A Compactness Result Related to the Stationary Boltzmann Equation in a Slab, with Applications to the Existence Theory

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Abstract. Weak compactness in $L^1([a,b]\times\mathbb{R}^3)$ is proved for any sequence of functions for which a weighted L^1-norm and the Boltzmann entropy production term are bounded, and which satisfy some very weak condition of local boundedness from below. The property holds for a wide class of collision kernels. This result is then used to solve the stationary Boltzmann equation in a slab, for given indata and for diffuse reflection with total inflow given under various small velocity truncations of the collision kernel.

Introduction. We study the stationary Boltzmann equation in a slab,

$$\xi\frac{\partial f}{\partial x} = Q(f,f),\quad x\in\,]0,a[,\quad v=(\xi,\eta,\zeta)\in\mathbb{R}^3, \tag{0.1}$$

with various boundary conditions at $x=0$ and $x=A$, which combine given inflows with specular and diffuse reflection. The collision operator Q is

$$Q(f,f)(x,v)=\int_{\mathbb{R}^3}\int_{S^2}B(v-v_*,\omega)[f'f'^*-ff^*]\,d\omega\,dv_*,$$

where

$$f^*=f(x,v_*),\quad f'=f(x,v'),\quad f'^*=f(x,v'_*),$$

and

$$v'=v-(v-v_*,\omega)\omega,\quad v'_*=v_*+(v-v_*\,,\omega)\omega. \tag{0.2}$$

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Here, $(v-v_*,\omega)$ denotes the Euclidean inner product in $\mathbb{R}^3$. Let ω be represented by the polar angle ϑ (with polar axis along $v-v_*$) and the azimuthal angle φ. We assume that

$$B(v-v_*,\omega)=|v-v_*|^\beta b(\vartheta), \tag{K1}$$

with

$$-3<\beta<2, \quad b\in L^\infty(S^2), \quad b(\vartheta)\geq c>0 \quad \text{for } \frac{\pi}{8}\leq\vartheta\leq\frac{3\pi}{8}.$$

(To the price of some technical complications, considerably more general B's can be used.) Truncations of the collision kernel B are sometimes introduced to reduce certain collision rates. This is obtained by introducing a function χ_α that is invariant under the collition transformation J defined by

$$J(v,\omega,v_*)=(v',-\omega,v'_*),$$

with v' and v'_* defined by (0.2), and using the collision kernel

$$B_\alpha(v,v_*,\omega)=B(v-v_*,\omega)\chi_\alpha(v,v_*,v',v'_*).$$

That gives instead of the collision operator Q, the following modified collision operator

$$Q_\alpha(f,f)(x,v)=\int_{\mathbb{R}^3}\int_{S^2}B(v,v_*,\omega)[f'f'^*-ff^*]\,d\omega\,dv_*. \tag{K2}$$

The entropy production is defined by

$$D(f,f)=\int_0^\alpha\int_{\mathbb{R}^3}\int_{S^2}B(v-v_*\,,\,\omega)[f'f'^*-ff^*]\ln\frac{f'f'^*}{ff^*}\,d\omega\,dv_*\,dv\,dx,$$

and correspondingly for B_α.

Certain results concerning solutions of the non-linear Boltzmann equation close to equilibrium, and solutions of the corresponding linearized equation are known. The solvability of boundary value problems for the Boltzmann equation in situations close to equilibrium is studied in [6], [7], and [8]. The unique solvability of internal stationary problems for the Boltzmann equation at large Knudsen numbers is established in [10]. Existence and uniqueness of stationary solutions of the linearized Boltzmann equation in a bounded domain is proved in [9]. A classification of the well-posedness of some boundary value problems for the linearized Boltzmann equation is made in [4]. Stationary solutions of the BGK model equation are derived in [11]. There are only few mathematical results on large data boundary-value problems for the non-linear stationary Boltzmann equation. A serious technical problem in such studies is the scarcity of known useful a-priori estimates. In particular, there is no entropy boundedness available, which in the time-dependent case, together with the boundedness of mass and energy, provide weak L^1-compactness. Measure solutions of the steady Boltzmann equation in a slab have been obtained in [3] for a collision kernel truncated for a small x-component of the velocity. The proof is based on the weak-* compactness of uniformly bounded measures and does not refer to any entropy argument. The entropy production term is used in [2] for proving, via non-standard arguments, that the same problem has solutions x a.e. in $L^1_{\xi^2}(\mathbb{R}^3)$.

