Moduli of symplectic manifolds

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I. Introduction

Recall the

<u>Definition</u>: A manifold X is called <u>irreducible symplectic</u> (or <u>irreducible hyperkähler</u>), if

(i) X is compact Kähler

(ii) X is simply connected

(iii) $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$ where σ is nowhere degenerate.

<u>Remarks</u>: (i) $\dim X = 2n$ is even

(ii) $0 \neq \sigma \wedge \ldots \wedge \sigma \in H^0(X, \Omega^{2n}_X) = H^0(X, \omega_X)$ is nowhere 0, i.e.

$$\omega_X = \mathcal{O}_X.$$

(iii) $h^{2,0}(X) = h^{0,2}(X) = 1$, $h^{1,0}(X) = h^{0,1}(X) = 0$

(iv) X is unobstructed (Tian, Todorov) and

$$T_{[X]} \operatorname{Def} X \cong H^1(X, T_X) \cong H^1(X, \Omega_X)$$

<u>Examples</u>: (i) $\dim X = 2$: $X = S$ = K3 surface.

(ii) $\operatorname{Hilb}^n S$

(iii) Generalised Kummer varieties

(iv) Examples by O'Grady

(v) Moduli spaces of sheaves on K3 surfaces

II Beauville form and Fujiki invariant

We have

$$H^2(X, \mathbb{C}) = \underbrace{H^{2,0}}_{\dim = 1} \oplus H^{1,1} \oplus \underbrace{H^{0,2}}_{\dim = 1}$$

One can define a <u>quadratic form</u> f on $H^2(X,\mathbb{C})$ by

$$f(\alpha) = \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\alpha^2 + (1-n)\int\sigma^{n-1}\bar\sigma^{n}\alpha\int\sigma^{n}\bar\sigma^{n-1}\bar\alpha .$$

For some suitable $\gamma\in\mathbb{R}$ one finds that

$$q_X := \gamma f : H^2(X,\mathbb{Z}) \to \mathbb{Z}$$

is a primitive integral quadratic form on $H^2(X,\mathbb{Z})$, resp. a symmetric bilinear form on $H^2(X,\mathbb{Z})$.

$$q_X = \text{Beauville form}.$$

<u>Remarks:</u> (i) $H^{2,0}\oplus H^{0,2}\perp H^{1,1}$

(ii) $\operatorname{sign} q_X = (3, b_2(X)-3)$

(iii) $q_X(\sigma)=0$.

For $\alpha\in H^2(X,\mathbb{Z})$ one can also define a form by

$$v(\alpha) := \alpha^{2n}\in\mathbb{Z}$$

By a result of Fujiki there exists a constant $c>0$ (<u>Fujiki constant</u>) such that

$$\boxed{v(\alpha)=c\,q_X(\alpha)^n} \qquad (\alpha\in H^2(X,\mathbb{Z})).$$

<u>Example.</u> $X = \mathrm{Hilb}^2 S$. Then

$$(H^2(X),q_X) = 3\underset{\begin{pmatrix}0&1\\1&0\end{pmatrix}}{U} \oplus 2E_8(-1)\oplus\langle-2\rangle \quad (\operatorname{sign}=(3,20$$

In this case $c=3$.

III. The period map

We fix X and an abstract lattice

$$L \cong (H^2(X), q_X).$$

Then L defines a homogeneous domain

$$\Omega_L = \{ [\omega] \in \mathbb{P}(L\otimes\mathbb{C}),\ (\omega,\omega)=0,\ (\omega,\bar\omega)>0 \}.$$

A marking of X is an isomorphism

$$\varphi : (H^2(X), q_X) \cong L$$

Given such a morphism we can associate to (X,φ) a period point:

$$\pi(X,\varphi) = [\varphi(\sigma)] \in \Omega_L.$$

Then one has:

- Local Torelli: The map
$$\pi : \mathrm{Def}\, X \to \Omega_L$$
$$t \mapsto \pi(X_t, \varphi_t)$$
is a local isomorphism (Beauville)
- Surjectivity of the period map (Huybrechts)

Remarks: (i) global Torelli does not hold in general

(ii) The "moduli space" $O(L)\backslash\Omega_L$ is badly behaved (e.g. non-Hausdorff).

For this reason it makes sense to consider polarized symplectic manifolds. We fix

L = lattice (which appears a $(H^2(X,\mathbb{Z}), q_X)$ for some irreducible symplectic X

c = Fujiki invariant for such X

For $\mathcal{L} \in \operatorname{Pic} X$ we set

$$h = c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}).$$

If $\mathcal{L}$ is ample, then $q(h) > 0$. Conversely, if $(X, \mathcal{L})$ is given with $q(h) > 0$, then a result of Huybrechts guarantees the existence of small deformations $(X_t, \mathcal{L}_t)$ of $(X, \mathcal{L})$ with $\mathcal{L}_t$ ample. We define

$$\mathcal{M}_{L,c,h} = \left\{ (X, \mathcal{L}) ;\ \begin{array}{l} X \text{ is irreducible symplectic with} \\ \text{Beauville lattice } L \text{ and Fujiki invariant } c, \\ \mathcal{L} \text{ ample with } c_1(\mathcal{L}) = h \end{array} \right\} \Big/ \text{Iso}$$

<u>Proposition</u>. The moduli space $\mathcal{M}_{L,c,h}$ exists and is a quasi-projective variety.

