Boltzmann asymptotics with diffuse reflection boundary conditions.

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Abstract. Strong L^1-convergence towards a stationary solution when time tends to infinity is established for the solutions of the time-dependent non-linear Boltzmann equation in a bounded domain $\Omega \subset I\!R^3$ with constant temperature on the boundary. The collisionless case is first investigated in the varying temperature case.

Introduction. The initial boundary value problem for the Boltzmann equation with large data when the behaviour at the boundary is either given by a pointwise reflection law or by a mixing of specular or reverse reflection with diffuse reflection was first studied by K. Hamdache [13]. In contrast to the specular reflection and periodic cases, the diffusion reflection boundary condition provides a well-defined boundary temperature. A later study by L. Arkeryd and C. Cercignani [2] deals with the case of general diffuse reflection with varying boundary temperature under a restriction to bounded velocities. The diffuse reflection case with unbounded velocities was solved by L. Arkeryd and N. Maslova [4]. A serious extra complication in this case in comparison with the specular reflection and periodic cases is the quite delicate trace behaviour due to the integral connecting ingoing and outgoing mass flows. A natural next question is the long time behaviour of the

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time-dependent solutions. Strong L^1 asymptotics for the Boltzmann equation in the periodic case when $t \to \infty$ was considered by L. Arkeryd [1] and P. L. Lions [14]. In the case of reverse and specular reflection, L. Desvillettes obtained weak convergence [10] when $t \to \infty$. For other boundary conditions the existence of stationary solutions becomes part of the problem. Under various cut-offs of the collision kernel for small velocities, the existence problem for a slab in the stationary case was studied in a measure setting by L. Arkeryd, C. Cercignani and R. Illner [3], and in an L^1 setting by L. Arkeryd and A. Nouri for given ingoing data as well as diffuse reflection boundary with varying temperature [5]. In the context of diffuse reflection on the boundary, C. Bose, P. Grzegorczyk and R. Illner studied the asymptotic behaviour of the discrete velocity model of the Boltzmann equation in the slab [6]. We are here going to consider the corresponding asymptotics for the full Boltzmann equation with diffuse reflection at the boundary when the boundary temperature is constant.

The first section of the present paper is devoted to the collisionless case in a bounded domain with maxwellian diffuse reflection and varying temperature at the boundary. Strong L^1-convergence of the solutions for the time-dependent equation inside the domain as well as on the boundary towards the associated stationary time-dependent solutions is established. The sole mechanism behind this convergence is the maxwellian diffuse reflection boundary conditions.

The asymptotics for the time-dependent Boltzmann solutions with maxwellian diffuse reflection and constant temperature on the boundary is studied in the second section. Here the entropy dissipation term is essential for the asympotic properties. First a global in time control of the energy is achieved by bounding the relative entropy with respect to the stationary solution, using a Darrozes & Guiraud inequality [8]. This is then used to prove the strong L^1-convergence to a global maxwellian (i.e. independent of x and t), uniquely determined by the boundary conditions and the conservation of mass. Such a uniqueness result is still open in the periodic and specular direct reflection boundary conditions cases.

Throughout the paper c denotes a constant which may have different values at different places.

1 The collisionless case.

The problem considered in this section is the asymptotic behaviour of the solution of

$$\frac{\partial f}{\partial t} + \xi . \nabla_x f = 0, \quad t \in I\!R_+, \quad x \in \Omega, \quad \xi \in I\!R^3, \tag{1.1}$$

where Ω is a bounded convex open set of $I\!R^3$, together with the initial condition

$$f(0, x, \xi) = f_0(x, \xi), \quad x \in \Omega, \quad \xi \in I\!R^3, \tag{1.2}$$

and the maxwellian diffuse reflection at the boundary

$$f(t, x, \xi) = M(x, \xi) \int_{\xi' \cdot n(x) < 0} | \xi' \cdot n(x) | f(t, x, \xi') d\xi',$$
$$t \in I\!R_+, \quad x \in \partial\Omega, \quad \xi \in I\!R^3, \quad \xi \cdot n(x) > 0, \tag{1.3}$$

where $n(x)$ is the inward normal at $x \in \partial\Omega$.

$$M(x, \xi) = (2\pi)^{-1} \theta^2(x) exp(-.5\theta(x) | \xi |^2) \tag{1.4}$$

is a maxwellian with prescribed inverse temperature $\theta(x)$, such that

$$0 < c_1 < \theta(x) < c_2 < \infty. \tag{1.5}$$

All along the paper, Ω is assumed to have a boundary $\partial\Omega$ of Lyapunov type, and $f_0 \geq 0$ is assumed to satisfy

$$\int_{\Omega \times I\!R^3} f_0 log f_0(x, \xi) dx d\xi < c, \quad \int_{\Omega \times I\!R^3} (1+ | \xi |^2) f_0(x, \xi) dx d\xi < c. \tag{1.6}$$

The existence of a unique solution of (1.1)-(1.6) can e.g.be deduced from [4]. Existence and uniqueness properties for the associated stationary problem are stated in the following theorem.

