Moduli Spaces of Holomorphic Bundles on Minimal Class VII Surfaces with $b_2 = 1$

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Outline

1. Introduction
   - Moduli Spaces of Holomorphic Bundles
   - Minimal Class VII Surfaces with $b_2 = 1$
2. Filtrable Holomorphic Bundles
3. The Moduli Spaces

Motivation

- DONALDSON theory: invariants
- new complex geometric applications
- construction of invariants for manifolds with *definite* intersection form [TELEMAN 2006]

Computation of Moduli Spaces of Holomorphic Bundles on Surfaces

- on algebraic surfaces
  - many examples [Okonek/Van de Ven, Friedman/Morgan, ...]
  - all bundles are filtrable
- on non-algebraic surfaces
  - problem: existence of non-filtrable bundles
    classification hopeless
  - examples only on elliptic fibrations: restriction to fibres
    [Braam/Hurtubise, Lübke/Teleman, Brînzănescu/Moraru, ...]
- minimal class VII surfaces with $b_2 = 1$:
  - non-Kähler ($\Rightarrow$ non-algebraic)
  - non-elliptic

Questions

1. Are non-filtrable bundles always very generic on non-algebraic surfaces?
   - as on KÄHLER surfaces
   - as for all known examples on non-KÄHLER surfaces
2. Is the UHLENBECK compactification of the moduli space of stable bundles always a complex space?
   - as on algebraic surfaces [LI]
   - as for all known examples on non-algebraic surfaces
3. Finitely many homeomorphism types of moduli spaces when g varies in the space of GAUDUCHON metrics?

Aim

We will compute a certain moduli space

- of polystable holomorphic structures
- on a particular topological vector bundle
- on *all* class VII surfaces with $b_2 = 1$
- for *all* possible GAUDUCHON metrics

In particular this will answers all our questions *negatively*!

The Moduli Space Under Consideration

- E topological vector bundle

$$\operatorname{rank} E = 2 \qquad c_1(E) = c_1(K) \qquad c_2(E) = 0$$

- moduli space

$$\mathcal{M}^{\text{simple}} := \{\, \mathcal{E} \text{ simple hol. str. on } E : \ \det \mathcal{E} \cong \mathcal{K} \,\} \,/\, \Gamma\big(S, \mathrm{SL}(E)\big)$$

- $\mathcal{E}$ simple $\quad :\Leftrightarrow \quad \operatorname{End}(\mathcal{E}) = \mathbb{C}\, \mathrm{id}_{\mathcal{E}}$
- idem: $\quad \mathcal{M}^{\text{stable}}, \quad \mathcal{M}^{\text{polystable}}$

Minimal Class VII Surfaces
Definition

- S surface
  - complex manifold of $\dim_{\mathbb{C}} S = 2$
  - connected
  - compact
- ENRIQUES-KODAIRA classification: seven classes
- last gap: class VII
  - first BETTI number: $b_1(S) = 1$
  - KODAIRA dimension: $\operatorname{kod}(S) = -\infty$
- minimal: S is not a blow-up

Minimal Class VII Surfaces
Classification?

- $b_2 = 0$:
  - HOPF or INOUE surfaces [BOGOMOLOV, TELEMAN, LI/YAU]
- $b_2 = 1$:
  - A. TELEMAN (2005): S contains a complex curve
  - consequence: S biholomorphic to either
    - the half INOUE surface or
    - an ENOKI surface or
    - the parabolic INOUE surface

  $\longrightarrow$ classification accomplished
- $b_2 = 2$:
  - existence of a complex curve
  - ?

Minimal Class VII Surfaces with $b_2 = 1$
Classification

- half INOUE surface
  - C singular rational curve with one node $\quad C^2 = -1$
  - $\mathcal{K} \cong \mathcal{F} \otimes \mathcal{O}(-C) \qquad \mathcal{F}^2 = \mathcal{O}$
- parabolic INOUE surface
  - C singular rational curve with one node $\quad C^2 = 0$
  - E elliptic curve $\quad E \cap C = \emptyset \qquad E^2 = -1$
  - $\mathcal{K} \cong \mathcal{O}(-C - E)$
- ENOKI surfaces
  - two-parameter family
  - C singular rational curve with one node $\quad C^2 = 0$
- $H^2(S, \mathbb{Z}) = \mathbb{Z}c_1(K) \quad c_1(K)^2 = -1 \qquad \mathrm{Pic}^0(S) \cong \mathbb{C}^*$

Filtrable Bundles
Definition

- $\operatorname{rank} \mathcal{E} = 2, \quad S$ surface
- definition simplifies to:

$$\mathcal{E} \text{ filtrable} \quad :\Leftrightarrow \quad \exists \quad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{R} \otimes \mathscr{I}_Z \longrightarrow 0$$

where

  - $\mathcal{L}, \mathcal{R}$ line bundles
  - Z locally complete intersection of dimension 0
  - $\mathscr{I}_Z$ ideal sheaf of Z
- problem: $\quad Z$ could be very complicated!

