Géométrie des variétés complexes I
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Nevanlinna theory and the degeneracy of holomorphic curves

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§1 Preliminaries

Griffiths, Bull. A.M.S. (1972)

Conjecture 1. X : cpt var. of general type
$\Rightarrow$ $^\forall f : \mathbb{C} \to X$, hol, is degenerate (= alg. deg. in this talk).

Conjecture 2. $D = \sum D_i$ ($\subset X$) s.n.c. divisor
X is smooth. $f : \mathbb{C} \to X$ non-deg.

$$\Rightarrow m_f(r, D) + T_f(r, K_X) \leqq \varepsilon T_f(r) ||_\varepsilon, \quad {}^\forall \varepsilon > 0.$$

Here

$$m_f(r, D) = \int_{|z|=r} \log \frac{1}{\|\sigma_D(f(z))\|} \frac{d\theta}{2\pi}$$

$$T_f(r, \Phi_X) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* \Phi_X$$

$$T_f(r, K_X) = T_f(r, c_1(K))$$

$$T_f(r) = T_f(r, \text{Herm. metric}_X)$$

N.B. · Conj. 2 implies Conj. 1.

· Conj. 2
$\Leftrightarrow T_f(r, L(D)) + T_f(r, K_X) \leqq N(r, f^*D) + \varepsilon T_f(r) ||$

I would like to strengthen it to

Conjecture 3. $D \subset X$, $f : \mathbb{C} \to X$ as above.

$$T_f(r, L(D)) + T_f(r, K_X) \leqq N_1(r, f^*D) + \varepsilon T_f(r) ||_\varepsilon.$$

Here $N_1(r, f^*D) = \int_1^r \frac{|f^{-1}(D) \cap \Delta(t)|}{t} dt$.

Conjecture 4 (Kobayashi, 1970). A "generic" hypersurface $X \subset \mathbb{P}^n$ is Kob. hyperbolic.

Conjecture 5 (Lang, 1974). X proj. alg. var$/k$, $k/\mathbb{Q}$, finite ext. If $k \overset{\exists}{\hookrightarrow} \mathbb{C}$ s.t. $X_\mathbb{C}$ is Kob. hyperbolic, then $|X(k)| < \infty$ (rat'l points).

Conjecture 6 (Cor. of Conj. 4+5). A "generic" hypersurface $X \subset \mathbb{P}^4_k \Rightarrow |X(k)| < \infty$.

Approaches for Kob. Conj.
① To find Examples / Existence
② To show the "generic" statement, now available only in $n \leq 3$.

For ①, Masuda - N. (1996). $^\exists d(n) \leq {}^\forall d$, $\exists$ hypersurface $X \subset \mathbb{P}^n$, $\deg = d$, Kob. hyperbolic.

N.B. This X carries some arithmetic finiteness property.
Siu-Yeung, Shiffman-Zaidenberg, $d(n) \sim n^2$.
Shirosaki, Fujimoto, $d(n) \sim 2^n$. -- arith. f. property

Example-Thm (N., 2002) $^\exists X \subset \mathbb{P}^n_\mathbb{Q}$, $\deg X = 13^{n-1}$ s.t. $X_\mathbb{C}$ is Kob. hyperbolic and $|X(k)| < \infty$ for $^\forall$ fixed $k/\mathbb{Q}$.

? Quest. $\exists X \subset \mathbb{P}^n$, Kob. hyperbolic and $|X(k)| = \infty$.

<u>Case</u>:
$n = 3$, $\dim X = 2$.

① Example, $X \subset \mathbb{P}^3$, Kob. hyperbolic, $\deg = 8$
Fujimoto (2001), Shiffman-Zaidenberg (2003 preprint)

? optimal $2 \times 3 + 1 = 7$

② "very generic" $\deg \geq 21$ (Demailly - Goul)
$\deg \geq 18$ (Paun, preprint)

§2 Jets and differential eqn.
The theory of holomorphic foliations comes in.

$\begin{cases} D \subset X \rightsquigarrow \text{log. jet bdle } J_k(X, D) \text{ (Nog. 1986)} \\ f: \mathbb{C} \to X \setminus D \rightsquigarrow J_k(f): \mathbb{C} \to J_k(X, D), \text{ jet lift} \end{cases}$

$\rightsquigarrow$ diff. eqn $\rightsquigarrow$ hol. foliation applied $\rightsquigarrow$ f is deg.
(McQuillan, Brunella, --)

⇓ allowing $f(\mathbb{C}) \cap D \neq \phi$

Nevanlinna theory, S.M.T.

