbb{C}^n \tag{3.5}$$

converges to a map $\beta\colon \bar{D}_1\times P_*\to\mathbb{C}^n$, and the sequence γ_k converges on $\bar{D}_{0,1}\times P_*$ to the map $(z,t)\to t$. (All convergences are in the $\mathcal{C}^{r,0}$-norms on the respective domains.) In the limit we obtain a desired splitting

$$\gamma\alpha = \beta \quad \text{on} \quad \bar{D}_{0,1}\times P_*.$$

We begin at $k=0$ with the given map $\gamma_0(z,t) = t + c_0(z,t)$ on $\bar{D}_{0,1}\times P_0$. Lemma 3.4, applied to c_0, gives maps a_0 and b_0 satisfying (1_0). If (2_0) holds (which is the case if $\epsilon_0 = ||c_0||_{\mathcal{C}^{r,0}(D_{0,1}\times P_0)} > 0$ is sufficiently small) then Lemma 3.5 furnishes a map $\gamma_1\colon \bar{D}_{0,1}\times P_1\to\mathbb{C}^n$ satisfying (3_0) and (4_0).

Assume inductively that for some $k \in \mathbb{N}$ we already have maps satisfying (1_j)–(4_j) for $j = 0, \ldots, k-1$, and consequently (3.3) holds with k replaced by $k-1$. Lemma 3.4, applied to $c_k(z,t) = \gamma_k(z,t) - t$ on $\bar{D}_{0,1} \times P_k$, gives maps a_k and b_k satisfying (1_k). If (2_k) holds (and we will show that it does if ϵ_0 is sufficiently small) then Lemma 3.5, applied with $\alpha = \alpha_k$, $\beta = \beta_k$, $\gamma = \gamma_k$ furnishes a map $\widetilde{\gamma} = \gamma_{k+1} \colon \bar{D}_{0,1} \times P_{k+1} \to \mathbb{C}^n$ satisfying (3_k) and (4_k). This completes the inductive step.

To make the induction work we must insure that the sequence $\epsilon_k = ||c_k||_{\mathcal{C}^{r,0}(D_{0,1} \times P_k)}$ satisfies (2_k) for every $k = 0, 1, 2, \ldots$. To control this process we set $N = \max\{\frac{4K_r}{\delta^*}, 1\}$ and define a sequence $\sigma_k > 0$ by

$$\sigma_0 = \epsilon_0; \quad \sigma_{k+1} = 2^k N \sigma_k^2, \quad k = 0, 1, 2, \ldots \tag{3.6}$$

Any sequence $\epsilon_k \geq 0$ beginning with $\epsilon_0 = \sigma_0$ and satisfying (4_k) for all $k \in \mathbb{Z}_+$ clearly satisfies $\epsilon_k \leq \sigma_k$. If we can insure (by choosing $\epsilon_0 > 0$ sufficiently small) that

$$\sigma_k < \frac{\delta^*}{2^{k+4}\sqrt{n}C_r}, \quad k \in \mathbb{Z} \tag{3.7}$$

then $4\sqrt{n}C_r\epsilon_k \leq 4\sqrt{n}C_r\sigma_k < 2^{-k-2}\delta^* = \delta_k$ and hence (2_k) holds.

We look for a solution in the form $\sigma_k = 2^{\mu_k} N^{\nu_k} {\epsilon_0}^{\tau_k}$. From (3.6) we get

$$\begin{aligned} \mu_{k+1} &= 2\mu_k + k, & \mu_0 &= 0; \\ \nu_{k+1} &= 2\nu_k + 1, & \nu_0 &= 0; \\ \tau_{k+1} &= 2\tau_k, & \tau_0 &= 1. \end{aligned}$$

Solutions are

$$\mu_k = 2^k \sum_{l=1}^{k} l 2^{-l} < 2^{k+1}, \quad \nu_k = 2^k - 1, \quad \tau_k = 2^k.$$

Therefore

$$\sigma_k < 2^{2^{k+1}} N^{2^k} {\epsilon_0}^{2^k} = (4N\epsilon_0)^{2^k}, \qquad k \in \mathbb{N}. \tag{3.8}$$

If $\epsilon_0 = ||c_0||_{\mathcal{C}^{r,0}(D_{0,1} \times P_0)} > 0$ is sufficiently small then this sequence converges to zero very rapidly and satisfies (3.7). (See Lemma 4.8 on p. 166 in [22] for more details.) For such ϵ_0 we have

$$||c_k||_{\mathcal{C}^{r,0}(D_{0,1} \times P_k)} = \epsilon_k \leq \sigma_k \leq (4N\epsilon_0)^{2^k} \to 0$$

and hence $\gamma_k(z,t) \to t$ in $\mathcal{C}^{r,0}(\bar{D}_{0,1} \times P_*)$ as $k \to \infty$.

To complete the proof of Theorem 3.2 we must show that the sequences (3.4) and (3.5) also converge in $\mathcal{C}^{r,0}(\bar{D}_0 \times P_*)$ resp. $\mathcal{C}^{r,0}(\bar{D}_1 \times P_*)$ provided that $\epsilon_0 > 0$ is sufficienly small. Write

$$\widetilde{\alpha}_k(z,t) = t + \widetilde{a}_k(z,t), \quad \widetilde{\beta}_k(z,t) = t + \widetilde{b}_k(z,t).$$

By Lemma 3.6 we have $\widetilde{a}_{k+1} = \widetilde{a}_k + a_{k+1} + e_{k+1}$ where

$$||e_{k+1}||_{\mathcal{C}^{r,0}(D_0\times P_{k+1})} \le \frac{L_r}{\delta_k} ||\widetilde{a}_k||_{\mathcal{C}^{r,0}(D_0\times P_k)} ||a_{k+1}||_{\mathcal{C}^{r,0}(D_0\times P_{k+1})}.$$

Assuming a priori that $||\widetilde{a}_k||_{\mathcal{C}^{r,0}(D_0\times P_k)} \le 1$ for all $k \in \mathbb{Z}_+$ we get the following estimates for the $\mathcal{C}^{r,0}(D_0 \times P_{k+1})$ norms:

$$||\widetilde{a}_{k+1} - \widetilde{a}_k|| \le ||a_{k+1}|| + ||e_{k+1}|| \le C_r \left(1 + \frac{L_r}{\delta_*} 2^{k+1}\right) \epsilon_{k+1} \le R 2^{k+1} \epsilon_{k+1}$$

with $R = C_r \left(1 + \frac{L_r}{\delta_*}\right)$. Note that $\widetilde{a}_0 = a_0$ and $||a_0|| \le C_r \epsilon_0$. Hence

$$||\widetilde{a}_0||_{\mathcal{C}^{r,0}(D_0\times P_0)} + \sum_{k=0}^{\infty} ||\widetilde{a}_{k+1} - \widetilde{a}_k||_{\mathcal{C}^{r,0}(D_0\times P_{k+1})} \le C_r \epsilon_0 + R \sum_{k=1}^{\infty} 2^k \epsilon_k.$$

Since $\epsilon_k \le \sigma_k \le (4N\epsilon_0)^{2^k}$ for $k \in \mathbb{N}$ (3.8), we see that $R\sum_{k=1}^{\infty} 2^k \epsilon_k < \epsilon_0$ if $\epsilon_0 > 0$ is sufficiently small. (See [22, Lemma 4.8, p. 166] for the details.) This justifies the assumption $||\widetilde{a}_k||_{\mathcal{C}^{r,0}(D_0\times P_k)} \le 1$ and implies that the sequence $\widetilde{a}_k = \widetilde{a}_0 + \sum_{j=1}^{k} (\widetilde{a}_j - \widetilde{a}_{j-1})$ converges on $\bar{D}_0 \times P_*$ to a limit $a = \lim_{k\to\infty} \widetilde{a}_k$ satisfying $||a||_{\mathcal{C}^{r,0}(D_0\times P_*)} \le (C_0 + 1)\epsilon_0$. Hence the estimate in Theorem 3.2 holds for $s = 0$ with the constant $M_{r,0} = C_0 + 1$.

The same proof shows convergence of the sequence $\widetilde{b}_k \to b$ on $\bar{D}_1 \times P_*$ and the estimate $||b||_{\mathcal{C}^{r,0}(D_1\times P_*)} \le (C_0 + 1)\epsilon_0$.

By shrinking the fiber domain $P_* = P_{-\delta^*/2}$ by an extra $\frac{\delta^*}{2}$ and applying the Cauchy estimates to the maps $a(z,\cdot\,)$ and $b(z,\cdot\,)$ we also obtain the estimates in the $\mathcal{C}^{r,s}$ norms in Theorem 3.2. In addition, if ϵ_0 is sufficiently small then the maps $\alpha(z,\cdot\,)\colon P_{-\delta^*} \to \mathbb{C}^n$ and $\beta(z,\cdot\,)\colon P_{-\delta^*} \to \mathbb{C}^n$ are injective holomorphic for each z in their respective domain $\bar{D}_0$ resp. $\bar{D}_1$.

This completes the proof of Theorem 3.2.

Remark 3.7. (Additions to Theorem 3.2.) Theorem 3.2 holds whenever $D_0, D_1, D_{0,1} = D_0 \cap D_1, D = D_0 \cup D_1$ are relatively compact domains with $\mathcal{C}^1$ boundaries satisfying the separation condition $\overline{D_0\backslash D_1} \cap \overline{D_1\backslash D_0} = \emptyset$ and there exists a linear operator $T\colon \mathcal{Z}^r_{0,1}(\bar{D}) \to \mathcal{C}^r(\bar{D})$ satisfying

$$\bar{\partial}(Tf) = f, \qquad ||Tf||_{\mathcal{C}^r(\bar{D})} \le C_r ||f||_{\mathcal{C}^r_{0,1}(\bar{D})}.$$

Pseudoconvexity of $D_{0,1}$ is not needed here, but will be used in the application to gluing sprays (Proposition 4.3).

The proof of Theorem 3.2 carries over to the *parametric case* when γ depends smoothly on real parameters $s = (s_1, \ldots, s_m) \in [0,1]^m \subset \mathbb{R}^m$. Indeed, the proof of Lemma 3.4 remains valid in the parametric case, and the estimates controlling the iteration process can be made uniform with respect to a finite number of s-derivatives. This gives a family of splittings $\gamma^s_z = \beta^s_z \circ (\alpha^s_z)^{-1}$ for $z \in \bar{D}_{0,1}$ with $\mathcal{C}^k$ dependence on $s \in [0,1]^m$ for a given $k \in \mathbb{N}$.