Filtrable Bundles

As Line Bundle Extensions

- $\mathcal{E}$ filtrable: $\quad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{R} \otimes \mathscr{I}_Z \longrightarrow 0$
- choice of the CHERN classes of E
  - $Z = \emptyset$
  - $\mathcal{L} \in \mathrm{Pic}^0(S)$ or $\mathcal{R} \in \mathrm{Pic}^0(S)$
- $\mathcal{L} \otimes \mathcal{R} \cong \det \mathcal{E} \cong \mathcal{K} \quad \Rightarrow \quad$ two types of extensions:

$$\mathcal{L} \in \mathrm{Pic}^0(S): \quad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \rightarrow \mathcal{L}^\vee \otimes \mathcal{K} \rightarrow 0$$

$$\mathcal{R} \in \mathrm{Pic}^0(S): \quad 0 \rightarrow \mathcal{R}^\vee \otimes \mathcal{K} \rightarrow \mathcal{E} \longrightarrow \mathcal{R} \longrightarrow 0$$

- non-trivial extensions?

Filtrable Bundles

Nontrivial Line Bundle Extensions

- define subsets of $\mathrm{Pic}^0(S)$

$$Q(S) := \{\ \mathcal{L} \in \mathrm{Pic}^0(S) : \quad H^0(\mathcal{L}^2 \otimes \mathcal{K}^\vee) \neq 0\ \}$$
$$R(S) := \{\ \mathcal{R} \in \mathrm{Pic}^0(S) : \quad H^0(\mathcal{R}^2) \neq 0\ \}$$

  at most countable

- $\forall\ \mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S) \qquad \exists$ non-trivial extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}_{\mathcal{L}} \longrightarrow \mathcal{L}^\vee \otimes \mathcal{K} \longrightarrow 0$$

  $\mathcal{E}_{\mathcal{L}}$ unique up to isomorphisms

- $\forall\ \mathcal{R} \in R(S) \qquad \exists$ non-trivial extension

$$0 \longrightarrow \mathcal{R}^\vee \otimes \mathcal{K} \longrightarrow \mathcal{A}_{\mathcal{R}} \longrightarrow \mathcal{R} \longrightarrow 0$$

  $\mathcal{A}_{\mathcal{R}}$ unique up to isomorphisms

Filtrable Bundles
Isomorphisms

$$\begin{array}{rcccccccccc} \mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S): & 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E}_{\mathcal{L}} & \to & \mathcal{L}^{\vee} \otimes \mathcal{K} & \to & 0 \\ \mathcal{R} \in R(S): & 0 & \to & \mathcal{R}^{\vee} \otimes \mathcal{K} & \to & \mathcal{A}_{\mathcal{R}} & \longrightarrow & \mathcal{R} & \longrightarrow & 0 \end{array}$$

- *bijective* parametrisation of filtrable bundles?
- find possible isomorphisms

$$\mathcal{E}_{\mathcal{L}'} \cong \mathcal{E}_{\mathcal{L}} \qquad \mathcal{A}_{\mathcal{R}'} \cong \mathcal{A}_{\mathcal{R}} \qquad \mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{L}}$$

- $\mathcal{E}_{\mathcal{L}}$ pairwise non-isomorphic
- S half INOUE: $\mathcal{A}_{\mathcal{O}} \cong \mathcal{A}_{\mathcal{F}}$
- S ENOKI: $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{R}^{\vee} \otimes \mathcal{O}(-C)}$

Simple Filtrable Bundles

- $\mathcal{M}^{\text{simple}}$ is a complex analytic space (possibly non-HAUSDORFF)
- $\mathcal{E}$ simple $\quad :\Leftrightarrow \quad \operatorname{End}(\mathcal{E}) = \mathbb{C}\,\mathrm{id}_{\mathcal{E}}$
- trivial extensions non-simple
- $\mathcal{E}_{\mathcal{L}}$ simple
- S parabolic INOUE:

$$\mathcal{A}_{\mathcal{R}} \cong \mathcal{R}(-E) \oplus \mathcal{R}^{\vee}(-C) \qquad \text{not simple}$$

