Classification of Extremal metrics on Geometrically Ruled Surfaces

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GEOMETRICALLY RULED SURFACES:

$$(M, J) = P(E) \to \Sigma$$

- $E \to \Sigma$: holomorphic rank 2 vector bundle.
- Σ compact connected Riemann surface of genus $\mathbf{g}$

Definition: A rank 2 holomorphic vector bundle $E \to \Sigma$ is **polystable** if it decomposes as a direct sum of stable vector bundles (in the sense of Mumford) so that if the summants are line bundles their degrees are equal.

By Narasimhan-Seshadri this is equivalent to E being projectively flat Hermitian.

So $(M, J) = P(E) \to \Sigma$ falls into three different cases:

CASE 1: $E \to \Sigma$ is polystable

CASE 2: $E = \mathcal{O} \oplus \mathcal{L} \to \Sigma$, where $\mathcal{L}$ is some holomorphic line bundle such that $deg(\mathcal{L}) \neq 0$. (E is not polystable)

CASE 3: $E \to \Sigma$ is indecomposable and not (poly)stable ($\mathbf{g} > 0$).

EXTREMAL KÄHLER METRICS:

For a particular Kähler class Ω, let $\mathcal{M}_\Omega$ denote the set of all Kähler forms in Ω.

Calabi functional: $\Phi : \mathcal{M}_\Omega \to \mathbb{R}$

$$\Phi(\omega) := \int_M Scal^2 d\mu$$

where $Scal$ and $d\mu$ is the scalar curvature respectively the volume form of the metric corresponding to the Kähler form $\omega \in \Omega$.

Proposition: (Calabi) $\omega \in \mathcal{M}_\Omega$ is an extremal point of Φ iff $\mathrm{grad}\, Scal$ is a **holomorphic real vector field**, that is,
$\mathcal{L}_{\mathrm{grad}\, Scal} J = 0$.

In this case we call g, corresponding to ω, an **extremal Kähler metric**. A Kähler metric with constant scalar curvature (CSC) is in particular extremal.

QUESTION: When may a geometrically ruled surface have a CSC Kähler metric?

- If we are in CASE 1, then there is a (local product) CSC Kähler metric in each Kähler class (Narasimhan-Seshadri).

- If we are in CASE 2, then there are no CSC Kähler metrics at all. (For instance, the Futaki invariant of each Kähler class is non-zero (LeBrun-Simanca, ACGT-F, T-F.)

THE QUESTION IS (WAS): Are there any CSC Kähler metrics in CASE 3?

- Burns and deBartolomeis (1988): Not in Kähler classes Ω with $c_1(M) \cdot \Omega = 0$.

- Fujiki (1992): Not when $\mathbf{g} = 1$.

- LeBrun (1995): Not when $\mathbf{g} > 1$ and in Kähler classes Ω with $c_1(M) \cdot \Omega < 0$.

- Apostolov and T-F (2004): If there is one in CASE 3 (and, without loss, $\mathbf{g} > 1$), then $E \rightarrow$ is simple, i.e, the only endomorphisms of E is the scalar multiplication.

but. . .

- Fujiki (1992): (paraphrased) If $\mathbf{g} > 1$, $E \to \Sigma$ is simple and non-polystable, and $P(E) \to \Sigma$ has a CSC Kähler metric, then, by a small deformation of E (to a stable bundle), we have an obstruction to uniqueness of CSC Kähler metrics in a fixed Kähler class.

- Chen and Tian (2005): Uniqueness does hold. . . for extremal metrics in fact.

so. . . no CSC Kähler metrics in CASE 3. . .

CONCLUSION: CSC Kähler metrics exist in CASE 1 and in CASE 1 only.

NON-CSC EXTREMAL KÄHLER METRICS:

Due to the fact that $\mathrm{Isom}_0(M, g)$ contains an S^1, E must split so such metrics cannot exist in CASE 3 and clearly there are none in CASE 1 (due to the vanishing of the Futaki invariant. So our focus is now on CASE 2:

Calabi (1982): If $\mathbf{g} = 0$ (M is a Hirzebruch surface) then each Kähler class has an extremal Kähler metrics.

Hwang (1994): This is also true if $\mathbf{g} = 1$.

T-F (1997): If $\mathbf{g} > 1$, then some Kähler classes do have extremal Kähler metrics - but not all (now that we know uniqueness holds).