Theorem 1.1 *The stationary equation*

$$\xi . \nabla_x f = 0, \quad x \in \Omega, \quad \xi \in I\!R^3, \tag{1.7}$$

together with the boundary condition

$$f(x, \xi) = M(x, \xi) \int_{\xi' \cdot n(x) < 0} | \xi' \cdot n(x) | f(x, \xi') d\xi',$$
$$x \in \partial\Omega, \quad \xi \in I\!R^3, \quad \xi \cdot n(x) > 0, \tag{1.8}$$

has a non-negative solution in $L^1 \cap L^\infty(\Omega \times I\!R^3)$ with given total mass μ, which is unique in $L^\infty(\Omega \times I\!R^3)$ as well as in $L^1(\Omega \times I\!R^3)$.

Proof of Theorem 1.1.
For $(x, \xi) \in \overline{\Omega} \times I\!R^3$, denote

$$s^+(x,\xi) = inf\{s > 0; x - s\xi \in \partial\Omega, \quad \xi \cdot n(x - s\xi) > 0\}.$$

Define a solution f_s of (1.7)-(1.8) by

$$f_s(x,\xi) = \kappa M(x - s^+(x,\xi)\xi, \xi), \quad x \in \partial\Omega, \quad \xi \cdot n(x) < 0,$$

and extend f_s so that it is constant along the characteristics ending at $(\partial\Omega \times I\!R^3)^- := \{(x,\xi) \in \partial\Omega \times I\!R^3 s.t. \ \xi \cdot n(x) < 0\}$, and choose κ such that

$$\int_{\Omega \times I\!R^3_\xi} f_s(x,\xi) dx d\xi = \mu.$$

Then, for $x \in \partial\Omega$,

$$\begin{array}{l} \int_{\xi' \cdot n(x) < 0} \mid \xi' \cdot n(x) \mid f_s(x,\xi') d\xi' = \kappa (2\pi)^{-1} \int_{\xi' \cdot n(x) < 0} \mid \xi' \cdot n(x) \mid \\ \theta^2 (x - s^+(x,\xi')\xi') exp(-.5\theta(x - s^+(x,\xi')\xi') \mid \xi' \mid^2) d\xi' \\ = \kappa \int_{e' \cdot n(x) < 0, |e'| = 1} \mid e' \cdot n(x) \mid de' \int_0^{+\infty} (2\pi)^{-1} exp(-.5\tau^2) \tau^3 d\tau \\ = \kappa. \end{array} \tag{1.9}$$

Hence $f_s(x,\xi) = \kappa M(x,\xi)$ for $x \in \partial\Omega, \xi \cdot n(x) > 0$, f_s satisfies (1.1) and (1.3), and f_s belongs to $L^1 \cap L^\infty(\Omega \times I\!R^3)$.
If there is another solution f in $L^1(\Omega \times I\!R^3)$ or $L^\infty(\Omega \times I\!R^3)$, then the boundary condition (1.8) implies that there exists a function K defined over $\partial\Omega$, such that

$$f(x,\xi) = K(x) M(x,\xi), \quad x \in \partial\Omega, \quad \xi \cdot n(x) > 0.$$

Here

$$K(x) = \int_{\xi \cdot n(x) < 0} K(x - s^+(x,\xi)\xi) M(x - s^+(x,\xi)\xi, \xi) \mid \xi \cdot n(x) \mid d\xi.$$

Hence K is continuous. Also

$$\int_{e \cdot n(x) < 0, |e| = 1} \int_0^\infty (2\pi)^{-1} \theta^2 exp(-.5\theta s^2) \mid e \cdot n(x) \mid s^3 ds de = 1$$

independently of x in $\partial\Omega$. It follows that K is a constant function, since at each point it is a convex combination of the other K-values.
We now derive energy and entropy bounds for the solutions of the time-dependent problem (1.1)-(1.6).

Theorem 1.2 *Let f be a solution of (1.1)-(1.3) under (1.4-6). Then*

$$\int_{\Omega\times I\!\!R^3} \mid \xi \mid^2 f(t,x,\xi)dxd\xi, \quad \int_{\Omega\times I\!\!R^3} f \mid logf \mid (t,x,\xi)dxd\xi$$

are uniformly bounded with respect to time t in $I\!\!R_+$. This also holds for the mass and energy flows

$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)>0} \xi\cdot n(x) f(t,x,\xi)dxd\xi d\tau,$$
$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)<0} \mid \xi\cdot n(x)\mid f(t,x,\xi)dxd\xi d\tau,$$
$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)>0} \xi\cdot n(x) \mid \xi\mid^2 f(t,x,\xi)dxd\xi d\tau,$$
$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)<0} \mid \xi\cdot n(x)\mid\mid \xi\mid^2 f(t,x,\xi)dxd\xi d\tau.$$