- otherwise $\mathcal{A}_{\mathcal{R}}$ simple

Degree of a Line Bundle

- $\exists$ g GAUDUCHON metric: HERMITian with $\partial\bar{\partial}\omega_g = 0$
- $\mathcal{L}$ line bundle
- h HERMITian metric in $\mathcal{L}$
- A_h CHERN connection
- $c_1(\mathcal{L}, A_h)$ first CHERN form

$$\begin{aligned} \deg_g : \quad \mathrm{Pic}(S) &\longrightarrow \mathbb{R} \\ \mathcal{L} &\longmapsto \deg_g \mathcal{L} := \int_S c_1(\mathcal{L}, A_h) \wedge \omega_g \end{aligned}$$

- independent of h
- LIE group morphism

Stable Filtrable Bundles

- $\mathcal{M}^{\text{stable}}$ is a HAUSDORFF complex analytic space
- $\operatorname{rank}\mathcal{E} = 2$, $\quad S$ surface $\quad\Rightarrow\quad$ definition simplifies to:

$$\mathcal{E} \quad g\text{-stable} \quad :\Leftrightarrow \quad \forall \mathcal{L} \subset \mathcal{E} \quad \text{line subbundle}: \quad \deg_g \mathcal{L} < \tfrac{1}{2} \deg_g \det \mathcal{E}$$

- stability condition "cuts out" a punctured disc

$$D^* \subset \mathrm{Pic}^0(S) \cong \mathbb{C}^* \subset \mathbb{C}$$

- $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ not both stable
- $\mathcal{E}$ $\;g$-polystable $\quad :\Leftrightarrow \quad \mathcal{E}$ $\;g$-stable $\quad$ or

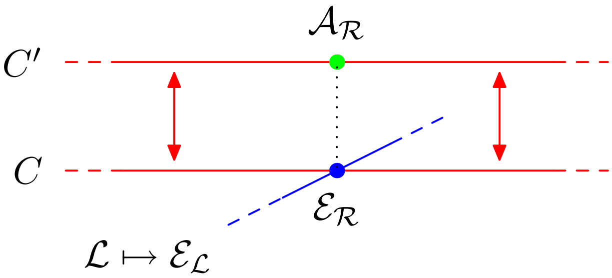
$$\mathcal{E} \cong \mathcal{L} \oplus \mathcal{M} \qquad \deg_g \mathcal{L} = \deg_g \mathcal{M} \qquad \text{(split polystable)}$$

Local Structure of the Moduli Space

$\mathcal{M}^{\text{simple}}$ is locally …

- a smooth complex curve at
  - non-filtrable bundles
  - $\mathcal{A}_{\mathcal{R}}$
  - $\mathcal{E}_{\mathcal{L}}$ if $\mathcal{L} \notin R(S)$, parametrised by $\mathcal{L}' \mapsto \mathcal{E}_{\mathcal{L}'}$
- a transverse intersection of two complex curves
  - at $\mathcal{E}_{\mathcal{R}}$ for $\mathcal{R} \in R(S)$
  - one of them parametrised by $\mathcal{L}' \mapsto \mathcal{E}_{\mathcal{L}'}$
- $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ not separable

The Moduli Space around $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$



locally: $C \setminus \{\mathcal{E}_{\mathcal{R}}\} \equiv C' \setminus \{\mathcal{A}_{\mathcal{R}}\}$

Moduli Spaces of Anti-Self-Dual Connections

- h HERMITian metric in E
- A h-unitary connection in E, F_A curvature of A

$$F_A \in \Omega^2\big(su(E)\big) = \Omega^2_+\big(su(E)\big) \oplus \Omega^2_-\big(su(E)\big)$$

- fix a connection a in $\det E$
- moduli space:

$$\mathcal{M}^{\mathrm{ASD}} := \{\, A \text{ } h\text{-unitary} : \ F_A^+ = 0,\ \det A = a \,\} \,/\, \Gamma\big(S, \mathrm{SU}(E)\big)$$

The KOBAYASHI-HITCHIN Correspondence

- relates moduli spaces in gauge theory to moduli spaces in complex geometry
- real analytic isomorphism

$$\begin{array}{ccc} \left(\mathcal{M}^{\text{ASD}}\right)^* & \xrightarrow{\cong} & \mathcal{M}^{\text{stable}} \\ [A] & \longmapsto & [\bar{\partial}_A] \end{array}$$

