Compact non-Kähler threefolds associated to real hyperbolic 3-manifolds

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Problem

G semisimple, connected,

complex linear algebraic group

Γ discrete subgroup Zariski dense,

torsion-free, and not co-compact

$U := G/\Gamma$ complex homogeneous space

Find a G-equivariant compactification

$$\iota : U \hookrightarrow X$$

into a compact complex G-manifold

Properties of X

- $S := X - U$ is a hypersuface
- $\exists T \in |-K|$ with supp $T = S$
- X admits no non-constant meromorphic functions [Huckleberry-Margulis '83]
- X is non-Kähler [Berteloot-Oeljeklaus '88] and $\notin \mathcal{C}$

Example: The twistor space associated to one of the following conformally-flat manifolds $V = (V, g)$ with $SO(3)$-actions give examples of a compactification for the group $G = PSL_2\boldsymbol{C}$ and $\Gamma = \pi_1(X)$.

(1) $V = \boldsymbol{P}^1 \times C_g, g \geq 2$.

(2) $V = C(r) = r(S^1 \times S^3), r \geq 1$.

Assume that dim $G = 3$:

Case 1: $G = PSL_2\boldsymbol{C}$

Case 2: $G = SL_2\boldsymbol{C}$

Recall:

$PSL_2\boldsymbol{C} \cong \mathrm{Isom}^+ H$

where H is a 3-dim. hyperbolic space form

$\Gamma \subseteq PSL_2\boldsymbol{C}$ is called a Kleinian group

H admits a natural compactification by adding

the sphere at ∞

$$\bar{H} = H \cup bH, \quad bH = S^2 = \boldsymbol{P}^1$$

on which G-action naturally extends.

Given Γ, we have the decomposition:

$$bH = \Omega \cup \Lambda$$

(Ω domain of discontinuity, Λ limit set)

$M := H/\Gamma$ is a complete hyperbolic manifold and vice versa.

Consider the case: Γ is <u>cofinite</u>

($\Leftrightarrow$ the volume of G/Γ is finite)

Theorem 1. If, further, Γ is co-finite, there exists no G-equivariant compactification of $U = G/\Gamma$.

The point: G/Γ cannot be compactified in cusp directions.

Speculation: In general G/Γ cannot be compactified in cusp directions whenever Γ is arithmetic.

Remark. When G is simple,

arithmetic $\Leftrightarrow$ cofinite, if $\dim G > 3$.

In general "$\Rightarrow$ " is true.

Assumption:

(1) Γ is geometrically finite

($\Leftrightarrow$ fundamental polyhedron is finite-sided.)

(2) purely loxodromic, i.e.,

contains no parabolic elements $\neq 1$

Theorem 2. Under the above assumptions there exists a natural G-equivariant compactification

$\iota : U := G/\Gamma \hookrightarrow X$

with the following properties:

Structure of S

$S = \coprod_{1 \le i \le k} S_i$ with $S_i \cong \boldsymbol{P}^1 \times C_i$,

(C_i a compact Riemann surface of genus $g_i \geq 2$)

G acts on the $\boldsymbol{P}^1$-factor naturally, and

trivially on the C_i-factor.

Anti-canonical bundle and rational curves

- $-K_X = 2[S],\ N_{S/X} = -K_S,$

  $N_{l/X} = O(1) \oplus O(1)$ (Case 1)
- $-K_X = 3[S],\ N_{S/X} = -\frac{1}{2}K_S,$

  $N_{l/X} = O \oplus O(1)$ (Case 2)

  where $l = \boldsymbol{P}^1 \times \{*\}, * \in C_i$.
- X is covered by nonsingular rational curves with normal bundle type as above.

X is a manifold of Class L in the sense of Ma. Kato in Case 1.

Topology

Let $g = \sum g_i$. Then we have:

- $\pi_1(X) \cong \Gamma$
- $b_1(X) = \mathrm{rank}\ \Gamma/[\Gamma, \Gamma]$
- $b_2(X) = b_1(X) + 2k - g - 1$
- $b_3(X) = 2g$
- $\chi(X) = \chi(S)$

Chern numbers

$c_1^3 = 2\chi(S)$ (Case 1), $= \frac{27}{2}\chi(S)$ (Case 2)

$c_1 c_2 = 6\chi(S)$

$c_3 = \chi(S)$

$\chi(O_X) = \frac{1}{4}\chi(S)$

Universal covering space $\tilde{X}$

- $\tilde{X}$ is a domain in Y with

  $Y = \boldsymbol{P}^3$ (in Case 1) and

  $= Q^3$ (hyperquadric in $\boldsymbol{P}^4$) (in Case 2)

- Its complement E is of the from:

  $\Lambda \times \boldsymbol{P}^1 \subseteq Q^2 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \subseteq \boldsymbol{P}^3$.

