Division theorems for the rational cohomology of discriminant complements and automorphism groups of projective hypersurfaces

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Main results

$n, k, 1 \leq k \leq n+1$.

$\underline{d} = (d_1, \ldots, d_k)$ = a collection of integers such that $2 \leq d_1 \leq \cdots \leq d_k$.

$\Pi_{\underline{d},n}$ = the $\mathbb{C}$-vector space of all k-tuples $(f_1, \ldots, f_k)$, where $f_i, i = 1, \ldots, k$, is a homogeneous polynomial in $n+1$ variables of degree d_i with coefficients in $\mathbb{C}$.

For $f = (f_1 \ldots, f_k) \in \Pi_{\underline{d},n}$ define $\operatorname{Sing} f$ = the (projectivisation of the) set of all $x \in \mathbb{C}^{n+1} \setminus \{0\}$ such that

- $f_i(x) = 0, i = 1, \ldots, k$,
- the gradients of $f_i, i = 1, \ldots, k$ at x are linearly dependent.

$\Sigma_{\underline{d},n} \subset \Pi_{\underline{d},n}$ is the set of all f such that $\operatorname{Sing} f \neq \varnothing$.

$\mathrm{GL}_{n+1}(\mathbb{C})$ acts on $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$.

Theorem 1. *Suppose $\underline{d} \neq (2)$. Then the geometric quotient of $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$ by $\mathrm{GL}_{n+1}(\mathbb{C})$ exists, and the Leray spectral sequence of the corresponding quotient map degenerates over $\mathbb{Q}$ (or modulo a sufficiently large prime) at the second term.*

This generalises a result by C. Peters and J. Steenbrink 2001 for $k = 1$.

Corollary 1.

$$H^*(\Pi_{d,n} \setminus \Sigma_{d,n}, \mathbb{Q}) \\ \cong H^*((\Pi_{d,n} \setminus \Sigma_{d,n})/\mathrm{Gl}_{n+1}(\mathbb{C}), \mathbb{Q}) \otimes H^*(\mathrm{Gl}_{n+1}(\mathbb{C}), \mathbb{Q}).$$

This isomorphism holds on the algebra level and respects the mixed Hodge structures.

By-product 1. *The order of the subgroup* $\mathrm{PGl}_{n+1}(\mathbb{C}), n \geq 1$, *consisting of the transformations that preserve a smooth hypersurface of degree* $d > 2$ *divides*

$$\frac{1}{n+1} \prod_{i=0}^{n-1} \left(\frac{1}{C_{n+1}^i}((-1)^{n-i} + (d-1)^{n-i+1}) \right. \\ \left. \mathrm{LCM}(C_{n+1}^i (d-1)^i, (n+1)(d-1)^n) \right). \quad (1)$$

By-product 2. *Let d be an integer > 2. Then the order of the subgroup of* $\mathrm{Gl}_{n+1}(\mathbb{C})$ *consisting of the transformations that fix* $f \in \Pi_{(d),n} \setminus \Sigma_{(d),n}$ *divides*

$$\prod_{i=0}^{n}((-1)^{n-i} + (d-1)^{n-i+1})(d-1)^i.$$

Actually, we prove an analogous statement for arbitrary d, but the resulting formula is a bit messy (and was therefore banned from here).

By-product 3. *Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a ramified covering of degree d^n. Then the order of the group formed by the automorphisms $g : \mathbb{C}P^n \to \mathbb{C}P^n$ such that $f \circ g = f$ divides*

$$d^{n^2-1} \prod_{i=2}^{n+1} \frac{1}{C_{n+1}^n} \mathrm{LCM}(C_{n+1}^i, (n+1)d^{i-1}).$$

Comments.

- (1) is majorated by $d^{\frac{3}{2}n(n+1)}(n+1)^{n-1}$; since (1) is divisible by the order of the projective automorphism group of any smooth hypersurface of degree d in $\mathbb{C}P^n$, it can hardly be expected to be sharp. Indeed, smaller bounds are known. The best one I know is

  $$J(n+1)d^n \tag{2}$$

  by A. Howard and A. J. Sommese 1980 where J is the Jordan function, i.e.,

  $$J(m) = \max_{G \subset \mathrm{Gl}_m(\mathbb{C}) \text{ finite}} \left(\min_{G' \triangleleft G \text{ Abelian}} (G : G') \right);$$

  $$J(m) \leq (m+1)! m^{a \ln m + b}$$

  for some $a, b \in \mathbb{R}$ (B. Weisfeiler 1984)).