NEW TERMINOLOGY: CASE 2 is a (very) special case of a **admissible manifold** (admits Kähler metrics with Hamiltonian 2-forms of order 1).

The extremal Kähler metrics constructed to prove the above existence results all admit a Hamiltonian 2-form of order 1.

THE QUESTION IS (WAS): Is this it? Or are there other types of extremal Kähler metrics in CASE 2?

THEOREM: (ACGT-F)
This IS it; there are no other types of extremal Kähler metrics in CASE 2.

Proof: By uniqueness, obviously the answer is "NO" when $\mathbf{g} \leq 1$. When $\mathbf{g} > 1$ we connect an ingredient in the construction - namely the **extremal polynomial** with a key ingredient in the uniqueness proof by Chen and Tian - namely the **modified K-energy**.

Modified K-energy: (Guan and Simanca)

- G: Maximal compact connected subgroup of $H_0(M, J)$
- $\mathcal{M}_\Omega$: Fréchet space of Kähler metrics in Kähler class Ω
- $\mathcal{M}_\Omega^G$: Subspace of G-inv. Kähler metrics
- $\mathrm{pr}_g^\perp$: L_2-projection orthogonal to the space of Killing potentials (defined on G inv. L^2-functions)
- The map $g \mapsto \mathrm{pr}_g^\perp Scal_g \mu_g$ is (by integration) a 1-form σ on $\mathcal{M}_\Omega^G$
- σ is closed
- $\forall \omega_0 \in \mathcal{M}_\Omega$, $\exists!$ functional $E_{\omega_0}^G : \mathcal{M}_\Omega^G \to \mathbb{R}$ with $dE_{\omega_0}^G = -\sigma$, $E_{\omega_0}^G(\omega_0) = 0$.
- Changing the base point $\omega_0 \in \mathcal{M}_\Omega$ would change $E_{\omega_0}^G$ by an additive constant.
- It agrees with the Mabuchi K-energy when G is trivial

• The critical points of $E^G_{\omega_0}$ are exactly the extremal Kähler metrics in $\mathcal{M}^G_\Omega$, since $\sigma = 0$ means that $Scal_g$ is a Killing potential.
• Any extremal Kähler metric $g \in \mathcal{M}_\Omega$ belongs to $\mathcal{M}^G_\Omega$ with $G = \mathrm{Isom}_0(M, g) \cap H_0(M)$ (Calabi)

Theorem(Chen-Tian) Extremal Kähler metrics in $\mathcal{M}_\Omega$ are unique up to automorphism and any extremal Kähler metric in $\mathcal{M}^G_\Omega$ realizes the absolute minimum of $E^G_{\omega_0}$ (for any $\omega_0 \in \mathcal{M}^G_\Omega$). IN PARTICULAR, if $\mathcal{M}^G_\Omega$ contains an extremal Kähler metric, then $E^G_{\omega_0}$ is bounded from below.

The construction:

- Let Σ be a compact connected Riemann surface with Kähler metric $(g_\Sigma, \omega_\Sigma)$.

- Let M be $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$, where $\mathcal{L} \to \Sigma$ is a holomorphic line bundle such that $c_1(\mathcal{L}) = [\omega_\Sigma / 2\pi]$.

- Let K be the vector field generating the canonical S^1 action on $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$.

- Let θ be a connection 1-form ($\theta(K) = 1$) with $d\theta = \omega_\Sigma$.

- Let $z \in [-1, 1]$.

- Let Θ be a smooth function on $[-1,1]$ satisfying
$$\Theta > 0 \qquad (1)$$
on $(-1,1)$,
$$\Theta(\pm 1) = 0, \quad \Theta'(\pm 1) = \mp 2. \qquad (2)$$

- Then
$$\begin{aligned} g &= \tfrac{1+xz}{x} g_\Sigma + \tfrac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \\ \omega &= \tfrac{1+xz}{x} \omega_\Sigma + dz \wedge \theta \end{aligned} \qquad (3)$$
defines a Kähler structure (ω, g, J) on the total space M^0 of $\mathcal{L} - \{0\} \to \Sigma$ which extends smoothly to M.

- Note that $K = J \operatorname{grad}_g z$ and $z : M \to [-1,1]$ should be interpreted as a moment map of K and ω.

- Note that (2) is neccesary for the smooth extention of (3) to M.

Definition: If g is as in (3) and g_Σ is a fixed Kähler metric of constant scalar curvature $Scal_\Sigma = 2s$, then we say that g is **admissible**.