$*$ = irreducible part:

$$A \text{ reducible} \quad :\Leftrightarrow \quad E = L \oplus M \quad \text{and} \quad A = A_L \oplus A_M$$

- choice of the CHERN classes of E
  - $\mathcal{M}^{\text{ASD}}$ compact!
  - no need to compactify

Compactification and Non-Filtrable Bundles

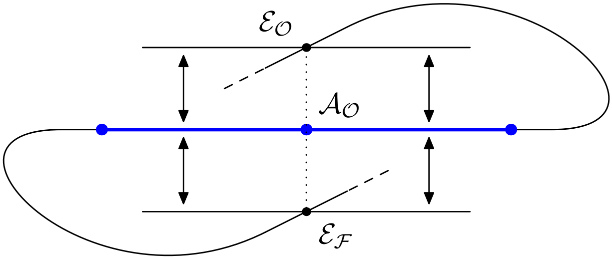
$$\begin{array}{ccccccc} \mathrm{Pic}^0(S) \supset & D^* & \longrightarrow & \mathcal{M}^{\text{stable}} & \xrightarrow[\text{K.-H.}]{\cong} & \left(\mathcal{M}^{\text{ASD}}\right)^* \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \supset D \cup \partial D & \longrightarrow & \mathcal{M}^{\text{polystable}} & \xrightarrow{\cong} & \mathcal{M}^{\text{ASD}} \end{array}$$

- inclusion $D^* \hookrightarrow \mathcal{M}^{\text{ASD}}$ extends to $D \cup \partial D$
  - $\partial D \cong S^1$ $\sim$ split polystables $\sim$ reducible connections
  - boundary structure of $\mathcal{M}^{\text{polystable}}$
  - center of D $\sim$ a bundle $\mathcal{E}$ verifying

$$\mathcal{E} \otimes \mathcal{F} \cong \mathcal{E} \qquad \text{where} \quad \mathcal{F}^{\otimes 2} = \mathcal{O}$$

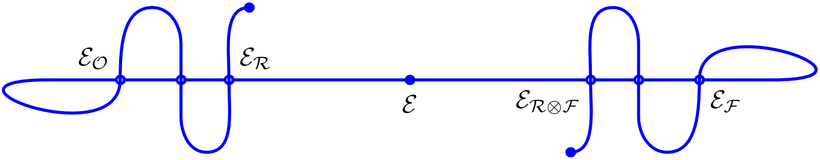
  - $\mathcal{E}$ splits on a double cover

The Moduli Space on the Half INOUE Surface



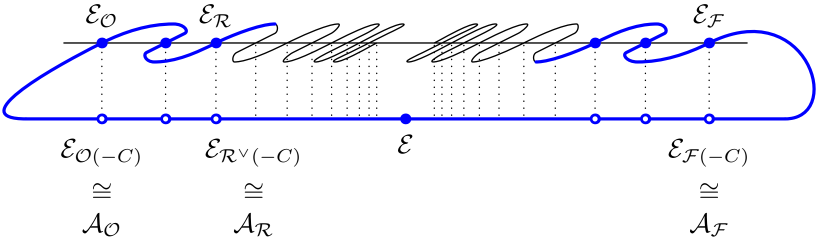
center filtrable: $\mathcal{E} \cong \mathcal{A}_{\mathcal{F}} \cong \mathcal{A}_{\mathcal{O}}$

The Moduli Space of Polystable Bundles on an ENOKI Surface



center $\mathcal{E}$ non-filtrable

The Moduli Space of Simple Bundles on an ENOKI Surface



center $\mathcal{E}$ non-filtrable

Other Connected Components in the Moduli Space?

- another connected component would
  - consist of unfiltrable bundles
  - be smooth of complex dimension one
  - be compact!
- M. TOMA (2006): no compact connected components in the moduli space on blown-up primary HOPF surfaces
- our surfaces are degenerations of blown-up primary HOPF surfaces
- deformation argument $\Rightarrow$ no other components

Summary

We computed moduli spaces
of simple/stable/polystable holomorphic bundles

- on *all* minimal class VII surfaces with $b_2 = 1$
- for *all* possible GAUDUCHON metrics

and saw that

- filtrable bundles can be generic
- the moduli space of polystable bundles
  is not a complex space (boundary!)
- there can be infinitely many homeomorphism types
  of moduli spaces when varying the GAUDUCHON metric
- the moduli space of simple bundles contains unseparable pairs
  consisting of a smooth and a singular point

This was possible only by combining
complex geometry and gauge theory!