  Here $\Lambda(\subseteq \boldsymbol{P}^1 \cong bH)$ is the limit set of Γ;

  an infinite set with $m(\Lambda) = 0$.

- $\tilde{X}$ is not Zariski open in Y and

  with Hausdorff measure $\mathcal{H}^4(E) = 0$.

Relation between Cases 1 and 2:

Γ_2 a discrete subgroup in Case 2.

Γ_1 its (isomorphic) image in $PSL_2\boldsymbol{C}$.

X_i the corresponding equivariant compactifications for Case i.

Then we have a natural equivariant double covering with branch locus S:

$$u : X_2 \to X_1$$

Their Betti numbers are the same.

Example

(1) $\Gamma \subseteq PSL_2\boldsymbol{R} \subseteq PSL_2\boldsymbol{C}$

a cocompact torsion-free Fuchsian group,

or more generally a quasi-Fuchsian group.

$k = 2,\ g_1 = g_2 =: p$

($C_1 \cong C_2$ with $C_i \cong H^2/\Gamma$ if Γ is Fuchsian)

$b_1(X) = 2p, \quad b_2(X) = 3, \quad b_3(X) = 4p$

$\chi(X) = 4(2 - p)$

$c_1^3 = 64(2 - p)$ (Case 1), $\quad = 54(2 - p)$ (Case 2)

$c_1 c_2 = 24(2 - p)$

$\Lambda = \boldsymbol{RP}^1 \subseteq \boldsymbol{P}^1$ (Fuchsian case)

(2) Γ (classical) Schottky group of rank $r \geq 2$

($\Leftrightarrow \Gamma$ is a free group of rank r

without parabolic elements $\neq 1$)

$k = 1, \quad g = r$

$b_1(X) = r, \quad b_2(X) = 1, \quad b_3(X) = 2r$

$\chi(X) = 4(1 - r)$

$c_1^3 = 64(1 - r)$ (Case 1), $\quad = 54(1 - r)$ (Case 2)

$c_1 c_2 = 24(1 - r)$

Λ totally disconnected, perfect set.

<u>Non-Zariski dense case</u>

$\Leftrightarrow \Gamma$ elementary ($\overset{def.}{\Leftrightarrow} \#\Lambda \leq 2$)

In this case

$\Gamma \cong \boldsymbol{Z} = \langle \gamma \rangle$, with γ loxodromic.

In Case 1

- X is a principal elliptic bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.
- algebraic dimension $a(X) = 2$,
- X is not in $\mathcal{C}$,
- $S = \boldsymbol{P}^1 \times C$, with C a smooth elliptic curve
- $\pi_1(X) \cong \boldsymbol{Z}$

  $b_1(X) = 1$, $b_2(X) = 1$, $b_3(X) = 2$
- For "real" γ

  X is a twistor space of a Hopf surface.

  $\Lambda = \{0, \infty\}$ and $\tilde{X}$ is Zariski open in $\boldsymbol{P}^3$.

Projective and quadric structures

In Case 1: X admits a (holomorphic) projective structure:

In Case 2: X admits a quadric structure:

Conversely,

Proposition. Our compactifications are characterized by the property that it admits a G-invariant projective (resp. quadric) structures.

Classifications of such structures:

for compact surfaces (Kobayashi-Ochiai '80, '82);

for projective threefolds (Jahnke-Radloff '04,'05).

The construction:

Consider only Case 1 $G = PSL_2\boldsymbol{C}$.

The basic diagram:

$$\begin{array}{ccccc} G & \cup & Q^2 & = & \boldsymbol{P}^3 \\ \downarrow & & \downarrow & & \downarrow \pi \\ K\backslash G & \cup & K\backslash Q^2 & = & K\backslash \boldsymbol{P}^3 \\ \| & & \| & & \| \\ H & \cup & bH & = & \bar{H}. \end{array}$$

Here

- $K = PSU(2)$.
- $\boldsymbol{P}^3 = \boldsymbol{P}(M_2(\boldsymbol{C}))$ is the projectivization of the space of 2×2 matrices.
- $Q^2 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is the quadric defined by the vanishing of the determinant.
- The left-right action of G on G extends naturally to one on $\boldsymbol{P}^3$ leaving Q^2 invariant.

- The left (resp. right) action on Q^2 is trivial on the first (resp. second) factor and via the natural action on the second (resp. first) factor.

Recall the decomposition: $bH = \boldsymbol{P}^1 = \Lambda \cup \Omega$ and restrict the above diagram to the Γ-invariant open subset $\tilde{X} := G \cup (\Omega \times \boldsymbol{P}^1) \subseteq \boldsymbol{P}^3$ and take the quotient by Γ.

$$\begin{array}{ccccc}
G/\Gamma & \cup & (\Omega \times \boldsymbol{P}^1)/\Gamma & =: & X \\
\downarrow & & \downarrow & & \downarrow \pi \\
K\backslash G/\Gamma & \cup & K\backslash(\Omega \times \boldsymbol{P}^1)/\Gamma & = & K\backslash X \\
\| & & \| & & \| \\
H/\Gamma & \cup & \Omega/\Gamma & =: & N
\end{array}$$

The manifold with boundary *Kleinian manifold* $N = (H \cup \Omega)/\Gamma$ is compact if and only if Γ satisfies the assumption of Theorem 2.