4. Gluing sprays on Cartan pairs

In this section X is a complex space and $h\colon X \to S$ is a holomorphic map to a complex manifold S. Its *branching locus* $\mathrm{br}(h) \subset X$ is the union of X_{sing} and the set of all those points in X_{reg} at which h fails to be a submersion; thus $X' = X \backslash \mathrm{br}(h)$ is a complex manifold and $h|_{X'}\colon X' \to S$ is a holomorphic submersion. For each $x \in X'$ we set $VT_x X = \ker dh_x$, the *vertical tangent space of* X.

A *section* of $h\colon X \to S$ over a subset $D \subset S$ is a map $f\colon D \to X$ satisfying $h(f(z)) = z$ for all $z \in D$. Let $D \subset\subset S$ be a smoothly bounded domain and $r \in \mathbb{Z}_+$. A section $f\colon \bar{D} \to X$ is of class $\mathcal{A}^r(D)$ if it is holomorphic in D and r times continuously differentiable on $\bar{D}$. (At points of $f(bD) \cap X_{sing}$ we use local holomorphic embeddings of X into a Euclidean space.)

Definition 4.1. (Notation as above) An *h-spray of class $\mathcal{A}^r(D)$ with the exceptional set $\sigma = \sigma(f) \subset \bar{D}$ of order $k \geq 0$* is a map $f\colon \bar{D} \times P \to X$, where P (the *parameter set* of f) is an open subset of a Euclidean space $\mathbb{C}^n$ containing the origin, such that the following hold:

(i) f is holomorphic on $D \times P$ and of class $\mathcal{C}^r$ on $\bar{D} \times P$,
(ii) $h(f(z,t)) = z$ for all $z \in \bar{D}$ and $t \in P$,
(iii) the maps $f(\cdot\,,0)$ and $f(\cdot\,,t)$ agree on σ up to order k for $t \in P$, and
(iv) for every $z \in \bar{D} \backslash \sigma$ and $t \in P$ we have $f(z,t) \notin \mathrm{br}(h)$, and the map
$$\partial_t f(z,t)\colon T_t\mathbb{C}^n = \mathbb{C}^n \to VT_{f(z,t)}X$$
is surjective (the *domination condition*).

For a product fibration $h\colon X = S \times Y \to S$, $h(z,y) = z$, we can identify an h-spray $\bar{D} \times P \to S \times Y$ with a *spray of maps* $\bar{D} \times P \to Y$ by composing with the projection $S \times Y \to Y$, $(z,y) \to y$. In this case (ii) is redundant and the domination condition (iv) is replaced by

(iv') if $z \in \bar{D} \backslash \sigma$ and $t \in P$ then $f(z,t) \in Y_{reg}$ and $\partial_t f(z,t)\colon T_t\mathbb{C}^n \to T_{f(z,t)}Y$ is surjective.

Condition (ii) means that $f_t = f(\cdot\,,t)\colon \bar{D} \to X$ is a section of h of class $\mathcal{A}^r(D)$ for every $t \in P$, and by (i) these sections depend holomorphically on the parameter t. We shall call f_0 the *core* (or *central*) *section* of the spray. Conditions (iii) and (iv) imply that the exceptional set $\sigma(f)$ is locally defined by functions of class $\mathcal{A}^r(D)$.

Unlike the sprays used in the Oka-Grauert theory which are defined for all values $t \in \mathbb{C}^n$ but are dominant only at the core section f_0, our sprays are local with respect to t and dominant everywhere for z outside of σ. In applications the parameter domain will be allowed to shrink.

Lemma 4.2. (Existence of sprays) *Let $h\colon X \to S$ be a holomorphic map of a complex space X to a complex manifold S. Let $r \geq 2$ and $k \geq 0$ be integers. Let D be a relatively compact domain with strongly pseudoconvex* *boundary of class $\mathcal{C}^2$ in a Stein manifold S, and let $\sigma \subset \bar D$ be the common zero set of finitely many functions in $\mathcal{A}^r(D)$. Given a section $f_0 \colon \bar D \to X$ of class $\mathcal{A}^r(D)$ such that the set $\{z \in \bar D \colon f(z) \in \mathrm{br}(h)\}$ does not intersect bD and is contained in σ, there exists an h-spray $f \colon \bar D \times P \to X$ of class $\mathcal{A}^r(D)$ with the core section f_0 and the exceptional set σ of order k.*

Proof. By Theorem 2.6 there exists a Stein open set $\Omega \subset X$ containing $f_0(\bar D)$. (This is the only place in the proof where the assumption $r \ge 2$ is used.) According to [21, Proposition 2.2] (for manifolds see [30, Lemma 5.3]) there exist an integer $n \in \mathbb{N}$, an open set $V \subset \Omega \times \mathbb{C}^n$ containing $\Omega \times \{0\}$, and a holomorphic *spray* map $s \colon V \to X$ satisfying the following:

(a) $s(x, 0) = x$ for $x \in \Omega$,
(b) $h(s(x, t)) = h(x)$ for $(x, t) \in V$,
(c) $s(x, t) = x$ when $(x, t) \in V$ and $x \in \mathrm{br}(h)$, and
(d) for each $x \in \Omega \backslash \mathrm{br}(h)$ the partial differential $\partial_t s(x, t)|_{t=0} \colon T_0\mathbb{C}^n \to VT_xX = \ker dh_x$ is surjective.

By the hypothesis we have $\sigma = \{z \in \bar D \colon g_1(z) = 0, \ldots, g_m(z) = 0\}$ where $g_1, \ldots, g_m \in \mathcal{A}^r(D)$. We can assume that $\sup_{z\in\bar D} |g_j(z)| < 1$ for $j = 1, \ldots, m$. Denote the coordinates on $(\mathbb{C}^n)^m = \mathbb{C}^{nm}$ by $t = (t_1, \ldots, t_m)$, where $t_j = (t_{j,1}, \ldots, t_{j,n}) \in \mathbb{C}^n$ for $j = 1, \ldots, m$. Let $l \in \mathbb{N}$. The map $\phi_l \colon \bar D \times (\mathbb{C}^n)^m \to \mathbb{C}^n$, defined by

$$\phi_l(z, t_1, \ldots, t_m) = \sum_{j=1}^{m} g_j(z)^{k+l}\, t_j,$$

is a linear submersion $\mathbb{C}^{nm} \to \mathbb{C}^n$ over each point $z \in \bar D\backslash\sigma$, and it vanishes to order $k + l$ on σ. Let $P \subset \mathbb{C}^{nm}$ be a bounded open set containing the origin. By choosing the integer l sufficiently large we can insure that the map

$$f(z, t) = s(f_0(z), \phi_l(z, t)) \in X$$

is a spray $\bar D \times P \to X$ with the core section f_0 and the exceptional set σ of order k. All conditions except (iv) are evident. Observe that the set Σ, consisting of all points $(x, t) \in V$ at which the spray s does not satisfy the domination condition (d), is a closed analytic subset of V satisfying $\Sigma \cap (\Omega \times \{0\}) = \mathrm{br}(h)$. The contact between Σ and $\Omega \times \{0\}$ is necessarily of finite order along their intersection, and by choosing $l \in \mathbb{Z}_+$ large enough we insure that $\phi_l(z, t) \in V\backslash\Sigma$ for every $z \in \bar D\backslash\sigma$ and $t \in P$. For such choices f also satisfies the property (iv) of a spray. $\square$

The following proposition provides the main tool for gluing holomorphic sections on Cartan pairs by preserving their boundary regularity.

Proposition 4.3. (Gluing sprays) *Let $h \colon X \to S$ be a holomorphic map from a complex space X onto a Stein manifold S. Let (D_0, D_1) be a Cartan pair of class $\mathcal{C}^\ell$ ($\ell \ge 2$) in S (Def. 3.1) and let $D = D_0 \cup D_1$, $D_{0,1} = D_0 \cap D_1$. Given integers $r \in \{0, 1, \ldots, \ell\}$, $k \in \mathbb{Z}_+$, and an h-spray $f \colon \bar D_0 \times P_0 \to$*

X of class $\mathcal{A}^r(D_0)$ with the exceptional set $\sigma(f)$ of order k and satisfying $\sigma(f) \cap \bar{D}_{0,1} = \emptyset$, there is an open set $P \Subset P_0$ containing $0 \in \mathbb{C}^n$ such that the following hold.

For every h-spray $f' \colon \bar{D}_1 \times P_0 \to X$ of class $\mathcal{A}^r(D_1)$ with the exceptional set $\sigma(f')$ of order k, with $\sigma(f') \cap \bar{D}_{0,1} = \emptyset$, such that f' is sufficiently $\mathcal{C}^r$ close to f on $\bar{D}_{0,1} \times P_0$ there exists an h-spray $g \colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ with the exceptional set $\sigma(g) = \sigma(f) \cup \sigma(f')$ of order k whose restriction $g \colon \bar{D}_0 \times P \to X$ is as close as desired to $f \colon \bar{D}_0 \times P \to X$ in the $\mathcal{C}^r$ topology. The core section $g_0 = g(\cdot\,, 0)$ is homotopic to f_0 on $\bar{D}_0$, and g_0 is homotopic to f_0' on $\bar{D}_1$. In addition, g_0 agrees with f_0 up to order k on $\sigma(f)$, and g_0 agrees with f_0' up to order k on $\sigma(f')$.

If f and f' agree to order $m \in \mathbb{N}$ along $\bar{D}_{0,1} \times \{0\}$ then g can be chosen to agree with f to order m along $\bar{D}_0 \times \{0\}$, and to agree with f' to order m along $\bar{D}_1 \times \{0\}$.

Proof. First we find a holomorphic transition map between the two sprays (Lemma 4.4); decomposing this map by Theorem 3.2 we can adjust the two sprays to match them over $\bar{D}_{0,1}$. The first step is accomplished by the following lemma applied on the strongly pseudoconvex domain $D_{0,1}$.

Lemma 4.4. *Let $D \Subset S$ be a strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$) in a Stein manifold S, let P_0 be a domain in $\mathbb{C}^n$ containing the origin, and let $f \colon \bar{D} \times P_0 \to X$ be a spray of class $\mathcal{A}^r(D)$ ($0 \leq r \leq \ell$) with trivial exceptional set. Choose $\epsilon^* > 0$. There exists an open set $P_1 \subset \mathbb{C}^n$, with $0 \in P_1 \Subset P_0$, satisfying the following. For every spray $f' \colon \bar{D} \times P_0 \to X$ of class $\mathcal{A}^r(D)$ which approximates f sufficiently closely in the $\mathcal{C}^r$ topology there exists a map $\gamma \colon \bar{D} \times P_1 \to \mathbb{C}^n$ of class $\mathcal{A}^{r,0}(D \times P_1)$ satisfying*

$$\gamma(z,t) = t + c(z,t), \qquad ||c||_{\mathcal{C}^{r,0}(D\times P_1)} < \epsilon^*, \tag{4.1}$$

$$f(z,t) = f'(z, \gamma(z,t)), \quad (z,t) \in \bar{D} \times P_1. \tag{4.2}$$

If f and f' agree to order m along $\bar{D} \times \{0\}$ then we can choose γ of the form $\gamma(z,t) = t + \sum_{|J|=m} \widetilde{c}_J(z,t) t^J$ with $\widetilde{c}_J \in \mathcal{A}^{r,0}(D \times P_1)^n$.

