Effective Local Finite Generation of Multiplier Ideal Sheaves[1]

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Abstract. Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$, and let $\mathcal{I}(m\varphi)$ be the associated multiplier ideal sheaves, $m \in \mathbb{N}^\star$. Motivated by global geometric issues, we establish an effective version of the coherence property of $\mathcal{I}(m\varphi)$ as $m \to +\infty$. Namely, given any $B \subset\subset \Omega$, we estimate the growth order in m of the number of generators over $\mathcal{O}_\Omega$ of $\mathcal{I}(m\varphi)_{|B}$, as well as the growth of the coefficients of sections in $\Gamma(B, \mathcal{I}(m\varphi))$ with respect to finitely many generators defined on Ω. Our approach relies on asymptotic integral estimates of Bergman kernels associated with singular weights which are obtained using Toeplitz concentration operators. In the latter half of the paper, we estimate asymptotically the additivity defect of multiplier ideal sheaves previously known to behave subadditively by a result of Demailly, Ein, and Lazarsfeld. As $m \to +\infty$, the decay rate of $\mathcal{I}(m\varphi)$ is proved to be almost linear if the singularities of φ are analytic.

0.1   Introduction

Let L be a holomorphic line bundle over a complex manifold X of complex dimension n. Suppose L is equipped with a possibly singular Hermitian metric h. This means that on each open subset of X on which L is trivial, the metric is defined as $h = e^{-\varphi}$ for some local weight function φ which is only assumed to be L^1_{loc} with respect to the Lebesgue measure. The induced metrics h^m on the tensor power line bundles L^m are then defined by the local weight functions $m\varphi$, $m \in \mathbb{N}^\star$.

In all that follows, the curvature current $i\Theta_h(L)$ of bidegree $(1, 1)$ associated with (L, h) is assumed to be ≥ 0 on X. Equivalently, the local weight functions φ are assumed to be plurisubharmonic (psh). Following Nadel ([Nad90]), one can associate with h a multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X$ defined as follows. If $h = e^{-\varphi}$ on a trivialising open set $\Omega \subset X$ for L, we set

$$\mathcal{I}(h)_{|\Omega} = \mathcal{I}(\varphi),$$

where the stalk $\mathcal{I}(\varphi)_x$ at any point $x \in \Omega$ is defined as the set of germs of holomorphic functions $f \in \mathcal{O}_{\Omega, x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates in a neighbourhood of x. The prime objective of the present paper is to study the variation of $\mathcal{I}(h^m)$ (or equivalently of $\mathcal{I}(m\varphi)$ on each trivialising open set Ω) as $m \to +\infty$.

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The sequence of multiplier ideal sheaves $\mathfrak{I}(h^m)$ is easily seen to be no-nincreasing as $m \to +\infty$. Indeed, to offset the possible nonintegrability of $e^{-2m\varphi}$ near points $x \in \Omega$ where $\varphi(x) = -\infty$ (singularities of the metric h), holomorphic germs $f \in \mathfrak{I}(m\varphi)_x$ may need to vanish to increasingly high orders at x as m increases. On the other hand, the Demailly-Ein-Lazarsfeld subadditivity property of multiplier ideal sheaves ([DEL00]) implies that

$$\mathfrak{I}(h^m) \subset \mathfrak{I}(h)^m, \quad (\text{resp.} \quad \mathfrak{I}(m\varphi) \subset \mathfrak{I}(\varphi)^m), \quad m \in \mathbb{N}^\star,$$

while the inclusion is strict in general. In other words, $\mathfrak{I}(h^m)$ may decrease more quickly than linearly as $m \to +\infty$. The main thrust of the ensuing development is to obtain an effective control of the decaying rate of $\mathfrak{I}(h^m)$ as $m \to +\infty$.

As the problems being dealt with are local in nature, we shall focus our attention on psh functions φ defined on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Examples of such functions are provided by the so-called psh functions with **analytic singularities**, namely those functions that can be locally written as

$$(\star) \qquad \qquad \varphi = \frac{c}{2} \log(|g_1|^2 + \cdots + |g_N|^2),$$

for some holomorphic functions $g_1, \ldots, g_N$ on Ω, and some constant $c > 0$. The $-\infty$-poles (or singularities) of φ are precisely the common zeroes of $g_1, \ldots, g_N$, and they only depend, up to an equivalence of singularities, on the ideal sheaf generated by $g_1, \ldots, g_N$.

The issue of the decaying rate of $\mathfrak{I}(m\varphi)$ is addressed in two ways.

First, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $|f|^2 e^{-2m\varphi}$ is integrable with respect to the Lebesgue measure on Ω. It is well-known that the ideal sheaf $\mathfrak{I}(m\varphi)$ is coherent and generated as an $\mathcal{O}_\Omega$-module by an arbitrary orthonormal basis $(\sigma_{m,j})_{j \in \mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ ([Nad90], see also [Dem93, Lemma 4.4]). By the strong Noetherian property of coherent sheaves, it is then generated, on every relatively compact open subset $\Omega' \subset\subset \Omega$, by only finitely many $\sigma_{m,j}$'s. Our first goal is to make this local finite generation property effective; in other words, to estimate the number N_m of generators needed, as well as the growth rate of the (holomorphic function) coefficients appearing in the decomposition of an arbitrary section of $\mathfrak{I}(m\varphi)$ on Ω' as a finite linear combination of $\sigma_{m,j}$'s, as $m \to +\infty$. The first set of results can be summed up as follows.

Theorem 0.1.1 *Let φ be a strictly psh function on $\Omega \subset \mathbb{C}^n$ such that $i\partial\bar{\partial}\varphi \geq C_0\,\omega$ for some constant $C_0 > 0$. Let $B := B(x, r) \subset\subset \Omega$ be an arbitrary open ball. Then, there exist a ball $B(x, r_0) \subset\subset B(x, r)$ and*

$m_0 = m_0(C_0) \in \mathbb{N}$, *such that for every* $m \geq m_0$ *the following property holds.* *Every* $g \in \mathcal{H}_B(m\varphi)$ *admits, with respect to some suitable finitely many elements* $\sigma_{m,1}, \ldots \sigma_{m,N_m}$ *in a suitable orthonormal basis* $(\sigma_{m,j})_{j \in \mathbb{N}^\star}$ *of* $\mathcal{H}_\Omega(m\varphi)$, *a decomposition :*

$$g(z) = \sum_{j=1}^{N_m} b_{m,j}(z)\, \sigma_{m,j}(z), \qquad z \in B(x, r_0),$$

with some holomorphic functions $b_{m,j}$ *on* $B(x, r_0)$ *satisfying :*

$$\sup_{B(x, r_0)} \sum_{j=1}^{N_m} |b_{m,j}|^2 \leq C\, N_m \int_B |g|^2\, e^{-2m\varphi} < +\infty,$$

where $C > 0$ *is a constant depending only on* n, r, *and the diameter of* Ω.

Moreover, if φ *has analytic singularities, then* $N_m \leq C_\varphi\, m^n$ *for* $m >> 1$, *where* $C_\varphi > 0$ *is a constant depending only on* φ, B, *and* n.

This can be seen as a local counterpart to the effective version of the global generation of multiplier ideal sheaves proved in [Siu02, Theorem 2.1] as a step towards the invariance of plurigenera for projective manifolds not necessarily of general type. Unlike in Siu's case, where spaces of global sections of coherent sheaves over compact manifolds were already finite dimensional, the corresponding spaces $\mathcal{H}_\Omega(m\varphi)$ of local sections are infinite dimensional. The thrust of our results here is to reduce an infinite dimensional situation to a finite dimensional one after restricting to a compact subset. This involves appropriate choices of generators with estimates obtained from an asymptotic study of Bergman kernels. The main idea is to concentrate the L^2 norms of holomorphic functions defined on Ω onto compact subsets of Ω by means of Toeplitz concentration operators.

Second, to complement the subadditivity property of Demailly, Ein, and Lazarsfeld, we obtain an upperadditivity result showing that the variation of $\mathcal{I}(m\varphi)$, though not necessarily linear in m, is almost linear if the singularities of φ are analytic. Namely, the following qualitative statement holds.

Theorem 0.1.2 *Let* φ *be psh with analytic singularities on* $\Omega \subset \mathbb{C}^n$. *Then, for every* $\Omega' \subset\subset \Omega$ *and every* $\delta > 0$, *we have :*

$$\mathcal{I}(m\varphi)_{|\Omega'}^p \subset \mathcal{I}(mp(1-\delta)\varphi)_{|\Omega'}, \qquad \text{for every } m \geq \tfrac{n+2}{c\,\delta}, \text{ and } p \in \mathbb{N}^\star.$$

A much stronger effective version of this result is obtained in Theorem 0.4.1 below. Specifically, for any sections $f_1, \ldots, f_p \in \Gamma(\Omega', \mathcal{I}(m\varphi))$ each with $e^{-2m\varphi}$-weighted L^2-norm equal to 1, the $e^{-2mp(1-\delta)\varphi}$-weighted L^2-norm of the

product $f_1 \cdots f_p \in \Gamma(\Omega', \mathcal{I}(mp(1 - \delta)\varphi))$ is estimated.

Here is an outline of our approach. In section 0.2, we recall the definition of Toeplitz concentration operators and observe some basic properties. Then we extend to the case of analytic singularities the asymptotic integral estimates for Bergman kernels previously obtained for smooth weight functions by Lindholm ([Lin01]) and Berndtsson ([Ber03]). Namely, if $(\sigma_{m,j})_{j \in \mathbb{N}^\star}$ is any orthonormal basis of the singularly weighted Bergman space $\mathcal{H}_\Omega(m\varphi)$, the trace of the associated Bergman kernel on the diagonal of $\Omega \times \Omega$ is defined as :

$$B_{m\varphi} := \sum_{j=0}^{+\infty} |\sigma_{m,j}|^2 \quad \text{on } \Omega, \tag{1}$$

the definition being independent of the orthonormal basis chosen. We obtain, among other things, the following.

Theorem 0.1.3 *Let φ be a psh function with analytic singularities on $\Omega \subset\subset \mathbb{C}^n$ such that $i\partial\bar{\partial}\varphi \geq C_0\omega$ for some constant $C_0 > 0$. Then, for any relatively compact open subset $B \subset\subset \Omega$, we have :*

$$\lim_{m \to +\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, d\lambda = \frac{2^n}{\pi^n} \int_B (i\partial\bar{\partial}\varphi)^n_{ac} < +\infty, \tag{2}$$

where $d\lambda$ denotes the Lebesgue measure in $\mathbb{C}^n$ and $(i\partial\bar{\partial}\varphi)_{ac}$ stands for the absolutely continuous part in the Lebesgue decomposition of the (complex measure) coefficients of the current $i\partial\bar{\partial}\varphi$ with respect to the Lebesgue measure.