Proof of Theorem 1.2.

$$(\partial_t + \xi\cdot\bigtriangledown_x)(flog\frac{f}{f_s}) = 0,$$

where f_s denotes the solution of the stationary problem introduced in Theorem 1.1. Integrating over $[0,t]\times\Omega\times I\!\!R^3$ leads to

$$\begin{aligned}&\textstyle\int_{\Omega\times I\!\!R^3} flog\frac{f}{f_s}(t,x,\xi)dxd\xi - \int_0^t\int_{\partial\Omega\times I\!\!R^3}\xi\cdot n(x)flog\frac{f}{f_s}(\tau,x,\xi)d\tau dxd\xi\\ &= \textstyle\int_{\Omega\times I\!\!R^3} f_0 log\frac{f_0}{f_s}(x,\xi)dxd\xi,\end{aligned} \tag{1.10}$$

so that, taking into account the non-positivity of the boundary term in (1.10) (see Darrozes & Guiraud inequality in [8]),

$$\int_{\Omega\times I\!\!R^3} flog\frac{f}{f_s}(t,x,\xi)dxd\xi < c. \tag{1.11}$$

Hence

$$\int_{\Omega\times I\!\!R^3} flog^+ f(t,x,\xi)dxd\xi - \int_{\Omega\times I\!\!R^3} f\mid log^- f\mid (t,x,\xi)dxd\xi$$
$$- \int_{\Omega\times I\!\!R^3} f(t,x,\xi)logf_s(x,\xi)dxd\xi < c.$$

It implies that

$$\int_{\Omega\times I\!\!R^3} flog^+ f(t,x,\xi)dxd\xi - \int_{\Omega\times I\!\!R^3} f\mid log^- f\mid (t,x,\xi)dxd\xi$$
$$+\tilde{c}\int_{\Omega\times I\!\!R^3} \mid\xi\mid^2 f(t,x,\xi)dxd\xi < c. \tag{1.12}$$

But for every positive real ϵ, (see [7]),

$$\int_{I\!R^3_\xi} f \mid log^- f \mid d\xi$$

$$= \int_{f \leq e^{-|\xi|^\epsilon}} f \mid logf \mid d\xi + \int_{e^{-|\xi|^\epsilon} < f \leq 1} f \mid logf \mid d\xi$$

$$< \int_{|\xi| \leq 1} d\xi + \int_{\xi > 1} e^{-|\xi|^\epsilon} \mid \xi \mid^\epsilon d\xi + \int f \mid \xi \mid^\epsilon d\xi,$$

so that

$$\int f \mid log^- f \mid d\xi < c + \int \mid \xi \mid^\epsilon f d\xi. \tag{1.13}$$

It follows from (1.12-13) that

$$\int_{\Omega \times I\!R^3} flog^+ f(t,x,\xi) dx d\xi - \int_{\Omega \times I\!R^3} \mid \xi \mid^\epsilon f(t,x,\xi) dx d\xi$$
$$+ \tilde{c} \int_{\Omega \times I\!R^3} \mid \xi \mid^2 f(t,x,\xi) dx d\xi < c.$$

Hence

$$\int_{\Omega \times I\!R^3} flog^+ f(t,x,\xi) dx d\xi$$
$$+ \frac{\tilde{c}}{2} \int_{\Omega \times I\!R^3, |\xi|^{2-\epsilon} \geq \frac{2}{\tilde{c}}} (\mid \xi \mid^2 - \frac{2}{\tilde{c}} \mid \xi \mid^\epsilon) f(t,x,\xi) dx d\xi$$
$$- \int_{\Omega \times I\!R^3, |\xi|^{2-\epsilon} \leq \frac{2}{\tilde{c}}} \mid \xi \mid^\epsilon f(t,x,\xi) dx d\xi$$
$$+ \frac{\tilde{c}}{2} \int_{\Omega \times I\!R^3, |\xi|^{2-\epsilon} \leq \frac{2}{\tilde{c}}} \mid \xi \mid^2 f(t,x,\xi) dx d\xi$$
$$+ \frac{\tilde{c}}{2} \int_{\Omega \times I\!R^3} \mid \xi \mid^2 f(t,x,\xi) dx d\xi < c,$$

which implies

$$\int_{\Omega \times I\!R^3} flog^+ f(t,x,\xi) dx d\xi + \frac{\tilde{c}}{2} \int_{\Omega \times I\!R^3} \mid \xi \mid^2 f(t,x,\xi) dx d\xi < c. \tag{1.14}$$

(1.13-14) end the proof of the boundedness of $\int_{\Omega \times I\!R^3} \mid \xi \mid^2 f(t,x,\xi) dx d\xi$ and $\int_{\Omega \times I\!R^3} f \mid logf \mid (t,x,\xi) dx d\xi$. Then bounds on mass and energy flows through $\partial\Omega$ follow from [4, Lemma 4.1].
The entropy flow of the trace of $f(t,.,.)$ on the boundary is studied in the following lemma.