Assuming Lemma 4.4 for the moment we conclude the proof of Proposition 4.3 as follows. Let P_1 be as in the conclusion of Lemma 4.4. Choose an open set $P \subset \mathbb{C}^n$ with $0 \in P \Subset P_1$. For $\epsilon^* > 0$ chosen sufficiently small, Theorem 3.2 applied to γ gives a decomposition

$$\gamma(z, \alpha(z,t)) = \beta(z,t), \qquad (z,t) \in \bar{D}_{0,1} \times P \tag{4.3}$$

where $\alpha \colon \bar{D}_0 \times P \to P_1 \subset \mathbb{C}^n$ and $\beta \colon \bar{D}_1 \times P \to P_1 \subset \mathbb{C}^n$ are maps of class $\mathcal{A}^{r,0}$. Replacing t by $\alpha(z,t)$ in (4.2) gives

$$f(z, \alpha(z,t)) = f'(z, \beta(z,t)), \quad (z,t) \in \bar{D}_{0,1} \times P. \tag{4.4}$$

Hence the two sides define a map $g \colon \bar{D} \times P \to X$ of class $\mathcal{C}^r(\bar{D} \times P)$ which is holomorphic in $D \times P$. Since the maps α and β are injective holomorphic on the fibers $\{z\} \times P$, g is a spray with the exceptional set $\sigma(g) = \sigma(f) \cup \sigma(f')$.

The estimates on α and β in Theorem 3.2 show that their distances from the identity map are controlled by the number ϵ^*, and hence (in view of Lemma 4.4) by the $\mathcal{C}^r$ distance of f' to f on $\bar{D}_{0,1} \times P_0$. Hence the new spray g approximates f in $\mathcal{C}^r(\bar{D}_0 \times P)$. On the other hand, we don't get any obvious control on the $\mathcal{C}^r$ distance between f' and g on $\bar{D}_1 \times P$, the problem being that the $\mathcal{C}^r$ norm of f' is not a priori bounded, and precomposing f' by a map β (even if it is close to the identity map) can still cause a big change. However, in our application in §6 we shall only need to control the range (location) of g, and this will be insured by the construction.

Finally, if f and f' agree to order m along $\bar{D}_{0,1} \times \{0\}$ then by Lemma 4.4 we can choose γ of the form $\gamma(z,t) = t + \sum_{|J|=m} \widetilde{c}_J(z,t)t^J$ with $\widetilde{c}_J \in \mathcal{A}^{r,0}(D_{0,1} \times P_1)^n$ for each multiindex J. Theorem 3.2 then gives a decomposition (4.3) where $\alpha(z,t) = t + \sum_{|J|=m} \widetilde{a}_J(z,t)t^J$ and $\beta(z,t) = t + \sum_{|J|=m} \widetilde{b}_J(z,t)t^J$, thereby insuring that the spray g (4.4) agrees with f resp. f' to order m at $t=0$. This proves Proposition 4.3 granted that Lemma 4.4 holds.

Proof of Lemma 4.4. This is essentially a consequence of the implicit function theorem. Let E denote the subbundle of $\bar{D} \times \mathbb{C}^n$ with fibers

$$E_z = \ker\left(\partial_t f(z,t)|_{t=0} \colon \mathbb{C}^n \to VT_{f(z,0)}X\right), \quad z \in \bar{D}.$$

This subbundle is holomorphic over D and of class $\mathcal{C}^r$ on $\bar{D}$. We claim that E is complemented, i.e., there exists a complex vector subbundle $G \subset \bar{D} \times \mathbb{C}^n$ of the same class as E such that $\bar{D} \times \mathbb{C}^n = E \oplus G$. For holomorphic vector bundles on open Stein manifolds this follows from Cartan's Theorem B [39, p. 256]. The same proof applies in the category of holomorphic vector bundles with continuous boundary values over a strongly pseudoconvex domain by using the corresponding version of Theorem B due to Leiterer [52] and Heunemann [45], thereby giving a direct sum complement G which is continuous on $\bar{D}$ and holomorphic over D. Finally we approximate G uniformly on $\bar{D}$ by holomorphic vector subbundles of $U \times \mathbb{C}^n$ over an open neighborhood $U \supset \bar{D}$ (Heunemann [44]) to obtain a desired splitting.

For each fixed $z \in U$ we write $\mathbb{C}^n \ni t = t'_z \oplus t''_z$ with $t'_z \in E_z$ and $t''_z \in G_z$. Note that the partial differential $\partial_t|_{t=0} f(\cdot\,,t)$ gives an isomorphism $G|_{\bar{D}} \to VT_{f_0(\bar{D})}X$ and it vanishes on E. The implicit function theorem now shows that there is an open neighborhood P_1 of $0 \in \mathbb{C}^n$, with $\bar{P}_1 \subset P_0$, such that for each spray $f' \colon \bar{D} \times P_0 \to X$ which is sufficiently $\mathcal{C}^r$ close to f on $\bar{D} \times P_0$ there is a unique map

$$\widetilde{\gamma}(z, t'_z \oplus t''_z) = t'_z \oplus (t''_z + \widetilde{c}(z,t)) \in E_z \oplus G_z = \mathbb{C}^n$$

of class $\mathcal{A}^{r,0}(D \times P_1)$ solving $f(z,\widetilde{\gamma}(z,t)) = f'(z,t)$, and $||\widetilde{c}||_{\mathcal{A}^{r,0}(D_{0,1} \times P_1)}$ is controlled by the $\mathcal{C}^r$ distance between f and f' on $\bar{D} \times P_0$. After shrinking P_1 the fiberwise inverse $\gamma(z,t) = t' \oplus (t''_z + c''(z,t))$ then satisfies (4.2), and $||c''||_{\mathcal{A}^{r,0}(D_{0,1} \times P_1)}$ is controlled by the $\mathcal{C}^r$ distance between f and f' on $\bar{D} \times P_0$. $\square$

Remark 4.5. The additions to Theorem 3.2, explained in Remark 3.7, yield corresponding addition and generalizations of Proposition 4.3. We indicate a few which we plan to use in the future.

First of all, one can relax the definition of a spray by omitting the condition regarding the exceptional set. The only essential condition needed in Proposition 4.3 is that the spray f is *dominating on* $\bar{D}_{0,1}$, in the sense that its t-differential is surjective on this set at $t = 0$. (This notion of domination agrees with the one introduced by Gromov [38].) Approximating such spray f sufficiently closely in the $\mathcal{C}^r$ topology on $\bar{D}_0 \times P$ (for some open neighborhood $P \subset \mathbb{C}^n$ of the origin) by another spray f', we can glue f and f' into a new spray g over $\bar{D}_0 \cup \bar{D}_1$ which is dominating over $\bar{D}_{0,1}$. The 'exceptional set' condition is only needed when one wishes to interpolate a given spray on a subvariety of $\bar{D}_0$.

The parametric version of Theorem 3.2 (see Remark 3.7) also gives the corresponding parametric version of Proposition 4.3 in which the two h-sprays f and f' depend smoothly on a real parameter $s \in [0,1]^m \subset \mathbb{R}^m$. The remaining ingredients of the proof (such as Lemma 4.4) carry over to the parametric case without difficulties.

5. Approximation of holomorphic maps to complex spaces

In this section we prove the following approximation theorem for maps of bordered Riemann surfaces to arbitrary complex spaces. This result is used in the proof of Theorem 1.1 to replace the initial map by another one which maps the boundary into the regular part of the space.

Theorem 5.1. *Let D be a connected, relatively compact, smoothly bounded domain in an open Riemann surface S, let X be a complex space, and let $f\colon \bar{D} \to X$ be a map of class $\mathcal{C}^r$ $(r \geq 2)$ which is holomorphic in D. Given finitely many points $z_1, \ldots, z_l \in D$ and an integer $k \in \mathbb{N}$, there is a sequence of holomorphic maps $f_\nu\colon U_\nu \to X$ in open sets $U_\nu \subset S$ containing $\bar{D}$ such that f_ν agrees with f to order k at z_j for $j = 1, \ldots, l$ and $\nu \in \mathbb{N}$, and the sequence f_ν converges to f in $\mathcal{C}^r(\bar{D})$ as $\nu \to +\infty$. If $f(D) \not\subset X_{sing}$, we can also insure that $f_\nu(bD) \subset X_{reg}$ for each $\nu \in \mathbb{N}$.*

Proof. We proceed by induction on $n = \dim X$. The result trivially holds for $n = 0$. Assume that it holds for all complex spaces of dimension $< n$ for some $n > 0$, and let $\dim X = n$. If $f(D) \subset X_{sing}$ then the conclusion holds by applying the inductive hypothesis with the complex space X_{sing}. Suppose now that $f(D) \not\subset X_{sing}$. The set

$$\sigma = \{z \in \bar{D}\colon f(z) \in X_{sing}\} \tag{5.1}$$

is compact, $\sigma \cap D$ is discrete, and $\sigma \cap bD$ has empty relative interior in bD. Indeed, as X_{sing} is an analytic subset of X and hence complete pluripolar, the existence of a nonempty arc in bD which f maps to X_{sing} would imply $f(\bar{D}) \subset X_{sing}$ in contradiction to our assumption.

Set $K = \{z_1, \ldots, z_l\}$. Let $bD = \cup_{j=1}^m C_j$ where each C_j is a closed Jordan curve. For each $j = 1, \ldots, m$ we choose a point $p_j \in C_j \backslash \sigma$ and an open set $U_j \subset S$ such that $p_j \subset U_j$ and $\overline{U}_j$ does not intersect $\sigma \cup K$. We choose the sets U_j so small that $f(\bar{D} \cap \overline{U}_j)$ is contained in a local chart of X_{reg}.

