Regularization of Currents with Mass Control and Singular Morse Inequalities

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Abstract. Let φ_0 be a subharmonic function on an open disc $D(x_0,\, r) \subset \mathbb{C}$. For every positive integer m, we construct a holomorphic function f_m on the disc $D(x_0,\, r)$ such that the integral of $|f_m|^2\, e^{-2m\varphi_0}$ over the disc is finite and has a very slow and well understood growth in m, while the number of zeroes of f_m in the disc is at most a polynomial of degree 1 in m whose leading coefficient is given explicitly in terms of the mass of the Laplacian $\Delta\varphi_0$ on the disc. As an application to several complex variables, we obtain a regularization theorem for (1, 1)-currents on compact, possibly non-Kähler manifolds, while keeping track of the Monge-Ampère masses of the regularizing currents. This strengthens Demailly's regularization-of-currents theorem and has geometric applications to the study of Moishezon manifolds and big line bundles via singular Morse inequalities.

0.1 Introduction

Let $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic (psh) function on an open subset $\Omega \subset \mathbb{C}^n$, and let $z = (z_1, \ldots, z_n)$ be the standard coordinates on $\mathbb{C}^n$. The Lelong number $\nu(\varphi,\, x)$ of φ at an arbitrary point $x \in \Omega$ is defined as the mass carried by the positive measure $dd^c\varphi \wedge (dd^c \log|z - x|)^{n-1}$ at x (see, for instance, Demailly's book [Dem97], chapter III). It is a well-known result of Skoda ([Sko72]) that the Lelong numbers of φ affect the local integrability of $e^{-2\varphi}$. Indeed, if $\nu(\varphi,\, x) < 1$, then $e^{-2\varphi}$ is integrable on some neighbourhood of x. On the contrary, if $\nu(\varphi,\, x) \geq n$, then $e^{-2\varphi}$ is not integrable near x. The integrability of $e^{-2\varphi}$ is unpredictable when $1 \leq \nu(\varphi,\, x) < n$.

Our first aim is to establish a potential-theoretic result in the case $n = 1$ when there is no unpredictability interval. Let $U \subset \mathbb{C}$ be an open set, $\varphi_0 : U \to \mathbb{R} \cup \{-\infty\}$ a subharmonic function, and $T = dd^c\varphi_0$ the associated closed positive current of bidegree $(1, 1)$. The current T can be identified with the Laplacian $\Delta\varphi_0$ of φ_0 computed in the sense of distributions. It defines a positive measure $\mu = dd^c\varphi_0$ on U. In one complex variable, the mass of $dd^c\varphi_0$ at a point x coincides with the Lelong number $\nu(\varphi_0,\, x)$. Let $D(x_0,\, r) \subset\subset U$ be an arbitrary disc of radius $0 < r < \frac{1}{2}$, and let

$$\gamma = \int\limits_{D(x_0,\, r)} dd^c\varphi_0,$$

be the mass carried by the measure $dd^c\varphi_0$ on this disc. Consider the decomposition :

$$\varphi_0 = N \star \Delta\varphi_0 + h_0, \qquad \text{on } D(x_0,\, r),$$

where $N(z) = \frac{1}{2\pi}\log|z|$ is the Newton kernel in one complex variable, and $h_0 = \mathrm{Re}\, g_0$ is a harmonic function expressed as the real part of a holomorphic function g_0.

In all that follows, we will be considering a method of neutralizing the $-\infty$-poles of φ_0 on a fixed disc in order to make the exponential $e^{-2m\varphi_0}$ integrable and to control the growth rate of its integral as $m \to +\infty$.

Theorem 0.1.1 *Let $\varphi_0 : U \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on an open set $U \subset \mathbb{C}$, and let $D(x_0,\, r) \subset\subset U$ be an open disc of radius $0 < r < \frac{1}{2}$.*

Then, for every small enough $\delta > 0$ and every $m \in \mathbb{N}^\star$, there exist finitely many points $a_1 = a_1(m), \dots, a_{N_m} = a_{N_m}(m) \in D(x_0,\, r)$, such that the positive integers m_j defined as :

$$m_j = \max\{[m\nu(\varphi_0, a_j)],\ 1\}, \qquad j = 1, \dots, N_m, \qquad \textit{([ ] is the integer part)},$$

and the holomorphic function $f_m(z) = e^{m\, g_0(z)} \prod_{j=1}^{N_m}(z - a_j)^{m_j}$ defined on $D(x_0,\, r)$, have the following properties :

(i) $\sum_{j=1}^{N_m} m_j \leq m\gamma(1+\delta)$, *where γ is the $dd^c\varphi_0$-mass of $D(x_0,\, r)$;*

(ii) There exists a constant $C = C(r) > 0$, independent of m, such that :

$$|a_j - a_k| \geq \frac{C}{m^2},$$

for all a_j, a_k, such that $j \neq k$ and $\nu(\varphi_0,\, a_j), \nu(\varphi_0,\, a_j) < \frac{1-\delta}{m}$.

(iii) $\displaystyle\int_{D(x_0,\, r)} |f_m(z)|^2 e^{-2m\varphi_0(z)}\, d\lambda(z) = o(m), \quad \text{when} \quad m \to +\infty,$

where $d\lambda$ is the Lebesgue measure in $\mathbb{C}$.

Higher dimensional analogues of this result have yet to be found. However, the Ohsawa-Takegoshi L^2 extension theorem (see [OT87], [Ohs88]) applied on a complex line enables us to derive geometric applications of Theorem 0.1.1 in several complex variables. The first application is a global regularization theorem for closed almost positive $(1,\, 1)$-currents in the spirit of Demailly (see [Dem92]), but with an additional control on the Monge-Ampère masses of the regularizing currents. Here is the set-up.

Let T be a d-closed current of bidegree $(1,\, 1)$ on a compact complex manifold X of dimension n. Assume that $T \geq \gamma$ for some real continuous

(1, 1)-form γ (i. e. T is almost positive). The current T can be globally written as $T = \alpha + dd^c\varphi$, with a global C^∞ (1, 1)-form α and an almost psh potential φ on X (i.e. φ can be locally expressed as the sum of a psh function and a C^∞ function). The notation $dd^c := \frac{i}{\pi}\partial\bar\partial$ will be used in all that follows. A variant of Demailly's regularization theorem (see [Dem92, Proposition 3.7]) asserts that T is the weak limit of currents $T_m = \alpha + dd^c\varphi_m$ lying in the $\partial\bar\partial$-cohomology class of T and having analytic singularities. These are, by definition, singularities for which φ_m can be locally written as

$$\frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + C^\infty, \tag{1}$$

with a constant $c > 0$, and holomorphic functions $g_1, \ldots, g_N$. Fix a Hermitian metric ω on X. Then T_m can be chosen such that :

$$T_m \geq \gamma - \varepsilon_m\,\omega, \qquad \text{for some sequence } \varepsilon_m \downarrow 0,$$

and the Lelong numbers satisfy $\nu(T,\, x) - \dfrac{n}{m} \leq \nu(T_m,\, x) \leq \nu(T,\, x), \quad x \in X.$

What this theorem does not specify, however, is whether there exist regularizations $T_m \to T$ with analytic singularities having the extra property that the growth in m of the Monge-Ampère masses of the wedge-power currents T_m^q is under control. In other words, if $V_m = \{\varphi_m = -\infty\}$ is the polar set of T_m, we would like to control the growth rate of the quantities :

$$\int\limits_{X\setminus V_m} (T_m - \gamma + \varepsilon_m\,\omega)^q \wedge \omega^{n-q}, \qquad q = 1, \ldots, n$$

as $m \to +\infty$. Using Theorem 0.1.1 we can modify Demailly's original construction to settle this question in the following form.

Theorem 0.1.2 *Let $T \geq \gamma$ be a d-closed current of bidegree* (1, 1) *on a compact complex manifold X, where γ is a continuous* (1, 1)*-form such that $d\gamma = 0$. Then, in the $\partial\bar\partial$-cohomology class of T, there exist closed* (1, 1)*-currents T_m with analytic singularities along an analytic set $V_m \subset X$ which converge to T in the weak topology of currents and satisfy :*

$(a)\quad T_m \geq \gamma - \dfrac{C}{m}\,\omega, \qquad m \in \mathbb{N}\,;$

$(b)\quad \nu(T,\, x) - \dfrac{n}{m} \leq \nu(T_m,\, x) \leq \nu(T,\, x), \quad x \in X,\, m \in \mathbb{N}\,;$

$(c)\quad \displaystyle\int\limits_{X\setminus V_m} (T_m - \gamma + \frac{C}{m}\,\omega)^q \wedge \omega^{n-q} \leq C\,(\log m)^q, \qquad q = 1, \ldots, n = dim_{\mathbb{C}}X,$

for some constant $C > 0$ independent of m.

This result can be used to prove new characterizations of big line bundles in terms of curvature currents. Let us briefly review a few basic facts. A holomorphic line bundle L over a compact complex manifold X of dimension n is said to be big if $\dim_{\mathbb{C}} H^0(X,\, L^m) \geq C\, m^n$ for some constant $C > 0$ and for all large enough $m \in \mathbb{N}$. This amounts to the global sections of L^m defining a bimeromorphic embedding of X into a projective space, for $m >> 0$. The compact manifold X is said to be Moishezon if the transcendence degree of its meromorphic function field equals $n := \dim_{\mathbb{C}} X$, or equivalently, if there exist n global meromorphic functions that are algebraically independent. A Moishezon manifold becomes projective after finitely many blow-ups with smooth centres. There is, moreover, a bimeromorphic counterpart to Kodaira's embedding theorem : a compact complex manifold X is Moishezon if and only if there exists a big line bundle $L \to X$. The asymptotic growth of the dimension of $H^0(X,\, L^m)$ as $m \to +\infty$ is actually measured by a birational invariant of L, the *volume*, defined as :

$$v(L) := \limsup_{m \to +\infty} \frac{n!}{m^n}\, h^0(X,\, L^m).$$

Clearly, L is big if and only if $v(L) > 0$. Switching now to the analytic point of view, recall that a singular Hermitian metric h on L is defined in a local trivialization $L_{|U} \simeq U \times \mathbb{C}^n$ as $h = e^{-\varphi}$ for some weight function $\varphi : U \to [-\infty,\, +\infty)$ which is only assumed to be locally integrable. In particular, the singularity set $\{x \in U,\, \varphi(x) = -\infty\}$ is Lebesgue negligible. The associated curvature current $T := i\Theta_h(L)$ is a closed current of bidegree $(1,\, 1)$ on X representing the first Chern class $c_1(L)$ of L. It is locally defined as $T = dd^c\varphi$ for weight functions φ of h.

Recall that an almost positive current T can be locally written in coordinates as $T = \sum_{j,\,k} T_{j,\,k}\, dz_j \wedge d\bar{z}_k$ for some complex measures $T_{j,\,k}$. The Lebesgue decomposition of the coefficients $T_{j,\,k}$ into an absolutely continous part and a singular part with respect to the Lebesgue measure induces a current decomposition as $T = T_{ac} + T_{sing}$. By the Radon-Nicodym theorem, the coefficients of the absolutely continous part are L^1_{loc} and thus the exterior powers T^m_{ac} are well defined (though not necessarily closed) currents for $m = 1, \dots,\, n$.