We then have $s = \frac{2(1-\mathrm{g})}{\deg(\mathcal{L})}$ by Gauss-Bonnet formula on Σ. So s is a parameter determined by the degree of the line bundle $\mathcal{L} \to \Sigma$, when M is $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$.

For given constants s and $0 < x < 1$ define the following polynomial:

$$F_x(z) = \frac{(1-z^2)(x^2(2-sx)z^2+x(6-2x^2)z+(6+sx^3-4x^2))}{2(3-x^2)}$$

Remark: $F_x(\pm 1) = 0$ and
$F_x'(\pm 1) = \mp 2(1 \pm x)$, so
$\Theta(z) := F_x(z)/(1+xz)$ satisfies (2) automatically.

Now an admissible metric is extremal exactly when $\Theta(z) = F_x(z)/(1+xz)$ (Calabi, Guan, Hwang).

MEANING OF x:

If (g, ω) is admissible then $\omega = \omega_\Sigma/x + \eta$ where $[\eta]$ is up to scale the Poincaré dual of the formal sum of the zero and infinity sections of $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$ and ω_Σ is viewed as the pullback to M of the corresponding form on Σ.

Conversely on $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$ with canonical complex structure J_0, any Kähler class is of the form $[\omega_\Sigma]/x + [\eta]$ where ω_Σ is some Kähler form on Σ and (necessarily) $0 < x < 1$. One may show that each class has a canonical admissible Kähler metric corresponding to $\Theta_0(z) = 1 - z^2$ whose complex structure is J_0.

So $x \in (0, 1)$ parametrizes the Kähler cone on (M, J_0).

Each Kähler class has an **extemal polynomial**, $F_x(z)$.

Remark: If $s \geq 0$ then $F_x(z)/(1+xz)$ satisfies (1) for all $0 < x < 1$ (Calabi, Guan, Hwang, Simanca). So admissible extremal Kähler metrics exhaust the Kähler cone when $\mathbf{g} < 2$.

For $s < 0$, $\exists 0 < x_s < 1$ such that

- For $0 < x < x_s$ $F_x(z)/(1+xz)$ satisfies (1)
- For $x = x_s$, $F_x(z)/(1+xz) \geq 0$ for $z \in (-1,1)$, but (1) fails.
- For $x_s < x < 1$, "$F_x(z)/(1+xz) \geq 0$ for $z \in (-1,1)$" fails.

So even though admissible extremal Kähler metrics do exist, they **do not** exhaust the Kähler cone when $\mathbf{g} > 1$.

MEANING OF $\Theta(z)$:

Now if we vary $\Theta(z)$ in the set of all functions from $[-1,1]$ satisfying (1) and (2) but keep all the other data fixed (for instance from a canonical metric), then the Kähler form is fixed but the complex structure J varies.

[Terminology: If $u(z)$ on $(-1,1)$ is such that $u''(z) = 1/\Theta(z)$, then u is the symplectic potential.]

However, via a Legendre transformation (to the Kähler potential of (ω, J)), there is a S^1-equivariant (fibre-preserving on M^0) diffeomorphism Ψ such that $\Psi^* J = J_0$ and $\Psi^*\omega \in [\omega]$.

Hence the moduli space $\mathcal{K}_x^{\mathrm{adm}}$ of admissible metrics in Ω determined by x is identified with the space of smooth functions Θ on $[-1,1]$ satisfying (1)–(2) or equivalently with $\{u \in C^0([-1,1]) : u - u_0 \in C^\infty([-1,1]),$ $u(\pm 1) = 0$ and $u'' > 0$ on $(-1,1)\}$.

(for simplicity $\mathbf{g} > 1$)

Remark:

- All admissible metrics are invariant under the same maximal compact connected subgroup G of $H_0(M)$. Namely, $G = S^1$, where the S^1 action is the natural action on $\mathcal{L} \to \Sigma$, generated by the vector field K.

- If $u_t(z)$ is a path of symplectic potentials in $\mathcal{K}_x^{\text{adm}}$, then there is a corresponding path in $\mathcal{M}_\Omega^G$ such that $\omega_t = \omega + dJ_0 d(h_t - h_0)$ and $\dot{u} = -\dot{h}$.

- WRT this G, the extremal vector field K_x of any Kähler class (which must be in the center of G) is a constant multiple of K.

- Actually $K_x = J\,\mathrm{grad}\,\mathrm{pr}_g Scal_g$, where $\mathrm{pr}_g Scal_g$ is the $L_2 - projection$ onto the space of Killing potentials wrt (e.g.) an admissible metric.