Lemma 5.2. *The map f can be approximated in $\mathcal{C}^r(\bar{D}, X)$ by maps $f' \colon \bar{D}' \to X$ of class $\mathcal{A}^r(D', X)$, where $D' \subset S$ is a smoothly bounded domain (depending on f') satisfying $D \cup \{p_j\}_{j=1}^m \subset D' \subset D \cup \left(\cup_{j=1}^m U_j\right)$. In addition we can choose f' such that it agrees with f to order k at z_j for $j \in \{1, \ldots, l\}$.*

Proof. By Theorem 2.1 the graph of f over $\bar{D}$ has an open Stein neighborhood in $S \times X$. It follows that the set σ (5.1) is the common zero set of finitely many functions in $\mathcal{A}^r(D)$. By Lemma 4.2 there is a spray $\widetilde{f} \colon \bar{D} \times P \to X$ ($P \subset \mathbb{C}^N$) of class $\mathcal{A}^r(D)$, with the core map $\widetilde{f}(\cdot\,, 0) = f$ and the exceptional set $\tilde{\sigma} = \sigma \cup K$ of order k.

After shrinking the parameter set $P \subset \mathbb{C}^N$ of $\widetilde{f}$ around $0 \in \mathbb{C}^N$ we may assume that $\widetilde{f}$ maps the set $E_j = (\overline{U}_j \cap \bar{D}) \times \bar{P}$ into a local chart $\Omega \subset X_{reg}$ for each $j = 1, \ldots, m$. Hence we can approximate the restriction of $\widetilde{f}$ to E_j as close as desired in the $\mathcal{C}^r$ sense by a spray $\widetilde{g}_j \colon \overline{V}_j \times P \to X_{reg}$, where V_j is an open set in S (depending on $\widetilde{g}_j$) satisfying $U_j \cap \bar{D} \subset V_j \subset U_j$.

If the approximations are sufficiently close, Lemma 4.4 furnishes a transition map γ_j between $\widetilde{f}$ and $\widetilde{g}_j$ for each j (we shrink P as needed), and Proposition 4.3 lets us glue $\widetilde{f}$ with the sprays $\widetilde{g}_j$ into a spray F of class $\mathcal{A}^r(D')$ over a domain $D' \subset S$ as in Lemma 5.2. By the construction F approximates $\widetilde{f}$ in the $\mathcal{C}^r(\bar{D} \times P)$ topology, and it agrees with $\widetilde{f}$ to a order k at the points $z_j \in K$. The core map $f' = F(\cdot\,, 0) \colon \bar{D}' \to X$ then satisfies the conclusion of the lemma.

A word is in order regarding the application of Proposition 4.3. Unlike in that proposition, the final domain D' in our present situation will have to depend on the choices of the sprays $\widetilde{g}_j$ (since the size of their z-domains in S depends on the rate of approximation). We can choose from the outset a fixed domain $D_1 \subset S$ such that (D, D_1) is a Cartan pair in S satisfying $\overline{D \cap D_1} \subset \cup_{j=1}^m (\bar{D} \cap U_j)$. Applying Theorem 3.2 gives maps α and β over $\bar{D}$ resp. $\bar{D}_1$; the new spray F is defined as $\widetilde{f}(z, \alpha(z,t))$ for $z \in \bar{D}$, and by $\widetilde{g}_j(z, \beta(z,t))$ for $z \in \bar{D}_1 \cap U_j$. Thus we are not using the map β on its entire domain of existence, but only over the domain of the sprays $\widetilde{g}_j$. $\square$

We continue with the proof of Theorem 5.1. Let $f' \colon \bar{D}' \to X$ be a map furnished by Lemma 5.2. In each boundary curve $C_j \subset bD$ we choose a closed arc $\lambda_j \subset C_j$ such that $C_j \backslash \lambda_j \subset D'$ (this is possible since D' contains the point $p_j \in C_j$). Let ξ_j be a holomorphic vector field in a neighborhood of λ_j in S such that $\xi(z)$ points to the interior of D for every $z \in \lambda_j$. More precisely, if $D = \{v < 0\}$, with $dv \neq 0$ on bD, we ask that $\Re(\xi_j \cdot v) < 0$ on λ_j; such fields clearly exist.

Choose a domain $D_0 \subset S$ with $\bar{D}' \subset D_0$ such that $\bar{D}$ is holomorphically convex in D_0. (This holds when $D_0 \backslash \bar{D}$ is connected.) The union of K with all the arcs λ_j is a compact holomorphically convex set in D_0. The tangent bundle of D_0 is trivial which lets us identify vector fields with functions. Hence there exists a holomorphic vector field ξ on D_0 which approximates the field ξ_j sufficiently closely on λ_j so that it remains inner radial to D there, and ξ vanishes to order k at the points $z_j \in K$. For sufficiently small $t > 0$ the flow ϕ_t of ξ carries each of the arcs λ_j into D, and hence $\phi_t(\bar{D}) \subset D'$ provided that $t > 0$ is small enough. (Recall that $C_j \backslash \lambda_j \subset D'$; hence the points of $\bar{D}$ which may be carried out of $\bar{D}$ by the flow ϕ_t along $C_j \backslash \lambda_j$ remain in D' for small $t > 0$.)

Since the set $\sigma' = \{z \in D' \colon f'(z) \in X_{sing}\}$ is discrete, a generic choice of $t > 0$ also insures that $\phi_t(bD) \cap \sigma' = \emptyset$. For such t the map $f' \circ \phi_t$ is holomorphic in an open neighborhood of $\bar{D}$, it maps bD to X_{reg}, it approximates f in the $\mathcal{C}^r(\bar{D})$ topology, and it agrees with f to order k at each point $z_j \in K$. This provides a sequence f_ν satisfying Theorem 5.1. $\square$

Remark 5.3. R. Chakrabarti proved the following approximation result in [8, Theorem 1.1.4]: *If D is a domain in $\mathbb{C}$ bounded by finitely many Jordan curves and X is a complex manifold then every continuous map $f \colon \bar{D} \to X$ which is holomorphic on D can be approximated uniformly on $\bar{D}$ by maps which are holomorphic in open neighborhoods of $\bar{D}$ in $\mathbb{C}$.* A comparison with Theorem 5.1 shows that there is a stronger hypothesis on X, but a weaker hypothesis on the map.

6. Proof of Theorem 1.1

We begin with the two main lemmas. The induction step in the proof of Theorem 1.1 is provided by Lemma 6.3, and the key local step is furnished by Lemma 6.2.

We denote by $d_{1,2}$ the partial differential with respect to the first two complex coordinates on $\mathbb{C}^n$.

Definition 6.1. Let A and B be relatively compact open sets in a complex space X. We say that B *is a 2-convex bump on* A (fig. 2) if there exist an open set $\Omega \subset X_{reg}$ containing $\bar{B}$, a biholomorphic map Φ from Ω onto a convex open set $\omega \subset \mathbb{C}^n$, and smooth real functions $\rho_B \leq \rho_A$ on ω such that

$$\Phi(A \cap \Omega) = \{x \in \omega \colon \rho_A(x) < 0\}, \ \Phi((A \cup B) \cap \Omega) = \{x \in \omega \colon \rho_B(x) < 0\},$$

ρ_A and ρ_B are strictly convex with respect to the first two complex coordinates, and $d_{1,2}(t\rho_A + (1-t)\rho_B)$ is non degenerate on ω for each $t \in [0, 1]$.

Let $\rho \colon X \to \mathbb{R}$ be a smooth function which is $(n-1)$-convex on an open subset $U \subset X$. If the set $\{x \in U \colon c_0 \leq \rho(x) \leq c_1\}$ is compact, contained in X_{reg}, and it contains no critical points of ρ then the set $\{x \in U \colon \rho(x) \leq c_1\}$ is obtained from $\{x \in U \colon \rho(x) \leq c_0\}$ by a finite process in which every step is an attachment of a 2-convex bump (Lemma 12.3 in [42]). The essential ingredient in the proof is Narasimhan's lemma on local convexification.

The following lemma was proved in [19] in the case when X is a complex manifold, D is the disc, and for holomorphic maps instead of sprays. Its proof in [19] was based on the solution of the non linear Cousin problem in [68]. This does not seem to suffice in the case of a complex space with singularities and an arbitrary bordered Riemann surface. Instead we shall use Proposition 4.3.

Since the complex space X is paracompact, it is metrizable. Fix a complete distance function d on X.

Lemma 6.2. *Let X be an irreducible complex space of $\dim X \geq 2$. Let $A \subset\subset X$ be relatively compact open subset of X and let B be a 2-convex bump on A (Def. 6.1). Let D be a bordered Riemann surface with smooth boundary, let P be a domain in $\mathbb{C}^N$ containing 0, and let $k \geq 0$ be an integer. Assume that $f\colon \bar{D} \times P \to X$ is a spray of maps of class $\mathcal{A}^2(D)$ with the exceptional set σ of order k (Def. 4.1) such that $f_0(bD) \cap \bar{A} = \emptyset$. (Here $f_0 = f(\cdot\,, 0)$ is the core map of the spray.) Further assume that K is a compact subset of A and U is an open subset of D such that $f_0(\bar{D}\backslash U) \cap K = \emptyset$.*

Given $\epsilon > 0$, there are a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g\colon \bar{D} \times P' \to X$ of class $\mathcal{A}^2(D)$, with the exceptional set σ of order k, such that g_0 is homotopic to f_0 and the following hold for all $t \in P'$:

(i) *$g_t(bD) \cap \overline{A \cup B} = \emptyset$,*
(ii) *$d(g_t(z), f_t(z)) < \epsilon$ for $z \in \overline{U}$,*
(iii) *$g_t(\bar{D}\backslash U) \cap K = \emptyset$, and*
(iv) *the maps f_0 and g_0 have the same k-jets at every point in σ.*

Proof. Let $\Phi\colon X \supset \Omega \to \omega \subset \mathbb{C}^n$ be a biholomorphic map as in Def. 6.1. By enlarging the set $U \subset\subset D$ we may assume that $\sigma \subset U$. For small $\lambda > 0$ set

$$\omega_\lambda = \{x \in \omega\colon \rho_B(x) < \lambda,\ \ \rho_A(x) > \lambda\}, \qquad \Omega_\lambda = \Phi^{-1}(\omega_\lambda).$$

Then $\omega_\lambda \Subset \omega$ and $\Omega_\lambda \Subset \Omega$.

Since $f_0(bD) \cap \bar{A} = \emptyset$, we have $\rho_A(\Phi(f_0(z))) > \lambda$ for every sufficiently small $\lambda > 0$ and for all $z \in bD$ with $f_0(z) \in \Omega$. A transversality argument shows that for almost every small $\lambda > 0$ the set $bD \cap f_0^{-1}(\overline{\Omega}_\lambda)$ is a finite union $\cup_{j=1}^{m'} I_j$ of pairwise disjoint closed arcs I_j ($j = 1, \ldots, m$) and simple closed curves I_j ($j = m+1, \ldots, m'$). Fix a λ for which the above hold.