In the first section of the present paper, a connection is established between lower bounds on the solutions and bounds on the integral $\int(1+|v|)^\beta f(x,v)\,dx\,dv$. Then we use the boundedness of the entropy production term to derive a compactness result in $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$ for general collision kernels. In the following sections we apply this compactness result to solve the stationary Boltzmann equation in a slab. The existence is proved for the stationary slab problems with:

- given indata f_0, f_a and reduced collision rate for small ξ in Section 2,
- given total inflow with diffuse reflection and reduced rate for small ξ in Section 3,
- reduced collision rate for small v, and rotation symmetry around the ξ-axis with respect to $\eta^2+\zeta^2$ for given indata in Section 4.

1. The main compactness result and some solution properties.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions on $[0,a]\times\mathbb{R}^3$.

Lemma 1.1. *Assume*

$$\text{(H1)}\qquad \exists c_1 : \forall n,\ \forall x\in[0,a],\quad \int_{\mathbb{R}^3}\xi^2 f_\alpha(x,v)\,dv\le c_1,$$

and

$$\text{(H2)}\qquad \exists c_2 : \forall n,\quad D(f_\alpha,f_\alpha)\le c_2.$$

If there exists a measurable bounded set $Z\subset\mathbb{R}^3$ with $|Z|>0$, and a constant $c>0$ such that for every $n\in\mathbb{N}$, ($f_\alpha(x,v)>c$ for a.e. x in $[0,a]$ and for v in a measurable subset $Z_n\subset Z$, with $|Z_n|>|Z|/2$, then

$$\text{(1.1)}\qquad \sup_{n\in\mathbb{N}}\int_{(0,a)\times\mathbb{R}^3} f_n(x,v)(1+|v|)^\beta\,dx\,dv<\infty.$$

Proof. We shall use the notations

$$a\wedge b := \min(a,b),\quad a\vee b := \max(a,b).$$

The domain $\mathbb{R}^3$ is split into three: $S_1 = \{v \in \mathbb{R}^3;\ |\xi|^2 \geq \varepsilon^2 \wedge \varepsilon^4(\eta^2+\zeta^2)\}$, $S_\varepsilon = \{v \in \mathbb{R}^3;\ |\xi| < \varepsilon,\ \eta^2+\zeta^2 < \varepsilon^{-2}\}$, and $\mathbb{R}^3 \setminus (S_1 \cup S_\varepsilon)$. For the first domain S_1, (1.1) follows from (H1). For the other two domains, we use the elementary inequality

$$y^2 \leq 2 + 2(y^2-1)\ln y.$$

It follows that

$$f_n f_n^* \leq 2 f_n' f_n'^* + (f_n f_n^* - f_n' f_n'^*) \ln \frac{f_n f_n^*}{f_n' f_n'^*}. \tag{1.2}$$

Then, keeping Z fixed in one half space $\xi > 0$ or $\xi < 0$ and away from $\xi = 0$, we can (if needed) decrease ε and let $v \in S_\varepsilon$, $v_* \in Z_n$, and, say, $|\vartheta - \pi/4| < \pi/16$. We take v' and v_*' in (1.2) defined by φ in a suitable sector of witdth, say, $\pi/8$ and ε so small that $\varepsilon \ll |\xi'| \wedge |\xi_*'|$ for all ϑ, φ in the set S_ω above. (1.1) for the domain S_ε follows from an integration of (1.2) multiplied by B over $(0,a) \times S_\varepsilon \times Z_n \times S_\omega$ with the help of (H1) and (H2). Here in the integration of $f_n' f_n'^*$ with respect to $S_\varepsilon \times Z_n \times S_\omega$ we switch from unprimed to primed velocity variables. The Jacobian of that transformation is one. Finally, for v in the remaining part of $\mathbb{R}^3$ and v_* in Z_n, $|v-v_*|^\beta \sim (1+|v|)^\beta$. And so the corresponding part of (1.1) can be estimated in the same way. □

All along the paper, $c_1, c_2, \ldots$ will denote positive constants independent of n, and subsequences of (f_n) will be denoted by (f_n). Now we state the main result of this section.