Lemma 1.3

$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)>0}\xi\cdot n(x)flogf(\tau,x,\xi)d\tau dxd\xi$$

and

$$\int_{[t,t+1]\times\partial\Omega}\int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid flogf(\tau,x,\xi)d\tau dxd\xi$$

are uniformly bounded with respect to t *in* $I\!R_+$.

Proof of Lemma 1.3.
By Theorem 1.2 and (1.10) the uniform boundedness of the outgoing flow follows from the uniform boundedness of the ingoing flow. Denote by q

$$q(t,x):=\int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid f(t,x,\xi)d\xi,\quad t\in I\!R_+,\quad x\in\partial\Omega. \quad (1.15)$$

Then, for $x\in\partial\Omega$ and $\xi\in I\!R^3$ such that $\xi\cdot n(x)>0$, the boundary condition (1.3) implies that

$$flogf(t,x,\xi)=f(t,x,\xi)logM(x,\xi)+M(x,\xi)(qlogq)(t,x).$$

By Theorem 1.2, the ingoing flow of $flogM$ is uniformly bounded as well as the ingoing and outgoing mass and energy flows. So

$$\int_t^{t+1}\int_{\partial\Omega}\int_{\xi\cdot n(x)>0}\xi\cdot n(x)flogf$$

is uniformly bounded in t if and only if

$$\int_t^{t+1}\int_{\partial\Omega}\int_{\xi\cdot n(x)>0}\xi\cdot n(x)Mqlogq$$

is uniformly bounded in t. The boundedness of the log^- part follows from $qlog^-q\leq\frac{1}{e}$. So it remains to prove that $\int_t^{t+1}\int_{\partial\Omega}\int_{\xi\cdot n(x)>0}\xi\cdot n(x)Mqlog^+q$ is uniformly bounded in t. But uniformly in x

$$\int_{\xi\cdot n(x)>0}\xi\cdot n(x)M(x,\xi)d\xi\leq c\int_{A_x}\xi\cdot n(x)M(x,\xi)d\xi$$

with

$$A_x=\{\xi;\frac{1}{\epsilon}>\mid\xi\mid>\epsilon,\xi\cdot n(x)>\epsilon\mid\xi\mid\}.$$

Hence

$$\int_t^{t+1} \int_{\partial\Omega} \int_{\xi\cdot n(x)>0} \xi \cdot n(x) Mqlog^+ q \le c \int_t^{t+1} \int_{\partial\Omega} \int_{A_x} \xi \cdot n(x) Mqlog^+ q.$$

For $(x,\xi) \in \overline{\Omega} \times I\!\!R^3$ denote

$$s^-(x,\xi) = inf\{s>0; x+s\xi \in \partial\Omega, \xi \cdot n(x+s\xi) < 0\}.$$

Then

$$\inf_A s^- > 0, \quad \sup_A s^- = S < \infty,$$

where

$$A = \{(x,\xi); x \in \partial\Omega, \xi \in A_x\}.$$

Now the lemma holds if

$$I = \int_t^{t+1} \int_A s^-(x,\xi)\xi \cdot n(x) Mqlog^+ q dxd\xi d\tau$$

is uniformly bounded in t. But

$$\begin{aligned} I &= \int_t^{t+1} \int_A \int_0^{s^-(x,\xi)} \xi \cdot n(x) flog^+ \frac{f}{M} dxd\xi d\tau ds \\ &\le \int_t^{t+1+S} \int_\Omega \int_{I\!\!R^3} flog^+ \frac{f}{M} dxd\xi d\tau, \end{aligned}$$

which is uniformly bounded in t by Theorem 1.2.
Let us now state the main result of this section.

Theorem 1.4 *Let f be a solution of (1.1) with the boundary condition (1.3) and initial data satisfying (1.6).*
Then for all $T>0$ the family of functions $f_\tau(t,x,\xi) := f(t+\tau,x,\xi)$ for $\tau \ge 0$ converges when $\tau \to \infty$ in $L^1([-T,T]\times\Omega\times I\!\!R^3)$ to the solution f_s of the stationary problem (1.7)-(1.8) with total mass $\int f_0 dxd\xi$. Moreover, the traces of f_τ on the boundary $\partial D^+ := [-T,T]\times\partial\Omega\times\{\xi\in I\!\!R^3; \xi\cdot n(x)>0\}$ (resp. $\partial D^- := [-T,T]\times\partial\Omega\times\{\xi\in I\!\!R^3; \xi\cdot n(x)<0\}$) converge in $L^1(\partial D^+)$ (resp. $L^1(\partial D^-)$) to the traces of f_s on ∂D^+ (resp. ∂D^-).