This can be seen as the first step towards a possible asymptotic expansion of the Bergman kernel associated with singular weights that would parallel the Tian-Yau-Zelditch asymptotic expansion obtained for smooth weights (cf. [Tia90], [Yau93], [Zel98]).

While these results may have an interest of their own, we apply them in our case at hand to estimate the number N_m of local generators needed for $\mathcal{I}(m\varphi)$. In section 0.3, Hörmander's L^2 estimates ([Hör65]) and Skoda's L^2 division theorem ([Sko72]) are used to prove an effective local finite generation property of $\mathcal{I}(m\varphi)$, and thus complete the proof of Theorem 0.1.1. Section 0.4 deals with the different but related issue of estimating the subadditivity defect of multiplier ideal sheaves. Demailly's local approximation of arbitrary psh functions by psh functions with analytic singularities ([Dem92], Proposition 3.1) is revisited to get improved effective estimates on fixed-sized subsets. The key ingredient is again Skoda's L^2 division theorem. Theorem 0.1.2 then follows as a corollary.

In [Pop06], we have applied the results of the present paper to the global geometry of compact (not necessarily Kähler) complex manifolds to get singular Morse inequalities estimating the asymptotic growth of the cohomology of

high tensor powers of holomorphic line bundles twisted by the corresponding multiplier ideal sheaves associated with metrics with arbitrary singularities (i.e. $\mathcal{O}_X(L^m) \times \mathcal{I}(h^m)$, $m >> 1$). In particular, a complete characterisation of the volume of a line bundle in terms of all (possibly singular) Hermitian metrics one can define thereon is obtained in [Pop06]. The case of positive volume gives a new metric characterisation of big line bundles, and implicitly of Moishezon manifolds, generalising previous results arisen from the resolution of the Grauert-Riemenschneider conjecture.

0.2 Toeplitz concentration operators

This section is intended to lay the groundwork for the proof of Theorem 0.1.1 by hinting at how a suitable orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ and finitely many local generators of $\mathcal{I}(m\varphi)$ will be chosen. The actual choices will be spelt out in the next section. The key point is an estimate of the growth order of the number N_m of local generators via asymptotic estimates on the Bergman kernel of independent interest.

Let $B \subset\subset \Omega$ be a relatively compact open subset, typically a ball. Following [Lin01] and [Ber03] (themselves inspired by [Lan67]), we consider, for every $m \in \mathbb{N}^\star$, the following Toeplitz concentration operator on B :

$$T_{B,m} : \mathcal{H}_\Omega(m\varphi) \to \mathcal{H}_\Omega(m\varphi), \qquad T_{B,m}(f) = P_m(\chi_B f),$$

where χ_B is the characteristic function of B, and $P_m : L^2(\Omega, e^{-2m\varphi}) \to \mathcal{H}_\Omega(m\varphi)$ is the orthogonal projection from the Hilbert space of (equivalence classes of) measurable functions f for which $|f|^2 e^{-2m\varphi}$ is Lebesgue integrable on Ω, onto the closed subspace of holomorphic such functions. It is easy to see that

$$T_{B,m}(f) = \chi_B f - u,$$

where u is the solution of the equation $\bar{\partial}u = \bar{\partial}(\chi_B f)$ of minimal $e^{-2m\varphi}$-weighted L^2-norm. Alternatively, if we consider the Bergman kernel :

$$K_{m\varphi} : \Omega \times \Omega \to \mathbb{C}, \qquad K_{m\varphi}(z, \zeta) = \sum_{j=1}^{+\infty} \sigma_{m,j}(z) \overline{\sigma_{m,j}(\zeta)},$$

its reproducing property shows that the concentration operator is also given as

$$T_{B,m}(f)(z) = \int_B K_{m\varphi}(z, \zeta) f(\zeta) e^{-2m\varphi(\zeta)} d\lambda(\zeta), \qquad z \in \Omega,$$

where $d\lambda$ is the Lebesgue measure. It is clear that $T_{B,m}$ is a compact operator

for it is defined by a square integrable kernel. Its eigenvalues $\lambda_{m,1} \geq \lambda_{m,2} \geq \ldots$ lie in the open interval $(0, 1)$. If $f \in \mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,m}$ corresponding to some eigenvalue λ, we see that $\langle\langle T_{B,m}(f), f\rangle\rangle = \lambda \|f\|^2$, and implicitly :

$$\lambda = \frac{\int_B |f|^2 \, e^{-2m\varphi}}{\int_\Omega |f|^2 \, e^{-2m\varphi}}.$$

Therefore, $f \in \mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,m}$ if and only if

$$f(z) = \frac{\int_\Omega |f|^2 \, e^{-2m\varphi}}{\int_B |f|^2 \, e^{-2m\varphi}} \int_B K_{m\varphi}(z, \zeta) \, f(\zeta) \, e^{-2m\varphi(\zeta)} \, d\,\lambda(\zeta), \qquad z \in \Omega,$$

which amounts to having :

$$f(z) = \frac{\int_\Omega |f|^2 \, e^{-2m\varphi}}{\int_B |f|^2 \, e^{-2m\varphi}} \sum_{j=1}^{+\infty} \sigma_{m,j}(z) \int_B f(\zeta) \, \overline{\sigma_{m,j}(\zeta)} \, e^{-2m\varphi(\zeta)} \, d\,\lambda(\zeta), \quad z \in \Omega.$$

On the other hand, every $f \in \mathcal{H}_\Omega(m\varphi)$ has a Hilbert space decomposition with Fourier coefficients as :

$$f(z) = \sum_{j=1}^{+\infty} \left(\int_\Omega f(\zeta) \, \overline{\sigma_{m,j}(\zeta)} \, e^{-2m\varphi(\zeta)} \, d\,\lambda(\zeta) \right) \sigma_{m,j}(z), \qquad z \in \Omega.$$

The uniqueness of the decomposition into a linear combination of elements in an orthonormal basis implies the following simple observation.

Lemma 0.2.1 *A function $f \in \mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,m}$ if and only if*

$$\frac{1}{\int_\Omega |f|^2 \, e^{-2m\varphi}} \int_\Omega f \, \bar{\sigma}_{m,j} \, e^{-2m\varphi} = \frac{1}{\int_B |f|^2 \, e^{-2m\varphi}} \int_B f \, \bar{\sigma}_{m,j} \, e^{-2m\varphi},$$

for every $j \geq 1$.

We will be now studying the behaviour of the eigenvalues of $T_{B,m}$ as $m \to +\infty$. Let us fix an orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ made up of eigenvectors of $T_{B,m}$ corresponding respectively to its eigenvalues $\lambda_{m,1} \geq \lambda_{m,2} \geq \ldots$ listed nonincreasingly. Let $\varepsilon > 0$. Since $T_{B,m}$ is a compact operator, there are (if any) at most finitely many eigenvalues $\lambda_{m,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,N_m} \geq 1 - \varepsilon$. In other words,

$$\int_B |\sigma_{m,1}|^2 \, e^{-2m\varphi} \geq \cdots \geq \int_B |\sigma_{m,N_m}|^2 \, e^{-2m\varphi} \geq 1 - \varepsilon > \int_B |\sigma_{m,k}|^2 \, e^{-2m\varphi},$$

for every $k \geq N_m + 1$.

Lemma 0.2.1 above shows, in particular, that the restrictions to B of the $\sigma_{m,j}$'s are still orthogonal to one another. If we let

$$B_{m\varphi}(z) := K_{m\varphi}(z, z) = \sum_{j=1}^{+\infty} |\sigma_{m,j}(z)|^2 = \sup_{f \in B(1)} |f(z)|^2, \quad z \in \Omega, \quad (3)$$

where $B(1)$ is the unit ball in $\mathcal{H}_\Omega(m\varphi)$, the traces of $T_{B,m}$ and $T^2_{B,m}$ are easily computed as :

$$\mathrm{Tr}\,(T_{B,m}) = \int_B B_{m\varphi}(z)\, e^{-2m\varphi(z)}\, d\,\lambda(z), \quad \text{and}$$

$$\mathrm{Tr}\,(T^2_{B,m}) = \int_{B \times B} |K_{m\varphi}(z, \zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, d\,\lambda(z)\, d\,\lambda(\zeta).$$

We are thus naturally led to a search for asymptotic estimates of the Bergman kernel, preferably in the general context of possibly singular psh functions φ. Bearing in mind the analogy with the L^2 volume of a line bundle defined on a compact complex manifold in terms of global sections and characterized in terms of curvature currents, we set the following.

Definition 0.2.2 *The volume on $B \subset\subset \Omega$ of a psh function φ on Ω is defined as*

$$v_B(\varphi) := \limsup_{m \to +\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, d\,\lambda,$$

where $d\,\lambda$ is the Lebesgue measure.

It is our intention to study for which psh functions φ the volume $v_B(\varphi)$ is finite. The outcome will be a control of the number N_m of eigenvalues of $T_{B,m}$ which are $\geq 1 - \varepsilon$. These eigenvalues will be seen to correspond to the local generators of $\mathfrak{I}(m\varphi)$ whose growth rate is estimated in Theorem 0.1.1. The motivation lies in the following asymptotic estimates for the Bergman kernel associated with a smooth φ, due to N. Lindholm (cf. Theorems 10, 11, and 13 in [Lin01]), and subsequently rewritten in a slightly different setting by B.Berndtsson (cf. Theorems 2.3, 2.4, and 3.1 in [Ber03]). The standard Kähler form on $\mathbb{C}^n$ will be denoted throughout by ω.