Theorem 1.2. *Let (f_n) be a sequence of nonnegative measurable functions on $[0,a] \times \mathbb{R}^3$ verifying* (H1), (H2), *and*

(H3) *For every measurable bounded set $Z \subset \mathbb{R}^3$, with $|Z| > 0$ and $\inf_{v \in Z} |\xi| > 0$, there exist a constant $c > 0$ and for every $n \in \mathbb{N}$, a measurable set $Z_n \subset Z$, with $|Z_n| > |Z|/2$, such that $f_n(x,v) > c$, $v \in Z_n$, $x \in [0,a]$.*

Then (f_n) is weakly relatively compact in $L^1_{\text{loc}}([0,a] \times \mathbb{R}^3)$.

Proof. The proof consists of two steps. Lemma 1.1 implies that a subsequence of $(f_n(1+|v|^\beta)$ converges to a measure μ for the weak–* topology on $M([0,a] \times \mathbb{R}^3)$. Here $M([0,a] \times \mathbb{R}^3)$ denotes the set of bounded measures on $[0,a] \times \mathbb{R}^3$. The first step proves that μ belongs to $L^1_{\text{loc}}([0,a] \times \mathbb{R}^3)$. The second step achieves the proof of the weak relative compactness of (f_n) in $L^1_{\text{loc}}([0,a] \times \mathbb{R}^3)$.

STEP 1. Suppose that μ does not belong to $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$. It means that μ has a singular part of strictly positive measure contained in an open set A of finite measure

$$A \subset [0,a]\times\{v;\ \xi\in[q_1,p_1],\ |\eta|<p_1,\ |\zeta|<p_1\}.$$

So $\exists c_3$ such that $\forall n$, $\exists A_n$ open $\subset A$ such that

$$|A_n| < \frac{1}{4n} \quad \text{and} \quad \int_{A_n} d\mu > c_3.$$

Then

$$\overline{\lim_{k\to\infty}} \int_{A_n} f_k(x,v)\,dx\,dv > c_3.$$

Let a subsequence of (f_n), constructed by a diagonal process, be such that

$$\forall n, \quad \int_{A_n} f_n(x,v)\,dx\,dv > \frac{c_3}{2}. \tag{1.3}$$

Let B_n be defined by

$$B_n = \{(x,v)\in A_n;\ f_n(x,v) > c_3 n\}.$$

Then

$$\int_{B_n} f_n(x,v)\,dx\,dv > \frac{c_3}{4}.$$

Let p_2 and p_3 be real numbers such that

$$p_1 - q_1 < p_3 - p_2 \ll p_2 - p_1. \tag{1.4}$$

Applying assumption (H3), with $Z=[p_2,p_3]\times[-p_1,p_1]^2$, leads for every n to the existence of a set $Z_n\subset Z$ such that $|Z_n|>|Z|/2$, and a constant c_4 such that

$$f_n(x,v) > c_4, \quad \text{a. e. } x\in[0,a],\ v\in Z_n. \tag{1.5}$$

For every $(x,v)\in B_n$, let us define

$$E_n(x,v) = \Big\{(v_*,\vartheta,\varphi);\ v_*\in Z_n,\ \vartheta\in\Big[\frac{\pi}{8},\frac{3\pi}{8}\Big],\ \varphi\in[0,2\pi]$$
$$\text{and } \sqrt{f_n(x,v)f_n(x,v_*)} \ge f_n(x,v')f_n(x,v'_*)\Big\}.$$

Assumption (H2) implies that

$$\int_{B_n}\int_{E_n(x,v)} B(v-v_*,\omega)(f_nf_n^*-f_n'f_n'^*)\ln\frac{f_nf_n^*}{f_n'f_n'^*}\,dv_*\,d\vartheta\,d\varphi\,dx\,dv < c_2. \tag{1.6}$$