  Asymptotically as $d \to \infty$,

  – (2) is smaller than (1);

– By-product 1 provides much more restrictions than (2), since the number of divisors of $x \in \mathbb{Z}$ grows more slowly than any positive power of x as $x \to \infty$.

- If $n \geq 3, d \geq 3$ and $(d, n) \neq (4, 3)$, then any automorphism of a smooth hypersurface of degree d in $\mathbb{C}P^n$ is known (Matsumura-Monsky 1964) to be the restriction of a projective transformation, so in these cases by-product 1 implies that the order of the full automorphism group divides (1).

- If $n = 1$, by-product 1 is equivalent to saying that the order of the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ that preserves a given subset of $d > 2$ points of $\mathbb{C}P^1$ divides

$$d(d-1)(d-2).$$

This is obvious. Optimal, if d is odd.

- If $n = 2$ and 3, then (1) becomes

$$d^2(d-1)^4(d^2-3d+3)(d-2)$$

and

$$\frac{1}{4}d^4(d-1)^{13}(d^4-5d^3+10d^2-10d+5)(d^3-4d^2+6d-4)$$
$$(d^2-3d+3)(d-2)\cdot \mathrm{LCM}(2,(d-1)^2)\mathrm{LCM}(2,d-1)$$

respectively.

- An explicit description of all projective automorphism groups of smooth projective hypersurfaces of given dimension and degree > 2 is rarely known. Three cases (to my knowledge):

  1. Cubic curves in $\mathbb{C}P^2$ can have 18, 36 or 54 projective automorphisms. LCM $= 2^2 \cdot 3^3$. (1)$=2^4 \cdot 3^3$.

  2. Quartic curves in $\mathbb{C}P^2$. The LCM of the orders of the projective automorphism groups is $2^5 \cdot 3^2 \cdot 7$ (I. Dolgachev's "Topics in classical algebraic geometry", available on I. Dolgachev's webpage). (1)$=2^5 \cdot 3^4 \cdot 7$.

  3. Cubic surfaces in $\mathbb{C}P^3$. The LCM of the orders of the projective automorphism groups is $2^3 \cdot 3^4 \cdot 5$ (B. Segre 1942 and T. Hosoh 1997). (1)$=2^{10} \cdot 3^4 \cdot 5$.

Idea of the proofs.

Linking numbers.

All (co)homology coefficients are $\mathbb{Z}$, unless indicated otherwise.

$\bar{H}_*(Y)$ = the Borel-Moore homology groups of Y (with integer coefficients). Two definitions: 1. the homology groups of the complex of locally finite chains; 2. the homology of the one-point compactification modulo the infinity.

M a smooth oriented manifold of $\dim_{\mathbb{R}} = p$, $X \subset M$ a closed subspace, $c \in \ker(\bar{H}_{p-q}(X) \to \bar{H}_{p-q}(M))$. Suppose $H_{q-1}(M) = H^{q-1}(M) = 0$. We define the linking number $\mathrm{lk}_{c,X,M} \in H^{q-1}(M \setminus X)$ by the following diagram

$$
\begin{array}{ccccccc}
\bar{H}_{p-q+1}(M) & \longrightarrow & \bar{H}_{p-q+1}(M,X) & \longrightarrow & \bar{H}_{p-q}(X) & \longrightarrow & \bar{H}_{p-q}(M) \\
\downarrow{\scriptstyle \text{Poincaré}} & & \downarrow{\scriptstyle \text{Poincaré}} & & & & \\
H^{q-1}(M) & \longrightarrow & H^{q-1}(M \setminus X) & & & &
\end{array}
$$

Another (equivalent) definition: represent c by a chain $\bar{c}$ and define a function

$$H_{q-1}(M \setminus X) = \ker(H_{q-1}(M \setminus X) \to H_{q-1}(M)) \to \mathbb{Z}$$

as follows. Take a class $z \in H_{q-1}(M \setminus X)$, represent it by a chain $\tilde{z}$, find a chain w such that $\partial w = \tilde{z}$, and map z to the intersection index $\#(w, \bar{c})$. The result will coincide with $(-1)^q \operatorname{lk}_{c,X,M}(c)$.