Theorem 0.2.3 *(Lindholm, Berndtsson) Let φ be a C^∞ psh function such that $i\partial\bar\partial\varphi \geq C_0\,\omega$ on $\Omega \subset\subset \mathbb{C}^n$ for some constant $C_0 > 0$. Then :*

(a) $\quad v_B(\varphi) = \lim_{m \to +\infty} \dfrac{n!}{m^n} \int_B B_{m\varphi}(z)\, e^{-2m\varphi(z)}\, d\,\lambda(z) = \dfrac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n \; ;$

(b) *the sequence of measures on $\Omega \times \Omega$ defined as*

$$\frac{n!}{m^n} \, |K_{m\varphi}(z, \, \zeta)|^2 \, e^{-2m\varphi(z)} \, e^{-2m\varphi(\zeta)} \, d\,\lambda(z) \, d\,\lambda(\zeta)$$

converges to $\dfrac{2^n}{\pi^n} \, (i\partial\bar{\partial}\varphi)^n \wedge [\Delta]$ in the weak topology of measures, as $m \to +\infty$, where $[\Delta]$ is the current of integration on the diagonal of $\Omega \times \Omega$;

(c) finally, $\displaystyle\lim_{m\to+\infty} \frac{n!}{m^n} \sum_{j=1}^{+\infty} \lambda_{m,\,j} = \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{j=1}^{+\infty} \lambda_{m,\,j}^2 = \frac{2^n}{\pi^n} \int_B (i\partial\bar{\partial}\varphi)^n,$

and since $\lambda_{m,\,1} \geq \cdots \geq \lambda_{m,\,N_m} \geq 1 - \varepsilon > \lambda_{m,\,k}$, for $k \geq N_m + 1$, we get :

$$\lim_{m\to+\infty} \frac{n!}{m^n} \, N_m = \frac{2^n}{\pi^n} \int_B (i\partial\bar{\partial}\varphi)^n \quad and \quad \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{k=N_m+1}^{+\infty} \lambda_{m,\,k} = 0.$$

We now single out the major steps in the proof given to this theorem in [Ber03, p. 4-6], as they will be subsequently adapted to a more general context. The proof of (a) hinges on two facts. First, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $i\partial\bar{\partial}\varphi$ with respect to ω, the following pointwise convergence is established :

$$\lim_{m\to+\infty} \frac{1}{m^n} \, B_{m\varphi}(z) \, e^{-2m\varphi(z)} = \frac{2^n}{\pi^n} \, \lambda_1(z) \ldots \lambda_n(z), \quad z \in \Omega. \tag{4}$$

Indeed, having fixed an arbitrary point $x \in \Omega$, the quadratic part φ_0 in the Taylor expansion of φ near x is diagonalized in some local coordinates centred at x. While the distance between φ and φ_0 is under control, the mean-value inequality is applied to φ_0 to get the upper estimate of the left-hand side by the right-hand side. The converse estimate is obtained by means of Hörmander's L^2 estimates allowing one to construct elements in $\mathcal{H}_\Omega(m\varphi)$ with a prescribed value at x and a global L^2 norm under control. These are local procedures carried out on small balls centred at x, and the desired estimates are obtained in the limit while shrinking the balls to x.

Second, the mean-value inequality argument alluded to above also gives the following uniform estimate on a relatively compact open subset :

$$0 < \frac{n!}{m^n} \, B_{m\varphi}(z) \, e^{-2m\varphi(z)} \leq C \frac{2^n}{\pi^n} \, \lambda_1(z) \ldots \lambda_n(z), \quad z \in B, \tag{5}$$

if m is big enough, for a constant $C > 0$ independent of m. This is possible since φ is C^∞ and B is relatively compact. One can then conclude by dominated convergence.

We will now prove that Theorem 0.2.3 is still essentially valid if we allow analytic singularities for φ (see definition $(\star)$ in the introduction), provided the current $i\partial\bar{\partial}\varphi$ is replaced throughout by its absoulutely continuous part $(i\partial\bar{\partial}\varphi)_{ac}$ with respect to the Lebesgue measure in the Lebesgue decomposition of its measure coefficients (into an absolutely continuous part and a singular part). We clearly have $(i\partial\bar{\partial}\varphi)_{ac} = i\partial\bar{\partial}\varphi$ if φ is C^∞.

We will split the analysis of the analytic singularity case into two steps according to whether the coefficient of these singularities is an integer or not.

(a) Analytic singularities with an integer coefficient

In this case, we have a complete analogue of Theorem 0.2.3.

Theorem 0.2.4 *Let $\varphi = \frac{p}{2} \log(|g_1|^2 + \cdots + |g_N|^2) + u$ for some holomorphic functions $g_1, \ldots, g_N$, some $p \in \mathbb{N}$, and some C^∞ function u on $\Omega \subset\subset \mathbb{C}^n$. Assume $i\partial\bar\partial\varphi \geq C_0\,\omega$ for some constant $C_0 > 0$. Then, for $B \subset\subset \Omega$, we have :*

(a) $\displaystyle v_B(\varphi) = \lim_{m\to+\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, d\lambda = \frac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty.$

(b) the analogue of (b) in Theorem 0.2.3 holds, with convergence to

$$\frac{2^n}{\pi^n} (i\partial\bar\partial\varphi)^n_{ac} \wedge [\Delta] \qquad \text{in the weak topology of measures;}$$

(c) the analogue of (c) in Theorem 0.2.3 holds, with

$$\lim_{m\to+\infty} \frac{n!}{m^n} N_m = \frac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty.$$

The key observation in this new setting is that $(i\partial\bar\partial\varphi)^n_{ac}$ is of locally finite mass, and thus the integrals involving $(i\partial\bar\partial\varphi)^n_{ac}$ above are finite. Indeed, we can resolve the analytic singularities of φ by blowing up the ideal sheaf $\mathfrak{I}$ generated as an $\mathcal{O}_\Omega$-module by $g_1, \ldots, g_N$. According to Hironaka, there exists a proper modification $\mu : \tilde\Omega \to \Omega$, arising as a locally finite sequence of smooth-centred blow-ups, such that $\mu^\star\mathfrak{I} = \mathcal{O}(-D)$ for an effective divisor D on $\tilde\Omega$. We thus get the following Zariski decomposition of the pull-back current :

$$\mu^\star(i\partial\bar\partial\varphi) = \alpha + c\,[D], \qquad \text{on } \tilde\Omega,$$

with a C^∞ closed $(1, 1)$-form $\alpha \geq 0$, where $[D]$ stands for the D-supported current of integration on D. If

$$V := \{g_1 = \cdots = g_N = 0\}$$

is the singular set of φ, we clearly have $\int_B (i\partial\bar\partial\varphi)^n_{ac} = \int_{\mu^{-1}(B)} \alpha^n$, and this quantity is finite since the smooth volume form α^n has locally finite mass. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $i\partial\bar\partial\varphi$ with respect to ω on $\Omega \setminus V$, this means that the product $\lambda_1 \ldots \lambda_n$ is integrable on $B \setminus V$.

Proof of Theorem 0.2.4. The overall idea is to run Berndtsson's proof of the smooth case (cf. [Ber03], p. 4-6) on $\Omega \setminus V$ where φ is C^∞. Thanks to the local nature of its proof, the pointwise convergence (4) still holds at points $x \in \Omega \setminus V$. The main difficulty stems from the uniform estimate (5) not being immediately clear near the singular set V. We claim, however, that the analogous uniform upper estimate does hold outside the singular set, namely :

$$0 < \frac{n!}{m^n}\, B_{m\varphi}(z)\, e^{-2m\varphi(z)} \le C\, \frac{2^n}{\pi^n}\, \lambda_1(z) \dots \lambda_n(z), \quad z \in B \setminus V, \qquad (6)$$

if m is big enough, for a constant $C > 0$ independent of m.

Assuming this for the moment, the strong Noetherian property satisfied by the coherent sheaf $\mathcal{I}(m\varphi)$ implies that $B_m\, e^{-2m\varphi}$ is integrable on $B \subset\subset \Omega$, since it is dominated there by a constant multiplied by a finite sum $\sum |\sigma_{m,j}|^2\, e^{-2m\varphi}$. Its integral on B equals therefore its integral on $B \setminus V$ (as V is Lebesgue-negligible). Using now the key observation that $\lambda_1(z) \dots \lambda_n(z)$ is integrable on $B \setminus V$, (a) follows by dominated convergence as in the smooth case.

The proof of (b) in [Ber03] (p. 6-7) can be repeated on $\Omega \setminus V$ and extended across V in a similar way. Explicitly, what we have to prove is that for every compactly supported continuous function g on $\Omega \times \Omega$, we have :

$$\lim_{m \to +\infty} \frac{1}{m^n} \int\limits_{\Omega \times \Omega} g(z,\, \zeta)\, |K_{m\varphi}(z,\, \zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, d\lambda(z)\, d\lambda(\zeta) =$$

$$= \frac{2^n}{\pi^n\, n!} \int\limits_{\Omega \setminus V} g(z,\, z)\, (i\partial\bar{\partial}\varphi)^n(z).$$

Again, $(i\partial\bar{\partial}\varphi)^n_{ac}$ having locally finite mass on Ω implies the well-definedness of $(i\partial\bar{\partial}\varphi)^n_{ac} \wedge [\Delta]$ as a complex measure on $\Omega \times \Omega$. If $M := \sup |g|$, we notice that :

$$\left| \frac{1}{m^n} \int\limits_{\Omega} g(z,\, \zeta)\, |K_{m\varphi}(z,\, \zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, d\lambda(z) \right| \le$$

$$\le \frac{M}{m^n} \int\limits_{\Omega} |K_{m\varphi}(z,\, \zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, d\lambda(z) = \frac{M}{m^n}\, B_m(\zeta)\, e^{-2m\varphi(\zeta)}$$

$$\le 2\, M\, \frac{2^n}{\pi^n}\, \lambda_1(\zeta) \dots \lambda_n(\zeta), \qquad \zeta \in \Omega \setminus V,$$

where the equality above follows from the reproducing property of the Bergman kernel, and the last expression is locally integrable on $\Omega \setminus V$, by our key observation. By dominated convergence, it is then enough to prove that :

$$\lim_{m \to +\infty} \frac{1}{m^n} \int_\Omega g(z, \zeta) \, |K_{m\varphi}(z, \zeta)|^2 \, e^{-2m\varphi(z)} \, e^{-2m\varphi(\zeta)} \, d\lambda(z) =$$

$$= \frac{2^n}{\pi^n} \, g(\zeta, \zeta) \, \lambda_1(\zeta) \dots \lambda_n(\zeta), \qquad \zeta \in \Omega \setminus V.$$

To this end, we can repeat Berndtsson's arguments showing the Bergman kernel to decay rapidly off the diagonal, namely that for every $\varepsilon > 0$ and every $\zeta \in \Omega \setminus V$, we have :

$$\lim_{m \to +\infty} \frac{1}{m^n} \int_{|z - \zeta| > \varepsilon} |K_{m\varphi}(z, \zeta)|^2 \, e^{-2m\varphi(z)} \, e^{-2m\varphi(\zeta)} \, d\lambda(z) = 0.$$

The reproducing property of the Bergman kernel then leads to (b). Point (c) is an easy consequence of (a) and (b).

The whole proof of Theorem 0.2.4 thus boils down to establishing the uniform upper estimate (6). We will proceed in several steps.

Step 1. Assume $\varphi = \psi + \log|g|$ on Ω, for some $g \in \mathcal{O}(\Omega)$ such that div g is a normal crossing divisor, and for some smooth and strictly psh ψ on Ω. Then $V = \{g = 0\}$ and $i\partial\bar{\partial}\varphi = i\partial\bar{\partial}\psi$ on $\Omega \setminus V$.

