ENERGY FUNCTIONALS AND KÄHLER-EINSTEIN METRICS

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JOINT WORK WITH

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The Problem of Kähler-Einstein Metrics

- (X, ω) compact Kähler manifold of dimension n

  $\omega = \frac{i}{2} g_{\bar{k}j} \, dz^j \wedge d\bar{z}^k$

  $R_{\bar{k}j} = - \partial_j \partial_{\bar{k}} \log \omega^n$ "Ricci curvature"

  $$\boxed{R_{\bar{k}j} = \mu \, g_{\bar{k}j}}$$ "Kähler-Einstein metrics"

- $\mu < 0$ (Yau, Aubin, early 1970's)
- $\mu = 0$ "Calabi conjecture" (Yau, 1976)

The Case $\mu > 0$

- Matsushima's Obstruction
- Futaki Invariant: $F : \{\text{Hol}^c \text{ Vector Fields}\} \to \mathbb{C}$

  $$F(V) = \int_X (Vh)\,\omega^n, \quad R_{\bar{k}j} = g_{\bar{k}j} + \partial_j \partial_{\bar{k}} h$$

  X admits a KE metric $\Rightarrow F \equiv 0$
- $\dim(X) = 2$: $\{\text{Futaki} = 0\} \Rightarrow \exists$ KE (Tian 1992)
- Counterexamples of Tian (1997) even when X has no hol^c vector fields

CONJECTURE OF YAU

(X,ω) admits a KE metric	$\Longleftrightarrow$	(X,ω) is "STABLE IN THE SENSE OF GEOMETRIC INVARIANT THEORY"

MANY COMPETING NOTIONS OF STABILITY

- Chow-Mumford, Hilbert-Mumford
- CM Stability (Tian, Paul-Tian), Slope stability (Ross-Thomas)
- K-Stability (Tian, Donaldson)

$$\nu_\omega(\varphi) = -\frac{1}{V}\int_X \dot\varphi\,(R-\mu n)\,\omega_\varphi^n \quad , \quad \omega_\varphi = \omega + i\partial\bar\partial\varphi$$

K-stability defined by asymptotic behavior of $\nu_\omega(\varphi_{\sigma(t)})$, with $\varphi_{\sigma(t)} = \imath^*_{\sigma(t)\cdot\underline{s}}(\omega_{FS})$, $\imath : X \to \mathbb{CP}^N$ Kodaira imbedding defined by a basis $\underline{s}$ of $H^0(X, K_X^{-m})$, $\sigma(t)$ 1-parameter subgroup of $GL(N_m$

(also, more generally, by asymptotic behavior of "test configurations"

- Desirable feature: Moduli of stable structures should be Hausdorff

NECESSARY CONDITIONS

- Tian (1997) : KE $\Rightarrow$ CM, K-stability
- Donaldson (2001): KE $\Rightarrow$ Chow-Mumford stability

SUFFICIENT CONDITIONS

No *general* result so far, except for

Analytic Approaches to Kähler-Einstein

...he Method of Continuity

(X,ω) compact Kähler , $R_{\bar{k}j} - \mu g_{\bar{k}j} = \partial_j \partial_{\bar{k}} h$

$$\boxed{\det\left(g_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi\right) = e^{h - t\mu\varphi} \det g_{\bar{k}j}} \qquad 0 \le t \le 1 \qquad (*)$$

- Openness: linear elliptic PDE's
- Closedness: key missing estimate $\|\varphi\|_{C^0}$
- Multiplier ideal obstructions: Siu, Nadel, Demailly-Kollar...