But, for (x,v,v_*), such that (x,v) and v_* belong to B_n and Z_n respectively,

$$f_n(x,v)f_n(x,v_*) \geq nc_3c_4.$$

Hence $f_n(x,v)f_n(x,v_*)$ is at least of the same order as n, which implies that nor n large enough, and (v',v_*') constructed from (x,v) in B_n and (v_*,ϑ,φ) in $E_n(x,v)$,

$$\begin{aligned}(f_nf_n^*-f_n'f_n'^*)\ln\frac{f_nf_n^*}{f_n'f_n'^*} &> \frac{1}{2}f_nf_n^*\ln\frac{f_nf_n^*}{f_n'f_n'^*}\\ &> \frac{1}{4}f_nf_n^*\ln(f_nf_n^*).\end{aligned} \tag{1.7}$$

Moreover, due to assumption (K1), and the fact that Z_n and the v-component of A are bounded disjoint sets,

$$B(v-v_*,\omega) > c_5,\quad (x,v)\in B_n,\ v_*\in Z_n,\ \vartheta\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]. \tag{1.8}$$

Then (1.5), (1.6), (1.7) and (1.8) imply that

$$\ln(n)\int_{B_n}|E_n(x,v)|f_n(x,v)\,dx\,dv < c_6. \tag{1.9}$$

(1.9) leads to

$$\int_{C_n} f_n(x,v)\,dx\,dv \leq \frac{c_3}{8}, \tag{1.10}$$

where $C_n = \{(x,v)\in B_n;\ |E_n(x,v)| > 8c_6/(c_3\ln(n))\}$. Hence by (1.4),

$$\int_{(x,v)\in B_n\setminus C_n} f_n(x,v)\,dx\,dv \geq \frac{c_3}{8}.$$

So there is (x_n,v_n) in B_n such that $|E_n(x_n,v_n)| \leq 8c_6/(c_3\ln(n))$, which implies

$$|E_n(x_n,v_n)| \leq \frac{c_7}{\ln(n)}.$$

Let F'_n and F'^{*}_n be defined by

$$F'_n = \{v';\ v' = v_n - (v_n - v_*\ ,\omega)\omega,\ v_* \in Z_n,\\ \vartheta \in \left[\frac{\pi}{8},\frac{3\pi}{8}\right],\ \varphi \in [0,2\pi],\ (v_*,\vartheta,\varphi) \notin E_n(x_n,v_n)\},$$

and

$$F'^{*}_n = \{v'_*;\ v'_* = v_* + (v_n - v_*\ ,\omega)\omega,\ v_* \in Z_n,\\ \vartheta \in \left[\frac{\pi}{8},\frac{3\pi}{8}\right],\ \varphi \in [0,2\pi],\ (v_*,\vartheta,\varphi) \in E_n(x_n,v_n)\}.$$

Then

$$f_n(x_n,v')f_n(x_n,v'_*) \geq \sqrt{f_n(x_n,v_n)f_n(x_n,v_*)} \geq \sqrt{nc_8},$$

and $v' \in F'_n$, $v'_* \in F'^{*}_n$ for $\vartheta \in [\pi/8\ ,\ 3\pi/8]$, $\varphi \in [0,2\pi]$, $(v_*,\vartheta,\varphi) \notin E_n(x_n,v_n)$. We recall that the diameter of the sphere in $\mathbb{R}^3$ with poles at v and v_* is bounded from below by $p_2 - p_1$. Also for each (ϑ,φ), v' and v'_* are antipodal on that sphere. Set

$$S_n = \{(v_*,\vartheta,\varphi);\ v_* \in Z_n,\ \vartheta \in \left[\frac{\pi}{8},\frac{3\pi}{8}\right],\ \varphi \in [0,2\pi]\}\backslash E_n.$$