When applied to curvature currents, the current regularization theorem 0.1.2 with controlled Monge-Ampère masses enables us to characterize the volume of a line bundle in terms of positive currents in $c_1(L)$. This gives in particular a bigness criterion for line bundles in terms of existence of singular Hermitian metrics satisfying positivity assumptions (and implicitly a characterization of Moishezon manifolds).

Theorem 0.1.3 *Let L be a holomorphic line bundle over a compact complex manifold X. Then the volume of L is characterized as :*

$$v(L) = \sup_{T\in c_1(L),\, T\geq 0} \int_X T_{ac}^n.$$

In particular, L is big if and only if there exists a possibly singular Hermitian metric h on L whose curvature current $T := i\Theta_h(L)$ satisfies the following positivity conditions :

$$(i) \quad T \geq 0 \quad \text{on } X\,; \qquad (ii) \quad \int_X T_{ac}^n > 0.$$

In the special case when the ambient manifold X is Kähler, this same result was obtained by Boucksom ([Bou02, Theorem 1.2]). This strengthens a previous (only sufficient) bigness criterion by Siu ([Siu85]) that solved affirmatively the Grauert-Riemenschneider conjecture ([GR70]). Condition (i) replaces Siu's stronger assumption that the curvature current T (or the metric h) be C^∞. Theorem 0.1.3 falls into the mould of ideas originating in Demailly's holomorphic Morse inequalities ([Dem85]). Its proof hinges on the regularization theorem 0.1.2 above and on Bonavero's singular version of Demailly's Morse inequalities ([Bon98]). It strengthens a bigness criterion in [Bon98] which required the curvature current T to have analytic singularities. On the other hand, Ji and Shiffman ([JS93]) proved that L being big is equivalent to L having a singular metric whose curvature current is strictly positive on X, (i.e. $\geq \varepsilon\omega$ for some small $\varepsilon > 0$). This implies, in particular, the "only if" part of the above Theorem 0.1.3. The thrust of the new "if" part of Theorem 0.1.3 is to relax the strict positivity assumption on the curvature current.

Let us finally stress that the main interest of Theorems 0.1.2 and 0.1.3 lies in X being an arbitrary compact manifold. Related results are known to exist for Kähler manifolds (e.g. [DP04, Theorem 0.4], [Bou02, Theorem 1.2]). The approach to the non-Kähler case treated here is quite different. The crux of the argument is modifying the existing procedure for regularizing $(1,1)$-currents to get an effective control on the Monge-Ampère masses (Theorem 0.1.2). If X is Kähler, the sequence of masses in the usual Demailly regularization of currents is easily seen to be bounded by applying Stokes's theorem and using the closedness of ω (see [Bou02). The situation is radically different in the non-Kähler case where a new regularization of currents is needed with a possibly unbounded, though only slowly growing, sequence of masses.

0.2 Preliminaries

Let us now focus on Theorem 0.1.1 and its setting described in the introduction. In this section we will be clearing the way to its proof. Fix $m \in \mathbb{N}^\star$ and $\delta > 0$. It is a trivial fact in one complex dimension that the upperlevel set for Lelong numbers :

$$E_{1-\delta}(m\, dd^c\varphi_0) := \{x \in \Omega\,;\, \nu(m\,\varphi_0,\, x) \geq 1-\delta\}$$

associated with a closed positive $(1,1)$-current, is analytic of dimension 0. Its intersection with a relatively compact disc is therefore finite. Let

$$E_{1-\delta}(m\, dd^c\varphi_0) \cap D(x_0,\, r) := \{a_1, \ldots,\, a_{p(m)}\}.$$

The function :

$$\psi_m(z) := m\, \varphi_0(z) - \sum_{j=1}^{p(m)} [m\, \nu(\varphi_0,\, a_j)]\, \log|z - a_j|,$$

is still subharmonic, since

$$dd^c\psi_m = m\, dd^c\varphi_0 - \sum_{j=1}^{p(m)} [m\, \nu(\varphi_0,\, a_j)]\, \delta_{a_j} \geq 0,$$

in the sense of currents. Moreover :

$$\nu(\psi_m,\, a_j) = m\, \nu(\varphi_0,\, a_j) - [m\, \nu(\varphi_0,\, a_j)], \qquad \text{for every } j.$$

Thus, after possibly replacing $m\, \varphi_0$ with ψ_m, and introducing a factor $(z - a_j)^{m_j}$ for every $[m\, \nu(\varphi_0,\, a_j)] \geq 1$ in the definition of f_m which is to be constructed in Theorem 0.1.1, we may assume in all that follows that :

$$m\, \nu(\varphi_0,\, x) < 1, \quad \text{for all} x \in D(x_0,\, r). \tag{2}$$

We have thus brought all point masses of the current $m\, dd^c\varphi_0$ below 1. Yet, there may still be some mass diffusely scattered over some regions of the disc $D(x_0,\, r)$ which could prevent the integral of $|f_m|^2\, e^{-2m\varphi_0}$ having the desired slow growth in m. The following lemma gives an upper bound for $e^{-2\varphi_0}$ in terms of the mass of the associated current $dd^c\varphi_0$. This lemma is akin to Skoda's result quoted in the introduction, but gives an estimate on a fixed disc instead of merely at a point.

Lemma 0.2.1 *With the notations in the introduction, if* $\gamma := \int_{D(x_0,\, r)} dd^c\varphi_0$, *the following estimate holds :*

$$e^{-2(\varphi_0(z) - h_0(z))} \leq \frac{1}{\int\limits_{D(x_0,\, r)} dd^c\varphi_0} \int\limits_{D(x_0,\, r)} \frac{1}{|\zeta - z|^{2\gamma}}\, dd^c\varphi_0(\zeta),$$

for all $z \in D(x_0,\, r)$.

Proof. Let $d\mu(\zeta) := \gamma^{-1}\, dd^c\varphi_0(\zeta)$ be a probability measure on $D(x_0,\, r)$. For all $z \in D(x_0,\, r)$, we have :

$$(\varphi_0 - h_0)(z) = \int_{D(x_0,\, r)} \log|\zeta - z|\, dd^c\varphi_0(\zeta),$$

or, equivalently,

$$-(\varphi_0 - h_0)(z) = \int_{D(x_0,\, r)} \gamma \log |\zeta - z|^{-1} d\mu(\zeta), \qquad z \in D(x_0,\, r).$$

Now, Jensen's convexity inequality entails :

$$e^{-2(\varphi_0 - h_0)(z)} \leq \int_{D(x_0,\, r)} e^{2\gamma \log|\zeta - z|^{-1}} d\mu(\zeta) = \gamma^{-1} \int_{D(x_0,\, r)} \frac{1}{|\zeta - z|^{2\gamma}} dd^c\varphi_0(\zeta).$$

This proves the lemma. $\square$

Lemma 0.2.1, when applied to the function $m\,\varphi_0$, yields the estimate :

$$e^{-2m(\varphi_0(z) - h_0(z))} \leq \frac{1}{\int\limits_{D(x_0,\, r)} dd^c\varphi_0} \int\limits_{D(x_0,\, r)} \frac{1}{|\zeta - z|^{2m\gamma}}\, dd^c\varphi_0(\zeta),$$

for all $z \in D(x_0,\, r)$.

The right-hand term of this inequality may not be integrable as a function of z when $m\gamma > 1$. To get around this, we will cut the disc $D(x_0,\, r)$ into pieces, each having a mass strictly less than 1 for the measure $m\, dd^c\varphi_0$. We thus see that it is essential for $m\, dd^c\varphi_0$ not to have point masses larger than 1. The number of pieces must not exceed $m\gamma(1+\delta)$. We will subsequently choose a point in each piece, intuitively its "centre", and will consider a holomorphic function on $D(x_0,\, r)$ whose only zeroes are these points. This is the desired function f_m of Theorem 0.1.1. The number of its zeroes does not exceed $m\gamma(1+\delta)$, by construction. Calculations developed in the next section 0.3 will show that the L^2 growth of f_m with weight $e^{-2m\,\varphi_0}$ is at most of order $o(m)$ as $m \to +\infty$.

Cutting the disc into pieces relies on the following lemma due to Yulmukhametov ([Yul85]) and, in a generalized form, to Drasin ([Dra01, Theorem 2.1]). It describes an atomization procedure for arbitrary positive measures μ in one complex variable. This is the main technical ingredient in the proof of Theorem 0.1.1.

Lemma 0.2.2 *([Yul85]; [Dra01].) Let μ be a positive measure supported in a square $R \subset \mathbb{R}^2$ with sides parallel to the coordinate axes. Suppose $\mu(R) = N > 1$, $N \in \mathbb{Z}$. Then, there exists a family of closed rectangles $(R_j)_{1 \leq j \leq N}$ with sides parallel to the coordinate axes, and a family of positive measures $(\mu_j)_{1 \leq j \leq N}$, such that :*

(a) $\mu = \sum_{j=1}^{N} \mu_j, \quad \mu_j(\mathbb{R}^2) = 1,$ *and* $\mathrm{Supp}\mu_j$ *is a convex subet of* R_j *;*

(b) $R = \bigcup_{j=1}^{N} R_j = \bigcup_{j=1}^{N} \mathrm{Supp}\, \mu_j$ *;*

(c) the interiors of the supports of the μ_j's are mutually disjoint;

(d) the ratio of the sides of each rectangle R_j lies in the interval $[\frac{1}{3}, 3]$ (i. e. R_j is an "almost square" in the terminology of [Dra01]);

(e) each point in $\mathbb{R}^2$ belongs to the interior of at most four distinct rectangles R_j;

(f) each $\operatorname{Supp}\mu_j$ is a rectangle, and the distance between the centres of any two such distinct rectangles is $\geq \frac{C}{N^2}$, where $C > 0$ is the side of the square R.

Idea of proof (according to [Dra01]). Yulmukhametov originally proved this result (see [Yul85]) for absolutely continuous measures μ. The generalization to the case of arbitrary measures is due to Drasin ([Dra01]). We summarize here the ideas of Drasin's proof. Conclusion (f) was not explicitly stated, but it can be easily inferred from the proof given there. The first idea is to reduce the problem to the case of a measure μ satisfying $\mu(p) < 1$ at every point $p \in R$. This is done by subtracting from the original measure μ the integer part $[\mu(p)]$ of each point mass $\mu(p) > 1$. We may also assume, after a possible rotation of the coordinate system of $\mathbb{R}^2$, that for every line L parallel to one of the coordinate axes, there exists at most one point $p \in L$ such that $\mu(p) > 0$, while $\mu(L \setminus p) = 0$.