...he Variational Approach

$$F_\omega(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int \varphi\,\omega^n - \log\left(\frac{1}{V}\int e^{h-\varphi}\omega^n\right)$$

$$J_\omega(\varphi) = \frac{i}{2V}\sum_{j=0}^{n-1}\left(\frac{n-j}{n+1}\right)\int \partial\varphi \wedge \bar\partial\varphi \wedge \omega^{n-1-j} \wedge \omega_\varphi^j$$

$$\boxed{(*) \iff \frac{\delta F_\omega}{\delta\varphi} = 0}$$

- Tian (1997): $\exists$ KE $\iff \exists\, \gamma > 0$ $\quad F_\omega(\varphi) \ge A_\gamma J_\omega(\phi)^\gamma - B$

Tian's Conjecture: $\gamma = 1$

- Analogous statements for the K-energy $\nu_\omega(\varphi)$, $\frac{\delta \nu_\omega}{\delta\varphi} = 0 \iff R = \mu n$

HE KÄHLER-RICCI FLOW

$$\boxed{\dot{g}_{\bar{k}j} = -\left(R_{\bar{k}j} - \mu g_{\bar{k}j}\right)}$$

- Flow exists for all times; Main issue = CONVERGENCE !
- Hamilton (1988): Convergence for $X = \mathbb{C}P^1$, $R > 0$ everywhere

  Chow (1991): removal of the condition $R > 0$
- Chen-Tian (2002): Convergence for $X = \mathbb{C}P^N$
- Perelman (2002):

$$\boxed{\exists \text{ KE} \implies \text{Convergence of the Kähler-Ricci flow}}$$

Furthermore

$$\left.\begin{array}{ll} \|R\|_{C^0} & \leq C \\ \operatorname{diam}(X) & \leq C \end{array}\right\} \text{ along the Kähler-Ricci flow}$$

- Tian-Zhu (2005): extension of Perelman's results to

  $\exists$ KE soliton $\implies$ convergence of KR flow to a KE soliton

BUT AS YET,

- No convergence result without an a priori assumption on the existence of a KE metric or soliton;
- No convergence result linked directly to STABILITY

THEOREM 1 (P., J. Song, J. Sturm, B. Weinkove)

- Assume that X has no non-trivial holomorphic vector fields, and that X admits a Kähler-Einstein metric ω_{KE}. Then

$$\boxed{F_{\omega_{KE}}(\varphi) \geq A\, J_{\omega_{KE}}(\varphi) - B} \qquad (*)$$

- More generally, assume that X admits a Kähler-Einstein metric ω_{KE} and let G be the subgroup of $\mathrm{Aut}_0(X)$ fixing ω_{KE}. Then the same inequality holds for all G-invariant potentials φ.

OBSERVATIONS

- The inequality $(*)$ holds for $F_{\omega_{KE}}(\varphi)$, $J_{\omega_{KE}}(\varphi)$ if and only if it holds for $F_\omega(\varphi)$, $J_\omega(\varphi)$ for any $\omega \in c_1(X)$.

- $X = \mathbb{CP}^1$: then $F_{\omega_{KE}}(\varphi) \geq 0$ is equivalent to

$$\frac{1}{2V}\int |\nabla\varphi|^2 - \frac{1}{V}\int \varphi - \log\left(\frac{1}{V}\int e^{-\varphi}\right) \geq 0$$

"Moser-Trudinger Inequality"

- ANALOGY WITH GÅRDING INEQUALITY:

$$\text{ENERGY INTEGRAL} \geq c\,\{\text{NORM}\} - \text{ERROR}$$

THEOREM 2 (P., J. Sturm) Assume that the Riemann curvature tensor is bounded along the flow. Let (A) and (B) be the conditions below.

- If condition (A) holds, then for any $s \geq 0$,

$$\lim_{t\to\infty} \| R_{\bar{k}j}(t) - \mu g_{\bar{k}j}(t) \|_{(s)} = 0$$

where $\|\cdot\|_{(s)}$ is the Sobolev norm with respect to the metric $g_{\bar{k}j}(t)$

- If both conditions (A) and (B) hold, and if diam(X) is uniformly bounded along the flow, then $g_{\bar{k}j}(t)$ converges exponentially fast in C^∞ to a Kähler-Einstein metric.