Thus $f \in \mathcal{O}(\Omega)$ satisfies $1 = \int_\Omega |f|^2 \, e^{-2m\varphi} = \int_\Omega \frac{|f|^2}{|g^m|^2} \, e^{-2m\psi}$ if and only if $f = g^m \, h$ for some $h \in \mathcal{O}(\Omega)$ satisfying $\int_\Omega |h|^2 \, e^{-2m\psi} = 1$. This means that $B_{m\varphi} = |g|^{2m} \, B_{m\psi}$, and we get :

$$\frac{n!}{m^n} \, B_{m\varphi} \, e^{-2m\varphi} = \frac{n!}{m^n} \, B_{m\psi} \, e^{-2m\psi}, \qquad \text{on } \Omega.$$

This last expression satisfies the uniform upper estimate claimed in (6) on $B \subset\subset \Omega$ thanks to the Berndtsson-Lindholm inequality (5) applied to the smooth function ψ. Notice that, in particular, this yields :

$$v_B(\psi + \log|g|) = v_B(\psi).$$

Step 2. Assume φ has locally divisorial singularities ; namely, every point in Ω has a neighbourhood on which $\varphi = \psi + \log|g|$ for some holomorphic function g such that div g has normal crossings, and for some smooth strictly psh function ψ.

Let $U \subset\subset \Omega$ be a pseudoconvex such neighbourhood. It is clear that the unit ball of $\mathcal{H}_\Omega(m\varphi)$ injects into the unit ball of $\mathcal{H}_U(m\varphi_{|U})$ under restriction to U. Therefore, in the light of (3), we get :

11

$$\frac{n!}{m^n} \, B_{m\varphi} \, e^{-2m\varphi} \leq \frac{n!}{m^n} \, B_{m\varphi_{|U}} \, e^{-2m\varphi} = \frac{n!}{m^n} \, B_{m\psi_{|U}} \, e^{-2m\psi}, \qquad \text{on } U.$$

The last term satisfies the uniform upper estimate claimed in (6) on $U \setminus V$ thanks again to the smooth case applied to ψ. The resulting constant $C > 0$ depends on U, but we get a constant independent of m for the estimate on $B \setminus V$ after taking a finite covering of $\bar{B}$ by such open sets U.

Step 3. Assume $\varphi = \frac{p}{2} \log(|g_1|^2 + \cdots + |g_N|^2) + u$ on Ω, for some holomorphic functions $g_1, \ldots, g_N$ and some smooth function u.

Let $J := (g_1, \ldots, g_N) \subset \mathcal{O}_\Omega$ be the ideal sheaf generated by $g_1, \ldots, g_N$, and let $\mu : \tilde{\Omega} \to \Omega$ be a proper modification such that

$$\mu^\star J = \mathcal{O}(-E),$$

$\tilde{\Omega}$ is a smooth manifold, and E is an effective normal crossing divisor on $\tilde{\Omega}$. The divisor E can be locally represented as $E = \operatorname{div} g$, for some locally defined holomorphic function g. We then get, locally on $\tilde{\Omega}$:

$$\varphi \circ \mu = \frac{p}{2} \log \left(\left| \frac{g_1 \circ \mu}{g} \right|^2 + \cdots + \left| \frac{g_N \circ \mu}{g} \right|^2 \right) + u \circ \mu + \log |g^p|$$

$$= \psi + \log |g^p|,$$

where ψ is a smooth strictly psh function. Thus $\varphi \circ \mu$ has locally divisorial singularities and satisfies the hypotheses of Step 2. In passing from Bergman kernels defined on bigger sets to Bergman kernels defined on smaller sets, we shall need the following comparison lemma.

Lemma 0.2.5 *If $U \subset\subset \Omega$ is a pseudoconvex open subset, the Bergman kernels associated with the weight $m\varphi$ on Ω and respectively U :*

$$B_{m\varphi, \Omega} := \sum_{k=0}^{+\infty} |\sigma_{m,k}|^2, \qquad B_{m\varphi, U} := \sum_{k=0}^{+\infty} |\mu_{m,k}|^2,$$

defined by orthonormal bases $(\sigma_{m,k})_{k\in\mathbb{N}}$ and $(\mu_{m,k})_{k\in\mathbb{N}}$ of the Hilbert spaces $\mathcal{H}_\Omega(m\varphi)$ and respectively $\mathcal{H}_U(m\varphi)$, can be compared, for every $m \in \mathbb{N}$, as :

$$B_{m\varphi, \Omega} \leq B_{m\varphi, B} \leq C_{n,d,r} \, B_{m\varphi, \Omega} \qquad \text{on any } U_0 \subset\subset U \subset\subset \Omega,$$

where $C_{n,d,r} > 0$ is a constant depending only on n, the diameter d of Ω, and the distance $r > 0$ between the boundaries of U_0 and U.

Proof. As the restriction to U defines an injection of unit ball of $\mathcal{H}_\Omega(m\varphi)$ into the unit ball of $\mathcal{H}_B(m\varphi)$, the former inequality follows. For the latter

inequality, fix $x \in U_0$ and let $f \in \mathcal{O}(U)$ such that $\int_U |f|^2 e^{-2m\varphi} = 1$ be an arbitrary element in the unit sphere of $\mathcal{H}_U(m\varphi)$. We use Hörmander's L^2 estimates ([Hor65]) to construct a holomorphic function $F \in \mathcal{H}_\Omega(m\varphi)$ such that $F(x) = f(x)$ and

$$\int\limits_\Omega |F|^2 e^{-2m\varphi} \leq 2\left(1 + C_n \frac{d^{2n} e^{2d^2}}{r^{2(n+1)}}\right) \int\limits_U |f|^2 e^{-2m\varphi} = C_{n,d,r}.$$

This is done by solving the equation :

$$\bar{\partial} u = \bar{\partial}(\theta f) \quad \text{on } \Omega$$

with a cut-off function $\theta \in C^\infty(\mathbb{C}^n)$, $\operatorname{Supp} \theta \subset\subset U$, $\theta \equiv 1$ on $U_0 \subset\subset U$, and with the strictly psh weight $m\varphi + n \log|z - x| + |z - x|^2$. There exists a C^∞ solution u satisfying the estimate :

$$\int\limits_\Omega \frac{|u|^2}{|z - x|^{2n}} e^{-2m\varphi} e^{-2|z-x|^2} \leq 2 \int\limits_\Omega \frac{|\bar{\partial}\theta|^2 |f|^2}{|z - x|^{2n}} e^{-2m\varphi} e^{-2|z-x|^2}.$$

Due to the non-integrability of $|z - x|^{-2n}$ near x, we have $u(x) = 0$. Thus F is obtained as :

$$F := \theta f - u \in \mathcal{O}(\Omega).$$

Now $F/\sqrt{C_{n,d,r}}$ belongs to the unit ball of $\mathcal{H}_\Omega(m\varphi)$, and we get :

$$|f(x)|^2 = |F(x)|^2 \leq C_{n,d,r} B_{m\varphi,\Omega}(x),$$

which proves the latter inequality by taking the supremum over all f in the unit sphere of $\mathcal{H}_U(m\varphi)$. $\qquad\square$

We now resume Step 3 of the proof of Theorem 0.2.4. We may assume, without loss of generality, that the Jacobian J_μ of μ is globally defined on $\tilde{\Omega}$. Otherwise we can work on smaller open subsets of $\tilde{\Omega}$ contained in coordinate patches and the previous Lemma 0.2.5 shows that the Bergman kernels are only distorted by an insignificant constant $C_{n,d,r}$ independent of m. A change of variable shows that, for each $\sigma_{m,j}$, we have :

$$1 = \int\limits_\Omega |\sigma_{m,j}|^2 e^{-2m\varphi} d\lambda = \int\limits_{\Omega\setminus V} |\sigma_{m,j}|^2 e^{-2m\varphi} d\lambda$$

$$= \int\limits_{\tilde{\Omega}\setminus \operatorname{Supp} E} |\sigma_{m,j} \circ \mu|^2 |J_\mu|^2 e^{-2m\varphi\circ\mu} d\tilde{V} = \int\limits_{\tilde{\Omega}} |\sigma_{m,j} \circ \mu|^2 |J_\mu|^2 e^{-2m\varphi\circ\mu} d\tilde{V},$$

for a suitable volume form $d\tilde{V}$ on $\tilde{\Omega}$. Consequently, if $B_{m\varphi\circ\mu}$ is the Bergman

13

kernel associated with $m \varphi \circ \mu$ and the volume form $d\tilde{V}$ on $\tilde{\Omega}$, we have

$$B_{m\varphi\circ\mu} = |J_\mu|^2 B_{m\varphi} \circ \mu.$$

Thus, proving the upper estimate claimed in (6) amounts to proving :

$$\frac{n!}{m^n} B_{m\varphi\circ\mu} e^{-2m\varphi\circ\mu} \leq C \frac{2^n}{\pi^n} |J_\mu|^2 \lambda_1 \circ \mu \dots \lambda_n \circ \mu, \qquad (7)$$

on $\mu^{-1}(B) \setminus \operatorname{Supp} E$, since μ defines an isomorphism between $\tilde{\Omega} \setminus \operatorname{Supp} E$ and $\Omega \setminus V$, and J_μ does not vanish on $\tilde{\Omega} \setminus \operatorname{Supp} E$. If $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ are the eigenvalues of $i\partial\bar{\partial}(\varphi \circ \mu)$, it can easily be seen that

$$|J_\mu|^2 \lambda_1 \circ \mu \dots \lambda_n \circ \mu = \tilde{\lambda}_1 \dots \tilde{\lambda}_n.$$

Meanwhile, $\varphi \circ \mu$ has locally divisorial singularities, and Step 2 can be applied to get the uniform estimate (7) which proves the uniform estimate claimed in (6) and completes the proof of Theorem 0.2.4. $\qquad\square$

(b) Analytic singularities with arbitrary coefficients

Let us first consider the case of divisorial singularities with noninteger coefficients. For the sake of simplicity, we assume that our domains are polydiscs, $\Omega = D^n$ and $B = D_{1-\varepsilon}^n$, where D (respectively $D_{1-\varepsilon}$) is the unit disc (respectively the disc of radius $1 - \varepsilon$) in $\mathbb{C}$.