Proof of Theorem 1.4.
It is enough to show that for every sequence t_n tending to infinity, there exists a subsequence t_{n_k} such that $f_{n_k}(t,x,\xi) := f(t+t_{n_k},x,\xi)$ converges

in $L^1([-T,T]\times\Omega\times I\!\!R^3)$ to the solution f_s of the stationary problem for all $T>0$ (where for a given T we only consider t_n with $t_n>T$). Theorem 1.2 and Lemma 1.3 imply for all $T>0$ the weak compactness in $L^1([-T,T]\times\Omega\times I\!\!R^3)$, $L^1(\partial D^+)$ and $L^1(\partial D^-)$ of $f(t_{n_k}+t,.,.)$ and its traces over ∂D^+ and ∂D^- respectively, so that subsequences converge to some g, g^+ and g^- respectively. Let us first prove that

$$\int_{\xi\cdot n(x)>0}\xi\cdot n(x)f_{n_k}d\xi,\quad (\text{resp. } \int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid f_{n_k}d\xi)$$

strongly converges in $L^1([-T,T]\times\partial\Omega)$ to

$$\int_{\xi\cdot n(x)>0}\xi\cdot n(x)g^+d\xi,\quad (\text{resp. } \int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid g^-d\xi).$$

Because of the boundedness of the energy flow through the boundary established in Theorem 1.2, and the boundedness of the entropy flow of Lemma 1.3 it is sufficient to prove that for arbitrary $\epsilon>0$

$$\int_{\xi\cdot n(x)\leq-\epsilon,\epsilon<|\xi|\leq\frac{1}{\epsilon}}\xi\cdot n(x)f_{n_k}d\xi$$

strongly converges in $L^1([-T,T]\times\partial\Omega)$ to

$$\int_{\xi\cdot n(x)\leq-\epsilon,\epsilon<|\xi|\leq\frac{1}{\epsilon}}\xi\cdot n(x)g^+d\xi.$$

It is enough to prove translational equicontinuity of the sequence in sL^1. This criterium is easily defined for translations in $I\!\!R^n$. A strong enough substitute can be defined with respect to $\partial\Omega$ as follows. Fix a sphere in Ω. Connect $x\in\partial\Omega$ to the centre of the sphere by a line segment, and let x' be the intersection with the sphere. Rotating the sphere takes x' to x'_t and thereby defines a corresponding image $x_t\in\partial\Omega$ for each $x\in\partial\Omega$. At an outgoing point (t,x,ξ) the value of f equals the one at the ingoing point $(t-s^+(x,\xi),x-s^+(x,\xi)\xi,\xi)$, where $s^+(x,\xi)=\inf\{s>0;x-s\xi\in\partial\Omega^+\}$. For x_t choose ξ_t so that $x_t-s^+(x,\xi)\xi_t=x-s^+(x,\xi)\xi$. Using the uniform continuity of M and a change of variables from ξ to ξ_t in the unperturbed integral gives the desired equicontinuity in sL^1. The argument for translation in time is similar. Since for x in Ω

$$f_{n_k}(t,x,\xi)=M(x-\xi s^+(x,\xi),\xi)$$
$$\int_{\xi'\cdot n(x-\xi s^+(x,\xi))<0}\mid\xi'\cdot n(x-\xi s^+(x,\xi))\mid f_{n_k}(t-s^+(x,\xi),x-\xi s^+(x,\xi),\xi')d\xi',$$

f_{n_k} converges in $L^1([-T,T]\times\Omega\times I\!R^3)$ to g, and the traces of g are g^+ and g^- which satisfy (1.3).
It remains to prove that g, g^+ and g^- are respectively equal to the stationary solution f_s with total mass $\mu=\int f_0(x,\xi)dxd\xi$, the trace of f_s on $\partial\Omega^+$ and the trace of f_s on $\partial\Omega^-$. From Egoroff's theorem, there exists for every positive real ϵ a subset A_ϵ of $[-T,T]\times\partial\Omega$, the complement of which is of measure smaller than ϵ, and such that $\int_{\xi\cdot n(x)>0}\xi\cdot n(x)f_{n_k}d\xi$ uniformly converges to $\int_{\xi\cdot n(x)>0}\xi\cdot n(x)g^+d\xi$ on A_ϵ. But

$$\int_t^{t+T}d\tau\int_{\partial\Omega}dx[\int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid f(t+\tau)log\frac{f(t+\tau)}{f_s}d\xi$$
$$-\int_{\xi\cdot n(x)>0}\xi\cdot n(x)f(t+\tau)log\frac{f(t+\tau)}{f_s}d\xi]+\int_{\Omega\times I\!R^3}f(t+T)log\frac{f(t+T)}{f_s}dxd\xi$$
$$=\int_{\Omega\times I\!R^3}f(t)log\frac{f(t)}{f_s}dxd\xi.$$