After these reductions, the key step is to prove that if an almost square R contains the support of a measure μ satisfying these properties, then there exist almost squares R_0 and R_1 and a decomposition $\mu = \mu_0 + \mu_1$ such that $\operatorname{Supp}\mu_j \subset R_j$, $j = 0, 1$, which satisfies conclusions $(b) - (d)$ of the lemma. The masses $\mu_j(R_j)$ are integers. If $\mu_j(R_j) > 1$, we repeat this procedure to obtain almost squares $R_{j,0}$, $R_{j,1}$ and a decomposition $\mu_j = \mu_{j,0} + \mu_{j,1}$. By repeatedly applying this procedure we get almost squares R_I and measures μ_I, indexed over multiindices $I = i_1, \ldots, i_l$ made up of digits 0 and 1. The procedure terminates when all masses $\mu_I(R_I) = 1$. A technical lemma then yields conclusion (e) and thus clinches the proof of this result. We refer for details to Drasin ([Dra01, §2, p. 165-171]). $\square$

0.3 Proof of Theorem 0.1.1.

Building on the preliminary developments in the previous section, we will now complete the proof of Theorem 0.1.1. The notation and set-up are unchanged. As previously explained, we may assume that $m\,\nu(\varphi_0, x) < 1$, for all $x \in D(x_0, r)$ (hypothesis (2)). Consider a square $R \subset \mathbb{C}$ of edge $2r$ which contains $D(x_0, r)$, and the positive measure $\mu := dd^c\varphi_0$ on R of total mass γ (or some $\gamma' > \gamma$.) Fix $0 < \delta < 1$ and, for $m >> 0$, choose an integer N_m such that

$$\frac{2}{2-\delta}\, m\,\gamma < N_m < m\,\gamma\,(1+\delta). \tag{3}$$

Such an integer exists as soon as $m\,\gamma\,(1+\delta-\frac{2}{2-\delta}) = m\,\gamma\,\frac{\delta(1-\delta)}{2-\delta} > 1$. We now apply the atomization lemma 0.2.2 to the measure $\frac{N_m}{\gamma}\,\mu = \frac{N_m}{\gamma}\,dd^c\varphi_0$ of total mass $N := N_m$. We thus get a covering of $D(x_0,\, r)$ by closed rectangles (almost squares) $R_j = R_j(m)$, and a decomposition $\frac{N_m}{\gamma}\,\mu = \sum\limits_{j=1}^{N_m}\nu_{m,\,j}$ such that $\nu_{m,\,j}(R_j) = 1$, satisfying the conclusions of Lemma 0.2.2. Set $\mu_{m,\,j} := \frac{m\,\gamma}{N_m}\,\nu_{m,\,j}$ and get a decomposition :

$$m\,\mu = dd^c(m\,\varphi_0) = \sum_{j=1}^{N_m}\mu_{m,\,j}, \qquad \text{with } \mu_{m,\,j}(R_j) = \tfrac{m\,\gamma}{N_m} \in (1-\delta,\, 1-\tfrac{\delta}{2}).$$

For every m, consider the rectangle $P_j = P_j(m) = \mathrm{int}\,(\mathrm{Supp}\,\mu_{m,\,j}) \subset R_j$, and let $a_j = a_j(m)$ be its centre. We will prove that the integer N_m and the points a_j satisfy the conclusions of Theorem 0.1.1. Hypothesis (2) implies :

$$m_j = \max\{[m\,\nu(\varphi_0,\, a_j)],\, 1\} = 1, \qquad \text{for } j = 1, \ldots,\, N_m,$$

which further gives : $\sum\limits_{j=1}^{N_m} m_j = N_m < m\gamma(1+\delta)$, being precisely the conclusion (i) of Theorem 0.1.1.

Conclusion (f) of Lemma 0.2.2 and the choice of N_m ensure that the points a_j satisfy the conclusion (ii) of Theorem 0.1.1.

Let us now consider the holomorphic function :

$$f_m(z) := e^{m\,g_0(z)}\prod_{j=1}^{N_m}(z-a_j), \qquad z \in D(x_0,\, r),$$

and let us study the growth in m of $\int\limits_{D(x_0,\,r)} |f_m|^2\, e^{-2m\varphi_0}\, d\lambda$. Since

$$\int\limits_{D(x_0,\,r)} |f_m|^2\, e^{-2m\varphi_0}\, d\lambda \leq \sum_{j=1}^{N_m}\int\limits_{P_j} |f_m|^2\, e^{-2m\varphi_0}\, d\lambda,$$

the analysis is reduced to finding a convenient upper bound for each integral on P_j. Fix $j \in \{1, \ldots, N_m\}$. Since $P_j \cap \bar{P}_k = \emptyset$, for all $j \neq k$, we get :

$$m\,\mu(P_j) = \mu_{m,\,j}(P_j) \leq \mu_{m,\,j}(R_j) = \tfrac{m\gamma}{N_m} < 1 - \tfrac{\delta}{2}.$$

We may assume, without loss of generality, that P_j is a disc $D(a_j,\, r_j)$. Conclusion (e) of Lemma 0.2.2 implies that the sum of the Euclidian areas of the P_j's is bounded above by four times the area of the square R of edge $2r$. This means that there is a constant $C_1(r) > 0$, depending only on r, such that

$$(e') \qquad \sum_{j=1}^{N_m} r_j^2 \leq C_1(r), \qquad \text{for all } m >> 0.$$

Lemma 0.2.1, when applied to the function $m\,\varphi_0$ on $P_j = D(a_j,\, r_j)$, yields the following estimate :

$$\begin{aligned} |f_m(z)|^2\, e^{-2m\varphi_0(z)} &= \prod_{k=1}^{N_m} |z-a_k|^2\, e^{-2m(\varphi_0(z)-h_0(z))} \\ &\leq (2r)^{2(N_m-1)}\, |z-a_j|^2\, \frac{1}{\int\limits_{P_j} dd^c\varphi_0} \int\limits_{P_j} \frac{1}{|\zeta - z|^{2\frac{m\gamma}{N_m}}}\, dd^c\varphi_0(\zeta), \end{aligned}$$

for all $z \in P_j$. We have used the obvious upper bound $|z-a_k|^2 \leq (2r)^2$, for all $k \neq j$.

When integrating above with respect to $z \in P_j$, Fubini's theorem yields :

$$(0.3.1) \quad \int\limits_{P_j} |f_m(z)|^2\, e^{-2m\varphi_0(z)} d\lambda(z) \leq \frac{(2r)^{2(N_m-1)}}{\int\limits_{P_j} dd^c\varphi_0} \int\limits_{P_j} \left(\int\limits_{P_j} \frac{|z-a_j|^2}{|z-\zeta|^{2\frac{m\gamma}{N_m}}} d\lambda(z) \right) dd^c\varphi_0(\zeta).$$

Let us now concentrate on the integral in z on the right-hand side. We get te following estimate :

$$(0.3.2) \quad \int\limits_{P_j} \frac{|z-a_j|^2}{|z-\zeta|^{2\frac{m\gamma}{N_m}}} d\lambda(z) = \int\limits_{D(a_j, r_j)} \frac{|z-a_j|^2}{|(z-a_j)-(\zeta-a_j)|^{2\frac{m\gamma}{N_m}}} d\lambda(z-a_j)$$

$$\leq 4\pi(|\zeta - a_j| + r_j)^{2(1-\frac{m\gamma}{N_m})} \cdot \left(\frac{(|\zeta-a_j|+r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{|\zeta-a_j|^2}{2(1-\frac{m\gamma}{N_m})} \right), \forall \zeta \in P_j.$$

Indeed, if we make the change of variable $x = z - a_j$ and set $\zeta - a_j = a$, we are reduced to estimating the integral :

$$\int\limits_{D(0,r)} \frac{|x|^2}{|x-a|^\tau} d\lambda(x),$$

where we have set $r_j := r$ and $\tau := 2\frac{m\gamma}{N_m}$ to simplify the notation. By the choice (3) of N_m, we get : $0 < \tau < 2$. The change of variable $x - a = y$, followed by a switch to polar coordinates with $|y| = \rho$, implies :

$$\begin{aligned}
\int\limits_{D(0,r)} \frac{|x|^2}{|x-a|^\tau} d\lambda(x) &= \int\limits_{D(-a,r)} \frac{|y+a|^2}{|y|^\tau} d\lambda(y) \le \int\limits_{D(-a,r)} \frac{(|y|+|a|)^2}{|y|^\tau} d\lambda(y) \\
&\le 2 \int\limits_{D(-a,r)} \frac{|y|^2+|a|^2}{|y|^\tau} d\lambda(y) \\
&= 2 \int\limits_{D(-a,r)} |y|^{2-\tau} d\lambda(y) + 2|a|^2 \int\limits_{D(-a,r)} |y|^{-\tau} d\lambda(y) \\
&\le 2\pi \cdot \left(2 \int_0^{|a|+r} \rho^{2-\tau} \rho \, d\rho + 2|a|^2 \int_0^{|a|+r} \rho^{-\tau} \rho \, d\rho \right) \\
&= 4\pi \, (|a|+r)^{2-\tau} \left(\frac{(|a|+r)^2}{4-\tau} + \frac{|a|^2}{2-\tau} \right).
\end{aligned}$$

For $r = r_j$, this gives the estimate (0.3.2). Relations (0.3.1) and (0.3.2) imply :

$$\begin{aligned}
&\int\limits_{P_j} |f_m(z)|^2 \, e^{-2m\varphi_0(z)} d\lambda(z) \le \\
&\frac{4\pi}{\int_{P_j} dd^c\varphi_0} (2r)^{2(N_m-1)} \int_{P_j} (|\zeta - a_j| + r_j)^{2(1-\frac{m\gamma}{N_m})} \left(\frac{(|\zeta - a_j| + r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{|\zeta - a_j|^2}{2(1-\frac{m\gamma}{N_m})} \right) dd^c\varphi_0(\zeta).
\end{aligned}$$

Let us now shift to polar coordinates with $|\zeta - a_j| = \rho$. This implies that $dd^c\varphi_0(\zeta) = dn(\rho)$, where $n(\rho) = \int_{D(a_j,\,\rho)} dd^c\varphi_0$, for all $\rho \ge 0$. Since P_j is assumed to be $D(a_j,\, r_j)$, we get :

$$\begin{aligned}
&\int\limits_{P_j} |f_m(z)|^2 \, e^{-2m\varphi_0(z)} d\lambda(z) \le \\
&\le C(r,\, r_j) \int_0^{r_j} (\rho + r_j)^{2(1-\frac{m\gamma}{N_m})} \left(\frac{(\rho + r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{\rho^2}{2(1-\frac{m\gamma}{N_m})} \right) n'(\rho) \, d\rho,
\end{aligned}$$

where $C(r,\, r_j) = \dfrac{8\pi^2}{\int_{P_j} dd^c\varphi_0} (2r)^{2(N_m-1)}$. The last expression can be successively written as :

$$\frac{C(r,\, r_j)}{2(2-\frac{m\gamma}{N_m})}\cdot\int_0^{r_j}(\rho+r_j)^{2(2-\frac{m\gamma}{N_m})}n'(\rho)\,d\rho+\frac{C(r,\, r_j)}{2(1-\frac{m\gamma}{N_m})}\cdot\int_0^{r_j}\rho^2(\rho+r_j)^{2(1-\frac{m\gamma}{N_m})}n'(\rho)\,d\rho$$

$$=\frac{C(r,\, r_j)}{2(2-\frac{m\gamma}{N_m})}\left(n(r_j)(2r_j)^{2(2-\frac{m\gamma}{N_m})}-2\left(2-\frac{m\gamma}{N_m}\right)\int_0^{r_j}n(\rho)(\rho+r_j)^{3-2\frac{m\gamma}{N_m}}\,d\rho\right)+$$

$$+\frac{C(r,\, r_j)}{2(1-\frac{m\gamma}{N_m})}\left(n(r_j)r_j^2(2r_j)^{2(1-\frac{m\gamma}{N_m})}-\right.$$

$$\left.-\int_0^{r_j}n(\rho)\left[2\rho(\rho+r_j)^{2(1-\frac{m\gamma}{N_m})}+2\left(1-\frac{m\gamma}{N_m}\right)(\rho+r_j)^{1-2\frac{m\gamma}{N_m}}\rho^2\right]\,d\rho\right).$$