Proposition 0.2.6 *Assume that $\varphi(z) = \psi(z) + c_1 \log|z_1| + \cdots + c_n \log|z_n|$, with $\psi(z) = \psi_1(z_1) + \cdots + \psi_n(z_n)$ for some C^∞ functions ψ_j on $\mathbb{C}$ depending only on $|z_j|$ respectively, and some constants $c_j > 0$, $j = 1, \dots, n$. If $i\partial\bar{\partial}\varphi \geq C_0\omega$ for some $C_0 > 0$, then :*

$$v_B\left(\psi + \sum_{j=1}^n c_j \log|z_j|\right) = v_B(\psi) = \frac{2^n}{\pi^n} \int_B (i\partial\bar{\partial}\psi)^n < +\infty.$$

Proof. Only the first equality needs a proof. The second equality follows from Theorem 0.2.3 for smooth functions. To begin with, we shall prove that :

$$\int_B B_{m\varphi} e^{-2m\varphi} d\lambda \geq \int_B B_{m\psi} e^{-2m\psi} d\lambda, \quad \text{for every} \quad m \in \mathbb{N}^\star, \qquad (8)$$

which clearly implies that

$$v_B\left(\psi + \sum_{j=1}^n c_j \log|z_j|\right) \geq v_B(\psi). \qquad (9)$$

14

Fix $m \in \mathbb{N}^\star$, and let $a_j := \{m\, c_j\}$ for $j = 1, \ldots, n$, where $\{\cdot\}$ stands for the fractional part. As already noticed at Step 1 in the proof of Theorem 0.2.4, we have :

$$B_{m\varphi}\, e^{-2m\varphi} = B_{m\psi + a_1\, \log|z_1| + \cdots + a_n\, \log|z_n|}\, e^{-2m\psi - a_1\, \log|z_1| - \cdots - a_n\, \log|z_n|}.$$

This already implies that $v_B(\psi + \sum\limits_{j=1}^{n} c_j\, \log|z_j|) = v_B(\psi + \sum\limits_{j=1}^{n} \{c_j\}\, \log|z_j|)$, and thus we may assume, without loss of generality, that $0 \le c_j < 1$ for $j = 1, \ldots, n$.

Since $a_j < 1$ for every j, the exponential $e^{-2m\varphi}$ is easily seen to be locally integrable (the integral being a product of integrals depending each on one complex variable). Thus all the monomials $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, make up a complete orthogonal set of $\mathcal{H}_\Omega(m\varphi)$, which becomes an orthonormal basis after each monomial is normalized to have norm 1. We thus get :

$$B_{m\varphi}\, e^{-2m\varphi} = \sum_{\alpha_1, \ldots, \alpha_n = 0}^{+\infty} \prod_{j=1}^{n} \frac{|z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j}}{\displaystyle\int_D |z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)},$$

and the analogous formula for $m\psi$:

$$B_{m\psi}\, e^{-2m\psi} = \sum_{\alpha_1, \ldots, \alpha_n = 0}^{+\infty} \prod_{j=1}^{n} \frac{|z_j|^{2\alpha_j}\, e^{-2m\psi_j}}{\displaystyle\int_D |z_j|^{2\alpha_j}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}.$$

It is then enough to prove that, for every j, we have :

$$\frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j}\, d\lambda_1(z_j)}{\displaystyle\int_{D} |z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j}\, d\lambda_1(z_j)} \ge \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2\alpha_j}\, e^{-2m\psi_j}\, d\lambda_1(z_j)}{\displaystyle\int_{D} |z_j|^{2\alpha_j}\, e^{-2m\psi_j}\, d\lambda_1(z_j)}. \tag{10}$$

Taking polar coordinates $z_j = r_j\, \rho_j$, $0 \le r_j \le 1$, $\rho_j \in S^1$, we are then reduced to proving that, given a C^∞ function $u \ge 0$ and a constant $0 \le c < 1$, we have :

$$\frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_0^1 x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx} \ge \frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, dx}{\displaystyle\int_0^1 x^{2k+1}\, u(x)\, dx},$$

or equivalently, that :

$$\frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, dx} \geq \frac{\displaystyle\int_{1-\varepsilon}^1 x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_{1-\varepsilon}^1 x^{2k+1}\, u(x)\, dx},$$

which is clear since the left-hand side is $\geq \frac{1}{(1-\varepsilon)^{2c}}$, and the right-hand side is $\leq \frac{1}{(1-\varepsilon)^{2c}}$. Inequalities (8) and (9) are thus proved.

We shall now prove that :

$$v_B\Big(\psi + \sum_{j=1}^n c_j\, \log|z_j|\Big) \leq v_B\Big(\psi + \sum_{j=1}^n \log|z_j|\Big). \tag{11}$$

Let $v(z) := \psi(z) + \log|z_1| + \cdots + \log|z_n|$. The monomials $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$, with $\alpha_1,\ldots,\alpha_n \geq m$, make up a complete orthogonal system of $\mathcal{H}_\Omega(mv)$. Therefore,

$$B_{mv}(z)\, e^{-2mv(z)} = \sum_{\alpha_1,\ldots,\alpha_n=1}^{+\infty} \frac{|z_1|^{2(\alpha_1-1)}\ldots|z_n|^{2(\alpha_n-1)}}{\displaystyle\int_\Omega |z_1|^{2(\alpha_1-1)}\ldots|z_n|^{2(\alpha_n-1)}\, e^{-2m\psi(z)}}\, e^{-2m\psi(z)}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=1}^{+\infty} \prod_{j=1}^n \frac{|z_j|^{2(\alpha_j-1)}\, e^{-2m\psi_j(z_j)}}{\displaystyle\int_D |z_j|^{2(\alpha_j-1)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)},$$

which entails :

$$\frac{1}{m^n}\int_B B_{mv}(z)\, e^{-2mv(z)}\, d\lambda(z) = \frac{1}{m^n}\prod_{j=1}^n \sum_{\alpha_j=1}^{+\infty} \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-1)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}{\displaystyle\int_D |z_j|^{2(\alpha_j-1)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}$$

$$\geq \frac{1}{m^n}\prod_{j=1}^n \left(\sum_{\alpha_j=0}^{+\infty} \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-c_j)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}{\displaystyle\int_D |z_j|^{2(\alpha_j-c_j)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)} - \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{-2c_j}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}{\displaystyle\int_D |z_j|^{-2c_j}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)} \right)$$

$$\geq \frac{1}{m^n}\prod_{j=1}^n \left(\sum_{\alpha_j=0}^{+\infty} \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-c_j)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)}{\displaystyle\int_D |z_j|^{2(\alpha_j-c_j)}\, e^{-2m\psi_j(z_j)}\, d\lambda_1(z_j)} - 1 \right)$$

$$\geq \frac{1}{m^n}\int_B B_{m\varphi}\, e^{-2m\varphi}\, d\lambda_n - \frac{1}{m}\sum_{k=1}^n \frac{1}{m^{n-1}}\int_{D_{1-\varepsilon}^{n-1}} B_{mu_k}\, e^{-2mu_k}\, d\lambda_{n-1}$$

$$+ \frac{1}{m^2}\sum_{k_1,\, k_2} \frac{1}{m^{n-2}}\int_{D_{1-\varepsilon}^{n-2}} B_{mu_{k_1,k_2}}\, e^{-2mu_{k,k_2}}\, d\lambda_{n-2} - \cdots + \frac{(-1)^n}{m^n},$$

16

where we have denoted $u_k(z_1, \ldots, \hat{z}_k, \ldots z_n) := \varphi(z) - \psi_k(z_k) - c_k \log |z_k|$, and the analogous expressions when several indices are missing. The k-dimensional Lebesgue measure has been denoted $d\lambda_k$. Note that as we assume $0 \leq c_j < 1$, the first inequality above is implied by estimate (10) with α_j replaced with $\alpha_j - c_j$, and c_j replaced with $1 - c_j$.

We can thus run an induction on the dimension n. If we assume the finiteness of the volume for psh functions of the form under consideration defined in less than n variables, all the terms appearing on the right-hand side, except the first one, tend to 0 as $m \to +\infty$. Inequality (11) is then what we get in the limit.

Now, as already noticed (e.g. at Step 1 above),

$$v_B(\psi) = v_B(\psi + \sum_{j=1}^n \log |z_j|),$$

which, alongside inequalities (9) and (11), completes the proof. $\qquad \square$

We can now round off the study of the finiteness of the volume of a psh function in the case of general analytic singularities with arbitrary (not necessarily integer) coefficients.

Proposition 0.2.7 *Let $\varphi = \frac{c}{2} \log(|g_1|^2 + \cdots + |g_N|^2) + u$, for some holomorphic functions $g_1, \ldots, g_N$, some $c > 0$, and some C^∞ function u on $\Omega \subset\subset \mathbb{C}^n$. Assume $i\partial\bar{\partial}\varphi \geq C_0\,\omega$ for some constant $C_0 > 0$. Then :*

$$v_B(\varphi) := \limsup_{m \to +\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, d\lambda < +\infty.$$

Proof. Let $J = (g_1, \ldots, g_N) \subset \mathcal{O}_\Omega$, and let $\mu : \tilde{\Omega} \to \Omega$ be a proper modification such that $\tilde{\Omega}$ is smooth and $\mu^\star J = \mathcal{O}(-E)$ for an effective normal crossing divisor E on $\tilde{\Omega}$. The change of variable formula shows, as at Step 3 in the proof of Theorem 0.2.4, that

$$v_B(\varphi) = v_{\mu^{-1}(B)}(\varphi \circ \mu).$$

If we cover $\overline{\mu^{-1}(B)}$ by finitely many open polydiscs U_k such that $E = \mathrm{div} g_k$ on U_k, and use the observation made at Step 2 in the proof of Theorem 0.2.4 on the behaviour of Bergman kernels under restrictions, we see that :

$$v_{\mu^{-1}(B)}(\varphi \circ \mu) \leq \sum_k v_{U_k}(\varphi \circ \mu_{|U_k}).$$

Now, $\varphi \circ \mu_{|U_k}$ can be written in the form of functions φ considered in the previous Proposition 0.2.6. Therefore, $v_{U_k}(\varphi \circ \mu_{|U_k}) < +\infty$ for every k. This completes the proof. $\qquad \square$

The conclusion of these considerations about the volume of a psh function with analytic singularities is that if $\lambda_{m,1}, \ldots, \lambda_{m,N_m}$ are those eigenvalues of the concentration operator $T_{B,m}$ that are $\geq 1 - \varepsilon$, for some $0 < \varepsilon < 1$, then their number can be estimated as :

$$N_m \leq \frac{v_B(\varphi)}{1 - \varepsilon} \frac{m^n}{n!} = O(m^n), \quad \text{for} \quad m >> 1. \tag{12}$$

0.3 Effective local finite generation

In this section, we will prove Theorem 0.1.1. Let $(\sigma_{m,j})_{j \in \mathbb{N}^\star}$ be an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ made up of eigenvectors of the Toeplitz concentration operator $T_{B,m}$ discussed in the previous section. Recall that we are looking for finitely many among these that generate the ideal sheaf $\mathcal{I}(m\varphi)$ on $B \subset\subset \Omega$.