$|S_n|$ is uniformly bounded away from zero. Hence for any n and some η^n_*, ζ^n_*, the set of $(\xi_*,\vartheta,\varphi)$ such that $((\xi_*,\eta^n_*,\xi^n_*),\vartheta,\varphi) \in S_n$ has positive Lebesgue measure uniformly in x bounded away from zero. Since $\lim_{n\to\infty}|E_n| = 0$, for n large, η^n_*, ζ^n_* can be chosen so that the corresponding set of ξ_* has Lebesgue measure uniformly bounded away from zero, and for each ξ_* the corresponding set of (ϑ,φ) maps into a set of v', v'_* covering at least, say, 90 percent of the area of a sphere with poles at v and at (ξ_*,η^n_*,ξ^n_*). Hence for n large and such v_*, ϑ, φ, due to geometry, either $|\{v' \in F'_n;\ f_n(x_n,v') \geq (nc_8)^{1/4}\}|$ or $|\{v'_* \in F'^{*}_n;\ f_n(x_n,v'_*) \geq (nc_8)^{1/4}\}|$ is bounded from below, say by $c_9 > 0$. Let us denote this set by F_n. The condition $p_1 - q_1 < p_3 - p_2 \ll p_2 - p_1$, together with $\vartheta \in (\pi/8\ ,\ 3\pi/8)$ imply that F'_n and F'^{*}_n are contained in $\{v;\ |\xi| \geq p_1\}$. Then, using assumption (H1) and the last remark,

$$(1.11)\qquad (nc_8)^{1/4}c_9 \leq \int_{F_n} f_n(x_n,v)\,dv \leq \frac{c_1}{p_1^2}.$$

The left hand side of (1.11) tends to ∞ as n tends to ∞, whereas the right hand side is uniformly bounded with respect to n. This leads to a contradiction and proves that μ belongs to $L^1_{\text{loc}}([0,a] \times \mathbb{R}^3)$.

STEP 2. Let us verify the assumptions of the Dunford-Pettis theorem. The uniform boundedness of (f_n) in $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$ is given by Lemma 1.1. Moreover, (f_n) is uniformly absolutely integrable in $L^1_{\text{loc}}(([0,a]\times\mathbb{R}^3)$. Else, restricting (x,v) to some $[b,d]\times V$, with $[b,d]\subset[0,a]$ and V a compact set of $\mathbb{R}^3$, there would exist a constant c, a sequence of sets $(A_n)\subset[b,d]\times V$ of measure respectively smaller than $1/(4n)$, and an increasing sequence of integers (j_n) such that

$$\int_{A_n} f_{jn}(x,v)\,dx\,dv > c,$$

which is the setting of (1.3), and was contradicted in Step 1. Hence the Dunford-Pettis theorem applies, which proves the weak relative compactness of (f_n) in $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$. □

Remark. It follows from Lemma 1.1 that, moreover, for $\beta=0$, (f_n) is uniformly bounded in $L^1([0,a]\times\mathbb{R}^3)$ and for $\beta>0$, (f_n) is weakly relatively compact in $L^1([0,a]\times\mathbb{R}^3)$.

For the rest of this section we only consider $(f_n)_{n\in\mathbb{N}}$ which are weak solutions (cf. (2.4) below) of (0.1) satisfying

$$f_0(v)\geq f_n(0,v)\geq\varepsilon f_0(v)\wedge n,\quad \xi>0,$$
$$f_a(v)\geq f_n(0,v)\geq\varepsilon f_a(v)\wedge n,\quad \xi<0,$$

where $\varepsilon\in\,]0,1]$. Throughout the paper we shall assume that f_0 and f_a are measurable, with

$$\int_{\xi>0}\xi(1+|v|^2)f_0(v)\,dv<\infty,\quad \int_{\xi<0}|\xi|(1+|v|^2)f_a(v)\,dv<\infty, \tag{1.12}$$

and

$$\int_{\xi>0}\xi f_0(v)\ln(f_0)(v)\,dv<\infty,\quad \int_{\xi<0}|\xi|f_a\ln(f_a)(v)\,dv<\infty. \tag{1.13}$$

A variant of the previous theorem will be neded in Section 2. Define χ_n by

$$\chi_n=1, \tag{1.14}$$

when

$$\min(|\xi|,|\xi_*|,|\xi'|,|\xi'_*|)>(\ln n)^{-1/8},\ |v-v_*|>(\ln n)^{-1/8},\ |v|^2+|v_*|^2\leq(\ln n)^{1/8},$$

and $\chi_n=0$ otherwise. Define χ_α by

$$\chi_\alpha(v,v_*,v',v'_*)=1\quad\text{if }\min(|\xi|,|\xi_*|,|\xi'|,|\xi'_*|)\geq\alpha, \tag{1.15}$$

and otherwise let χ_n equal one multiplied by ξ_j^2/α^2 when $|\xi_j|<\alpha$, and $\xi_j=\xi$, ξ_*, ξ', ξ'_*. (Several factors may be incorporated simultaneously).