The boundary integral in the left hand side is positive and increasing with T by the Darrozes & Guiraud inequality, which guarantees that its inner bracket [...] is non-negative. It follows that $\int_{\Omega\times I\!R^3}f(t)log\frac{f(t)}{f_s}dxd\xi$ is decreasing with time. But the mass of f is conserved, and by Theorem 1.2 its energy uniformly bounded in time. So by [7] $\int_{\Omega\times I\!R^3}f(t)logf(t)dxd\xi$ is bounded from below. Using (1.5) and Theorem 1.2 it follows that also $\int_{\Omega\times I\!R^3}f(t)log\frac{1}{f_s}dxd\xi$ is bounded from below. Hence the decreasing function $\int_{\Omega\times I\!R^3}f(t)log\frac{f(t)}{f_s}dxd\xi$ has a finite limit when t tends to infinity. It follows that

$$\lim_{k\to\infty}\int_{-T}^{T}d\tau\int_{\partial\Omega}dx[\int_{\xi\cdot n(x)<0}(f_{n_k}log\frac{f_{n_k}}{f_s}+f_s-f_{n_k})d\xi$$
$$-\int_{\xi\cdot n(x)>0}(f_{n_k}log\frac{f_{n_k}}{f_s}+f_s-f_{n_k})]=0.$$

Since the inner bracket [...] is non-neggative the same holds if $[-T,T]\times\partial\Omega$ is substituted by A_ϵ. Hence with q given by (1.15) for $f=g^-$ we have

$$\int_{A_\epsilon}\int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid(g^-log\frac{g^-}{f_s}+f_s-g^-)(\tau,x,\xi)d\tau dxd\xi$$
$$\leq\lim_{k\to\infty}\int_{A_\epsilon}\int_{\xi\cdot n(x)<0}\mid\xi\cdot n(x)\mid(f_{n_k}log\frac{f_{n_k}}{f_s}+f_s-f_{n_k})(\tau,x,\xi)d\tau dxd\xi$$
$$=\lim_{k\to\infty}\int_{A_\epsilon}\int_{\xi\cdot n(x)>0}\xi\cdot n(x)(f_{n_k}log\frac{f_{n_k}}{f_s}+f_s-f_{n_k})d\tau dxd\xi$$

$$= \int_{A_\epsilon} \int_{\xi \cdot n(x) > 0} \xi \cdot n(x) (Mqlog \frac{q}{\kappa} + \kappa M - Mq) d\tau dx d\xi. \tag{1.16}$$

From the Darrozes & Guiraud inequality,

$$\int_{A_\epsilon} \int_{\xi \cdot n(x) > 0} \xi \cdot n(x) (g^+ log \frac{g^+}{f_s} + f_s - g^+) d\tau dx d\xi$$
$$\leq \int_{A_\epsilon} \int_{\xi \cdot n(x) < 0} \mid \xi \cdot n(x) \mid (g^- log \frac{g^-}{f_s} + f_s - g^-) d\tau dx d\xi,$$

so that (1.16) implies that

$$\int_{A_\epsilon} \int_{\xi \cdot n(x) > 0} \xi \cdot n(x) (g^+ log \frac{g^+}{f_s} + f_s - g^+) d\tau dx d\xi$$
$$+ \int_{A_\epsilon} \int_{\xi \cdot n(x) < 0} \xi \cdot n(x) (g^- log \frac{g^-}{f_s} + f_s - f^-) d\tau dx d\xi = 0.$$

Moreover,

$$\int_{[-T,T] \times \partial\Omega} \int_{\xi \cdot n(x) > 0} \xi \cdot n(x) (g^+ log \frac{g^+}{f_s} + f_s - g^+) d\tau dx d\xi < c,$$

and

$$\int_{[-T,T] \times \partial\Omega} \int_{\xi \cdot n(x) < 0} \mid \xi \cdot n(x) \mid (g^- log \frac{g^-}{f_s} + f_s - g^-) d\tau dx d\xi < c.$$

Therefore

$$\int_{[-T,T] \times \partial\Omega} \int_{\xi \cdot n(x) > 0} \xi \cdot n(x) (g^+ log \frac{g^+}{f_s} + f_s - g^+) d\tau dx d\xi$$
$$+ \int_{[-T,T] \times \partial\Omega} \int_{\xi \cdot n(x) < 0} \xi \cdot n(x) (g^- log \frac{g^-}{f_s} + f_s - g^-) d\tau dx d\xi = 0.$$

And so by the equality case for the Darrozes & Guiraud inequality there is a function $q(t,x)$ defined over $[-T,T] \times \partial\Omega$, such that

$$g^+(t,x,\xi) = f_s(x,\xi) q(t,x), \quad (t,x,\xi) \in \partial D^+,$$

and

$$g^-(t,x,\xi) = f_s(x,\xi) q(t,x), \quad (t,x,\xi) \in \partial D^-.$$

Since Ω is a convex set, and g satisfies

$$(\partial_t + \xi \bigtriangledown_x) g = 0,$$

q is independent of x and t. Moreover, because of the mass conservation, this constant is equal to one.