The Leray-Hirsch principle.

In order to prove theorem 1, it suffices to construct classes

$$\mathbf{a}_i^{d,n} \in H^{2i-1}(\Pi_{d,n} \setminus \Sigma_{d,n}), i = 1, \ldots, n+1$$

that give nonzero multiples of the canonical multiplicative generators

$$\mathbf{c}_i^n \in H^{2i-1}(\mathrm{GL}_{n+1}(\mathbb{C}))$$

when pulled back under (any) orbit mapping $g \mapsto g_* f$.

For a subvariety X of $\mathbb{C}P^n$ we set $V_{d,n,X}$ to be the subset of $\Pi_{d,n}$ consisting of all $(f_1, \ldots, f_k)$ such that $\operatorname{Sing}(f_1, \ldots, f_k) \cap X \neq \varnothing$.

$$\Sigma_{d,n} = V_{d,n,\mathbb{C}P^n}.$$

$$\operatorname{codim}_{\Pi_{d,n}} V_{d,n,\mathbb{C}P^i} = n + i - 1.$$

Set

$$\mathbf{a}_i^{d,n} = \mathrm{lk}_{\left[V_{d,n,\mathbb{C}P^{n+i-1}}\right]}\Sigma_{d,n}, \Pi_{d,n}.$$

We compute the pullbacks of $\mathbf{a}_i^{d,n}$'s under an orbit mapping $g \mapsto g \cdot f$. For simplicity suppose $k = 1$ (projective hypersurfaces in $\mathbb{C}P^n$); $\underline{d} = (d), d > 2$.

Induction on n.

The case $n = 0$ (empty hypersurfaces in $\mathbb{C}P^0$).

$$\Pi_{(d),0} = \{x_0^d\} \cong \mathbb{C}, \Sigma_{d,0} = \{0\} = V_{(d),\{0\},\mathbb{C}P^0}.$$

The action of $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ is

$$z \cdot x_0^d = z^d x_0^d, z \in \mathbb{C}^*.$$

The pullback of $\mathbf{a}_1^{(d),0} = \mathrm{lk}_{\left[V_{(d),0,\mathbb{C}P^{0+1-1}}\right]}\Sigma_{(d),0}, \Pi_{(d),0}$ under an orbit mapping is $d\mathbf{c}_1^1$. Set $\mathbf{m}_1^{(d),0} = d$.

How to pass from n to $n + 1$?

Suppose that the pullback of $\mathbf{a}_i^{(d),n}, i = 1, \ldots, n+1$ is $\mathbf{m}_i^{(d),n}\mathbf{c}_i^n$; with $\mathbf{m}_i^{(d),n} \neq 0$. Define the suspension map $S : \Pi_{(d),n} \to \Pi_{(d),n+1}$ by

$$S(f) = f + x_{n+1}^d.$$

Set

$$L_1 = \{(0 : \cdots : 0 : z_{i-1} : \cdots : z_n)\} \cong \mathbb{C}P^{n-i+1} \subset \mathbb{C}P^n,$$

$$L_2 = \{(0 : \cdots : 0 : z_{i-1} : \cdots : z_{n+1}\} \cong \mathbb{C}P^{(n+1)-i+1} \subset \mathbb{C}P^{n+1}.$$

Set-theoretically,

$$S^{-1}(V_{(d),n+1,L_2}) = V_{(d),n,L_1}.$$

But the intersection multiplicity of $V_{(d),n+1,L_2}$ and the image of S is $d-1$. Due to the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_{n+1}(\mathbb{C}) & \xrightarrow{\text{orbit}} & \Pi_{(d),n} \\ \downarrow & & \downarrow S \\ \mathrm{GL}_{n+2}(\mathbb{C}) & \xrightarrow{\text{orbit}} & \Pi_{(d),n+1} \end{array}$$

we obtain

$$S^* \left(\mathrm{lk}_{[V_{(d),n+1,L_2}]} \Sigma_{(d),n+1}, \Pi_{(d),n+1} \right) = (d-1)\, \mathrm{lk}_{[V_{(d),n,L_1}]} \Sigma_{(d),n}, \Pi_{(d),n}.$$