PROBLEM

(1) $\exists$ an equivariant compactification when Γ is not geometrically finite ?

(2) $\exists$ an exotic equivariant compactification when Γ is geometrically finite ?

Remark. By blowing up any lines $* \times \boldsymbol{P}^1 \subseteq S$ we get another equivariant compactification. In this case some connected component of S has more than one irreducible components and some irreducible components of S have open orbits.

G-equivariant deformations

Theorem 3. Let $\Gamma \subseteq G$ be a geometrically finite Kleinian group without nontrivial parabolic element and X the associated G-equivariant compactification of G/Γ as in Theorem 2. Then any small G-equivariant deformation X' of X is obtained from a quasi-conformal deformation Γ' of X by the method of Theorem 2.

- Γ' is a *quasi-conformal deformation* of $\Gamma \overset{def.}{\Leftrightarrow} \Gamma' = f\Gamma f^{-1}$ for some quasi-conformal homeomorphism f of $\boldsymbol{P}^1$.
- Γ geometrically finite without parabolic element $\Leftrightarrow$ any small "deformation" of Γ is obtained by quasi-conformal deformations [Sullivan '85]

- The theory of quasi-conformal deformations is equivalent to the deformation theory of the curve $\Omega/\Gamma = C_1 \coprod \ldots \coprod C_k$. So we have $3g - 3k$ dimensional natural deformation of X.

<u>Infinitesimal description</u>:

$$\begin{array}{ccccccccc}
 & & H^1(\Theta_X(-S))^G & = & H^1(\Theta_X(-S))^G & & & & \\
 & & \downarrow \Vert & & \downarrow & & & & \\
0 & \longrightarrow & (H^1(O_X) \otimes sl_2)^G & \stackrel{a}{\longrightarrow} & H^1(\Theta_X(-\log S))^G & \stackrel{b}{\longrightarrow} & H^1(\Theta_C) & \longrightarrow & 0 \\
 & & & & \downarrow & & \Vert & & \\
 & & & & H^1(\Theta_S)^G & \stackrel{\sim}{\longrightarrow} & H^1(\Theta_C) & &
\end{array}$$

where $sl_2 =$ Lie algebra of $PSL_2\boldsymbol{C}$.

Higher dimensional examples

Theorem 4 For any positive integer m we can find complex Schottky groups of arbitrary rank $r > 0$ and Γ in $G = PSL_{2m}\boldsymbol{C}$ such that $U := G/\Gamma$ admits a G-equivariant compactification $\iota : U \hookrightarrow X$. The complement $S := X - U$ is an irreducible hypersuface in X with singularities in codimension 3 (if $m > 1$). We have $-K = 2m[S]$.

- A complex Schottky group in general is a subgroup of $PSL_{2n+2}\boldsymbol{C}$, which is a free group and is a generalization of the classical Schottky groups for the case $n = 0$. [Nori '84, Larusson '98, Seade-Verjovsky '01]

  It has a domain of discontinuity Ω in $\boldsymbol{P}^{2n+1}$ with compact quotient Ω/Γ, called a Schottky manifold.

- We consider a higher dimensional analogue of the construction for $PSL_2\boldsymbol{C}$ above, and observe that the construction is compatible with that of Schottky manifolds if Γ is taken suitably.
- The examples in Theorem 4 are higher dimensional analogue of those in 2) of Example 2 for the classical Schottky groups.

On the proof

- For any positive integer r consider r pairs (L_i, L'_i) of mutually disjoint linear subspace of dimension n in $\boldsymbol{P}^{2n+1}$. For each such pair one associates an element γ_i of $PSL_{2n+2}\boldsymbol{C}$ such that these γ_i generate a free group Γ (Schottky group) of rank r and that the limit set Λ of this action is the closure of the unions of Γ orbit of the union of all the L_i and L'_j.
- With respect to the natural Zariski-open embedding of $G = PSL_{2m}\boldsymbol{C}$ into $\boldsymbol{P}^{(2m)^2-1}$ the complement is stratified by a $(2m-1)$ $G \times G$ orbits M_k, the set of $2m \times 2m$ complex matrices of rank m upto projectivization. We then take the L_i and L'_i in such a way that they are contained in $\overline{M}_m$, the closure of M_m, and left G-invariant. Since Γ acts from the right, G also leaves invariant the limit set Λ. This gives us the desired compactification.