⇓ $f: \mathbb{C} \to X \setminus D$ is deg.

∘ There is some advantage to introduce the truncated counting function $N_1(r, f^*D)$.

To explain this we look at

<u>Example</u> (observation by Yamanoi)

For $f: \mathbb{C} \to \mathbb{P}^1$, mer. and $a_1, \ldots, a_q \in \mathbb{P}^1$, distinct

$(q-2) T_f(r) \leq \sum_i N_1(r, f^*a_i) + \varepsilon T_f(r) \|_E$, $\forall \varepsilon > 0$.

(Nevanlinna's S. M. T.)

" $N_1(r, f^*a_i)$ " makes it possible to deal with curves X of higher genus. For instance,

$\pi : X \to \mathbb{P}^1$, hyperelliptic curve, branching over $a_1, \dots, a_q$ (q: even). genus $g = \frac{q-2}{2}$

$g \geq 2 \iff q \geq 6$

Suppose $g : \mathbb{C} \to X$, $\neq$ const.

$f = \pi \circ g$

$$
\begin{aligned}
(q-2) T_f(r) &\leq \sum_{i=1}^{q} N_1(r, f^*a_i) + \varepsilon T_f(r) \\
&\leq \frac{1}{2} \sum_{i=1}^{q} N(r, f^*a_i) + \varepsilon T_f(r) \\
&\leq \frac{q}{2} T_f(r) + \varepsilon T_f(r)
\end{aligned}
$$

$\Rightarrow 2(q-2) \leq q \Rightarrow q \leq 4 \Rightarrow g \leq 2$.

The "General case" is handled by Hurwitz formula.

- Conj. 3 for $X = \mathbb{P}^n \Rightarrow$ Conj. 1.

$$
\begin{tikzcd}
f : \mathbb{C} \arrow[r] \arrow[dr, "g"'] & X \supset E \text{ ramification} \arrow[d] \\
 & \mathbb{P}^n \supset D, \text{ branch}
\end{tikzcd}
$$

$T_g(r, L(D)) + T_g(r, K_{\mathbb{P}^n}) \leq N_1(r, g^*D) + \varepsilon T_g(r) \,\|$

$\rightsquigarrow T_f(r, K_X) \leq \frac{1}{2} T_f(r, K_X) + \varepsilon T_f(r)$,

contradiction.

We will give a new result based on this method for hol. curv

AJM.2

$\exists$ some related result due to Min Ru (preprint).

Thm (Min Ru) $D = \sum_1^q D_i \subset \mathbb{P}^n$, in general posit
$f: \mathbb{C} \to \mathbb{P}^n$, non deg.

$$\Rightarrow (q-n-1) T_f(r) \leq \sum_1^q \frac{1}{\deg D_j} N(r, f^*D_j) + \varepsilon T_f(r) ||_\varepsilon$$

N.B. No truncation ~~for~~ $N(r, f^*D_j)$.

§3. Fibration by order (growth) functions.

X, alg. var., $f: \mathbb{C} \to X$ non-deg.

$\mathbb{C}(X) =$ the rational function field of X.

$K = \{ \varphi \in \mathbb{C}(X) ;\ T_{\varphi \circ f}(r) \leq \varepsilon T_f(r) ||_\varepsilon, \forall \varepsilon > 0 \}$

$K \subset \mathbb{C}(X)$, subfield.

$K \ni \varphi_1, \dots, \varphi_l$, generators $/ \mathbb{C}$.

$\Phi : x \in X \longrightarrow (\varphi_1(x), \dots, \varphi_l(x)) \in \Phi(X) = Y_f (\subset \mathbb{P}^l)$, dominant.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[dr, "g"', "\text{non-deg}"] & X \arrow[d, "\Phi"] \\ & Y_f \end{tikzcd} \qquad T_g(r) \leq \varepsilon T_f(r) ||_\varepsilon$$

(3.1) $\boxed{\begin{array}{l} \text{If } \Phi \text{ is finite, then } T_f(r) = O(T_g(r)) \\ \Rightarrow T_f(r) \leq \varepsilon T_f(r) ||_\varepsilon, \text{ Contradiction} \\ \Rightarrow f \text{ is deg.} \end{array}}$

Example. $\Phi_{K_X^l} : X^n \to \mathbb{P}^N$, pluri-canonical system

~~dim X = 1~~, $f: \mathbb{C}^m \to X$, hol.

$$\Rightarrow T_{\Phi_{K_X^l} \circ f}(r) \leq \varepsilon T_f(r) ||_\varepsilon.$$

Thus, if K_X is big, then $\det(df) \equiv 0$.
(Picard, Griffiths, Kobayashi-Ochiai).
$\dim X = 1$ K_X, ample K_X, big.