Corollary 1.5 *Theorem 1.4 holds under the weaker assumption that the initial data only satisfy the mass and energy bound of (1.6).*

Proof of Corollary 1.5.
The problem is linear and monotone. The truncated initial values $f_0^n = f_0 \wedge n$ for $| v | \leq n$, $f_0^n = 0$ otherwise, satisfy (1.6). The corresponding solutions are a monotone increasing sequence with mass conservation $\int f_t^n = \int f_0^n$. This implies that the L^1-limit $f(t) = \lim_{n \to \infty} f^n(t)$ is uniform in time, hence that f satisfies Theorem 1.4.

Remark.
The above approach via (1.6) was chosen (instead of more regular f_0^n *'s*, such as $f_0^n \in L^\infty$) to connect with kinetic aspects also useful in more general contexts.

2 Asymptotics for the Boltzmann equation

This section studies the asymptotic behaviour of the solution of the Boltzmann equation in an open bounded convex domain $\Omega \subset I\!\!R^3$, with Lyapunov type boundary,

$$(\partial_t + \xi \cdot \bigtriangledown_x) f = Q(f, f), \quad t \in I\!\!R^3, \quad x \in \Omega, \quad \xi \in I\!\!R^3, \tag{2.1}$$

where Q denotes the collision operator, together with an initial condition

$$f(t, x, \xi) = f_0(x, \xi), \quad x \in \Omega, \quad \xi \in I\!\!R^3 \tag{2.2}$$

satisfying (1.6), and maxwellian diffuse reflection on the boundary

$$f(t, x, \xi) = M(\xi) \int_{\xi' \cdot n(x) < 0} | \xi' \cdot n(x) | f(t, x, \xi') d\xi',$$
$$t \in I\!\!R^+, \quad x \in \partial\Omega, \quad \xi \cdot n(x) > 0. \tag{2.3}$$

M is a normalized maxwellian with a constant temperature $\frac{1}{\theta} > 0$,

$$M(\xi) = (2\pi)^{-1} \theta^2 exp(-.5\theta \mid \xi \mid^2). \tag{2.4}$$

The relevant equilibrium solution is $f_s = cM$ with

$$c = \frac{1}{\mid \Omega \mid} \int_{\Omega \times I\!R^3} f_0(x,\xi) dx d\xi. \tag{2.5}$$

An existence result for (2.1-3) from [4] is recalled in the following theorem, where the collision operator is of the full generality in [4].

Theorem 2.1 *There is a function*

$$f \in C(I\!R^+, L^1(\Omega \times I\!R^3)), \quad f \geq 0,$$

satisfying (2.1-2) in DiPerna-Lions sense and (2.3) for the traces, possibly with inequality, the left hand side being greater than or equal to the right hand side.

Actually the relevant result from [4] Theorem 6.1 states slightly less, but the proof of Theorem 6.1 also implies the trace inequality as formulated here. Energy and entropy bounds are also derived in [4], but on finite intervals of time, the bounds depending exponentially on time. Bounds uniform on $I\!R_+$ are derived in the following theorem.

Theorem 2.2 *Let* f *be a solution of* $(2.1-4)$*. Then*

$$\int_{\Omega \times I\!R^3} \mid \xi \mid^2 f(t,x,\xi) dx d\xi, \quad \int_{\Omega \times I\!R^3} f \mid logf \mid (t,x,\xi) dx d\xi,$$

$$\int_{[t,t+T]\times\partial\Omega} \int_{\xi\cdot n(x)>0} \mid \xi \mid^2 \xi \cdot n(x) f(\tau,x,\xi) d\tau dx d\xi,$$
$$\int_{[t,t+T]\times\partial\Omega} \int_{\xi\cdot n(x)<0} \mid \xi \mid^2 \mid \xi \cdot n(x) \mid f(\tau,x,\xi) d\tau dx d\xi,$$

$$\int_{[t,t+T]\times\partial\Omega} \int_{\xi\cdot n(x)>0} \xi \cdot n(x) f(\tau,x,\xi) d\tau dx d\xi,$$
$$\int_{[t,t+T]\times\partial\Omega} \int_{\xi\cdot n(x)<0} \mid \xi \cdot n(x) \mid f(\tau,x,\xi) d\tau dx d\xi \tag{2.6}$$

are uniformly bounded for t *varying in* $I\!R_+$*.*

Proof of Theorem 2.2.
Formally

$$(\partial_t + \xi \cdot \nabla_x)(f log\frac{f}{M}) = Q(f,f) log\frac{f}{M} + Q(f,f).$$