THE CONDITIONS (A) AND (B)

(A) The Mabuchi functional $\nu_\omega(\varphi)$ is bounded from below on the space

$$\mathcal{P}(X,\omega) = \{\varphi \in C^\infty(X);\ \omega + i\partial\bar{\partial}\varphi > 0\}$$

(B) Let J = complex structure of X, viewed as a tensor $J^p{}_q$,

$\mathcal{O}(J) = C^\infty$ closure of orbit of J under diffeomorphism group

Then

$$\tilde{J} \in \mathcal{O}(J) \Rightarrow \dim \left\{ \begin{array}{c} \text{Hol'c vector fields} \\ \text{with respect to } \tilde{J} \end{array} \right\} = \dim \left\{ \begin{array}{c} \text{Hol'c vector fields} \\ \text{with respect to } J \end{array} \right\}$$

Observations

- Perelman : $|R|$, $\operatorname{diam}(X) \leq C$ along Kähler-Ricci flow
- In $\dim X = 2$, the condition

  (C) The Ricci curvature is non-negative and
  the traceless curvature operator is 2-nonnegative

  is preserved by the Kähler-Ricci flow
- In $\dim X = 2$, and if the initial metric satisfies (C), then the Riemann curvature tensor and the diameter are bounded along the flow
- Thus, under the above conditions, if (X, ω) satisfies (A) and (B) then the flow converges to a KE metric.

Proof of Theorem 2

The Ricci Potential h

$$R_{\bar{k}j} - \mu g_{\bar{k}j} = \partial_j \partial_{\bar{k}} h$$

The Potential φ and the Ricci Potential

$$g_{\bar{k}j} = g_{\bar{k}j}(0) + \partial_j \partial_{\bar{k}} \varphi \Rightarrow \partial_j \partial_{\bar{k}} \dot{\varphi} = \dot{g}_{\bar{k}j}$$
$$= -(R_{\bar{k}j} - \mu g_{\bar{k}j}) = -\partial_j \partial_{\bar{k}} h$$

$$\boxed{\dot{\varphi} = -h + c}$$

The Mabuchi Energy $\nu_\omega(\varphi)$ and the Kähler-Ricci Flow

$$\dot{\nu} = -\frac{1}{V} \int_X \dot{\varphi}(R - \mu n)\, \omega_\varphi^n$$

$$= + \frac{1}{V} \int_X (h - c)(\Delta h)\, \omega_\varphi^n$$

$$= -\frac{1}{V} \int_X |\nabla h|^2\, \omega_\varphi^n < 0$$

$$\frac{1}{V} \int_0^T \int_X |\nabla h|^2 \omega_\varphi^n = \nu_\omega(\varphi(0)) - \nu_\omega(\varphi(T)) \underbrace{\leq C}_{\text{by condition (A)}} < \infty$$

$$\Rightarrow \boxed{\exists\, t_j \to +\infty \qquad \frac{1}{V} \int_X |\nabla h(t_j)|^2 \omega_{\varphi_{t_j}}^n \to 0}$$

• DIFFERENTIAL INEQUALITY FOR $|\nabla h|^2$

$$\partial_t |\nabla h|^2 = \Delta(|\nabla h|^2) + \mu |\nabla h|^2 - |\bar\nabla\nabla h|^2 - |\nabla\nabla h|^2$$

Set

$$Y(t) = \int_X |\nabla h|^2 \omega_\varphi^n$$

$$\Rightarrow \dot Y = \mu(n+1) Y - \int_X |\nabla h|^2 R \omega_\varphi^n - \int_X |\bar\nabla\nabla h|^2 \omega_\varphi^n - \int_X |\nabla\nabla h|^2 \omega_\varphi^n$$

$$\Rightarrow \qquad \dot Y \leq [\mu(n+1) + C] Y \qquad \text{, assuming } |R| \leq C$$

$$\left.\begin{array}{l}\text{Convergence to } 0 \text{ along } t_i \to +\infty \\ \qquad + \\ \text{Differential inequality}\end{array}\right\} \Longrightarrow \lim_{t\to 0^+} Y(t) = 0$$