Applying (3.1) for jet lifts, we have

- log Bloch-Ochiai's Theorem. $f : \mathbb{C} \to X \subset A$ (semi-abelian)
  X is not semi-abelian $\Rightarrow f$ is deg.
- Lang's Conj. $f : \mathbb{C} \to A \setminus D$ (D = divisor)
  $\Rightarrow f$ is deg.

Siu-Yeung : A = abelian, jet + Θ-Wronskian (1996)
N. — : A = Semi-abelian, jet of jets (1998)

§ 4. Semi-abelian varieties

$X = A$, semi-abelian variety

$$0 \to (\mathbb{C}^*)^t \to A \to A_0 \text{ (abelian var.)} \to 0.$$

$f : \mathbb{C} \to A$, hol. non-deg.

$J_k(f) : \mathbb{C} \to J_k(A)$, k-jet lift. $k = 0, 1, 2, \ldots$

$X_k(f) = \overline{J_k(f)(\mathbb{C})}^{\,\mathrm{Zar}}$

Theorem (4.1) (N.-Winkelmann-Yamanoi, to appear in Forum Math)
For sub var. $Z \subset X_k(f)$, $\exists$ compactification $\bar{Z} \subset \overline{X_k(f)}$
such that

$$(*) \quad T_f(r, \omega_{\bar{Z}, J_k(f)}) \leqq N_1(r, J_k(f)^* Z) + \varepsilon T_f(r) \|_\varepsilon$$

Moreover, if codim $Z \geqq 2$, then

$$(**) \quad T(r, \omega_{\bar{Z}, J_k(f)}) \leqq \varepsilon T_f(r) \|_\varepsilon.$$

Proof. $k=0$ for simplicity. $\exists \ell \geq 1$ $(\ell < \infty)$.

- $T_f(r, \omega_{\bar{Z}, \partial f}) \leqq N_\ell(r, f^* Z) + \varepsilon T_f(r) \|_E$

$\mathrm{ord}_\zeta f^* Z \geqq 2 \Rightarrow J_1(f) \in X_1(f) \cap J_1(Z) = Z'$

<u>Claim 1</u>. $\operatorname{codim}_{X_1(f)} Z' \geqq 2$

<u>Claim 2</u> $N_1(r, J_1(f)^* Z') \leqq \varepsilon T_f(r) \|_E$

$\Rightarrow$ Tow inequalities, $(*) + (**)$

N.B. The case of $k=0$, abelian A done by Yamanoi. (2004).

Thm 4.2 (Yamanoi) A, abelian and simple, $\pi : X \to A$, ramified $(R_\pi \neq \phi)$

$\Rightarrow$ X is Kob. hyperbolic

N.B. $\dim X = 2$ by C. Grant (1986).

Thm 4.3 (N.-W.-Y.) $f : \mathbb{C} \to \mathbb{P}^n \setminus D$

D : $x_i = 0$, $1 \leq i \leq n$, linear, and

$$x_0^q + \cdots + x_n^q = 0, \quad q \geqq 2$$

$\Rightarrow$ f is deg

N.B. $n = q = 2$ and f of finite order by M. Green (197?) who conjectured for general f and $n = q = 2$.

§5 Degeneracy Theorem (application of Thm (4.1))

Theorem (5.1) (N.-W.-Y., preprint)
A, semi-abelian, $\pi : X \to A$, (surjective) prop
$\bar{\kappa}(X) > 0 \Rightarrow {}^{\forall} f : \mathbb{C} \to X$ is degenerate.

By Kawamata's thm.
$$\begin{array}{ccc} X & \xrightarrow[/B]{} & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow[/B]{} & A/B \end{array}$$
(after finit étale cover