Integrating this over $[0,t]\times\Omega\times I\!R^3$ and using (2.2-3) gives

$$\int_{\Omega\times I\!R^3}(flog\frac{f}{M})(t,x,\xi)dxd\xi-\int_0^t\int_{\partial\Omega\times I\!R^3}\xi\cdot n(x)(flog\frac{f}{M})d\tau dxd\xi$$
$$+\int_0^t\int_{\Omega\times I\!R^3}e(f)d\tau dxd\xi$$
$$\leq\int_{\Omega\times I\!R^3}f_0log\frac{f_0}{M}(x,\xi)dxd\xi,$$

where

$$e(f)=\frac{1}{4}\int_{I\!R^3}\int_{B^+}B(\mid\xi-\xi_*\mid,u)(f'f'_*-ff_*)log\frac{f'f'_*}{ff_*}d\xi_*du.$$

This inequality strictly holds (see [4]). Since $e(f)\geq 0$ and Darrozes & Guiraud's inequality holds for the boundary term, it follows that

$$\int_{\Omega\times I\!R^3}flog\frac{f}{M}(t,x,\xi)dxd\xi<c, \tag{2.7}$$

and

$$0\leq\int_0^{+\infty}\int_{\Omega\times I\!R^3}e(f)(t,x,\xi)dtdxd\xi<c. \tag{2.8}$$

From here the proof proceeds as the proof of Theorem 1.2 from (1.11) on. For the solutions of Theorem 2.1, the asymptotic behaviour of a solution of the Boltzmann problem (2.1-3) can be derived, if we require that the collision kernel B of Q is nowhere vanishing.

Theorem 2.3 *Let f be a solution in the sense of Theorem 2.1 of the initial boundary value problem $(2.1-3)$ with nowhere vanishing collision kernel. Then when t tends to infinity $f(t,.,.)$ converges strongly in $L^1(\Omega\times I\!R^3)$ to the global maxwellian cM, where M is defined in the boundary condition (2.3) and c gives the conservation of mass ($c=\frac{\int f_0}{\int M}$).*

Proof of Theorem 2.3.
It is enough to show that for every sequence t_n tending to infinity there exists a subsequence t_{n_k} such that $f_{n_k}(t,x,\xi):=f(t+t_{n_k},x,\xi)$ converges in $L^1([0,T]\times\Omega\times I\!R^3)$ to cM for all $T>0$. The weak $L^1([0,T]\times\Omega\times I\!R^3)$ convergence of a subsequence of f_n follows from Theorem 2.2. Given (2.8) and Theorem 2.2 and arguing as in [1] or [14] we may conclude that the limit is of strong L^1-type, that it satisfies the Boltzmann equation in mild sense,

and is a local maxwellian $m(t,x,\xi)$ since the collision kernel B is nowhere vanishing. By [10] it follows that a maxwellian solution of the Boltzmann equation has the form

$$m(t,x,\xi) = exp\{d_0 + C_1 \cdot (x-\xi t) + c_3 \mid x-\xi t \mid^2 \tag{2.9}$$

$$+C_0 \cdot \xi + c_2(x-\xi t)\cdot \xi + c_1 \mid \xi \mid^2 + \Lambda_0(x)\cdot \xi\}, \tag{2.10}$$

where $d_0, c_1, c_2, c_3 \in I\!R^+, C_0, C_1 \in I\!R^3$, and Λ_0 is a skew-symmetric tensor. The proof of Theorem 6.1 in [4] can also be applied with the f_{n_k} satisfy (2.3) with inequality

$$f_{n_k}(t,x,\xi) \geq M(\xi) \int_{\xi'\cdot n(x)<0)} \mid \xi' \cdot n(x) \mid f_{n_k}(t,x,\xi')d\xi'$$

on the boundary. The conclusion is that also the traces of m satisfy the same inequality

$$m(t,x,\xi) \geq M(\xi) \int_{\xi'\cdot n(x)<0} \mid \xi' \cdot n(x) \mid m(t,x,\xi')d\xi',$$

$$t \in I\!R^+, \quad x \in \partial\Omega, \quad \xi \cdot n(x) > 0.$$

But the collision term of the Boltzmann equation for m is L^1-integrable (actually zero). Since the mass $\int m(t,x,\xi)dxd\xi$ is time-independent, Green's formula (see [4]) gives that the inflow of mass of m on $\partial\Omega$ over a time interval $[0,T]$ equals the corresponding outflow. Hence there is for a.e. (x,t) equality in (2.3). This together with (2.9) gives that $m(t,x,\xi)$ is a global maxwellian.

Remarks.

In contrast to the periodic and specular or direct reflection boundary condition cases, here the maxwellian is uniquely determined by the initial value and the boundary condition. This uniqueness of the maxwellian under diffuse reflection at the boundary was first noticed by C. Cercignani in [9]

The driving mechanism behind the maxwellian asymptotic behaviour for the Boltzmann equation with a strictly non-vanishing collision kernel in Theorem 2.3 (see also [15] for the linear case) is the entropy dissipation term. This should be compared with the collisionless case of Theorem 1.4 where the maxwellian diffuse reflection boundary condition is the sole underlying agent.

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