Theorem 1.3. *Assume that $-1<\beta<0$ and $f_n(0,\cdot)\le n$, $f_n(a,\cdot)\le n$. For any $\alpha>0$, if (f_n) verify* (H1), (H3), *and if each f_n is a solution to* (2.5) *below, verifying* (H2) *for $\tilde{D}_n$ instead of D, where*

$$\tilde{D}_n(f,f)=\int \chi_n\chi_\alpha B(v-v_*\,,\omega)\left(\frac{ff^*}{1+\dfrac{1}{n}ff^*}-\frac{f'f'^*}{1+\dfrac{1}{n}f'f'^*}\right)\ln\frac{ff^*}{f'f'^*},$$

then $(\xi^2 f_n)$ is weakly relatively compact in $L^1_{\rm loc}([0,a]\times\mathbb{R}^3)$.

Proof. We shall concentrate on the deviations from the proof of the previous theorem. (H1) implies that a susequence of $(\xi^2 f_n)$ converges to a measure μ for the weak–* topology on $M([0,a]\times\mathbb{R}^3)$. Lets us prove that μ belongs to $L^1([0,a]\times\mathbb{R}^3)$. The weak relative compactness of $(\xi^2 f_n)$ in $L^1_{\rm loc}([0,a]\times\mathbb{R}^3)$ then follows as in Step 2 of the previous theorem. Suppose that μ does not belong to $L^1([0,a]\times\mathbb{R}^3)$. It means that μ has a singular part of strictly positive measure contained in an open set A of finite measure

$$A\subset[0,a]\times\{v;\ \xi\in[q_1,p_1],\ |\eta|<p_1,\ |\zeta|<p_1\}.$$

By elementary computations, in the present case

$$\sup_j\sup_{0\le x\le a}\int_{|\xi|\le\varepsilon}\xi^2 f_j(x,v)\,dv=O(\varepsilon^{1+\beta})$$

when ε tends to 0. Indeed, define φ as a regularisation of the indicatrix function of $[-1,1]$ and $\varphi_\varepsilon(x)=\varphi(x/\varepsilon)$, so that $|\nabla\varphi|\sim 1/\varepsilon$. Multiplying (0.1) by φ and integrating on $(0,x)\times\mathbb{R}^3$ leads to

$$\begin{aligned}
&\int_{0<\xi<\varepsilon}\varphi_\varepsilon(\xi)\xi^2 f(x,v)\,dv\\
&=\int_{0<\xi<\varepsilon}\varphi_\varepsilon(\xi)\xi^2 f_0(v)\,dv+\int_0^x\int_{0<\xi<\varepsilon}\int_{S^2\times\mathbb{R}^3}\xi(v-v_*)^\beta b(\vartheta)\\
&\qquad\times[f(y,v')f(y,v'_*)-f(y,v)f(y,v_*)]\varphi_\varepsilon(\xi)\,dy\,dv\,dv_*d\omega\\
&=\int_{0<\xi<\varepsilon}\varphi_\varepsilon(\xi)\xi^2 f_0(v)\,dv+\int_0^x\int_{0<\xi<\varepsilon}(v-v_*)^\beta b(\vartheta)\\
&\qquad\times[\varphi_\varepsilon(\xi')\xi'-\varphi_\varepsilon(\xi)\xi]f(y,v)f(y,v_*)\,dy\,dv\,dv_*d\omega,
\end{aligned}$$

and the last two terms are respectively of order ε and $\varepsilon^{1+\beta}$. This implies that for a suitable $\varepsilon>0$, we can either choose $A\subset\{v;\ \xi>\varepsilon\}$, or $A\subset\{v;\ \xi<-\varepsilon\}$. Let us discuss the first case, i.e. with $q_1>0$. Then

$$\exists c_3 \text{ such that } \forall n,\ \exists A_n \text{ open } \subset A \text{ such that } |A_n|<\frac{1}{4n} \text{ and } \int_{A_n}d\mu>c_3.$$