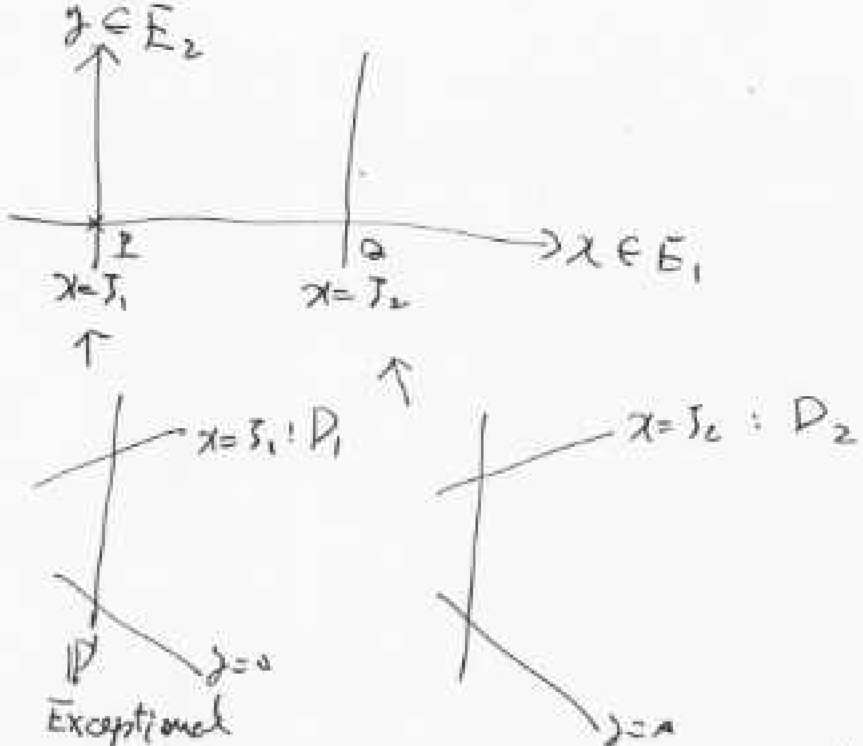
$\bar{\kappa}(Y) = \dim Y = \bar{\kappa}(X) > 0$.
Reduced to $\bar{\kappa}(X) = \dim X$.

N.B. Related results by Dethloff-Lu (to appear
Fourier): $\dim X = 2$
(a) $\bar{\kappa}(X) = 2$ and f is Brody.
(b) $\bar{\kappa}(X) = 1$, a weaker condition than "prope
to keep the above reduction by Kawamata

Problem. What is a Brody curve?

Counter example for the "proper" condition.
(due to Dethloff - Lu)

E_i, $i = 1, 2$, elliptic curves, $A = E_1 \times E_2$
$\zeta_1, \zeta_2 \in E_1$, distinct
$P = (\zeta_1, 0)$, $Q = (\zeta_2, 0)$
$\tilde{A} \to A$, blow up at P and Q.



$X \to A$ is not pr

$$X = \tilde{A} \setminus (D_1 + D_2), \quad \bar{\kappa}(X) = 1.$$

By ~~a result~~ general result of Buzzard-Lu

$\exists\ \tilde{g}: \mathbb{C}^2 \longrightarrow X$, $\det(d\tilde{g}) \not\equiv 0$

$\exists\ f: \mathbb{C} \longrightarrow X$, non-deg.

~~Explicit~~ Simple construction of g.

$\pi_1 : \mathbb{C} \longrightarrow E_1$, univ. covering

$\pi_1^{-1}\{\zeta_1, \zeta_2\} = \{z_\nu\}_{\nu=1}^{\infty} \subset \mathbb{C}$, discrete.

Take an entire function F s.t.

$F(z_\nu) = F'(z_\nu) = 0$, $\nu = 1, 2, \dots$, no other zeros (on $F \not\equiv 0$)

$\pi_2 : \mathbb{C} \longrightarrow E_2$, univ. covering

Set $g : (z, w) \in \mathbb{C}^2 \to (\pi_1(z), \pi_2(wF(z))) \in A$.

If $\pi_1(z) = \zeta_1$ or ζ_2, $dg\, T_{(z,w)}(\mathbb{C}^2) \subset T_{\zeta_j}(E_1) \times \{0\}$.

$\exists$ lifting $\tilde{g} : \mathbb{C}^2 \longrightarrow X$, hol.

In this example,

$$N(r, f^*(P_1 + P_2)) \sim T_{p_1 \circ f}(r),$$

where $p_i : X \longrightarrow E_i$, projections.

By Theorem (4.1)

$$N(r, f^*(P_1 + P_2)) \leq \epsilon T_f(r) \|_E.$$

Thus $\quad T_{p_1 \circ f}(r) \leq \epsilon T_f(r) \|_E$

In terms of fibrations in §3

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[dr, "p_1 \circ f"'] & X \arrow[d, "p_1"] \\ & E_1 = Y_f \end{tikzcd}$$

$\nexists\, f : \mathbb{C} \longrightarrow X$ with $\dim Y_f = 0$.