Since the terms appearing with a " $-$" sign are all negative, for $1-\frac{m\gamma}{N_m}>0$ and therefore $2-\frac{m\gamma}{N_m}>0$, they can be ignored. We thus get the following upper estimate :

$$\begin{aligned}\int\limits_{P_j}|f_m(z)|^2\,e^{-2m\varphi_0(z)}d\lambda(z) \ &\leq \\ &\leq \ C(r,\, r_j)\cdot n(r_j)\cdot\left(\frac{(2r_j)^{2(2-\frac{m\gamma}{N_m})}}{2(2-\frac{m\gamma}{N_m})}+\frac{r_j^2(2r_j)^{2(1-\frac{m\gamma}{N_m})}}{2(1-\frac{m\gamma}{N_m})}\right).\end{aligned}$$

Since $n(r_j)=\int_{P_j}dd^c\varphi_0$, the previous upper bound and the formula of $C(r,\, r_j)$ show that :

$$(0.3.3)\qquad \int\limits_{P_j}|f_m(z)|^2\,e^{-2m\varphi_0(z)}d\lambda(z)\leq C\left(r,\,\frac{m\gamma}{N_m}\right)\cdot r_j^{2(2-\frac{m\gamma}{N_m})},$$

where the constant $C(r,\,\frac{m\gamma}{N_m})$ is given by the formula :

$$C\left(r,\,\frac{m\gamma}{N_m}\right)=8\pi^2\,(2r)^{2(N_m-1)}\left(\frac{2^{2(2-\frac{m\gamma}{N_m})}}{2(2-\frac{m\gamma}{N_m})}+\frac{2^{2(1-\frac{m\gamma}{N_m})}}{2(1-\frac{m\gamma}{N_m})}\right).$$

Since the estimate (0.3.3) holds for all indices $j\in\{1,\dots,\,N_m\}$, we get, after summing over j, that :

$$\int_{D(x_0,\,r)}|f_m(z)|^2\,e^{-2m\varphi_0(z)}\,d\lambda(z)\leq C\big(r,\,\frac{m\gamma}{N_m}\big)\sum_{j=1}^{N_m}r_j^{2(2-\frac{m\gamma}{N_m})}.$$

The choice of N_m was made in such a way that $1-\delta < \frac{1}{1+\delta} < \frac{m\gamma}{N_m} < 1-\frac{\delta}{2}$ (cf. (3)), which implies :

$$\frac{\delta}{2} < 1-\frac{m\gamma}{N_m} < \delta \quad \text{and} \quad 1+\frac{\delta}{2} < 2-\frac{m\gamma}{N_m} < 1+\delta.$$

Since $0 < 2r < 1$, there exists a constant $C_2(r) > 0$ depending only on r, such that $C(r, \frac{m\gamma}{N_m}) \leq C_2(r)$, for all $m \in \mathbb{N}$. Since $r_j \leq 2r < 1$, we have :

$$r_j^{2(2-\frac{m\gamma}{N_m})} < r_j^2, \quad \text{for} \quad 2(2-\tfrac{m\gamma}{N_m}) > 2.$$

Thus, the estimate (e') (inferred above from (e) of lemma 0.2.2) implies :

$$\int_{D(x_0,\, r)} |f_m(z)|^2\, e^{-2m\varphi_0(z)}\, d\lambda(z) \leq C(r), \quad \forall m >> 0,$$

where $C(r) = C_1(r)\, C_2(r) > 0$ is a constant depending only on the radius r of the disc $D(x_0,\, r)$ on which we are working. This yields conclusion (iii) of Theorem 0.1.1 and completes its proof. $\square$

0.4 Local regularization with mass control

In this section, we will use Theorem 0.1.1, combined with the Ohsawa-Takegoshi L^2 extension theorem (see [OT87], [Ohs88]), to introduce a new local approximation procedure for psh functions with zero Lelong numbers. The main new outcome is an additional control of the Monge-Ampère masses. This can be seen as a local version of Theorem 0.1.2 under the stronger assumption that all the Lelong numbers vanish.

Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. A well-known result by Demailly (cf. [Dem92, Proposition 3.1]) asserts that φ can be approximated pointwise and in $L^1_{loc}(\Omega)$ topology by psh functions φ_m with analytic singularities (see definition (1) in the introduction), constructed as :

$$\varphi_m = \frac{1}{2m}\,\log\sum_{j=0}^{+\infty} |\sigma_{m,\,j}|^2, \tag{4}$$

where $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$ of holomorphic functions f on Ω such that $|f|^2\, e^{-2m\varphi}$ is integrable on Ω. They even satisfy the estimates :

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z|<r} \varphi(\zeta) + \frac{1}{m}\,\log\frac{C_2}{r^n}, \tag{5}$$

for every $z \in \Omega$ and every $r < d(z,\, \partial\Omega)$. In particular, the sequence $dd^c\varphi_m$ converges to $dd^c\varphi$ in the weak topology of currents, and the corresponding Lelong numbers satisfy :

$$\nu(\varphi, x) - \frac{n}{m} \leq \nu(\varphi_m, x) \leq \nu(\varphi, x), \quad x \in \Omega. \tag{6}$$

For analytic singularities, the Lelong number $\nu(\varphi_m, x)$ at an arbitrary point x equals $\frac{1}{m} \min_{j \geq 0} \mathrm{ord}_x \sigma_{m,j}$, where ord_x is the vanishing order at x. The sequence $(\varphi_m)_{m \in \mathbb{N}}$ defined in (4) has come to be referred to as the Demailly approximation of φ.

Let us now suppose that φ has zero Lelong numbers everywhere (see [Dem97, chapter III] for a comprehensive discussion of Lelong numbers). In other words,

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(x)}{\log |z - x|} = 0, \qquad \text{for every } x \in \Omega.$$

Psh functions φ for which there are points x such that $\varphi(x) = -\infty$ and $\nu(\varphi, x) = 0$ do exist! For instance, $\varphi(z) := -\sqrt{-\log|z|}$ has an isolated singularity with a zero Lelong number at the origin. These singularities, very different to analytic ones, are usually hard to grasp as the familiar tools at hand intended to handle singularities, viz. multiplier ideal sheaves and Lelong numbers, are trivial at such points.

We can alter the Demailly approximation to get the following Monge-Ampère mass control.

Theorem 0.4.1 *Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Suppose, furthermore, that φ has a zero Lelong number at every point $x \in \Omega$. Then, there exists a sequence $(\psi_m)_{m \in \mathbb{N}}$ of smooth psh functions on Ω such that $dd^c \psi_m$ converges to $dd^c \varphi$ in the weak topology of currents as $m \to +\infty$, and such that, for any relatively compact open subset $B \subset\subset \Omega$, we have :*

$$\int_B (dd^c \psi_m)^q \wedge \beta^{n-q} \leq C\, (\log m)^q, \qquad q = 1, \ldots, n,$$

where β is the standard Kähler form on $\mathbb{C}^n$, and $C > 0$ is a constant independent of m.

Proof. The idea is to modify Demailly's original construction of regularizing functions by taking into account not only the elements $\sigma_{m,j}$ in an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, but also their first order partial derivatives. Set therefore :

$$\psi_m := \frac{1}{2m} \log \left(\sum_{j=0}^{+\infty} |\sigma_{m,j}|^2 + \sum_{j=0}^{+\infty} \left| \frac{\partial \sigma_{m,j}}{\partial z_1} \right|^2 + \cdots + \sum_{j=0}^{+\infty} \left| \frac{\partial \sigma_{m,j}}{\partial z_n} \right|^2 \right),$$

where $z = (z_1, \ldots, z_n)$ is the standard coordinate on $\mathbb{C}^n$. We can easily infer from Demailly's estimate (5) combined with Parseval's formula that :

$$\varphi(z) - \frac{C_1}{m} \leq \psi_m(z) \leq \sup_{|\zeta - z| < 2r} \varphi(\zeta) - \frac{1}{m}\, \log r + \frac{1}{m} \log \frac{C_3}{r^n}, \tag{7}$$

for every $z \in \Omega$ and every $r < \frac{1}{2}\, d(z,\, \partial\Omega)$. This means that ψ_m still converges to φ pointwise and in $L^1_{loc}(\Omega)$ topology, and thus $dd^c\psi_m$ converges to $dd^c\varphi$ in the weak topology of currents. Moreover, as the Lelong numbers of φ are assumed to be zero at every point, and as, thanks to (6),

$$\nu(\psi_m,\, x) \leq \nu(\varphi_m,\, x) \leq \nu(\varphi,\, x), \qquad x \in \Omega,$$

for every m, each ψ_m has zero Lelong numbers everywhere. This means that the $\sigma_{m,\,j}$'s and their first order derivatives have no common zeroes, and therefore ψ_m is C^∞ on Ω.
Our aim is to control the Monge-Ampère masses of the new regularizing smooth forms $dd^c\psi_m$ on a given open set $B \subset\subset \Omega$. To this end, we can apply the Chern-Levine-Nirenberg inequalities (see [CLN69] or [Dem97, chapter III, page 168]) to get :

$$\int_B (dd^c\psi_m)^q \wedge \beta^{n-q} \leq C\, (\sup_{\tilde{B}} |\psi_m|)^q, \qquad q = 1, \ldots,\, n,$$

where $\tilde{B} \subset\subset \Omega$ is an arbitrary relatively compact open subset containing $\bar{B}$, and $C > 0$ is a constant depending only on B and $\tilde{B}$. Note that $\sup_{\tilde{B}} |\psi_m| < +\infty$ since ψ_m is smooth. The proof is then reduced to settling the following.