If I_j is an arc, we choose a smooth simple closed curve $\Gamma_j \subset \bar{D}\backslash U$ such that $\Gamma_j \cap bD$ is a neighborhood of I_j in bD, and Γ_j bounds a simply connected domain $U_j \subset D\backslash\overline{U}$ (fig. 3). Choose a smooth diffeomorphism $h_j\colon \bar{\triangle} \to \overline{U}_j$ which is holomorphic on $\triangle$, and choose a compact set $V_j \subset \overline{U}_j$ containing a neighborhood of I_j in $\bar{\triangle}$.

If I_j is a simple closed curve, there is a collar neighborhood $\overline{U}_j \subset \bar{D}\backslash\overline{U}$ of I_j in $\bar{D}$ whose boundary $bU_j = I_j \cup I'_j$ consists of two smooth simple

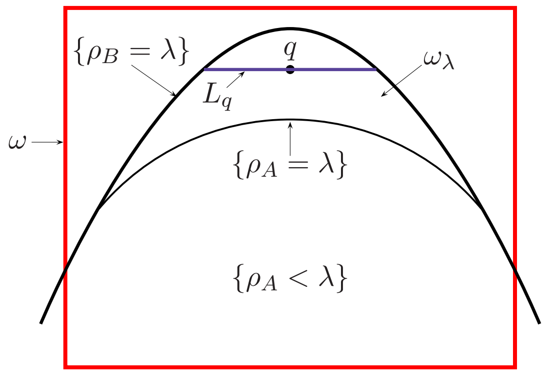

FIGURE 2. A 2-convex bump

closed curves. Let $\Gamma_j = I_j$. There are an open subset W_j of $\triangle$ and a diffeomorphism $h_j\colon \bar{\triangle}\backslash W_j \to \bar{U}_j$ which is holomorphic on $\triangle\backslash\overline{W}_j$ such that $h_j(b\triangle) = \Gamma_j$. Choose a compact annular neighborhood V_j of Γ_j in $U_j \cup \Gamma_j$.

By choosing the sets $U_1, \ldots, U_{m'}$ sufficiently small we can insure that their closures are pairwise disjoint and don't intersect $\overline{U}$, and we have

$$f_0(\overline{U}_j) \subset \{x \in \Omega\colon \rho_A(\Phi(x)) > \lambda\}, \quad j = 1, \ldots m'.$$

Denote by D_1 the union $\cup_{j=1}^{m'} U_j$. There is a smoothly bounded open set D_0, with $D\backslash D_1 \subset D_0 \subset D\backslash \cup_{j=1}^{m'} V_j$, such that (D_0, D_1) is a Cartan pair (Def. 3.1; see fig. 3). Let $D_{0,1} = D_0 \cap D_1$.

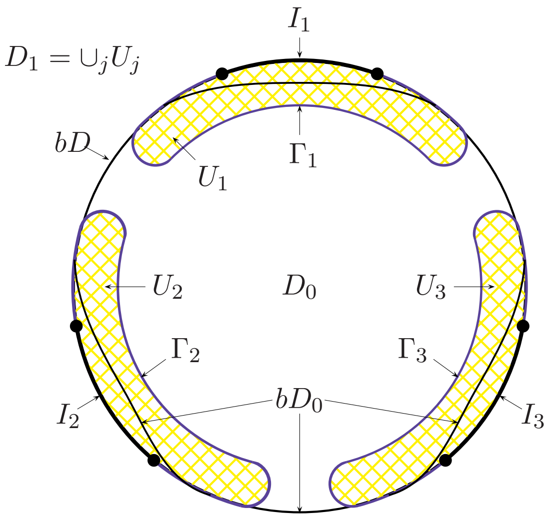

FIGURE 3. Cartan pair (D_0, D_1)

Our goal is to approximate f in the $\mathcal{C}^2$ topology on $\bar{D}_{0,1}$ by a spray f' over $\bar{D}_1$ such that the maps f'_t will satisfy properties (i) and (iii) on its domain. (The final spray g over $\bar{D}$ will be obtained by gluing the restriction of f to $\bar{D}_0$ with the spray f', using Proposition 4.3.) To this end we shall now find a suitable family of holomorphic discs which will be used to increase the value of $\rho \circ f_0$ on the part of bD which is mapped by f_0 into Ω_λ.

Consider the homotopy $\rho_s \colon \omega \to \mathbb{R}$ defined by

$$\rho_s = (1-s)(\rho_A - \lambda) + s(\rho_B - \lambda), \quad s \in [0,1].$$

The function ρ_s is strictly convex with respect to the first two coordinates (since it is a convex combination of functions with this property), and $d_{1,2}\rho_s$ is non degenerate on ω by the definition of a 2-convex bump. As the parameter s increases from $s=0$ to $s=1$, the sets $\{\rho_s \le 0\}$ increase smoothly from $\{\rho_A \le \lambda\}$ to $\{\rho_B \le \lambda\}$. (Inside ω_λ these sets are strictly increasing.) For each point $q \in \omega_\lambda$ we have $\rho_A(q) > \lambda$ while $\rho_B(q) < \lambda$; hence there is a unique $s \in [0,1]$ such that $\rho_s(q) = 0$. Write $q = (q_1, q_2, q'')$, with $q'' \in \mathbb{C}^{n-2}$. The set

$$M_{s,q''} = \{(x_1, x_2, q'') \in \omega \colon \rho_s(x_1, x_2, q'') = 0\}$$

is a real three dimensional submanifold of $\mathbb{C}^2 \times \{q''\}$. Let $T_q M_{s,q''}$ denote its real tangent space at q; then $E_q = T_q M_{s,q''} \cap i\, T_q M_{s,q''}$ is a complex line in $T_q\mathbb{C}^n = \mathbb{C}^n$. By strict convexity of ρ_B with respect to the first two variables the intersection

$$L_q = (q + E_q) \cap \{x \in \omega \colon \rho_B(x) \le \lambda\}$$

is a compact, connected, smoothly bounded convex subset of $q + E_q$ with $bL_q \subset \{\rho_B = \lambda\}$ (fig. 2). The sets L_q depend smoothly on $q \in \omega_\lambda$ and degenerate to the point $L_q = \{q\}$ for $q \in b\,\omega_\lambda \cap \{\rho_A > \lambda\}$. We set $L_q = \{q\}$ for all points $q \in \omega$ with $\rho_B(q) \ge \lambda$.

Given a point $z \in \Gamma_j \subset bD_1$ for some $j \in \{1, \ldots, m'\}$, we set

$$\widetilde{L}_z = L_q \ \text{ with } \ q = \Phi(f_0(z)).$$

The definition is good since $\rho_A(\Phi(f_0(z))) > \lambda$ for all $z \in \bar{D}_1$.

An elementary argument (see e.g. [33, Section 4]) gives for each $j \in \{1, \ldots m'\}$ a continuous map $H_j \colon \Gamma_j \times \bar{\triangle} \to \omega$ such that for each $z \in I_j$ the map $\bar{\triangle} \ni \eta \mapsto H_j(z,\eta) \in \widetilde{L}_z$ is a holomorphic parametrization of $\widetilde{L}_z$ and $H_j(z,0) = \Phi(f_0(z))$; if $z \in \Gamma_j \backslash I_j$ then $H_j(z,\eta) = \Phi(f_0(z))$ for all $\eta \in \bar{\triangle}$.

Recall that h_j is a parametrization of $\overline{U}_j$ by a $\bar{\triangle}$ if $j \in \{1, \ldots, m\}$, resp. by an annular region in $\bar{\triangle}$ if $j \in \{m+1, \ldots, m'\}$. Let $G_j \colon b\triangle \times \bar{\triangle} \to \mathbb{C}^n$ be defined by

$$G_j(\zeta, \eta) = H_j(h_j(\zeta), \eta) - \Phi(f_0(h_j(\zeta))), \quad \zeta \in b\triangle,\ \eta \in \bar{\triangle}.$$

Observe that $G_j(\zeta,\eta) = 0$ if $\zeta \in h_j^{-1}(\Gamma_j \backslash I_j)$ and $\eta \in \bar{\triangle}$.

Let $\mathbb{B} \subset \mathbb{C}^n$ denote the unit ball and $\delta\,\mathbb{B}$ the ball of radius δ. For each $j \in \{1, \ldots, m'\}$ and each $\delta > 0$ we solve approximately the Riemann-Hilbert problem for the map G_j, using [33, Lemma 5.1], to obtain a holomorphic polynomial map $Q_{\delta,j}\colon \mathbb{C} \to \mathbb{C}^n$ satisfying the following properties:

$$Q_{\delta,j}(\zeta) \in G_j(\zeta, b\triangle) + \delta\,\mathbb{B} \quad \text{for } \zeta \in b\triangle, \tag{6.1}$$

$$|D^2 Q_{\delta,j}(\zeta)| < \delta \quad \text{for } \zeta \in h_j^{-1}(\overline{U_j \backslash V_j}), \tag{6.2}$$

$$Q_{\delta,j}(\zeta) \in G_j(b\triangle, \bar{\triangle}) + \delta\,\mathbb{B} \quad \text{for } \zeta \in h_j^{-1}(\overline{U}_j). \tag{6.3}$$

Here $D^2Q = (Q, Q', Q'')$ is the second order jet of Q. Although Lemma 5.1 in [33] only gives a uniform estimate in (6.2), we can apply it to a larger disc containing $h_j^{-1}(\overline{U_j \backslash V_j})$ in its interior to obtain the estimates of derivatives.