- $\mathrm{pr}_g Scal_g$ must be an affine function of z.

CLAIM:

$$\mathrm{pr}_g^{\perp} Scal_g = \frac{F_x''(z) - (\Theta(z)(1+xz))''}{(1+xz)}, \qquad (4)$$

Proof: For any admissible metric with CSC g_Σ we have $Scal_g = \frac{2sx-(\Theta(z)(1+xz))''}{1+xz}$ and so r.h.s. of (4) is seen to be equal to

$$Scal_g + \frac{6((sx^2-2x)z + x^2 - sx - 1)}{3-x^2}.$$

Since this turns out to be orthogonal to the Killing potentials 1 and z, it must be equal to $\mathrm{pr}_g^{\perp} Scal_g$.

So for a Kähler class determined by x, we may now consider the modified K-energy restricted to $\mathcal{K}_x^{\text{adm}}$:

$$\begin{aligned}
dE^G_{\omega_0} &= \int_M \mathrm{pr}_g^{\perp} Scal_g \dot{u} d\mu \\
&= \int_M \frac{F_x''(z) - (\Theta(z)(1+xz))''}{(1+xz)} \dot{u} d\mu \\
&= C\int_{-1}^{1} (F_x''(z) - (\Theta(z)(1+xz))'')\dot{u} dz \\
&= C\int_{-1}^{1} (F_x(z) - (\Theta(z)(1+xz)))\dot{u}'' dz,
\end{aligned}$$

where C is a positive constant (depending on s and x) and the last equality is gotten by integrating twice by parts and using (2).

So now we have:

Propostion: Let Ω be a Kähler class corresponding to some x on M. Then the K-energy restricted to the space of admissible Kähler metrics is (up to an additive constant) a positive multiple of the functional

$$\begin{aligned} \mathcal{E}_{g_0} : u(z) &\mapsto \int_{-1}^{1} F_x(z)(u''(z) - u_0''(z))dz \\ &- \int_{-1}^{1}(1 + xz) \log\left(\frac{u''(z)}{u_0''(z)}\right)dz, \end{aligned}$$

where $u(z)$ is the symplectic potential.

Corollary: If there is an extremal Kähler metric in Ω corresponding to x, then $F_x \geq 0$ on $[-1, 1]$.

Proof: If there is an extremal Kähler metric in Ω, then by the Chen-Tian theorem, the modified K-energy is bounded from below. We now apply an argument due to Donaldson: take any nonnegative smooth function $f(z)$ with supp$(f) \subset (-1, 1)$ and consider the sequence $u_k(z)$ with $u_k''(z) = u_0''(z) + kf(z)$ of symplectic potentials for admissible Kähler metrics. We therefore get

$$\begin{aligned} \mathcal{E}_{g_0}(u_k) &= -\int_{-1}^{1}(1+xz)\log(1+k\frac{f(z)}{u_c''(z)})dz \\ &+ k\int_{-1}^{1} F_x(z)f(z)dz. \end{aligned}$$

This will tend to $-\infty$ if $\int_{-1}^{1} F_x(z)f(z)dz < 0$ for some f.

SUMMARY:

So for $s < 0$ (genus $\mathbf{g} > 1$) we have that in the Kähler classes determined by x such that $x_s < x < 1$, there are no extremal Kähler metrics.

By uniqueness, the openness-of-the-extremal-cone result of LeBrun and Simanca, and the fact that a convergent sequence of (up to automorphism) admissible metrics converges to an admissible (up to automorphism) metric, there are no extremal metrics in the class corresponding to x_s either.

Thus, for CASE 2, the only extremal Kähler metrics are indeed the admissible ones. This finishes the proof of our theorem.

The notion of K-**stability** first introduced by G. Tian has now been considered by several people using similar definitions. The overarching principle (following a conjecture by Yau) is in it's full generality that existence of extremal Kähler metrics should be equivalent to K-stability in some appropiate form.

G. Székelyhidi has developed a notion of relative K-polystability of a polarized variety, which he conjectures is equivalent to the existence of extremal Kähler metrics. He considered the stability for CASE 2 surfaces and observed that non stability happens if the polarization (equivalent to a choice of Hodge Kähler class) does not admit an extremal Kähler metric (with hamilitonian 2-form of order 1).

David Calderbank will be discussing generalizations of this tomorrow morning.