Definition 0.3.1 *A nonzero function $f \in \mathcal{H}_\Omega(m\varphi)$ is said to be δ-concentrated on B, for some $\delta > 0$, if $\dfrac{\int_B |f|^2 \, e^{-2m\varphi}}{\int_\Omega |f|^2 \, e^{-2m\varphi}} \geq 1 - \delta$.*

Since the restrictions to B of the $\sigma_{m,j}$'s are still orthogonal to one another (cf. lemma 0.2.1), an element f in the subspace of $\mathcal{H}_\Omega(m\varphi)$ generated by $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$ is ε-concentrated on B, like its generators. The following simple observation shows that the converse is not far from being true.

Lemma 0.3.2 *Let $f \in \mathcal{H}_\Omega(m\varphi)$ be ε^2-concentrated on B. Then, there is an element g in the subspace of $\mathcal{H}_\Omega(m\varphi)$ generated by the ε-concentrated $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$ such that $\int_\Omega |f - g|^2 \, e^{-2m\varphi} < \varepsilon \int_\Omega |f|^2 \, e^{-2m\varphi}$.*

Proof. We may assume that $\int_\Omega |f|^2 \, e^{-2m\varphi} = 1$. Let $f = \sum\limits_{j=1}^{N_m} a_j \, \sigma_{m,j} + \sum\limits_{k=N_m+1}^{+\infty} a_k \, \sigma_{m,k}$ be the decomposition of f with respect to the chosen orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, where $a_j, a_k \in \mathbb{C}$. Then, $\sum\limits_{j=1}^{+\infty} |a_j|^2 = 1$, and

$$\int_B |f|^2 \, e^{-2m\varphi} = \sum_{j=1}^{+\infty} |a_j|^2 \int_B |\sigma_{m,j}|^2 \, e^{-2m\varphi}. \text{ We then get :}$$

$$\int_{\Omega \setminus B} |f|^2 \, e^{-2m\varphi} = \sum_{j=1}^{+\infty} |a_j|^2 \int_{\Omega \setminus B} |\sigma_{m,j}|^2 \, e^{-2m\varphi} \leq \varepsilon^2.$$

Since $\int_{\Omega \setminus B} |\sigma_{m,k}|^2 \, e^{-2m\varphi} > \varepsilon$, for every $k \geq N_m + 1$, we get $\sum\limits_{k=N_m+1}^{+\infty} |a_k|^2 < \varepsilon$.

If we set $g := \sum\limits_{j=1}^{N_m} a_j \, \sigma_{m,j}$, the lemma is proved. $\qquad\qquad \square$

This strongly suggests where to turn for the most likely choice of finitely many local generators for $\mathfrak{I}(m\varphi)$. For a small given $\varepsilon > 0$, let $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$ be those elements which are ε-concentrated on B (cf. Definition 0.3.1). We will now show that, if ε is well chosen and m is big enough, the ideal sheaf $\mathfrak{I}(m\varphi)$ is generated, on a relatively compact open subset, by $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$, with an effective control of the coefficients. We shall proceed inductively. The crux is the following approximation to order one of a local section of $\mathfrak{I}(m\varphi)$ by a finite linear combination of $\sigma_{m,j}$'s. The standard Kähler form on $\mathbb{C}^n$ is denoted throughout by ω.

Lemma 0.3.3 *Let φ be a strictly psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$ such that $i\partial\bar{\partial}\varphi \geq C_0\,\omega$ for some constant $C_0 > 0$. Let $B := B(x, r) \subset\subset \Omega$ be an arbitrary open ball. Then, there exist $\varepsilon > 0$ and $m_0 = m_0(C_0) \in \mathbb{N}$ such that, for every $m \geq m_0$, the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to the ε-concentrated $\sigma_{m,j}$'s, a decomposition :*

$$g(z) = \sum_{j=1}^{N_m} c_j\,\sigma_{m,j}(z) + \sum_{l=1}^{n}(z_l - x_l)\,h_l(z), \qquad z \in B,$$

with some $c_j \in \mathbb{C}$ satisfying $\displaystyle\sum_{j=1}^{N_m} |c_j|^2 \leq C\,N_m \int_B |g|^2\,e^{-2m\varphi}$, and some holomorphic functions h_l on B, satisfying :

$$\sum_{l=1}^{n} \int_B |h_l|^2\,e^{-2m\varphi} \leq C \int_B |g|^2\,e^{-2m\varphi},$$

where $C > 0$ is a constant depending only on n, on r, and on the diameter of Ω.

Proof. We may assume, without loss of generality, that $x = 0$. Let $\theta \in C^\infty(\Omega), 0 \leq \theta \leq 1$, be a cut-off function such that $\mathrm{Supp}\,\theta \subset B$, $\theta \equiv 1$ on some arbitrary $B' := B(0, r') \subset\subset B$, and $|\bar{\partial}\theta| \leq \frac{3}{r}$. Let us fix some $g \in \mathcal{H}_B(m\varphi)$ such that $C_g := \int_B |g|^2\,e^{-2m\varphi} < +\infty$. We will now use Hörmander's L^2 estimates ([Hor65]) to solve the equation :

$$\bar{\partial}u = \bar{\partial}(\theta g), \qquad \text{on } \Omega,$$

with the weight $m\varphi(z) + (n+1)\log|z|$. We get a solution $u \in C^\infty(\Omega)$ such that :

$$\int_\Omega \frac{|u|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi} \leq \frac{1}{C_0\,m} \int_\Omega \frac{|\bar{\partial}\theta|^2\,|g|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi} \leq \frac{9}{r^2}\,\frac{1}{C_0\,m} \int_{B\setminus B'} \frac{|g|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi}.$$
$$(13)$$

Put $F_m := \theta g - u \in \mathcal{H}_\Omega(m\varphi)$ and get the decomposition $g = F_m + (g - F_m)$ on B. The above estimate for u implies :

$$\int_B \frac{|g - F_m|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} \leq 2 \int_{B\setminus B'} \frac{|1 - \theta|^2 |g|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} + 2 \int_B \frac{|u|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} \quad (14)$$

$$\leq C(r,\, r',\, d)\, C_g,$$

with a constant $C(r,\, r',\, d) > 0$ depending only on r, r', and the diameter d of Ω, if m is chosen so big that $\frac{1}{C_0\, m} < 1$.

We shall now apply Skoda's L^2 division theorem (cf. [Sko72]) to obtain :

$$g(z) - F_m(z) = \sum_{l=1}^n z_l\, v_l(z), \quad z \in B, \quad (15)$$

for some holomorphic functions v_l on B satisfying $\sum_{l=1}^n \int_B \frac{|v_l|^2}{|z|^{2n}}\, e^{-2m\varphi} \leq 2\, C(r,\, r',\, d)\, C_g$. Since $|z|^{2n} \leq r^{2n}$, for $z \in B$, we get :

$$\sum_{l=1}^n \int_B |v_l|^2\, e^{-2m\varphi} \leq C_1(r,\, r',\, d)\, C_g, \quad (16)$$

where $C_1(r,\, r',\, d) = 2\, r^{2n}\, C(r,\, r',\, d)$. As for F_m, the obvious pointwise inequality $|F_m|^2 \leq 2(|\theta\, g|^2 + |u|^2)$, combined with estimate (13) for u, gives :

$$\int_B |F_m|^2\, e^{-2m\varphi} \leq C(r,\, r',\, d)\, C_g,$$

after absorbing an extra $d^{2(n+1)}$ in the constant $C(r,\, r',\, d)$. On the other hand, the factor $\frac{1}{C_0\, m}$ in the estimate (13) for u shows that if m is chosen big enough, the L^2 norm of u on Ω is very small in comparison with the L^2 norm of g on B. This is where the strict psh assumption on φ comes in. Since $F_m = g - u$ on B' and $F_m = -u$ on $\Omega \setminus B$, we get, for m big enough and some constant $C_1 = C_1(C_0) > 0$ independent of m :

$$1 - \frac{C_1}{m} \leq \frac{\int_B |F_m|^2\, e^{-2m\varphi}}{\int_\Omega |F_m|^2\, e^{-2m\varphi}} < 1.$$

In other words, F_m is $\frac{C_1}{m}$-concentrated on B. Fix some small $\varepsilon > 0$ whose choice will be explained later. If m_ε is chosen such that $\frac{C_1}{m} < \varepsilon^2$ for $m \geq m_\varepsilon$, then F_m is, in particular, ε^2-concentrated on B. Then, lemma 0.3.2 shows that in the decomposition :

$$F_m(z) = \sum_{j=1}^{N_m} a_j\, \sigma_{m,\, j}(z) + \sum_{k=N_m+1}^{+\infty} a_k\, \sigma_{m,\, k}(z), \quad z \in \Omega,$$

of F_m with respect to the chosen orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, we have :

$$\sum_{k=N_m+1}^{+\infty} |a_k|^2 \le C(r,\, r',\, d)\, \varepsilon\, C_g, \qquad \sum_{j=1}^{N_m} |a_j|^2 \le C(r,\, r',\, d)\, C_g. \qquad (17)$$

If we set $g_1(z) := \sum\limits_{k=N_m+1}^{+\infty} a_k\, \sigma_{m,k}(z)$, (15) gives the decomposition :

$$g(z) = \sum_{j=1}^{N_m} a_j\, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l\, v_l(z) + g_1(z), \qquad z \in B, \qquad (18)$$

with an effective control of the a_j's and v_l's (see (17) and (16)), and such that :

$$\int_B |g_1|^2\, e^{-2m\varphi} \le \int_\Omega |g_1|^2\, e^{-2m\varphi} = \sum_{k=N_m+1}^{+\infty} |a_k|^2 \le C(r,\, r',\, d)\, \varepsilon\, C_g.$$

Our aim is clearly to get rid of this "small" g_1 in (18). The idea is to iterate the previous procedure with g_1 instead of g such that the unwanted term in the decomposition (18) becomes smaller and smaller at each step. We shall see that if m is chosen big enough and fixed, this error term disappears as the number of iterations tends to $+\infty$.