Define a map $Q_\delta\colon \bar{D}_1 = \cup_{j=1}^{m'} \overline{U}_j \to \mathbb{C}^n$ by

$$Q_\delta(z) = Q_{\delta,j}(h_j^{-1}(z)), \quad z \in \overline{U}_j.$$

By (6.2) the map Q_δ and its first two derivatives have modulus bounded by δ on $\cup_{j=1}^{m'} \overline{U_j \backslash V_j}$, and hence on $\bar{D}_{0,1}$. If $z \in \Gamma_j \cap bD$ then (6.1) gives

$$|Q_\delta(z) + \Phi(f_0(z)) - H_j(z,\eta)| < \delta \quad \text{for some} \quad \eta \in b\triangle,$$

and hence the point $Q_\delta(z) + \Phi(f_0(z))$ is contained in the δ-neighborhood of $b\widetilde{L}_z$. Recall that for $z \in I_j$ we have $b\widetilde{L}_z \subset \{\rho_B = \lambda\}$, and for $z \in \Gamma_j \backslash I_j$ we have $\widetilde{L}_z = \{\Phi(f_0(z))\}$. By choosing $\delta_0 > 0$ sufficiently small we insure that

$$\rho_B\big(Q_\delta(z) + \Phi(f(z,t))\big) > 0$$

for all $z \in \Gamma_j \cap bD$, $j = 1, \ldots, m'$, $0 < \delta < \delta_0$, and all t in a certain neigborhood $P_0 \subset P$ of $0 \in \mathbb{C}^N$. For such choices (and a fixed $\delta \in (0, \delta_0)$) the map $f' = f'_\delta\colon \bar{D}_1 \times P_0 \to X$, defined by

$$f'(z,t) = \Phi^{-1}\big(Q_\delta(z) + \Phi(f(z,t))\big), \quad z \in \bar{D}_1,\ t \in P_0,$$

is a spray of maps of class $\mathcal{A}^2(D_1)$, with trivial (empty) exceptional set, whose boundary values on $bD_1 \cap bD$ lie outside of $\overline{A \cup B}$. By choosing $\delta > 0$ small enough we insure that f' approximates the spray f as close as desired in the $\mathcal{C}^2$ norm on $\bar{D}_{0,1} \times P_0$.

By Proposition 4.3 we can glue f and f' into a spray of maps $g\colon \bar{D} \times P' \to X$ approximating f on $\bar{D}_0 \times P'$; hence the central map $g_0 = g(\cdot\,, 0)$ satisfies property (ii) in Lemma 6.2, and also property (i) on $bD_0 \cap bD$. For $z \in \bar{D}_1$ we have $g(z,t) = f'(z, \beta(z,t))$ by (4.4), where the $\mathcal{C}^2$ norm of β is controlled by δ. Choosing $\delta > 0$ sufficiently small we insure that for each $z \in bD_1 \cap bD$ we have $g_0(z) = g(z,0) \in X \backslash \overline{A \cup B}$, so (i) holds also on $bD_1 \cap bD$. Similarly, since $f'_t(\bar{D}_1)$ does not intersect $\bar{A} \supset K$, we see that g_0 satisfies property (iii). By shrinking P' we obtain the same properties for all maps g_t, $t \in P'$. Finally, property (iv) holds by the construction (this does not depend on the choice of the constants). $\square$

Lemma 6.3. *Let X be an irreducible complex space of dimension $n \geq 2$, and let $\rho\colon X \to \mathbb{R}$ be a smooth exhaustion function which is $(n-1)$-convex on $\{x \in X\colon \rho(x) > M_1\}$. Let D be a finite Riemann surface, let P be an open* *set in $\mathbb{C}^N$ containing 0, and let $M_2 > M_1$. Assume that $f\colon \bar{D}\times P \to X$ is a spray of maps of class $\mathcal{A}^2(D)$ with the exceptional set $\sigma \subset D$ of order $k \in \mathbb{Z}_+$, and that $U \Subset D$ is an open subset such that $f_0(z) \in \{x \in X_{reg}\colon \rho(x) \in (M_1, M_2)\}$ for all $z \in \bar{D}\backslash U$. Given $\epsilon > 0$ and a number $M_3 > M_2$, there exist a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g\colon \bar{D}\times P' \to X$ of class $\mathcal{A}^2(D)$, with exceptional set σ of order k, satisfying the following properties:*

(i) $g_0(z) \in \{x \in X_{reg}\colon \rho(x) \in (M_2, M_3)\}$ *for* $z \in bD$,
(ii) $g_0(z) \in \{x \in X\colon \rho(x) > M_1\}$ *for* $z \in \bar{D}\backslash U$,
(iii) $d(g_0(z), f_0(z)) < \epsilon$ *for* $z \in \overline{U}$, *and*
(iv) f_0 *and* g_0 *have the same k-jets at each of the points in* σ.

Moreover, g_0 can be chosen homotopic to f_0.

Proof. The idea is the following. Lemma 6.2 allows us to push the boundary of our curve out of a 2-convex bump in X. By choosing these bumps carefully we can insure that in finitely many steps we push the boundary of the curve to a given higher super level set of ρ (i); at the same time we take care not to drop substantially lower anywhere (ii), and to approximate the given map on the compact subset $\overline{U} \subset D$ (iii). In the construction we always keep the boundary of the image curve in the regular part of X. Special care must be taken to avoid the critical points of ρ. We now turn to details.

By [12, Lemma 5] there exists an *almost plurisubharmonic function* v on X (i.e., a function whose Levi form has bounded negative part on each compact in X) which is smooth on X_{reg} and satisfies $v = -\infty$ on X_{sing}. We may assume that $v < 0$ on $\{\rho \le M_3 + 1\}$.

For every sufficiently small $\delta > 0$ the function $\tau_\delta = \rho - M_1 + \delta v$ is $(n-1)$-convex on $\{\rho \le M_3\}$, and its Levi form is positive on the linear span of the eigenspaces corresponding to the positive eigenvalues of the Levi form of ρ at each point. Note that $X_{sing} \cup \{\rho \le M_1\} \subset \{\tau_\delta < 0\}$. Since $\rho(f_0(z)) > M_1$ and $f_0(z) \in X_{reg}$ for all $z \in bD$, we have $\tau_\delta(f_0(z)) > 0$ for all $z \in bD$ and all small $\delta > 0$. Fix $\delta > 0$ for which all of the above hold and write $\tau = \tau_\delta$.

Choose a number $M \in (M_2, M_3)$. (The central map g_0 of the final spray will map bD close to $\{\rho = M,\ \tau > 0\}$.) Since $\tau = -\infty$ on X_{sing}, the set

$$\Omega = \{x \in X\colon \rho(x) < M_3,\ \tau(x) > 0\}$$

is contained in the regular part of X. By a small perturbation one can in addition achieve that 0 is a regular value of τ, M is a regular value of ρ, and the level sets $\{\rho = M\}$ and $\{\tau = 0\}$ intersect transversely. Denote their intersection manifold by Σ. There is a neighborhood U_Σ of Σ in X with $\overline{U}_\Sigma \subset \{\rho > M_2\} \cap X_{reg}$.

We are now in the same geometric situation as in [22, Subsection 6.5]. (See especially the proof of Lemma 6.9 in [22]. The fact that our X is not necessarily a manifold is unimportant since $\overline{\Omega} \subset X_{reg}$.) For $s \in [0,1]$ set

$$\rho_s = (1-s)\tau + s(\rho - M), \quad G_s = \{\rho_s < 0\} \cap \{\rho < M_3\}.$$

The Levi form of ρ_s, being a convex combination of the Levi forms of τ and ρ, is positive on the linear span of the eigenspaces corresponding to the positive eigenvalues of the Levi form of ρ. Therefore G_s is strongly $(n-1)$-convex at each smooth boundary point for every $s \in [0,1]$. As the parameter s increases from $s=0$ to $s=1$, the domains $G_s \cap \{\rho < M\}$ increase from $\{\tau < 0,\ \rho < M\}$ to $G_1 = \{\rho < M\}$. (The sets $G_s \cap \{M < \rho < M_3\}$ decrease with s, but that part will not be used.) All hypersurfaces $\{\rho_s = 0\} = bG_s$ intersect along Σ. Since $d\rho_s = (1-s)d\tau + sd\rho$ and the differentials $d\tau$ and $d\rho$ are linearly independent along Σ, each hypersurface bG_s is smooth near Σ. By a generic choice of ρ and τ we can insure that only for finitely many values of $s \in [0,1]$ does the critical point equation $d\rho_s = 0$ have a solution on $bG_s \cap \Omega$, and in this case there is exactly one solution. Therefore bG_s has non smooth points only for finitely many s.

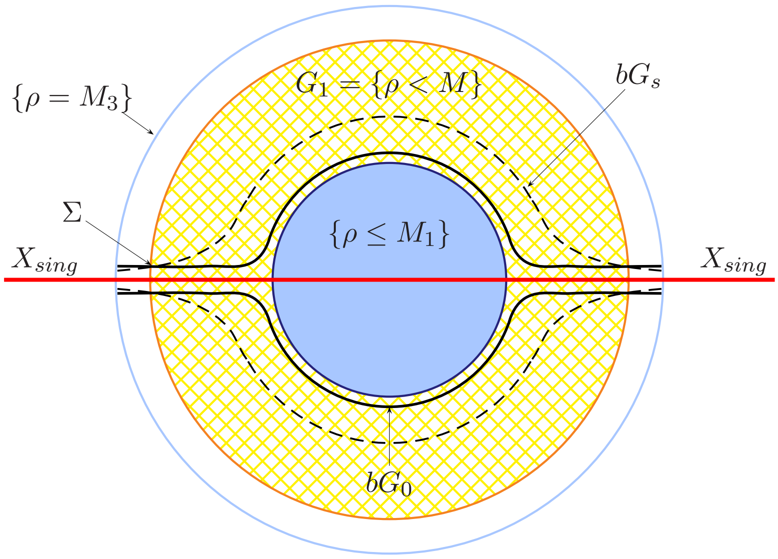

Figure 4. The sets G_s.

Fix two values of the parameter, say $0 \le s_0 < s_1 \le 1$. Consider first the *non critical case* when $d\rho_s \neq 0$ on $bG_s \cap \Omega$ for all $s \in [s_0, s_1]$, and hence all boundaries bG_s for $s \in [s_0, s_1]$ are smooth. By attaching to G_{s_0} finitely many small 2-convex bumps of the type used in Lemma 6.2 and contained in $G_1 \cup U_\Sigma$ we cover the set $G_{s_1} \cap \Omega$. (See [22, p. 180] for a more detailed description.) Using Lemma 6.2 at each bump we push the boundary of the central map in the spray outside the bump while keeping control on the compact subset $\overline{U} \subset D$. After a finite number of steps the boundary of the central map lies outside $G_{s_1} \cap \Omega$ and inside $G_1 \cup U_\Sigma$. In the sequel this will be called the *non critical procedure*.

It remains to consider the values $s \in [0,1]$ for which bG_s has a non smooth point (the *critical case*). We begin by discussing the most difficult case $\dim X = 2$ when there is least space to avoid the critical points. The functions ρ and τ are then 1-convex and hence strongly plurisubharmonic. As in [22, p. 180] we introduce the function $h(x) = \frac{\tau(x)}{\tau(x)-\rho(x)}$, $x \in \Omega$. A generic perturbation of τ insures that h is a Morse function. Note that $\{h = s\} = \{\rho_s = 0\} = bG_s$. The critical points of h coincide with critical points of ρ_s on $\{\rho_s = 0\}$, and the Levi form of h at a critical point is positive definite [22, p. 180].