Claim 0.4.2 *There is a constant $C > 0$ independent of m such that*

$$\sup_{\tilde{B}} |\psi_m| \leq C\, \log m, \qquad \textit{for every } m.$$

The upper bound for ψ_m given in (7) is clearly sufficient for our purposes. The delicate point in estimating $|\psi_m|$ is finding a finite lower bound (possibly greatly negative) for ψ_m. If $B_m(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$, it is easy to see that :

$$\psi_m(z) \geq \sup_{F_m \in B_m(1)} \frac{1}{2m}\, \log \sum_{|\alpha| \leq 1} |D^\alpha F_m(z)|^2, \quad z \in \Omega, \tag{8}$$

where D^α stands for the partial derivative with respect to the multi-index $\alpha = (\alpha_1, \ldots,\, \alpha_n)$, and $|\alpha|$ is the length of the multi-index.
Fix $x \in \Omega$. To find a uniform lower bound for $\psi_m(x)$, we need produce an element $F_m \in B_m(1)$ for which we can uniformly estimate below one of the first order partial derivatives at x. The Lelong number of φ at x is known to be equal to the Lelong number at x of the restriction $\varphi_{|L}$ to almost every complex line L passing through x (see [Siu74]). Choose such a line L and

coordinates $z = (z_1, \ldots, z_n)$ centred at x such that $L = \{z_2 = \cdots = z_n\}$. Consider, as in the introduction, the decompostion :

$$\varphi_{|L} = N \star \Delta\varphi_{|L} + h, \qquad \text{on } \Omega \cap L,$$

where N is the one-dimensional Newton kernel, and $h = \mathrm{Re}\, g$ is a harmonic function equal to the real part of a holomorphic function g. Theorem 0.1.1 gives the existence of a holomorphic function f_m on $\Omega \cap L$ such that :

$$f(z_1) = e^{mg(z_1)} \prod_{j=1}^{N_m} (z_1 - a_{m,\,j}), \quad z_1 \in \Omega \cap L,$$

with $N_m \leq C_0\, m$, for a constant $C_0 > 0$ independent of m, and

$$C_m := \int\limits_{\Omega \cap L} |f_m|^2\, e^{-2m\varphi}\, dV_L = o(m).$$

where dV_L is the volume form on L. As the Lelong numbers of φ (and implicitly those of $m\varphi$) are assumed to be zero, the retriction of $e^{-2m\varphi}$ to L is locally integrable on $\Omega \cap L$. Conclusion (ii) of Theorem 0.1.1 shows that the points $a_{m,\,j}$ can be chosen such that $|a_{m,\,j} - a_{m,\,k}| \geq \frac{C_1}{m^2}$ for some constant $C_1 > 0$ independent of m and L.

The Ohsawa-Takegoshi L^2 extension theorem (cf. [Ohs88, Corollary 2, p. 266]) can now be applied to get a holomorphic extension $F_m \in \mathcal{H}_\Omega(m\varphi)$ of f_m from the line $\Omega \cap L$ to Ω, satisfying the estimate :

$$\int\limits_{\Omega} |F_m|^2\, e^{-2m\varphi} \leq C \int\limits_{\Omega \cap L} |f_m|^2\, e^{-2m\varphi} = C\, C_m,$$

for a constant $C > 0$ depending only on Ω and n. Thus the function $\frac{F_m}{\sqrt{C\, C_m}}$ belongs to the unit ball $B_m(1)$ of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$. Thanks to (8), we get the following lower bound for ψ_m :

$$\psi_m(z_1) \geq \tfrac{1}{m} \log |f'_m(z_1)| - \tfrac{1}{2m} \log(C\, C_m), \qquad z_1 \in \tilde{B} \cap L.$$

In particular, for $z_1 = a_{m,\,j}$, we get :

$$\begin{aligned} \psi_m(a_j) \;&\geq\; h(a_j) + \tfrac{1}{m} \sum_{k \neq j} \log |a_{m,\,k} - a_{m,\,j}| - \tfrac{1}{2m} \log(C\, C_m) \\ &\geq\; h(a_j) + \tfrac{N_m - 1}{m}\, \log \tfrac{C_1}{m^2} - \tfrac{1}{2m} \log(C\, C_m). \end{aligned}$$

Since h is C^∞ (for it is harmonic), it is locally bounded (by constants independent of L). Therefore, there exists a constant $C_2 > 0$ independent of m and L such that $\psi_m \geq -C_2 \log m$ on $\tilde{B} \cap L$ for every m. In particular, $\psi_m(x) \geq -C_2 \log m$. This proves the claim 0.4.2 and completes the proof of

Theorem 0.4.1. $\square$

Remark 0.4.3 Theorem 0.4.1 constructs a regularization of currents for which the Monge-Ampère masses have an at most slow (logarithmic) growth. It is worth stressing that the sequence of these masses may not bounded above. To see this, suppose φ is C^∞ in the complement of an analytic set $V \subset \Omega$. If $(\varphi_m)_{m\in\mathbb{N}}$ is the Demailly approximation of φ, it is shown in [DPS01, p. 701-702] that the sequence $(\varphi_{2^m} + 2^{-m})_{m\in\mathbb{N}}$ is decreasing using an effective version of the subadditivity property of multiplier ideal sheaves. The same proof shows that the corresponding sequence $(\psi_{2^m} + 2^{-m})_{m\in\mathbb{N}}$ in the new regularization defined in the previous theorem is also decreasing. Then the C^∞ (1, 1)-forms $(dd^c\psi_{2^m})^q$ are well-defined on $\Omega \setminus V$ and converge in the weak topology of currents to $(dd^c\varphi)^q$ for every $q = 1, \ldots, n$ (see [Dem97, chapter III, Theorem 3.7]). If $K \subset\subset B \setminus V$ and $0 \leq \chi \leq 1$ is a C^∞ function with compact support in $B \setminus V$ such that $\chi \equiv 1$ on K, then :

$$\int_{B\setminus V} \chi(dd^c\psi_{2^m})^q \wedge \beta^{n-q} \leq \int_{B\setminus V} (dd^c\psi_{2^m})^q \wedge \beta^{n-q}, \quad m \in \mathbb{N},$$

and taking $\liminf_{m\to+\infty}$, the weak convergence implies :

$$\int_{B\setminus V} \chi(dd^c\varphi)^q \wedge \beta^{n-q} \leq \liminf_{m\to+\infty} \int_{B\setminus V} (dd^c\psi_{2^m})^q \wedge \beta^{n-q}, \quad q = 1, \ldots, n.$$

Clearly $\int_K (dd^c\varphi)^q \wedge \beta^{n-q} \leq \int_{B\setminus V} \chi(dd^c\varphi)^q \wedge \beta^{n-q}$, and letting $K \subset\subset B \setminus V$ increase, we get :

$$\int_{B\setminus V} (dd^c\varphi)^q \wedge \beta^{n-q} \leq \liminf_{m\to+\infty} \int_{B\setminus V} (dd^c\psi_{2^m})^q \wedge \beta^{n-q}, \quad q = 1, \ldots, n.$$

Now, there are examples of psh functions φ for which the Monge-Ampère mass in the left-hand side above is infinite for $q = n$ (see Kiselman's example in [Kis84, p.141-143] of a φ with zero Lelong numbers, or the Shiffman-Taylor example in [Siu 75, p.451-453]). Thus the last inequality shows that for such functions the sequence of Monge-Ampère masses associated with the above regularization is unbounded.

0.5 Global regularization with mass control

In this section we patch together the local regularizations constructed in the previous section to prove Theorem 0.1.2 under the extra assumption that the original current T has vanishing Lelong numbers everywhere. For the sake of simplicity we assume X is compact. The result actually holds for any manifold X which can be covered by finitely many coordinate patches on which the local regularization theorem 0.4.1 can be applied.

Theorem 0.5.1 *Let $T \geq \gamma$ be a d-closed current of bidegree $(1,\,1)$ on a compact complex manifold X, where γ is a continuous $(1,\,1)$-form such that $d\gamma = 0$. Assume T has a zero Lelong number at every point in X. Then, there exist C^∞ $(1,\,1)$-forms T_m in the same $\partial\bar\partial$-cohomology class as T which converge to T in the weak topology of currents and satisfy :*

$$(a) \quad T_m \geq \gamma - \frac{C}{m}\,\omega\,;$$

$$(b) \quad \int\limits_X (T_m - \gamma + \frac{C}{m}\,\omega)^q \wedge \omega^{n-q} \leq C\,(\log m)^q, \qquad q = 1, \ldots, n = dim_{\mathbb{C}} X,$$

for a fixed Hermitian metric ω on X and some $C > 0$ independent of m.

Proof. As the patching procedure is essentially well-known (see, for instance, [Dem92] or [Pop04]), we will only outline the main points and new aspects.

Let $T = \alpha + dd^c\varphi \geq \gamma$ be a closed almost positive current of bidegree $(1,\,1)$ on a compact Hermitian manifold $(X,\,\omega)$. Assume T has a zero Lelong number at every point in X. The set-up is the one described in the introduction. After possibly replacing T with $T - \alpha$ and γ with $\gamma - \alpha$, we can assume $T = dd^c\varphi \geq \gamma$. Let us fix $\delta > 0$, and four finite coverings of X by concentric coordinate balls $(B_j^{(3)})_j$, $(B_j')_j$, $(B_j'')_j$ and $(B_j)_j$ of radii $\frac{\delta}{2}$, δ, $\frac{3}{2}\delta$, and respectively 2δ. Since $d\gamma = 0$, γ is locally exact and we can assume that, for every j, $\gamma = dd^c h_j$ on B_j for some C^∞ function h_j. The function :

$$\psi_j := \varphi - h_j,$$

is psh on B_j for every j. Theorem 0.4.1 can then be applied to each ψ_j on B_j to get approximations with analytic singularities :

$$\psi_{j,\,m} := \frac{1}{2m}\,\log\left(\sum_{l=0}^{+\infty}|\sigma_{j,\,m,\,l}|^2 + \sum_{r=1}^{n}\sum_{l=0}^{+\infty}\left|\frac{\partial\sigma_{j,\,m,\,l}}{\partial z_r}\right|^2\right),$$

with an arbitrary orthonormal basis $(\sigma_{j,\,m,\,l})_{l\in\mathbb{N}}$ of the Hilbert space $\mathcal{H}_{B_j}(m\psi_j)$ (see notation in the previous section). Then $\varphi_{j,m} := \psi_{j,m} + h_j$ converges pointwise and in L^1_{loc} topology to φ as $m \to +\infty$ on B_j, and these local approximations can be glued together into a global approximation of φ defined as :

$$\varphi_m(z) := \sup_{B_j'' \ni z}\left(\varphi_{j,\,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^j|^2)\right),$$

with a constant $C_1(\delta) > 0$ depending only on δ which will be specified below, and a local holomorphic coordinate system z^j centred at the centre of B_j. The currents $T_m := dd^c\varphi_m$ satisfy the conclusions of Theorem 0.5.1 if the following patching condition holds :

$$\varphi_{j,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^j|^2) \le \varphi_{k,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^k|^2), \tag{9}$$

for $z \in (\overline{B}''_j \setminus B'_j) \cap B^{(3)}_k$. One can then prove the existence of a constant $C_1(\delta) > 0$ satisfying this patching condition by means of Hörmander's L^2 estimates ([Hor65]). One need only estimate the difference $\psi_{j,m} - \psi_{k,m}$ on $B''_j \cap B''_k$ and show that $\varphi_{j,m} - \varphi_{k,m}$ is uniformly bounded above on $B''_j \cap B''_k$ by $\mathcal{O}(\frac{1}{m})$ as $m \to +\infty$. Now, for every fixed $z \in B_j$, the norms of the linear maps $f \mapsto f(z)$ and $f \mapsto \frac{\partial f}{\partial z_r}(z)$, $r = 1, \ldots, n$, defined on the Hilbert space $\mathcal{H}_{B_j}(m\psi_j)$, can be expressed in terms of an orthonormal basis, and we get :

$$\frac{1}{2m}\log \sup_{f\in\overline{B}_{j,m}}\left(|f(z)|^2 + \sum_{r=1}^{n}\left|\frac{\partial f}{\partial z_r}(z)\right|^2\right) \le \psi_{j,m}(z)$$
$$\le \tfrac{1}{2m}\log\left((n+1)\sup_{f\in\overline{B}_{j,m}}\left(|f(z)|^2 + \sum_{r=1}^{n}\left|\tfrac{\partial f}{\partial z_r}(z)\right|^2\right)\right),$$

where $\overline{B}_{j,m}$ is the unit ball of $\mathcal{H}_{B_j}(m\psi_j)$. We also have the analogous relations for $\psi_{k,m}$ on B_k. This means that to compare $\psi_{j,m}$ and $\psi_{k,m}$ at a fixed point $x_0 \in B''_j \cap B''_k$, it is enough to show that for every holomorphic function f_j on B_j such that $\int_{B_j} |f_j|^2\, e^{-2m\psi_j} = 1$, there exists a holomorphic function f_k on B_k having an L^2-norm under control and satisfying :

$$f_k(x_0) = f_j(x_0), \quad \text{and} \quad \frac{\partial f_k}{\partial z_r}(x_0) = \frac{\partial f_j}{\partial z_r}(x_0), \quad \text{for } r = 1, \ldots, n.$$