If we replace g by g_1, we get the following decomposition analogous to (18) :

$$g_1(z) = \sum_{j=1}^{N_m} a_{1,j}\, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l\, v_{1,l}(z) + g_2(z), \qquad z \in B, \qquad \text{with}$$

$$\int_B |g_2|^2\, e^{-2m\varphi} \le \Big(C(r,\, r',\, d)\, \varepsilon \Big)^2 C_g,$$

$$\sum_{j=1}^{N_m} |a_{1,j}|^2 \le C(r,\, r',\, d) \Big(C(r,\, r',\, d)\, \varepsilon \Big) C_g,$$

$$\sum_{l=1}^{n} \int_B |v_{1,l}|^2\, e^{-2m\varphi} \le C_1(r,\, r',\, d) \Big(C(r,\, r',\, d)\, \varepsilon \Big) C_g,$$

which further gives :

$$g(z) = \sum_{j=1}^{N_m} \Big(a_j + a_{1,j} \Big)\, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l \Big(v_l(z) + v_{1,l}(z) \Big) + g_2(z),$$

for every $z \in B$. We can now indefinitely iterate this procedure with g_2 instead of the original g. After p iterations, we get the decomposition :

$$g(z) = \sum_{j=1}^{N_m} \Big(a_j + a_{1,j} + \cdots + a_{p-1,j} \Big)\, \sigma_{m,j}(z) +$$

$$+ \sum_{l=1}^{n} z_l \Big(v_l(z) + v_{1,l}(z) + \cdots + v_{p-1,l}(z) \Big) + g_p(z),$$

for every $z \in B$, with estimates :

$$\int_B |g_p|^2 \, e^{-2m\varphi} \leq \Big(C(r, \, r', \, d) \, \varepsilon \Big)^p C_g, \qquad (19)$$

$$\sum_{j=1}^{N_m} |a_{p-1,j}|^2 \leq C(r, \, r', \, d) \Big(C(r, \, r', \, d) \, \varepsilon \Big)^{p-1} C_g,$$

$$\sum_{l=1}^{n} \int_B |v_{p-1,l}|^2 \, e^{-2m\varphi} \leq C_1(r, \, r', \, d) \Big(C(r, \, r', \, d) \, \varepsilon \Big)^{p-1} C_g,$$

Let us now set $\varepsilon_0 := C(r, \, r', \, d) \, \varepsilon$, $c_j^{(p)} := a_j + a_{1,j} + \cdots + a_{p-1,j}$, and $v_l^{(p)} := v_l + v_{1,l} + \cdots + v_{p-1,l}$. On the one hand, we get :

$$\Big(\sum_{j=1}^{N_m} |c_j^{(p)}|^2 \Big)^{\frac{1}{2}} \leq \sum_{j=1}^{N_m} |c_j^{(p)}| \leq \sum_{j=1}^{N_m} |a_j| + \sum_{j=1}^{N_m} |a_{1,j}| + \cdots + \sum_{j=1}^{N_m} |a_{p-1,j}|$$

$$\leq \Big(N_m \, C(r, \, r', \, d) \, C_g \Big)^{\frac{1}{2}} (1 + \varepsilon_0^{\frac{1}{2}} + \cdots + \varepsilon_0^{\frac{p-1}{2}}) = \Big(N_m \, C(r, \, r', \, d) \, C_g \Big)^{\frac{1}{2}} \frac{1 - \varepsilon_0^{\frac{p}{2}}}{1 - \varepsilon_0^{\frac{1}{2}}}.$$

On the other hand, if $|| \cdot ||$ stands for the $e^{-2m\varphi}$-weighted L^2 norm, we get :

$$\sum_{l=1}^{n} ||v_l^{(p)}|| \leq \sqrt{n} \left[(\sum_{l=1}^{n} ||v_l||^2)^{\frac{1}{2}} + (\sum_{l=1}^{n} ||v_{1,l}||^2)^{\frac{1}{2}} + \cdots + (\sum_{l=1}^{n} ||v_{p-1,l}||^2)^{\frac{1}{2}} \right]$$

$$\leq \sqrt{n \, C_1(r, \, r', \, d) \, C_g} \, (1 + \varepsilon_0^{\frac{1}{2}} + \cdots + \varepsilon_0^{\frac{p-1}{2}}) = \sqrt{n \, C_1(r, \, r', \, d) \, C_g} \, \frac{1 - \varepsilon_0^{\frac{p}{2}}}{1 - \sqrt{\varepsilon_0}},$$

by using the Cauchy-Schwarz inequality and the previous estimates on $v_{k,l}$'s. We finally get :

$$\sum_{l=1}^{n} ||v_l^{(p)}||^2 \leq \Big(\sum_{l=1}^{n} ||v_l^{(p)}|| \Big)^2 \leq n \, C_1(r, \, r', \, d) \left(\frac{1 - \varepsilon_0^{\frac{p}{2}}}{1 - \sqrt{\varepsilon_0}} \right)^2 C_g.$$

for every $p \in \mathbb{N}^\star$. Let us now choose ε so small that $\varepsilon_0 := C(r, \, r', \, d) \, \varepsilon = \frac{1}{4}$. By letting $p \to +\infty$, the estimate (19) shows that g_p disappears from the expression of g, and we get :

$$g(z) = \sum_{j=1}^{N_m} c_j \, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l h_l(z), \qquad z \in B,$$

with $c_j \in \mathbb{C}$ obtained as limits of $c_j^{(p)}$, and holomorphic functions h_l on B obtained as limits of $v_l^{(p)}$, satisfying the estimates in the statement, after possibly replacing $4 \, C(r, \, r', \, d)$ and $4 \, n \, C_1(r, \, r', \, d)$ by the maximum C of these two constants. Lemma 0.3.3 is proved. $\qquad \square$

We can now run an induction argument using lemma 0.3.3 repeatedly to get, at every step p, an approximation to order p of the original local section g of $\mathcal{I}(m\varphi)$ by a finite linear combination of $\sigma_{m,j}$'s. The following is a slightly more precise rewording of the first part of Theorem 0.1.1.

Theorem 0.3.4 *Given φ such that $i\partial\bar\partial\varphi \geq C_0\,\omega$, and having fixed a ball $B := B(x,\,r) \subset\subset \Omega$, there exist a ball $B(x,\,r_0) \subset\subset B(x,\,r)$, $\varepsilon > 0$, and $m_0 = m_0(C_0) \in \mathbb{N}$ such that for every $m \geq m_0$ the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to the ε-concentrated $\sigma_{m,j}$'s in an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ made up of eigenvectors of $T_{B,m}$, a decomposition :*

$$g(z) = \sum_{j=1}^{N_m} b_{m,j}(z)\,\sigma_{m,j}(z), \qquad z \in B(x,\,r_0),$$

with some holomorphic functions $b_{m,j}$ on $B(x,\,r_0)$, satisfying :

$$\sup_{B(x,r_0)} \sum_{j=1}^{N_m} |b_{m,j}|^2 \leq C\,N_m \int_B |g|^2\,e^{-2m\varphi} < +\infty,$$

where $C > 0$ is a constant depending only on n, r, and the diameter of Ω.

Proof. We may assume, without loss generality, that $x = 0$. Fix $\varepsilon > 0$ as in lemma 0.3.3, associated with an arbitrary fixed open ball $B(0,\,r') \subset\subset B(0,\,r)$. Let $g \in \mathcal{H}_B(m\varphi)$ with $C_g := \int_B |g|^2\,e^{-2m\varphi} < +\infty$. Lemma 0.3.3 gives a decomposition $g(z) = \sum_{j=1}^{N_m} a_j\,\sigma_{m,j}(z) + \sum_{l_1=1}^{n} z_{l_1}\,h_{l_1}(z)$ for every $z \in B(0,\,r)$, with coefficients under control. We now apply lemma 0.3.3 to every function h_{l_1}, $l_1 = 1,\ldots,n$, and get :

$$h_{l_1}(z) = \sum_{j=1}^{N_m} a_{j,l_1}\,\sigma_{m,j}(z) + \sum_{l_2=1}^{n} z_{l_2}\,h_{l_1,l_2}(z), \qquad z \in B(0,\,r),$$

with constant coefficients $a_{j,l_1} \in \mathbb{C}$, satisfying :

$$\sum_{l_1=1}^{n}\sum_{j=1}^{N_m} |a_{j,l_1}|^2 \leq C\,N_m \sum_{l_1=1}^{n} \int_B |h_{l_1}|^2\,e^{-2m\varphi} \leq C^2\,N_m\,C_g,$$

and holomorphic functions h_{l_1,l_2} on B, satisfying :

$$\sum_{l_1=1}^{n}\sum_{l_2=1}^{n} \int_B |h_{l_1,l_2}|^2\,e^{-2m\varphi} \leq C \sum_{l_1=1}^{n} \int_B |h_{l_1}|^2\,e^{-2m\varphi} \leq C^2\,C_g.$$

We thus obtain, after p applications of lemma 0.3.3, the decomposition :

$$g(z) = \sum_{j=1}^{N_m} \left(a_j + \sum_{\nu=1}^{p-1} \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} \, z_{l_1} \ldots z_{l_\nu} \right) \sigma_{m,j}(z) + \sum_{l_1,\ldots,l_p=1}^{n} z_{l_1} \ldots z_{l_p} \, v_{l_1,\ldots,l_p}(z),$$

for every $z \in B(0, r)$, with coefficients $a_{j,l_1,\ldots,l_\nu=1} \in \mathbb{C}$ and $v_{l_1,\ldots,l_p} \in \mathcal{O}(B(0, r))$, satisfying the estimates :

$$\sum_{l_1,\ldots,l_\nu=1}^{n} \sum_{j=1}^{N_m} |a_{j,l_1,\ldots,l_\nu}|^2 \leq C^{\nu+1} \, N_m \, C_g, \qquad \nu = 1,\ldots,p-1,$$

$$\sum_{l_1,\ldots,l_p=1}^{n} \int_B |v_{l_1,\ldots,l_p}|^2 \, e^{-2m\varphi} \leq C^p \, C_g.$$

We now put $b_{m,j}(z) := a_j + \sum_{\nu=1}^{+\infty} \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} \, z_{l_1} \ldots z_{l_\nu}$, for $j = 1,\ldots,N_m$, and will prove that the series defining $b_{m,j}$ converges to a holomorphic function on some smaller ball $B_0 := B(0, r_0)$, and that, with an appropriate choice of r_0, we have :

$$\sup_{B(0,r_0)} \sum_{j=1}^{N_m} |b_{m,j}|^2 \leq \frac{1}{(1-r/d)^2} \, C \, N_m \, C_g.$$