• HIGHER DERIVATIVES

$$\partial_t(|\nabla^s \bar\nabla^r h|^2) = \Delta |\nabla^s\bar\nabla^r h|^2 + 2(\mu - r - s)|\nabla^s\bar\nabla^r h|^2$$
$$- |\nabla^{s+1}\bar\nabla^r h|^2 - |\bar\nabla\nabla^s\bar\nabla^r h|^2 + O(R_{\bar k j \bar\ell m})$$

$$\Rightarrow \qquad \lim_{t\to\infty} \int_X |\nabla^s \bar\nabla^r h|^2 \omega_\varphi^n = 0$$

Since

$R_{\bar k j} - \mu g_{\bar k j} = \partial_j \partial_{\bar k} h$, this implies $\boxed{\lim_{t\to\infty} \|R_{\bar k j} - \mu g_{\bar k j}\|_{C^\infty} = 0}$

CONSEQUENCES OF BOUNDED GEOMETRY

"BOUNDED GEOMETRY" = Bounded diameter, injectivity radius, curvature

$$\Downarrow$$

Bounded Sobolev constants

$$\Downarrow$$

$$\sup_X |D^p (R_{\bar{t}_j} - \mu g_{\bar{t}_j}(t))|_t \to 0$$

However, the metrics $g_{\bar{t}_j}(t)$ are not necessarily equivalent !

• CHEEGER-GROMOV-HAMILTON COMPACTNESS THEOREM

$$\left.\begin{array}{l} g_{t_j}(t) \text{ have uniformly} \\ \text{bounded geometry} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists \text{ diffeomorphisms } F_{t_j} : X \to X \\ \text{so that} \\ (F_{t_j})^*(g_{t_j}) \text{ converges in } C^\infty \end{array}\right.$$

However, we have no control over the diffeomorphisms F_{t_j} !

• KEY POINT : MORE PRECISE ESTIMATES ON THE RATE $Y(t_j) \to 0$

Recall $\quad Y(t) = \int_X |\nabla h|^2 \omega_\varphi^n$

and $\quad \dot{Y} \leq \underbrace{[\mu(n+1) + C]}_{\text{positive}} Y \quad \leftarrow$ This has to be improved !

Two Key Estimates

- Set λ_t = lowest strictly positive eigenvalue of Δ_t on $T^{1,0}$ vector fields

  π_t = projection on holomorphic vector fields

  $F : \{\text{Holo vector fields}\} \to \mathbb{C}$ "Futaki Invariant"

$$\boxed{\begin{aligned}\dot{Y} \leq & -2\lambda_t Y + 2\lambda_t F(\pi_t(\nabla h)) \\ & - \int_X |\nabla h|^2 (R - \mu n)\omega_\varphi^n - \int_X \nabla^j h \nabla^{\bar{k}} h (R_{\bar{k}j} - \mu g_{\bar{k}j})\omega_\varphi^n\end{aligned}}$$

- Let J be a complex structure satisfying condition (B).

  Fix $V>0$, $D>0$, $\delta>0$, C_k. Then $\exists C>0$ so that

$$\boxed{\|\bar{\partial} W\|^2 \geq C \|W\|^2, \text{ for all } W \perp H^0(X, T^{1,0})}$$

and all Kähler metrics $g_{\bar{k}j}$ with

$$\begin{aligned} \text{Volume} &\leq V \\ \text{Diameter} &\leq D \\ \text{Injectivity radius} &\geq \delta \\ \sup_{|\beta|\leq k} |D^\beta \text{Riem}| &\leq C_k . \end{aligned}$$