This is done using Hörmander's L^2 estimates ([Hor65]). Let θ be a cut-off function supported in a neighbourhood of x_0 such that $\theta \equiv 1$ near x_0, and solve the equation

$$\bar\partial g = \bar\partial(\theta f_j)$$

on B_k with a weight containing the term $2(n+1)\log|z - x_0|$ which forces the solution g to vanish to order at least 2 at x_0. Specifically, if h_{jk} is a holomorphic function on $B_j \cup B_k$ such that $h_j - h_k = \mathrm{Re}\, h_{jk}$ on $B_j \cap B_k$, we can find a solution g to the above equation on B_k satisfying Hörmander's L^2 estimates with the strictly psh weight :

$$2m(\psi_k - \mathrm{Re}\, h_{jk}) + 2(n+1)\log|z - x_0| + |z - x_0|^2.$$

Now set $f_k := \theta f_j - g$ which is easily seen to satisfy the requirements. The precise estimate of the solution g gives the uniform upper estimate of $\varphi_{j,m} - \varphi_{k,m}$ on $B''_j \cap B''_k$ by $\mathcal{O}(\frac{1}{m})$ which implies the existence of a constant $C_1(\delta) > 0$ satisfying the patching condition (9). The details are left to the reader.

The loss of positivity incurred in T_m with respect to the original T can be seen to be at most $\frac{C}{m}$ as in [Pop04] thanks to the form γ being closed. This proves (a). That the approximating currents $T_m := dd^c\varphi_m$ constructed through this patching procedure satisfy the condition (b) on Monge-Ampère masses follows from the local Theorem 0.4.1 proved in the previous section. Theorem 0.5.1 is thus proved. □

0.6 Regularization with mass control : the general case

It is now a matter of putting together a few observations and techniques to derive the general case from the case of zero Lelong numbers. In all that follows β will denote the standard Kähler form and Ω a bounded pseudoconvex open subset of $\mathbb{C}^n$. To begin with, we investigate the complementary situation in which the singularities of the original function are analytic.

Let $u = \frac{1}{2}\log(|g_1|^2 + \cdots + |g_N|^2)$ be psh with analytic singularities on Ω, and let $V := \{g_1 = \cdots = g_N = 0\}$ be its singularity set. Let $\mathcal{J} = (g_1, \dots, g_N) \subset \mathcal{O}_\Omega$ be the ideal sheaf generated by $g_1, \dots, g_N$, and let $\mu : \tilde{\Omega} \to \Omega$ be a proper modification such that $\tilde{\Omega}$ is a smooth variety and $\mu^\star\mathcal{J} = \mathcal{O}(-D)$ for a normal crossing divisor D on $\tilde{\Omega}$. After possibly shrinking Ω, we can find finite coverings of Ω by open balls $B'_l \subset\subset B_l, l = 1, \dots, p$, such that the restriction of $\mu^\star\mathcal{J}$ to each $\tilde{B}_l := \mathrm{int}\,(\overline{\mu^{-1}(B_l \setminus V)})$ is generated by some holomorphic function f_l. Then

$$g_j \circ \mu = h_j\, f_l, \quad \text{on} \quad \tilde{B}_l, \quad j = 1, \dots, N, \tag{10}$$

for some holomorphic functions $h_1, \dots, h_N$ with no common zeroes. Let $u_m^{(l)}$, $m \in \mathbb{N}$, be the Demailly approximations (cf. (4) in section 0.4) of the restriction of u to B_l. This means that

$$u_m^{(l)} = \frac{1}{2m}\log\sum_{j=0}^{+\infty}|\sigma_{m,j}^{(l)}|^2, \qquad m \in \mathbb{N},$$

where $(\sigma_{m,j}^{(l)})_{m\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_{B_l}(m\,u)$. In particular, we have :

$$1 = \int_{B_l}\frac{|\sigma_{m,j}^{(l)}|^2}{(|g_1|^2 + \cdots + |g_N|^2)^m} = \int_{\tilde{B}_l}\frac{|\sigma_{m,j}^{(l)} \circ \mu|^2\,|J_\mu|^2}{|f_l|^{2m}\,(|h_1 \circ \mu|^2 + \cdots + |h_N \circ \mu|^2)^m},$$

where J_μ denotes the Jacobian of μ. This implies the existence of holomorphic functions $b_{m,j}^{(l)}$ on $\tilde{B}_l$, without common zeroes, such that

$$J_\mu\,\sigma_{m,j}^{(l)} \circ \mu = f_l^m\,b_{m,j}^{(l)} \qquad \text{on } \tilde{B}_l, \text{ for every } j \in \mathbb{N}.$$

Consequently $\frac{1}{2m}\log\sum_{j=0}^{+\infty}|b_{m,j}^{(l)}|^2$, $m \in \mathbb{N}$, are the Demailly approximations of the smooth psh function $\frac{1}{2}\log(|h_1|^2 + \cdots + |h_N|^2)$ on $\tilde{B}_l$. In particular, they satisfy inequalities (5) of section 0.4 and are thus locally uniformly bounded with respect to m by the bounds of $\frac{1}{2}\log(|h_1|^2+\cdots+|h_N|^2)$. Moreover,

$$\mu^\star(dd^c u_m^{(l)}) = dd^c\Big(\frac{1}{2m}\log\sum_{j=0}^{+\infty}|b_{m,j}^{(l)}|^2\Big)+[D]-\frac{1}{m}[\operatorname{div}J_\mu], \tag{11}$$

since $D=\operatorname{div}f_l$ on $\tilde{B}_l$ and the corresponding integration current satisfies $[D]=dd^c\log|f_l|$. Let $\tilde{U}_l$ be an open set such that $\tilde{B}'_l\subset\subset\tilde{U}_l\subset\subset\tilde{B}_l\subset\subset\tilde{\Omega}$. The supports of the currents D and $\operatorname{div}J_\mu$ are included in $\mu^{-1}(V)$. This fact can be combined with the Chern-Levine-Nirenberg inequalities (see e.g. [Dem97, chapter III,p.168]) to get the following estimates for the Monge-Ampère masses :

$$\int_{B'_l\setminus V}(dd^c u_m^{(l)})^q\wedge\beta^{n-q}=\int_{\mu^{-1}(B'_l\setminus V)}\mu^\star(dd^c u_m^{(l)})^q\wedge\mu^\star\beta^{n-q}$$
$$=\int_{\tilde{B}'_l}\Big(\frac{1}{2m}\log\sum_{j=0}^{+\infty}|b_{m,j}^{(l)}|^2\Big)^q\wedge\mu^\star\beta^{n-q}\leq C_l\Big(\sup_{\tilde{U}_l}\Big|\frac{1}{2m}\log\sum_{j=0}^{+\infty}|b_{m,j}^{(l)}|^2\Big|\Big)^q,$$

with a constant $C_l>0$ independent of m (depending only on $\tilde{B}'_l$ and $\tilde{U}_l$). As noticed above, the last term is bounded independently of m. Now, the locally defined psh functions $(u_m^{(l)})_{l=1,\ldots,p}$ can be glued together into global regularizing functions $u_m\to u$ on Ω by the patching procedure described in the proof of Theorem 0.5.1 (as there are only finitely many local pieces and the restriction on Lelong numbers imposed there has no bearing on the patching procedure). The loss of positivity in the Hessian $dd^c u_m$ is no more than $\frac{C}{m}\beta$. The boundedness of the sequence of Monge-Ampère masses and the property of the Lelong numbers analogous to (6) of section 0.4 survive the patching procedure. Moreover, if $u=\frac{c}{2}\log(|g_1|^2+\cdots+|g_N|^2)$ for some constant $c>0$, we set $u_m:=c\,v_m$, where v_m are the approximations of u/c we have just constructed. We have thus proved the following.

Lemma 0.6.1 *If $u=\frac{c}{2}\log(|g_1|^2+\cdots+|g_N|^2)$ has analytic singularities on Ω, there exist almost psh functions with analytic singularities $(u_m)_{m\in\mathbb{N}}$ on Ω such that u_m converges to u pointwise and in L^1_{loc} topology as $m\to+\infty$, $dd^c u_m\geq-\frac{C}{m}\beta$ on Ω, $\nu(u,x)-\frac{n}{m}\leq\nu(u_m,x)\leq\nu(u,x)$ at every point $x\in\Omega$, and the Monge-Ampère masses are bounded on any open subset $B\subset\subset\Omega$, to wit :*

$$\int_{B\setminus V_m}(dd^c u_m+\frac{C}{m}\beta)^q\wedge\beta^{n-q}\leq C,\qquad q=1,\ldots,n,\qquad \text{all } m\in\mathbb{N},$$

where $V_m:=\{u_m=-\infty\}$, and $C>0$ is a constant independent of m.

We can now combine these two complementary cases to regularize sums of a psh function with zero Lelong numbers and a psh function with analytic singularities and have controlled Monge-Ampère masses. The idea is simply to regularize separately the two terms. Let $\varphi = \psi + u$, where ψ is psh with zero Lelong numbers and $u = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2)$ has analytic singularities on $\Omega \subset\subset \mathbb{C}^n$. Theorem 0.4.1 applied to ψ gives smooth psh functions $\psi_m \to \psi$, while Lemma 0.6.1 applied to u furnishes psh functions with analytic singularities $u_m \to u$.

Lemma 0.6.2 *The Monge-Ampère masses in the regularization $dd^c\psi_m + dd^c u_m$ of $dd^c\varphi$ satisfy the following estimate on any open subset $B \subset\subset \Omega$:*

$$\int_{B\setminus V_m} (dd^c\psi_m + dd^c u_m)^q \wedge \beta^{n-q} \leq C\,(\log m)^q, \quad q = 1, \dots, n,$$

where $V_m := \{u_m = -\infty\}$, and $C > 0$ is a constant independent of m.