<u>Proof</u>. Standard GIT construction using Viehweg's work.

We also have a period map. Let

$$L_h = h^{\perp} \subset L, \quad \operatorname{sign} L = (2, \operatorname{rank} L - 2)$$

and we have a homogeneous domain

$$\Omega_{L_h} = h^{\perp} \cap \Omega_L = \mathcal{D}_{L_h} \amalg \mathcal{D}'_{L_h}.$$

We have the groups

$$\Gamma_h = \{ g \in O(L) ;\ g(h) = \pm h \}$$

and

$$\underset{\substack{| \\ \text{'stable' orthogonal} \\ \text{group of } L_h}}{\tilde{O}(L_h)} = \ker \big(O(L_h) \to \underset{\substack{| \\ \text{discriminant group}}}{O(L_h^{\vee}/L_h)} \big)$$

One then has

$$\boxed{\tilde{O}(L_h) \cong \Gamma_h}$$

Let

$$\mathcal{M}_{L,h} = \tilde{O}(L_h) \backslash \Omega_{L_h}.$$

Then the <u>period map</u> defines an étale morphism

$$\mathcal{M}_{L,ch} \longrightarrow \mathcal{M}_{L,h}.$$

As a consequence we obtain

$$k(\mathcal{M}_{L,ch}) \geq k(\mathcal{M}_{L,h}).$$

IV. <u>A special case</u>

We now assume that

(*) X is a deformation of Hilb^{2}S.

In this case

$$L = L_{K3} \oplus \langle -2 \rangle = 3U \oplus 2E_8(-1) \oplus \langle -2 \rangle,$$
$$c! = 3.$$

Let $h \in L$. Then we have the following two invariants of h

- $q(h)$
- $\mathrm{div}(h) = (h, L) =$ pos. generator of $\{q_X(h,l),\ l \in$

Then by a theorem of Eichler

$$h \sim h' \bmod O(L) \iff (q(h), \mathrm{div}(h)) = (q(h'), \mathrm{div}(h')).$$

<u>Lemma</u> If $h \in L$ is primitive with $q(h) = 2d > 0$ then there are two possibilities.

(1) $\operatorname{div}(h) = 1$. In this case

$$L_h = h^\perp \cong 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle$$

(2) $\operatorname{div}(h) = 2$ In this case

$$L_h = h^\perp \cong 2U \oplus 2E_8(-1) \oplus \begin{pmatrix} -2d & 1 \\ 1 & -2 \end{pmatrix}.$$

As before, we consider

$$M_{L_h} = M_h = \widetilde{O}(L_h) \backslash \Omega_{L_h}$$

<u>Theorem:</u> Let $h \in L$ be primitive with $q(h) = 2d > 0$ and $\operatorname{div} h = 1$. Then

(i) If $d \geq 11$ then $\kappa(M_h) \geq 0$

(ii) If $d = 12$ or $d \geq 13$, then M_h is of general type.

<u>Corollary</u> The corresponding statements also hold for the moduli spaces of symplectic manifolds.

<u>Remark.</u> (i) Similar result for $\operatorname{div}(h) = 2$

(ii) Also works for deformations of $\mathrm{Hilb}^n S$.

<u>Idea of proof</u>

① We have

$$\Omega_{L_h} = \mathcal{D}_{L_h} \sqcup \mathcal{D}_{L_h}'$$

and

$$M_h = \widetilde{O}(L_h) \backslash \Omega_{L_h} = \widetilde{O}^+(L_h) \backslash \mathcal{D}_{L_h}$$

where $\tilde{O}^+(L_h)$ is the index 2 subgroup of $\tilde{O}(L_h)$ which fixes the connected components.

(2) <u>Claim</u> For $a = 17$ and $d > 15$ there exist a modular form F_a with respect to the group $\tilde{O}^+(L_a)$ such that:

- weight $F_a < 20 = \dim M_h$
- F_a is a cusp form
- F_a vanishes on the fixed loci of all reflections in $\tilde{O}^+(L_h)$.

The idea is to construct suitable embeddings

$$L_h = 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle \hookrightarrow II_{2,26} = 2U \oplus 3E_8$$

This induces embeddings

$$D_{L_h} \subset D_{II_{2,26}}$$

On $D_{II_{2,26}}$ one has Borcherds' modular form Φ_{12} of weight 12. Then

$$F_a = \left. \frac{\Phi_{12}}{\prod F_h} \right|_{D_{L_h}}$$

where the F_h are equations of Heegner divisors in $D_{II_{2,26}}$ containing D_{L_h}.

(3) Consider

$$V_h = F_a^k \, M_{\underbrace{(20-a)}_{>0}k}(\tilde{O}^+(L_h)) \subset S_{20k}(\tilde{O}^+(L_h$$

$$(\sim 20^k)$$

Then

$G_k \in V_k \implies G_k (dz)^k$ is a modular form on
$\mathcal{H}_k^\circ = \mathcal{H}_k \setminus$ fixed points of $\hat{O}^+(L_k)$
which extends to any smooth
projective model of $\mathcal{H}_k$. $\square$