Since $\sup_{B(0,r_0)} |z_{l_1} \ldots z_{l_\nu}|^2 \leq r_0^{2\nu}$, we get, for every $1 \leq \nu < +\infty$ and every $z \in B(0, r_0)$, the estimate :

$$\sum_{j=1}^{N_m} \left| \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} \, z_{l_1} \ldots z_{l_\nu} \right|^2 \leq r_0^{2\nu} \sum_{j=1}^{N_m} \left(\sum_{l_1,\ldots,l_\nu=1}^{n} |a_{j,l_1,\ldots,l_\nu}| \right)^2$$

$$\leq r_0^{2\nu} \, n^\nu \sum_{j=1}^{N_m} \sum_{l_1,\ldots,l_\nu=1}^{n} |a_{j,l_1,\ldots,l_\nu}|^2 \leq \left(n \, r_0^2 \, C \right)^\nu C \, N_m \, C_g$$

$$= \left(\frac{r}{d} \right)^{2\nu} C \, N_m \, C_g,$$

if we choose $r_0 = \dfrac{r}{d} \dfrac{1}{\sqrt{n\,C}}$. The remaining arguments are purely formal. Put :

$$F_{\nu,j} := \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} \, z_{l_1} \ldots z_{l_\nu}, \quad \text{for } \nu \geq 1 \text{ and } j = 1,\ldots,N_m,$$

$$F_{0,j} = a_j, \quad \text{for } j = 1,\ldots,N_m.$$

Since $b_{m,j} = \sum_{\nu=0}^{+\infty} F_{\nu,j}$, for $j = 1,\ldots,N_m$, we get :

$$\left(\sum_{j=1}^{N_m} |b_{m,j}|^2\right)^{\frac{1}{2}} \leq \sum_{\nu=0}^{+\infty} \left(\sum_{j=1}^{N_m} |F_{\nu,j}|^2\right)^{\frac{1}{2}} \leq \sum_{\nu=0}^{+\infty} \left(\frac{r}{d}\right)^{\nu} \sqrt{C\, N_m\, C_g}$$

$$= \frac{1}{1 - r/d}\, \sqrt{C\, N_m\, C_g},$$

at every point in $B(0, r_0)$. If we absorb the denominator in the constant $C > 0$, the proof is complete. $\square$

Remark 0.3.5 *In retrospect, we see that estimate (12) which concluded section 0.2 clearly proves the last statement of Theorem 0.1.1 in the introduction. The proof of this theorem is now complete.*

Corollary 0.3.6 *Under the hypotheses of Theorem 0.3.4, the following estimate holds :*

$$\sum_{j=1}^{+\infty} |\sigma_{m,j}(z)|^2 \leq C\, N_m \sum_{j=1}^{N_m} |\sigma_{m,j}(z)|^2, \qquad z \in B(x, r_0).$$

Proof. As $\displaystyle\sum_{j=1}^{+\infty} |\sigma_{m,j}(z)|^2 = \sup_{f \in \bar{B}_m(1)} |f(z)|^2$, where $\bar{B}_m(1)$ is the closed unit ball in $\mathcal{H}_\Omega(m\varphi)$, and as every such f has a decomposition as in Theorem 0.3.4 on $B(x, r_0)$, the estimate follows from the Cauchy-Schwarz inequality. $\square$

0.4 Regularization of plurisubharmonic functions

We now turn to the second part of the paper. A by now classical result of Demailly's ([Dem92], *Proposition 3.1.*) states that a psh function φ with arbitrary singularities can be approximated by psh functions φ_m with analytic singularities (cf. $(\star)$), constructed as

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{j=1}^{+\infty} |\sigma_{m,j}(z)|^2 = \sup_{f \in \bar{B}_m(1)} \frac{1}{m} \log |f(z)|, \quad z \in \Omega, \qquad (20)$$

where $(\sigma_{m,j})_{j \in \mathbb{N}^\star}$ is an arbitrary orthonormal basis, and $\bar{B}_m(1)$ is the unit ball, of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$ considered in the previous sections. We even have :

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log(\frac{C_2}{r^n}), \qquad (21)$$

for every $z \in \Omega$ and every $r < d(z, \partial\Omega)$. The lower estimate is a consequence of the Ohsawa-Takegoshi L^2 extension theorem. The upper estimate is far

easier, coming from an application of the submean value inequality satisfied by squares of absolute values of holomorphic functions. Our aim is to improve the upper estimate by replacing the supremum by a pointwise upper bound which affords a much better understanding of singularities. This is not possible for an arbitrary φ, but the following proposition shows it to be possible if φ is assumed to have analytic singularities.

Theorem 0.4.1 *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and let $\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2)$ be a plurisubharmonic function with analytic singularities on Ω. If $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ are relatively compact open subsets, then for every $\delta > 0$ and every $m \geq \frac{n+2}{c\,\delta}$, we have :*

$$\varphi_m(z) \leq (1-\delta)\,\varphi(z) + c\,\delta\,\log A + \frac{\log(C_n\,(m\,c-n))}{m}, \qquad z \in \Omega',$$

where $A := \max\{\sup_{\Omega''}\sqrt{|g_1|^2 + \cdots + |g_N|^2}, 1\}$, and $C_n > 0$ is a constant depending only on Ω', Ω'', and n.

Remark 0.4.2 The upper bound given in Theorem 0.4.1, combined with Demailly's lower bound in inequality 21, implies that φ and its regularizations φ_m have the same $-\infty$ poles on Ω', if m is big enough. Only the Lelong numbers may be slightly different. Since the set of poles of an arbitrary psh function φ is not necessarily analytic, whereas the polar set of φ_m is always analytic, we see that some analyticity assumption on φ is necessary. We may ask for the weakest such assumption on φ under which Theorem 0.4.1 holds.

Proof. Fix $0 < \delta < 1$. Let $m \geq \frac{n+2}{c\,\delta}$, and let f be an arbitrary holomorphic function on Ω such that :

$$\int_\Omega |f|^2\,e^{-2m\varphi} = 1 \quad \Leftrightarrow \quad \int_\Omega \frac{|f|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc}} = 1.$$

We clearly have $mc \geq n + 2$; put $q := [mc] - (n+1) \geq 1$. Then Skoda's L^2 division theorem ([Sko72]) gives the existence of holomorphic functions $h_{i_1,\ldots,i_q}$ on Ω, for all multiindices $(i_1,\ldots,i_q) \in \{1,\ldots,N\}^q$, such that :

$$f(z) = \sum_{i_1,\ldots,i_q=1}^{N} h_{i_1,\ldots,i_q}(z)\,g_{i_1}(z)\ldots g_{i_q}(z), \qquad z \in \Omega,$$

and

$$\sum_{i_1,\ldots,i_q=1}^{N} \int_\Omega \frac{|h_{i_1,\ldots,i_q}|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc-q}} \leq$$

$$\leq \frac{m\,c-n}{m\,c-[m\,c]+1}\int_\Omega \frac{|f|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc}} \leq m\,c-n.$$

Put $|g|^2 := |g_1|^2 + \cdots + |g_N|^2$. As Ω'' satisfies $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, we get, for every $z \in \Omega'$:

$$|f(z)|^2 \leq |g(z)|^{2q} \sum_{i_1,\ldots,i_q=1}^{N} |h_{i_1,\ldots,i_q}(z)|^2 \leq$$

$$\leq C_n \, |g(z)|^{2q} \sum_{i_1,\ldots,i_q=1}^{N} \int_{\Omega''} \frac{|h_{i_1,\ldots,i_q}|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc-q}} \, |g|^{2(mc-q)}$$

$$\leq C_n \, (m \, c - n) \, (\sup_{\Omega''} |g|)^{2(mc-q)} \, |g(z)|^{2q},$$

where $C_n > 0$ is the constant involved in the submean value inequality applied to every $|h_{i_1,\ldots,i_q}|^2$ at z, and thus depending only on the open sets Ω', Ω'', and on n, but independent of m. We can see, since $A = \max\{\sup_{\Omega''} |g|, 1\}$, that :

$$C_n \, (m \, c - n) \, (\sup_{\Omega''} |g|)^{2(mc-q)} \leq C_n \, (m \, c - n) \, A^{2(mc-q)}.$$

We thus get :

$$|f(z)|^2 \, e^{-2m(1-\delta)\varphi(z)} \leq C_n \, (m \, c - n) \, A^{2(mc-q)} \frac{|g(z)|^{2([mc]-n-1)}}{|g(z)|^{2mc(1-\delta)}}$$

$$= C_n \, (m \, c - n) \, A^{2(mc-q)} \, |g(z)|^{2(mc\delta-n-1-(mc-[mc]))}, \qquad z \in \Omega'.$$

Since m has been chosen such that $mc\delta - n - 2 \geq 0$, the exponent $2(mc\delta - n - 1 - (mc - [mc]))$ above is positive, and we thus finally get :

$$|f(z)|^2 \, e^{-2m(1-\delta)\varphi(z)} \leq C_n \, (m \, c - n) \, A^{2(mc-q)} \, A^{2(mc\delta-n-1-(mc-[mc]))}$$

$$= C_n \, (m \, c - n) \, A^{2(mc\delta-n-1+[mc]-q)} = C_n \, (m \, c - n) \, A^{2mc\delta}, \qquad z \in \Omega'.$$

By taking log and dividing by $2m$ on both sides, we get :

$$\varphi_m(z) \leq (1 - \delta) \, \varphi(z) + c\delta \, \log A + \frac{\log(C_n(m \, c - n))}{m}, \qquad z \in \Omega',$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 0.4.3 *If in Theorem 0.4.1 the psh function with analytic singularities is given, more generally, as $\varphi = \frac{c}{2} \log(|g_1|^2 + \cdots + |g_N|^2) + v$, for some C^∞ psh function v, then the estimate is modified as :*

$$\varphi_m(z) \leq (1-\delta) \, \varphi(z) + c \, \delta \, \log A + \frac{\log(C(m \, c - n))}{m} + \sup_{\Omega''} v - (1-\delta) \inf_{\Omega''} v,$$

for every $z \in \Omega'$ and every $m \geq \frac{n+2}{c\,\delta}$.

Theorem 0.1.2 stated in the Introduction appears now as a consequence of Theorem 0.4.1.

Proof of Theorem 0.1.2. As repeatedly pointed out above, the ideal sheaf $\mathcal{I}(m\varphi)$ is generated as an $\mathcal{O}_\Omega$-module by the Hilbert space $\mathcal{H}_\Omega(m\varphi)$. Now, by the definition (20) of φ_m, the estimate obtained in Theorem 0.4.1 implies that every $f \in \mathcal{H}_\Omega(m\varphi)$ with norm 1 satisfies :

$$|f(z)|^2 \leq C\,(m\,c - n)\,A^{2\,m\,c\,\delta}\,e^{2m(1-\delta)\varphi(z)}, \qquad z \in \Omega',\, m \geq \tfrac{n+2}{c\,\delta}.$$

If $f_1, \ldots, f_p \in \mathcal{H}_\Omega(m\varphi)$ have norm 1, then $|f_1 \ldots f_p|^2\,e^{-2mp(1-\delta)\varphi}$ is bounded and therefore integrable on Ω', proving that $f_1, \ldots, f_p$ is a section of $\mathcal{I}(mp(1-\delta)\varphi)$ on Ω'. The proof is complete. $\qquad\square$

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