HE TWO KEY ESTIMATES IMPLY THEOREM 2

- $\nu_\omega(\varphi) \geq -C \implies F(V) \equiv 0$

- $$\left.\begin{array}{r} \|R_{\bar{k}j} - \mu g_{\bar{k}j}\|_{(s)} \to 0 \\ \lambda_t \geq C > 0 \\ F(\pi_t(\nabla h)) = 0 \end{array}\right\} \implies \dot{Y} \leq (-2\lambda_t + \varepsilon) Y$$

$$\Downarrow$$

$$Y(t) \leq C e^{-ct}, \quad c > 0$$

- Recall $Y(t) = \int_X |\nabla h|^2 \omega_\varphi^n$

By induction, we obtain in the same way

$$\sup_X |\nabla^k h|^2 \leq C_k e^{-ct}$$

$$\Downarrow$$

$$\int_T^\infty \sup_X |\dot{g}_{\bar{k}j}(t)| \, dt = \int_T^\infty \sup_X |R_{\bar{k}j} - \mu g_{\bar{k}j}| \, dt \leq \int_T^\infty \sup_X |\nabla^k h| < \infty$$

$$\Downarrow$$

All metrics $g_{\bar{k}j}(t)$ are equivalent

$$\Downarrow$$

$\|R_{\bar{k}j} - \mu g_{\bar{k}j}\|_{g(0)} \to 0$ exponentially. Q.E.D

Proof of Theorem 1

T-Up

Fix $\varphi \in P(X, \omega_{KE})$, and set $\omega = \omega_{KE} + i\partial\bar\partial\varphi$

$$\boxed{(\omega + i\partial\bar\partial\varphi_t)^n = e^{h - t\varphi_t}\,\omega^n} \qquad 0 \le t \le 1$$

- Because KE exists, equation exists for $0 \le t \le 1$ (Bando-Mabuchi)

$$\omega + i\partial\bar\partial\varphi_1 = \omega_{KE} \implies \varphi_1 = -\varphi$$

- Set $\omega_t = \omega + i\partial\bar\partial\varphi_t$. Then

$$Ric(\omega_t) - \omega_t = i\partial\bar\partial h_t \quad \text{with} \quad \boxed{h_t = (t-1)\varphi_t}$$

- $\omega_t = \omega + i\partial\bar\partial\varphi_t = \omega_{KE} + i\partial\bar\partial(\varphi_t - \varphi_1)$
- Equivalent form of the Monge-Ampere equation, but with ω_{KE} as reference metric:

$$\boxed{\log \frac{\omega_{KE}^n}{(\omega_{KE} + i\partial\bar\partial(\varphi_t - \varphi_1))^n} + (\varphi_1 - \varphi_t) = h_t}$$

ist Estimate

$$F_{\omega_{KE}}(\varphi) \geq (1-t)\, J_{\omega_{KE}}(\varphi) - (1-t)\, \|\varphi_t - \varphi_1\|_{C^0} \qquad (**)$$

$\forall t,\ 0 \leq t \leq 1$

ESTIMATES FROM LINEARIZATION

$$\log \frac{\omega_{KE}^n}{(\omega_{KE} - i\partial\bar{\partial}\psi)^n} + \psi = h$$

Then there exists $\varepsilon > 0$ so that

$$\|h\|_{C^0} \leq \varepsilon \Rightarrow \|\psi\|_{C^0} \leq \|\psi\|_{L^p_2} \leq C\|h\|_{C^0}$$

In the present case, $h = (t-1)\varphi_t$, $\psi = \varphi_1 - \varphi_t$

$$\boxed{(1-t)\|\varphi_t\|_{C^0} \leq \varepsilon \Rightarrow \|\varphi_1 - \varphi_t\|_{C^0} \leq C\,(1-t)\|\varphi_t\|_{C^0}}$$

APPLICATION TO (**)

$$F_{\omega_{KE}}(\varphi) \geq \varepsilon \frac{J_{\omega_{KE}}(\varphi)}{\|\varphi_t\|} - \varepsilon' \sim \varepsilon \frac{J_{\omega_{KE}}(\varphi)}{\|\varphi_1\|_{C^0}} - \varepsilon'$$

since $(1-t) \ll 1 \Rightarrow \|\varphi_1\|_{C^0} \sim \|\varphi_t\|_{C^0}$.