To push the boundary over a critical level of h we shall apply Lemma 6.7 in [22, p. 177] (see also [28, §4]). Let p be a critical point of h, with $h(p) = c \in (0,1)$. (Our h corresponds to ρ in [22].) It suffices to consider the case when the Morse index of p is either 1 or 2 since we cannot approach a minimum of h by the non critical procedure. Choose a neighborhood $W \subset X$ of p on which h is strongly plurisubharmonic. Lemma 6.7 in [22] furnishes a new function $\widetilde{h}$ (denoted τ in [22]) which is strongly plurisubharmonic on W, while outside of W each level set $\{\widetilde{h} = \epsilon\}$ (for values ϵ close to 0) coincides with a certain level set $\{h = c(\epsilon)\}$, such that $\widetilde{h}$ satisfies the following properties (see fig. 5). The sublevel set $\{\widetilde{h} \le 0\}$ is contained in the union of the sublevel set $\{h \le c_0\}$ for some $c_0 < c$ (close to c) and a totally real disc E (the unstable manifold of the critical point p with respect to the gradient flow of h). Furthermore, for a small $d > 0$ with $c_0 < c - d$ we have

$$\{h \le c + d\} \subset \{\widetilde{h} \le 2d\} \subset \{h < c + 3d\}, \tag{6.4}$$

$\widetilde{h}$ has no critical values on $(0, 3d)$, and h has no critical values on $[c-d, c+3d]$ except for $h(p) = c$.

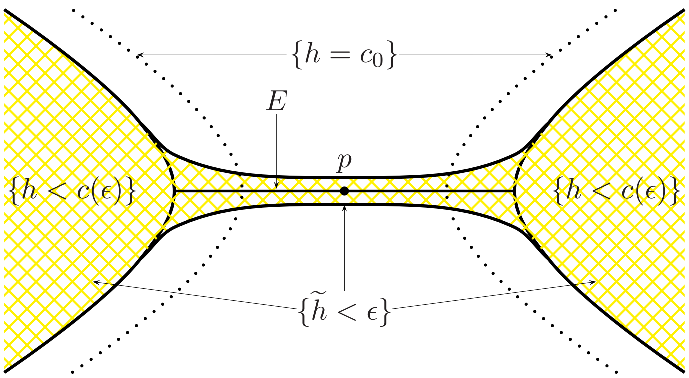

Figure 5. The level sets of $\widetilde{h}$

By the non critical procedure applied with the function h we push the boundary of the central map of the spray into the set $\{c - d < h < c\}$. Let $\widetilde{f}$ denote the new spray. For parameters $t \in \mathbb{C}^N$ sufficiently close to 0 the map $\widetilde{f}_t$ also has boundary values in $\{c-d<h<c\}$. Since $\dim_{\mathbb{R}} E \le 2$, we can find t arbitrarily close to 0 such that $\widetilde{f}_t(bD)\cap E=\emptyset$. By translation in the t variable we can choose $\widetilde{f}_t$ as the new central map of the spray.

Since $\{\widetilde{h}\le 0\}\subset\{h\le c_0\}\cup E\subset\{h\le c-d\}\cup E$, the above insures that $\widetilde{h}>0$ on $\widetilde{f}_t(bD)$. Since $\widetilde{h}$ has no critical values on $(0,3d)$, we can use the non critical procedure with $\widetilde{h}$ to push the boundary of the central map into the set $\{\widetilde{h}>2d\}$, appealing to Lemma 6.2. As $\{\widetilde{h}>2d\}\subset\{h>c+d\}$ by (6.4), we have thus pushed the image of bD across the critical level $\{h=c\}$ and avoided running into the critical point p. Now we continue with the non critical procedure applied with h to reach the next critical level of h.

This concludes the proof for $n=2$. The same procedure can be adapted to the case $n=\dim_{\mathbb{C}} X>2$ by considering the appropriate two dimensional slices on which the function ρ is strongly plurisubharmonic. Alternatively, we can apply the same geometric construction as in [19] to keep the boundary of the central map at a positive distance from the critical points of ρ. $\square$

Proof of Theorem 1.1. Let d denote a complete distance function on X. We denote the initial map in Theorem 1.1 by $f_0\colon \bar{D}\to X$. By Theorem 5.1 we may assume that f_0 is holomorphic in a neighborhood of $\bar{D}$ in an open Riemann surface $S\supset\bar{D}$ and $f_0(bD)\subset(X_c)_{reg}$. Here $X_c=\{\rho>c\}$ is the set on which ρ is assumed to have at least two positive eigenvalues.

Choose an open relatively compact subset $U\Subset D$ and a number $\epsilon>0$. It suffices to find a proper holomorphic map $g\colon D\to X$ such that $\sup_{z\in U} d(f_0(z),g(z))<\epsilon$ and such that g agrees with f_0 to order k at each of the given points $z_j\in D$; a sequence of proper maps g_ν as in Theorem 1.1 is then obtained by Cantor's diagonal process.

Let σ denote the union of $\{z\in D\colon f_0(z)\in X_{sing}\}$ and the finite set $\{z_j\}\subset D$ on which we wish to interpolate to order $k\in\mathbb{N}$; thus σ is a finite subset of D. Lemma 4.2 furnishes a spray of maps $f\colon\bar{D}\times P\to X$ of class $\mathcal{A}^2(D)$, with the given central map f_0 and the exceptional set σ of order k, such that $f_t(bD)\subset(X_c)_{reg}$ for each $t\in P\subset\mathbb{C}^N$.

Set $f^0=f$, $c=c_0$, and choose an open subset $P_0\Subset P$ containing the origin $0\in\mathbb{C}^N$. Choose a number $c_1>c_0$ such that $c_0<\rho(f^0_t(z))<c_1$ for all $z\in bD$ and $t\in P_0$, and then choose an open subset $U_0\subset\subset D$ containing $\sigma\cup U$ such that $f^0_t(\bar{D}\backslash U_0)\subset\{x\in X\colon c_0<\rho(x)<c_1\}$ for all $t\in P_0$. Choose a sequence $c_0<c_1<c_2\cdots$ with the given initial numbers c_0 and c_1 such that $\lim_{j\to\infty}c_j=+\infty$. Also choose a decreasing sequence $\epsilon_j>0$ with $0<\epsilon_1<\epsilon$ such that for each $j\in\mathbb{N}$ we have

$$x,y\in X,\ \rho(x)<c_{j+1},\ d(x,y)<\epsilon_j\Rightarrow|\rho(x)-\rho(y)|<1.$$

We shall inductively find a sequence of sprays $f^j\colon\bar{D}\times P_j\to X$ of class $\mathcal{A}^2(D)$ with the exceptional set σ of order k, with $P=P_0\supset P_1\supset P_2\supset\cdots$, and a sequence of open sets $U_0 \subset U_1 \subset \cdots \subset \cup_{j=1}^{\infty} U_j = D$ satisfying the following properties for each $j \in \mathbb{Z}_+$ and $t \in P_j$:

(i) $f_t^j(bD) \subset \{x \in X_{reg} \colon c_j < \rho(x) < c_{j+1}\}$,
(ii) $f_t^j(\bar{D} \backslash U_j) \subset \{x \in X \colon c_j < \rho(x) < c_{j+1}\}$,
(iii) $f_t^j(\bar{D} \backslash U_{j-1}) \subset \{x \in X \colon c_{j-1} < \rho(x) < c_{j+1}\}$,
(iv) $d\big(f_0^j(z), f_0^{j-1}(z)\big) < \epsilon_j 2^{-j}$ for $z \in U_{j-1}$, and
(v) f_0^j and f_0^{j-1} are homotopic, and they have the same k-jets at each of the points in σ.

For $j = 0$ the properties (i) and (ii) hold while the remaining properties are vacuous. (In (iii) we take $U_{-1} = U_0$ and $c_{-1} = c_0$.) Assuming that we already have sprays $f^0, \ldots, f^j$ satisfying these properties, Proposition 6.3 applied to $f = f^j$ furnishes a new spray f^{j+1} (called g in the statement of that Proposition) satisfying (i), (iii), (iv) and (v). Choose an open set $U_{j+1} \subset\subset D$ with $U_j \subset U_{j+1}$ such that (ii) holds (this is possible by continuity since (i) already holds and we are allowed to shrink the parameter set P_{j+1}). Hence the induction proceeds. When choosing the sets U_j we can easily insure that they exhaust D.

Conditions (i)–(v) imply that the sequence of central maps $f_0^j \colon \bar{D} \to X$ ($j \in \mathbb{Z}_+$) converges uniformly on compacts in D to a proper holomorphic map $g \colon D \to X$ satisfying $d(f_0(z), g(z)) < \epsilon$ ($z \in \overline{U}_0$) and such that the k-jet of g agrees with the k-jet of f_0 at every point of σ. In addition, we can combine the homotopies from f_0^j to f_0^{j+1} to find a homotopy from $f_0|_D$ to g. For the details see e.g. [33]. This completes the proof of Theorem 1.1.

Acknowledgements. The first named author wishes to thank the Laboratoire de Mathématiques E. Picard, Université Paul Sabatier de Toulouse, for its hospitality, and the EGIDE program for support during a part of the work. The second named author thanks J.-P. Demailly, C. Laurent-Thiébaut, J. Leiterer, I. Lieb, N. Øvrelid and J.-P. Rosay for helpful discussions, and the Institut Fourier, Université de Grenoble, for support and hospitality during a part of the work.

References

1. Ahlfors, L., Open Riemann surfaces and extremal problems on compact subregions. *Comment. Math. Helv.*, 24 (1950), 100–134.
2. Andreotti, A., Grauert, H., Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90 (1962), 193–259.
3. Berndtsson, B., Rosay, J.-P., Quasi-isometric vector bundles and bounded factorization of holomorphic matrices. *Ann. Inst. Fourier (Grenoble)*, 53 (2003), 885–901.
4. Barlet, D., How to use the cycle space in complex geometry. *Several complex variables (Berkeley, CA, 1995–1996)*, 25–42, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
5. Behnke, H., Sommer, F., *Theorie der analytischen Funktionen einer komplexen Veränderlichen.* 3rd. ed. Springer, Berlin, 1965.
6. Bishop, E., Mappings of partially analytic spaces. *Amer. J. Math.*, 83 (1961), 209–242.