Any $\mathbf{a}_i^{(d),n+1}, i = 1, \ldots, n+1$ pulls back under an orbit map to

$$\mathbf{m}_i^{(d),n+1} \mathbf{c}_i^{(d),n+1}$$

with

$$\mathbf{m}_i^{(d),n+1} = (d-1)\mathbf{m}_i^{(d),n}.$$

The case of $\mathbf{a}_{n+2}^{(d),n+1}$ has to be considered separately. Set

$$x_0 = (0, \ldots, 0, 1),$$

and define

$$F : \Pi_{(d),n+1} \to \mathbb{C}^{n+2}$$

by $f \mapsto df|_{x_0}$. We have

$$V_{(d),n+1,\{(0\cdots 0:1)\}} = F^{-1}(0).$$

Hence, $\mathbf{a}_{n+2}^{(d),n+1}$ is the restriction of $F^*(\mathrm{lk}_{\{\{0\}\},\mathbb{C}^{n+2}})$ to $\Pi_{(d),n+1} \setminus \Sigma_{(d),n+1}$. Fix an $f_0 \in \Pi_{(d),n+1} \setminus \Sigma_{(d),n+1}$.

The image of $\mathbf{a}_{n+2}^{(d),n+1}$ under the orbit mapping $g \mapsto g \cdot f_0$ is just the image of the canonical generator of

$$H^{2n+1}(\mathbb{C}^{n+2} \setminus \{0\})$$

under the mapping

$$g \mapsto d(g \cdot f)|_{x_0} = g^T \cdot df|_{g \cdot x_0}.$$

This mapping factorises as follows.

$$g \mapsto (g^T, \text{the last column of } g) \mapsto (g^T, df_0|_{\text{the last column of } g}) \mapsto g^T df_0|_{\text{the last column of } g}.$$

Here the second arrow is

$$\mathrm{Id} \times (x \mapsto df_0|_x)$$

and the third one is $(g, x) \mapsto g \cdot x$.

The image of the canonical generator of $H^{2n+1}(\mathbb{C}^{n+2} \setminus \{0\})$ under the above composition is $((d-1)^{n+2} + (-1)^{n+1})\mathbf{c}_{n+2}^{n+2}$, hence the image of $\mathbf{a}_{n+2}^{(d),n+1}$ under any orbit mapping is $\mathbf{m}_{n+2}^{(d),n+1}\mathbf{c}_{n+2}^{n+2}$ with

$$\mathbf{m}_{n+2}^{(d),n+1} = (d-1)^{n+2} + (-1)^{n+1}.$$

What has all this to do with automorphism groups?

Suppose that

- $R : \mathrm{GL}_l(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ is a representation such that $R(\lambda I_l) = \lambda^s I_N$ where I_l and I_N are the identity operators.
- $U \subset \mathbb{C}^N$ is an open $\mathrm{GL}_l(\mathbb{C})$-invariant subset that does not contain 0.
- $a_i \in H^{2i-1}(U), i = 1, \ldots, l$ are classes that pull back to $m_i \mathbf{c}_i^l, i = 1, \ldots, l$ under an orbit map.
- For any $x \in U$ and $\bar{x} \in U/\mathbb{C}^*$ the stabilisers $\mathrm{Stab}(x) \subset \mathrm{GL}_l(\mathbb{C})$ and $\mathrm{Stab}(\bar{x}) \subset \mathrm{PGL}_l(\mathbb{C})$ are finite.

Then

- For any $x \in U$, $\#\mathrm{Stab}(x)$ is a divisor of

$$\prod_{i=1}^{l} m_i.$$

- If $l \geq 2$ and for any $i = 2, \ldots, l$ there exists a class $b_i \in H^{2i-1}(U/\mathbb{C}^*)$ that pulls back to $r_i b_i$ under $U \to U/\mathbb{C}^*$ with $r_i \neq 0$, then $\#\mathrm{Stab}(\bar{x})$ divides

$$\frac{1}{l} \prod_{i=2}^{l} m_i r_i.$$