However, $\|\varphi_1\|_{C^0} \sim J_{\omega_{KE}}(\varphi)$, so we get only

$$F_{\omega_{KE}}(\varphi) \geq -C$$

NOT GOOD ENOUGH!

KEY IMPROVEMENT (Tian) $\exists \beta > 0$

$$\boxed{(1-t)^{1+\beta} \|\varphi_t\|_{C^0} \leq \varepsilon \Rightarrow \|\varphi_1 - \varphi_t\|_{C^0} \leq C\left\{(1-t)\|\varphi_t\|_{C^0} + 1\right\}}$$

Observe

- $(1-t) \ll 1 \Rightarrow \|\varphi_1\|_{C^0} \sim \|\varphi_t\|_{C^0}$

- $$(1-t)\|\varphi_1 - \varphi_t\|_{C^0} \leq C\left\{(1-t)^2 \|\varphi_t\|_{C^0} + 1\right\}$$
$$\leq C\left\{(1-t)^{1+\beta}\|\varphi_t\|_{C^0} + 1\right\} \leq C$$

- Thus $$F_{\omega_{KE}}(\varphi) \geq \varepsilon \frac{J_{\omega_{KE}}(\varphi)}{\|\varphi_t\|^{1/1+\beta}} - C$$
$$\sim \varepsilon \frac{J_{\omega_{KE}}(\varphi)}{\|\varphi_1\|^{1/1+\beta}} - C$$
$$\sim \varepsilon J_{\omega_{KE}}^{1-\frac{1}{1+\beta}}(\varphi) - C$$

and we obtain

$$F_{\omega_{KE}}(\varphi) \geq A_\gamma J_{\omega_{KE}}^{\gamma}(\varphi) - B_\gamma$$

with $\gamma = 1 - \frac{1}{1+\beta} = \frac{\beta}{1+\beta}$.

Smoothing by the Kähler-Ricci Flow

$$\omega_{\varphi_t} \longrightarrow \omega_{\tilde{\varphi}_t} \qquad \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\partial}{\partial s}\omega(s) = -\left(Ric(\omega(s)) - \omega(s)\right) \\ \omega(0) = \omega_{\varphi_t} \end{array} \right\} \\ \\ \omega_{\tilde{\varphi}_t} \equiv \omega(1) \end{array} \right.$$

$$\varphi_1 - \varphi_t \longrightarrow \varphi_1 - \tilde{\varphi}_t$$

$$\log \frac{\omega_{KE}^n}{(\omega_{KE} - i\partial\bar{\partial}(\varphi_1 - \tilde{\varphi}_t))^n} + \varphi_1 - \tilde{\varphi}_t = \tilde{h}_t$$

The parabolic smoothing effect produces $\|\nabla \tilde{h}_t\|_{C^0} \le \|h_t\|_{C^0}$, and whence,

$$\|\tilde{h}_t\| \le C(1-t)^\beta \|h_t\|_{C^0}^{1-\beta}$$

$$\begin{aligned} \|\varphi_1 - \varphi_t\|_{C^0} &\le \|\varphi_1 - \tilde{\varphi}_t\|_{C^0} + \|h_t\|_{C^0} + 1 \\ &\le C\left((1-t)\|\varphi_t\|_{C^0} + 1\right) \end{aligned}$$

Key New Estimate (to get the exponent $\gamma = 1$)

$$F_{\omega_{KE}}(\varphi_t - \varphi_1) \le (1-t)\left(J_{\omega_{KE}}(\varphi_t - \varphi_1) + 1\right)$$

This implies : $\exists C, c > 0$ depending only on ω_{KE}, $\exists t$ with

$$\boxed{1 - t \ge c \quad \text{and} \quad J_{\omega_{KE}}(\varphi_t - \varphi_1) \le C}$$