Proof. The set-up is the one described before Lemma 0.6.1 in which the regularization $(u_m)_{m\in\mathbb{N}}$ of u was defined. Relations (10) show that

$$u \circ \mu = \tfrac{c}{2}\log(|h_1|^2 + \cdots + |h_N|^2) + c\,\log|f_l| \qquad \text{on } \tilde{B}_l,$$

and gluing these local pieces together and taking Hessians, we get :

$$\mu^\star(dd^c u) = dd^c(u \circ \mu) = \alpha + c\,[D] \qquad \text{on } \tilde{\Omega},$$

with a C^∞ (1, 1)-form α, as the integration current $[D] = dd^c \log|f_l|$ locally on $\tilde{\Omega}$. On the other hand, gluing local pieces together and taking Hessians, the decomposition (11) can be globalized to :

$$\mu^\star(dd^c u_m) = dd^c(u_m \circ \mu) = \alpha_m + [E_m] \qquad \text{on } \tilde{\Omega},$$

for some C^∞ (1, 1)-form α_m, and the $\mathbb{R}$-divisor $E_m := c\,D - \dfrac{1}{m}\,E$, where E denotes the zero divisor of J_μ. In other words, the modification μ simultaneously resolves the singularities of u and u_m for all $m \in \mathbb{N}^\star$. Now, if $\tilde{B}$ is an open set such that $B \subset\subset \tilde{B} \subset\subset \Omega$, the smoothness of ψ_m and the above decompositions combined with the Chern-Levine-Nirenberg inequalities (cf. [Dem97, chapter III, p.168]) imply :

$$\begin{aligned}
&\int_{B\setminus V_m} (dd^c\psi_m)^p \wedge (dd^c u_m)^{k-p} \wedge \beta^{n-k} = \\
&= \int_{\mu^{-1}(B)\setminus \operatorname{Supp} E_m} (dd^c(\psi_m \circ \mu))^p \wedge \mu^\star(dd^c u_m)^{k-p} \wedge \mu^\star\beta^{n-k} \\
&= \int_{\mu^{-1}(B)} (dd^c(\psi_m \circ \mu))^p \wedge \alpha_m^{k-p} \wedge \mu^\star\beta^{n-k}
\end{aligned}$$

$$\leq C_{B,\tilde{B},\mu}\left(\sup_{\mu^{-1}(\tilde{B})}|\psi_m\circ\mu|\right)^p\int_{\mu^{-1}(\tilde{B})}\alpha_m^{k-p}\wedge\mu^\star\beta^{n-k+p}$$
$$=C_{B,\tilde{B},\mu}\left(\sup_{\tilde{B}}|\psi_m|\right)^p\int_{\tilde{B}\setminus V_m}(dd^cu_m)^{k-p}\wedge\beta^{n-k+p},$$

where $C_{B,\tilde{B},\mu}>0$ is a constant independent of m. Now, by the proof of Theorem 0.4.1 (Claim 0.4.2), the growth of $\sup_{\tilde{B}}|\psi_m|$ is no more than $\mathcal{O}(\log m)$, and an application of Lemma 0.6.1 completes the proof. □

Lemma 0.6.3 *Suppose $\varphi=\sum_{k=1}^{p}\lambda_k\log|g_k|$ for some holomorphic functions $g_1,\ldots,g_p\in\mathcal{O}(\Omega)$ and constants $\lambda_1,\ldots,\lambda_p>0$ such that $\sum_{k=1}^{p}\lambda_k\,div\,g_k$ is a normal crossing divisor. Then there exist psh functions with analytic singularities $(\varphi_m)_{m\in\mathbb{N}}$ such that $\varphi_m\to\varphi$ pointwise and in L^1_{loc} topology on Ω as $m\to\infty$, $\nu(\varphi,x)-\frac{n}{m}\leq\nu(\varphi_m,x)\leq\nu(\varphi,x)$ for every $x\in\Omega$, and for any relatively compact open subset $B\subset\subset\Omega$ we have :*

$$\int_{B\setminus V_m}(dd^c\varphi_m)^q\wedge\beta^{n-q}\leq C\,(\log m)^q,\quad q=1,\ldots,n,$$

where $V_m:=\{\varphi_m=-\infty\}$, and $C>0$ is a constant independent of m.

Proof. The L^2 condition defining $\mathcal{H}_\Omega(m\varphi)$ reads

$$\int_\Omega|f|^2/\prod_{k=1}^{p}|g_k|^{2m\lambda_k}<+\infty,$$

implying that every $f\in\mathcal{H}_\Omega(m\varphi)$ decomposes as $f=\prod_{k=1}^{p}g_k^{[m\lambda_k]}h$ with $h\in\mathcal{H}_\Omega\left(m\sum_{k=1}^{p}\frac{\{m\lambda_k\}}{m}\log|g_k|\right)$, where [ ] (resp. { }) denotes the integer (resp. fractional) part. An orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m\varphi)$ can then be written as

$$\sigma_{m,j}=\prod_{k=1}^{p}|g_k|^{[m\lambda_k]}\,h_{m,j},\quad j\in\mathbb{N},$$

where $(h_{m,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_\Omega\left(m\sum_{k=1}^{p}\frac{\{m\lambda_k\}}{m}\log|g_k|\right)$. If $\lambda_1,\ldots,\lambda_p\in\mathbb{N}$, φ has analytic singularities and we can define $(\varphi_m)_{m\in\mathbb{N}}$ as the Demailly approximations of φ (cf. (4)). This case is actually covered by Lemma 0.6.1 and we obtain a bounded sequence of Monge-Ampère masses. If the coefficients λ_k are not integers, we set $\varphi_m=\sum_{k=1}^{p}\frac{[m\lambda_k]}{m}\log|g_k|+\psi_m$, with

$$\psi_m := \frac{1}{2m}\log\left(\sum_{j=0}^{+\infty}|h_{m,j}|^2 + \sum_{j=0}^{+\infty}\left|\frac{\partial h_{m,j}}{\partial z_1}\right|^2 + \cdots + \sum_{j=0}^{+\infty}\left|\frac{\partial h_{m,j}}{\partial z_n}\right|^2\right),$$

where $z = (z_1, \ldots, z_n)$ is the standard coordinate on $\mathbb{C}^n$. The local integrability of $1/\prod_{k=1}^{p}|g_k|^{2\{m\lambda_k\}}$ ensures that $h_{m,j}$, $j \in \mathbb{N}$, have no common zeroes and thus ψ_m is smooth. Since $dd^c\varphi_m = dd^c\psi_m$ outside the singular locus consisting of the divisors $\operatorname{div} g_k$, $k = 1, \ldots, p$, the proof of Theorem 0.4.1 can be repeated to give the result. $\square$

We can now dispense with the vanishing restriction imposed on the Lelong numbers in Theorem 0.4.1 to get a general approximation result with controlled Monge-Ampère masses. It should be noted, however, that the proof given below reduces the general case to the case of zero Lelong numbers which thus turns out to be the crucial case. Recall that for every $c > 0$, the Lelong number upperlevel set

$$E_c := \{x \in X,\ \nu(T, x) \geq c\}$$

is known to be analytic by a by now classical result of Siu ([Siu74]). Thus the set

$$E_+(T) := \bigcup_{c>0} E_c(T) = \bigcup_{c\in\mathbb{Q}_+^\star} E_c(T)$$

where T has positive Lelong numbers is always a countable union of analytic sets. Given an irreducible analytic set A, the Lelong number of T at almost all points $x \in A$ (i.e. outside a countable union of proper analytic subsets of A) equals the infimum of the Lelong numbers of T on A which is therefore called the generic Lelong number on A. Thus the set of Lelong numbers of T on X is at most countable. We shall now momentarily suppose this set to be finite only to lift this restriction later on. This assumption covers currents whose local potentials are of the form :

$$\varphi = \sum_j \log\left(\sum_k \prod_l |f_{j,k,l}|^{\alpha_{j,k,l}}\right), \qquad \alpha_{j,k,l} > 0,$$

with holomorphic functions $f_{j,k,l}$, as well as currents with isolated singularities.

Theorem 0.6.4 *Let φ be a psh function on $\Omega \subset \mathbb{C}^n$. Suppose that the Lelong numbers of φ (or equivalently of $dd^c\varphi$) assume finitely many values in Ω. Then there exist almost psh functions φ_m with analytic singularities on Ω such that $dd^c\varphi_m$ converges to $dd^c\varphi$ in the weak topology of currents as $m \to +\infty$, and the following hold :*

(a) $dd^c\varphi_m \geq -\frac{C}{m}\beta^n$ *on* Ω, $m \in \mathbb{N}$ *;*

(b) $\nu(\varphi, x) - \frac{n}{m} \leq \nu(\varphi_m, x) \leq \nu(\varphi, x), \quad x \in \Omega, m \in \mathbb{N}$ *;*

(c) for any relatively compact open subset $B \subset\subset \Omega$ *we have :*

$$\int_{B\setminus V_m} (dd^c\varphi_m)^q \wedge \beta^{n-q} \leq C\,(\log m)^q, \quad q = 1, \ldots, n,$$

where $V_m := \{\varphi_m = -\infty\}$*, and* $C > 0$ *is a constant independent of* m*.*

Proof. By the finiteness assumption on the set of Lelong numbers, $E_+(\varphi)$ is analytic as a finite union of analytic sets. If $E_+(\varphi)$ has irreducible components A_j of codimension 1 on which φ has positive generic Lelong numbers λ_j, then the Siu decomposition formula (see e.g. [Dem97, chapter III, p.207]) gives $dd^c\varphi = \sum_j \lambda_j\,[A_j] + R$, where R is a closed positive (1, 1)-current whose positive Lelong numbers may only occur in codimension ≥ 2. After regularizing the part $\sum_j \lambda_j\,[A_j]$ independently using Lemma 0.6.3 and replacing $dd^c\varphi$ with R if necessary, we may assume that $\operatorname{codim} E_+(\varphi) \geq 2$. Let $\mathcal{I}_{E_+(\varphi)} \subset \mathcal{O}_\Omega$ be the coherent ideal sheaf of germs of holomorphic functions vanishing on $E_+(\varphi)$, and let $\mu : \tilde{\Omega} \to \Omega$ be a proper modification such that $\tilde{\Omega}$ is smooth and $\mu^\star\mathcal{I}_{E_+(\varphi)}$ is invertible and associated with a normal crossing divisor. Let $E = \mu^{-1}(E_+(\varphi)) = \bigcup_j Z_j$ with reduced structure and irreducible components Z_j, all smooth of codimension 1 and meeting transversally. The Siu decomposition applied to the current $\mu^\star(dd^c\varphi) = dd^c(\varphi \circ \mu)$ gives :

$$\mu^\star(dd^c\varphi) = \sum_j \lambda_j\,[Z_j] + R \qquad \text{on } \tilde{\Omega},$$

with $\lambda_j \geq 0$, and a closed positive (1, 1)-current R with positive Lelong numbers only in codimension ≥ 2. We may assume without loss of generality that R has vanishing Lelong numbers everywhere. Indeed, if R does have positive Lelong numbers, by our assumption they appear along finitely many analytic sets which can in turn be blown up to become divisors and isolated by Siu's decomposition formula. After finitely many blowups we get a current R with only zero Lelong numbers. We can now apply Theorem 0.5.1 to R to get smooth closed (1, 1)-forms R_m on $\tilde{\Omega}$ converging weakly to R such that the associated Monge-Ampère masses have an at most logarithmic growth as $m \to +\infty$. Indeed, the noncompacity of $\tilde{\Omega}$ is no obstacle since the modification μ is obtained as the composition of finitely many blow-ups with smooth centres and thus $\tilde{\Omega}$ can be covered by finitely many coordinate patches in which the local approximation theorem 0.4.1 applies. Moreover, the forms R_m can be chosen in the $\partial\bar{\partial}$-cohomology class of R, i.e. $\{R_m\} = \{R\}$. Taking direct images we infer :

$$\mu_\star(\sum_j \lambda_j\,[Z_j] + R_m) = \mu_\star R_m \longrightarrow dd^c\varphi, \quad \text{weakly on } \Omega \text{ as } m \to +\infty.$$

The currents $(\mu_\star R_m)_{m\in\mathbb{N}}$ furnish thus an approximation of $dd^c\varphi$ in the same (zero) $\partial\bar\partial$-cohomology class. In particular, $\mu_\star R_m = dd^c\varphi_m$ for some almost psh function φ_m on Ω. Moreover, these currents satisfy the desired condition on Monge-Ampère masses on $B\setminus V$. To see this, note that

$$\mu^\star\mu_\star R_m = R_m + \sum_j r_{m,\,j}\,[Z_j]$$

for some nonnegative real numbers $r_{m,\,j}$ (see [Fuj81, Lemma 2.4, p.743-744]), and therefore the masses of $(dd^c\varphi_m)^q$ on $B\setminus V$ are the same as the masses of R_m^q on $\mu^{-1}(B)\setminus \operatorname{Supp} E$ which have an at most logarithmic growth in m by construction.