7. Campana, F., Peternell, T., Cycle spaces. *Several complex variables, VII*, 319–349, *Encyclopaedia Math. Sci.*, 74, Springer, Berlin, 1994.
8. Chakrabarti, R., Approximation of maps with values in a complex or almost complex manifold. *Ph. D. thesis*, University of Wisconsin, Madison, 2006.
9. Chen, S.-C., Shaw, M.-C., *Partial Differential Equations in Several Complex Variables.* Amer. Math. Soc. and International Press, Providence, RI, 2001.
10. Colţoiu, M., Complete locally pluripolar sets. *J. Reine Angew. Math.*, 412 (1990), 108–112.
11. Colţoiu, M., Q-convexity. A survey. *Complex analysis and geometry (Trento, 1995)*, 83–93, Pitman Res. Notes Math. Ser., 366, Longman, Harlow, 1997.
12. Demailly, J.-P., Cohomology of q-convex spaces in top degrees. *Math. Z.*, 204 (1990), 283–295.
13. Demailly, J.-P., Lempert, L., Shiffman, B., Algebraic approximations of holomorphic maps from Stein domains to projective manifolds. *Duke Math. J.*, 76 (1994), 333–363.
14. Dor, A., Immersions and embeddings in domains of holomorphy. *Trans. Amer. Math. Soc.*, 347 (1995), 2813–2849.
15. Dor, A., A domain in $\mathbb{C}^m$ not containing any proper image of the unit disc. *Math. Z.*, 222 (1996), 615–625.
16. Douady, A., Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Ann. Inst. Fourier (Grenoble)*, 16 (1966), 1–95.
17. Drinovec-Drnovšek, B., Discs in Stein manifolds containing given discrete sets. *Math. Z.*, 239 (2002), 683–702.
18. Drinovec-Drnovšek, B., Proper discs in Stein manifolds avoiding complete pluripolar sets. *Math. Res. Lett.*, **11** (2004), no. 5, 575-581.
19. Drinovec-Drnovšek, B., On proper discs in complex manifolds. *Ann. Inst. Fourier (Grenoble)*, to appear. [arXiv: math.CV/0503449]
20. Eisenman, D. A., Intrinsic measures on complex manifolds and holomorphic mappings. *Memoirs of the Amer. Math. Soc.*, 96, Providence, Rhode Island, 1970.
21. Forstnerič, F., The Oka principle for multivalued sections of ramified mappings. *Forum Math.*, 15 (2003), 309–328.
22. Forstnerič, F., Noncritical holomorphic functions on Stein manifolds. *Acta Math.*, 191 (2003), 143–189.
23. Forstnerič, F., Extending holomorphic mappings from subvarieties in Stein manifolds. *Ann. Inst. Fourier (Grenoble)*, 55 (2005), 733–751.
24. Forstnerič, F., Runge approximation on convex sets implies Oka's property. *Annals of Math.*, 163 (2006), 689–707.
25. Forstnerič, F., Globevnik, J., Discs in pseudoconvex domains. *Comment. Math. Helv.*, 67 (1992), 129–145.
26. Forstnerič, F., Globevnik, J. Proper holomorphic discs in $\mathbb{C}^2$. *Math. Res. Lett.*, 8 (2001), 257–274.
27. Forstnerič, F., Globevnik, J., Stensønes, B., Embedding holomorphic discs through discrete sets. *Math. Ann.*, 305 (1996), 559–569.
28. Forstnerič, F., Kozak, J., Strongly pseudoconvex handlebodies. *J. Korean Math. Soc.*, 40 (2003), 727-745.
29. Forstnerič, F., Løw, E., Øvrelid, N., Solving the d and $\bar\partial$-equations in thin tubes and applications to mappings. *Michigan Math. J.*, 49 (2001), 369–416.
30. Forstnerič, F., Prezelj, J., Oka's principle for holomorphic fiber bundles with sprays. *Math. Ann.*, 317 (2000), 117–154.
31. Forstnerič, F., Slapar, M., Stein structures and holomorphic mappings. *Math. Z.*, to appear. [arXiv: math.CV/0507212]
32. Forstnerič, F., Winkelman, J., Holomorphic discs with dense images. *Math. Res. Lett.*, 12 (2005), 265–268.
33. Globevnik, J., Discs in Stein manifolds. *Indiana Univ. Math. J.*, 49 (2000), 553–574.

34. Gohberg, I., Lancaster, P., Rodman, L., *Invariant subspaces of matrices with applications.* John Wiley and Sons, New York, 1986.
35. Grauert, H., Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135 (1958), 263–273.
36. Grauert, H., Theory of q-convexity and q-concavity. *Several complex variables, VII*, 259–284, *Encyclopaedia Math. Sci.*, 74, Springer, Berlin, 1994.
37. Greene, R. E., Wu, H., Embedding of open Riemannian manifolds by harmonic functions. *Ann. Inst. Fourier (Grenoble)*, 25 (1975), 215–235.
38. Gromov, M., Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2 (1989), 851-897.
39. Gunning, R. C., Rossi, H., *Analytic functions of several complex variables.* Prentice–Hall, Englewood Cliffs, 1965.
40. Henkin, G. M., Integral representation of functions which are holomorphic in strictly pseudoconvex regions, and some applications. (Russian) *Mat. Sb. (N.S.)*, 78 (120), (1969), 611–632.
41. Henkin, G. M., Leiterer, J., *Theory of Functions on Complex Manifolds.* Akademie-Verlag, Berlin, 1984.
42. Henkin, G. M., Leiterer, J., *Andreotti-Grauert theory by integral formulas.* Birkhäuser, Boston, 1988.
43. Henkin, G. M., Leiterer, J., The Oka-Grauert principle without induction over the basis dimension. *Math. Ann.*, 311 (1998), 71–93.
44. Heunemann, D., An approximation theorem and Oka's principle for holomorphic vector bundles which are continuous on the boundary of strictly pseudoconvex domains. *Math. Nachr.*, 127 (1986), 275–280.
45. Heunemann, D., Theorem B for Stein manifolds with strictly pseudoconvex boundary. *Math. Nachr.*, 128 (1986), 87–101.
46. Hörmander, L., *An Introduction to Complex Analysis in Several Variables.* Third ed. North Holland, Amsterdam, 1990.
47. Kaliman, S., Zaidenberg, M., Non-hyperbolic complex space with a hyperbolic normalization. *Proc. Amer. Math. Soc.*, 129 (2001), 1391–1393.
48. Kerzman, N., Hölder and L^p estimates for solutions of $\bar{\partial} u = f$ in strongly pseudoconvex domains. *Comm. Pure Appl. Math.* 24 (1971), 301–379.
49. Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings.* Marcel Dekker, New York, 1970.
50. Kobayashi, S., Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.*, 82 (1976), 357–416.
51. Leiterer, J., Analytische Faserbündel mit stetigem Rand über streng-pseudokonvexen Gebieten. I: *Math. Nachr.*, 71 (1976), 329–344. II: *Math. Nachr.*, 72 (1976), 201–217.
52. Leiterer, J., Theorem B für analytische Funktionen mit stetigen Randwerten. *Beiträge zur Analysis*, 8 (1976), 95–102.
53. Lempert, L., Algebraic approximations in analytic geometry. *Invent. Math.*, 121 (1995), 335–354.
54. Lieb, I., Solutions bornées des équations de Cauchy-Riemann. *Fonctions de plusieurs variables complexes (Sém. François Norguet, 1970–1973; à la mémoire d'André Martineau)*, pp. 310–326. Lecture Notes in Math., Vol. 409, Springer, Berlin, 1974.
55. Lieb, I., Michel, J., *The Cauchy-Riemann complex. Integral formulæ and Neumann problem.* Aspects of Mathematics, E34. Friedr. Vieweg & Sohn, Braunschweig, 2002.
56. Lieb, I., Range, R. M., Lösungsoperatoren für den Cauch-Riemann-Komplex mit $\mathcal{C}^k$-Abschätzungen. *Math. Ann.*, 253 (1980), 145–165.
57. Lieb, I., Range, R. M., Integral representations and estimates in the theory of the $\bar{\partial}$-Neumann problem. *Annals of Math.*, (2) 123 (1986), 265–301.
58. Lieb, I., Range, R. M., Estimates for a class of integral operators and applications to the $\bar{\partial}$-Neumann problem. *Invent. Math.*, 85 (1986), 415–438.

59. Michel, J., Perotti, A., $\mathbb{C}^k$-regularity for the $\overline{\partial}$-equation on strictly pseudoconvex domains with piecewise smooth boundaries. *Math. Z.*, 203 (1990), 415–427.
60. Narasimhan, R., Imbedding of holomorphically complete complex spaces. *Amer. J. Math.*, 82 (1960), 917–934.
61. Narasimhan, R., The Levi problem for complex spaces. *Math. Ann.*, 142 (1961), 355–365.
62. Nash, J., Real algebraic manifolds. *Annals of Math.*, (2) 56 (1952), 405–421.
63. Ohsawa, T., Completeness of noncompact analytic spaces. *Publ. Res. Inst. Math. Sci.*, 20 (1984), 683–692.
64. Peternell, M., q-completeness of subsets in complex projective space. *Math. Z.*, 195 (1987), 443–450.
65. Ramírez de Arelano, E., Ein Divisionsproblem und Randintegraldarstellung in der komplexen Analysis. *Math. Ann.*, 184 (1970), 172–187.
66. Range, R. M., *Holomorphic functions and integral representations in several complex variables.* Graduate Texts in Math., 108. Springer-Verlag, New York, 1986.
67. Range, M., Siu, Y.-T., Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundary. *Math. Ann.*, 206 (1973), 325–354.
68. Rosay, J.-P., Approximation of non-holomorphic maps, and Poletsky theory of discs. *J. Korean Math. Soc.*, 40 (2003), 423–434.
69. Royden, H. L., The extension of regular holomorphic maps. *Proc. Amer. Math. Soc.*, 43 (1974), 306–310.
70. Sebbar, A., Principe d'Oka-Grauert dans A^∞. *Math. Z.*, 201 (1989), 561–581.
71. Siu, J.-T., Every Stein subvariety admits a Stein neighborhood. *Invent. Math.*, 38 (1976), 89–100.
72. Springer, G., *Introduction to Riemann surfaces.* Addison-Wesley, Reading, Mass., 1957.
73. Stolzenberg, G., Polynomially and rationally convex sets. *Acta Math.*, 109 (1963), 259–289.
74. Wermer, J., The hull of a curve in $\mathbb{C}^n$. *Annals of Math.*, (2) 68 (1958), 550–561.

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: barbara.drinovec@fmf.uni-lj.si

E-mail address: franc.forstneric@fmf.uni-lj.si