We have yet to prove that the currents $\mu_\star R_m$ have analytic singularities. To see this, let $Z_j = \operatorname{div} g_j$ and $R = dd^c\psi$ on some open subset $\tilde U \subset \tilde\Omega$ included in some coordinate patch, with g_j holomorphic and ψ psh with zero Lelong numbers. Then

$$\varphi\circ\mu = \sum_j \lambda_j\,\log|g_j| + \psi \qquad \text{on } \tilde U,$$

and R_m is obtained by gluing together locally defined smooth forms of the form $dd^c\psi_m$, with

$$\psi_m := \frac{1}{2m}\,\log\left(\sum_{j=0}^{+\infty}|\sigma_{m,\,j}|^2 + \sum_{j=0}^{+\infty}\left|\frac{\partial\sigma_{m,\,j}}{\partial w_1}\right|^2 + \cdots + \sum_{j=0}^{+\infty}\left|\frac{\partial\sigma_{m,\,j}}{\partial w_n}\right|^2\right) \qquad \text{on } \tilde U,$$

where $w = (w_1,\ldots,w_n)$ is the standard coordinate on $\tilde U$, and $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_{\tilde U}(m\psi)$ (see notation after (4) in section 0.4). On the other hand

$$\mu_\star(dd^c\psi_m) = dd^c(\psi_m\circ\mu^{-1}) \qquad \text{in the complement of } V,$$

since $\tilde\Omega\setminus E$ and $\Omega\setminus V$ are biholomorphic under μ. The holomorphic functions $\sigma_{m,\,j}\circ\mu^{-1}$ and $\frac{\partial\sigma_{m,\,j}}{\partial w_l}\circ\mu^{-1}$, defined in the complement of the analytic set V of codimension ≥ 2, extend holomorphically across V. Thus the current $dd^c(\psi_m\circ\mu^{-1})$ extends to a closed positive current across V which has analytic singularities along V. Since $\operatorname{codim} V \geq 2$ and since closed positive $(1,\,1)$-currents cannot carry mass on analytic sets of codimension ≥ 2 (see e.g. [Dem97, chapter III, corollary 2.11, p. 163]), it follows that $\mu_\star(dd^c\psi_m) = dd^c(\psi_m\circ\mu^{-1})$ everywhere. This proves that $\mu_\star R_m$ has (if any) analytic singularities along V for every m. The proof is complete. $\square$

Theorem 0.6.4 we have just proved is a special case of Theorem 0.1.2

stated in the introduction for $X = \Omega \subset\subset \mathbb{C}^n$ and $\gamma = 0$. It is then enough to apply once more the patching procedure described in the proof of Theorem 0.5.1 to complete the proof of Theorem 0.1.2.

0.7 Singular hermitian metrics and big line bundles

We are now in a position to prove the analytic characterization of the volume of a line bundle spelt out in Theorem 0.1.3 as a geometric application of our current regularization Theorem 0.1.2 with controlled Monge-Ampère masses. The use of singular Morse inequalities to tackle such questions was outlined by Bonavero in his thesis ([Bon95], p. 41-43). It was subsequently implemented by Boucksom in [Bou02] under the extra assumption that the ambient manifold X be Kähler. It relies on regularizations of currents with controlled Monge-Ampère masses which are comparatively easily obtained in the Kähler case. We derive here the non-Kähler counterpart as an application of our Monge-Ampère mass control in the general case.

First, we briefly review the set-up. For more details on multiplier ideal sheaves associated with singular metrics and psh functions, the reader is referred to [Dem01]. Let $(L,\, h)$ be a holomorphic line bundle over a compact Hermitian manifold $(X,\, \omega)$ equipped with a possibly singular Hermitian metric h. Let $T := i\Theta_h(L)$ be the curvature current associated with h. No positivity assumption is made on T. There is a global representation of T as $T = \alpha + dd^c\varphi$ with a global C^∞ $(1,\,1)$-form α on X. For every $q = 1, \dots, n$, define the q-index set of T as the open subset $X(q,\, T)$ of X consisting of those points x such that $T_{ac}(x)$ has precisely q negative and $n-q$ positive eigenvalues. Let $X(\leq q,\, T) := X(0,\, T) \cup \dots \cup X(q,\, T)$. For every $m \in \mathbb{N}^\star$, consider the singular metric h^m on L^m induced by h. This means that if $h = e^{-\varphi}$ on an open subset $U \subset X$ on which L is trivial, h^m is defined as $h^m = e^{-m\varphi}$ on U. If $T := i\Theta_h(L) \geq -C\,\omega$ for some constant $C > 0$ (i.e. T is almost positive and φ is almost psh), the associated multiplier ideal sheaf $\mathcal{I}(h^m)$ is the coherent subsheaf of $\mathcal{O}_X$ defined locally as $\mathcal{I}(h^m)_{|U} = \mathcal{I}(m\varphi)$, where the multiplier ideal sheaf $\mathcal{I}(m\varphi) \subset \mathcal{O}_U$ is in turn defined at every point $x \in U$ as :

$$\mathcal{I}(m\varphi)_x := \{f \in \mathcal{O}_{U,\,x},\; |f|^2\, e^{-2m\varphi} \text{ is Lebesgue-integrable near } x\}.$$

Demailly's holomorphic Morse inequalities (see [Dem85]) for smooth metrics h were generalized by Bonavero ([Bon98]) to the case of singular metrics h with analytic singularities in the form of the following asymptotical estimates for the cohomology group dimensions of the twisted coherent sheaves $\mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)$:

$$\sum_{j=0}^{q} (-1)^{q-j}\, h^j(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)) \leq \frac{m^n}{n!} \int\limits_{X(\leq q,\, T)} (-1)^q\, T_{ac}^n + o(m^n),$$

as $m \to \infty$, for all $q = 1, \ldots, n$. For $q = 1$, we get :

$$h^0(X, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) - h^1(X, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) \geq \frac{m^n}{n!}\int\limits_{X(\leq 1,\, T)} T_{ac}^n + o(m^n).$$

As $\quad h^0(X, \mathcal{O}_X(L^m)) \geq h^0(X, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m))$

$$\geq h^0(X, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) - h^1(X, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)),$$

we infer the following lower bound for the volume of L :

$$v(L) \geq \int\limits_{X(\leq 1,\, T)} T_{ac}^n, \tag{12}$$

for every closed current $T \geq 0$ with only analytic singularities (if any) in $c_1(L)$.

Proof of theorem 0.1.3. It is clearly enough to prove the equality characterizing the volume as the bigness criterion is an immediate consequence of it. The inequality "$\leq$" bounding the volume above can be proved as in [Bou02] since X is Moishezon when $v(L) > 0$ and can be modified into a projective manifold. If $v(L) = 0$, the inequality "$\leq$" is obvious.

Thus proving Theorem 0.1.3 boils down to obtaining the lower bound for the volume of L in terms of curvature currents. In the light of the above explanations, this can be seen as singular Morse inequalities for arbitrary singularities. Let $T := i\Theta_h(L) \geq 0$ be the curvature current associated with a singular Hermitian metric h with arbitrary singularities on L. If no positive current exists in $c_1(L)$, there is nothing to prove. Apply Theorem 0.1.2 to get regularizing currents with analytic singularities $T_m \to T$ in $c_1(L)$ such that $T_m \geq -\frac{C}{m}\,\omega$ for some constant $C > 0$ independent of m. Furthermore, Theorem 2.4 in [Bou02, p. 1050] asserts that a regularizing sequence of currents with analytic singularities can be combined with a regularizing sequence of smooth forms constructed in [Dem82] to produce yet another regularizing sequence of currents retaining all its previous properties and getting an additional grip on the absolutely continuous part of T. In other words, after modifying our sequence $(T_m)_{m\in\mathbb{N}}$ by means of Theorem 2.4 in [Bou02, p. 1050], we may assume that besides all its properties, it also satisfies :

$$T_m(x) \to T_{ac}(x) \quad \text{as} \quad m \to +\infty, \quad \text{for almost every} \quad x \in X. \tag{13}$$

As explained above, by the Morse inequalities applied to L with $T_m \in c_1(L)$ as curvature current with analytic singularities, we get (cf. (12)) :

$$v(L) \geq \int\limits_{X(\leq 1,\, T_m)} T_{m,\,ac}^n = \int\limits_{X(0,\, T_m)} T_{m,\,ac}^n + \int\limits_{X(1,\, T_m)} T_{m,\,ac}^n, \qquad \text{for every } m \in \mathbb{N}.$$

On the other hand, the proof of Proposition 3.1. in [Bou02, p. 1052-53] uses the Fatou lemma to derive the following inequality from property (13) :

$$\liminf_{m\to+\infty}\int_{X(0,\,T_m)} T^n_{m,\,ac} \geq \int_{X(T,\,0)} T^n_{ac} = \int_X T^n_{ac}.$$

Thus, to prove the Morse-type inequality "$\geq$" it is enough to show that $\lim\limits_{m\to+\infty}\int\limits_{X(1,\,T_m)} T^n_{m,\,ac} = 0$. Note that on the open set $X(1,\,T_m)$ we have :

$$0 \leq -T^n_{m,\,ac} \leq n\,\frac{C}{m}\,(T_{m,\,ac} + \frac{C}{m}\,\omega)^{n-1}\wedge\omega.$$

It is thus enough to show that

$$\lim_{m\to+\infty}\frac{C}{m}\int\limits_X (T_{m,\,ac} + \frac{C}{m}\,\omega)^{n-1}\wedge\omega = 0.$$

Since $\int\limits_X (T_{m,\,ac} + \frac{C}{m}\,\omega)^{n-1}\wedge\omega = \int\limits_{X\setminus V_m}(T_m + \frac{C}{m}\,\omega)^{n-1}\wedge\omega$, this is immediate from the control of the Monge-Ampère masses obtained in Theorem 0.1.2 (cf. conclusion (c)). The proof is thus the same as in the Kähler case settled in [Bou02] once we have obtained Theorem 0.1.2 which is new in the non-Kähler context. $\square$

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