And so

$$\overline{\lim_{j\to\infty}} \int_{A_n} f_j(x,v)\,dx\,dv > c_4.$$

Let a subsequence (f_{j_n}) of the original sequence (f_j), constructed by a diagonalization process, be such that

$$\forall n, \int_{A_n} f_{j_n}(x,v)\,dx\,dv > \frac{c_4}{2}.$$

Write the subsequence as (f_n). Let B_n be defined by

$$B_n = \{(x,v) \in A_n;\ f_n(x,v) > c_4 n\}.$$

Then

$$\int_{B_n} f_n(x,v)\,dx\,dv > \frac{c_4}{4}.$$

Let p_2 and p_3 be real numbers such that

$$p_1 < p_3 - p_2 \ll p_2 - p_1.$$

Applying assumption (H3) with $Z = [p_2,p_3] \times [-p_1,p_1]^2$ leads, for every n, to the existence of a set $Z_n \subset Z$ such that $|Z_n| > |Z|/2$, and a constant c_5 such that

$$f_n(x,v) > c_5, \quad \text{a. e. } x \in [0,a],\ v \in Z_n.$$

Also, using (H1) and Tchebycheff's inequality, we can moreover require that $f_n(x,v) \leq \bar{c}_5$, $(x,v) \in Z_n$, for some (large) $\bar{c}_5$ independent of n. For every $(x,v) \in B_n$, let us define

$$E_n(x,v) = \Big\{(v,\vartheta,\varphi);\ v_* \in Z_n,\ \vartheta \in \Big[\frac{\pi}{8}, \frac{3\pi}{8}\Big],\ \varphi \in [0,2\pi] \text{ and } \sqrt{f_n(x,v)f_n(x,v_*)} \geq f_n(x,v')f_n(x,v'_*)\Big\}.$$

Assumption (H2) implies that for n large enough

$$\int_{b_n}\int_{E_{n(x,v)}} B\left(\frac{f_n f_n^*}{1+\frac{1}{j_n}f_n f_n^*} - \frac{f'_n f'^*_n}{1+\frac{1}{j_n}f'_n f'^*_n}\right) \ln\frac{f_n f_n^*}{f'_n f'^*_n}\,dv_*\,d\vartheta\,d\varphi\,dx\,dv < c_2.$$

But, for (x,v,v_*) such that (x,v) and v_* belong to B_n and Z_n respectively,

$$f_n(x,v)f_n(x,v_*) \geq n c_4 c_5.$$

Hence $f_n(x,v)f_n(x,v_*)$ is at least of the same order as n. Since

$f_n(0,\cdot) \le j_n,\ f_n(a,\cdot) \le j_n,\ |v_*| \le |\ln j_n|^{1/8},\ \chi_n\chi_\alpha B \le c(\ln j_n)^{|\beta|/8}$,

$$\frac{f_n f_n^*}{1+\dfrac{f_n f_n^*}{j_n}} \le j_n,$$

it follows from the solution written in mild form that

$$f_n \le c j_n (\ln j_n)^{5/16}$$

in the present case. So either $cn \le f_n \le j_n(\ln n)^{-1/32}$ or $j_n(\ln j_n)^{-1/3} < f_n < c j_n(\ln j_n)^{5/16}$. In the first case, for large n,

$$\frac{f_n f_n^*}{1+\dfrac{f_n f_n^*}{j_n}} \ge \frac{f_n f_n^*}{1+c(\ln j_n)^{-1/32}}.$$

And so

$$\left(\frac{f_n {f_n}^*}{1+\dfrac{f_n {f_n}^*}{j_n}} - \frac{f'_n {f'_n}^*}{1+\dfrac{f'_n {f'_n}^*}{j_n}}\right)\ln\frac{f_n {f_n}^*}{f'_n {f'_n}^*} \ge \left(0.9 f_n {f_n}^* - \sqrt{f_n {f_n}^*}\right)\ln\sqrt{f_n f_{n^*}}$$

$$\ge \frac{1}{4} f_n {f_n}^* \ln(f_n f_n^*).$$

In the second case, for n large,

$$\left(\frac{f_n f_n^*}{1+\dfrac{f_n f_n^*}{j_n}} - \frac{f'_n {f'_n}^*}{1+\dfrac{f'_n {f'_n}^*}{j_n}}\right)\ln\frac{f_n f_n^*}{f'_n {f'_n}^*}$$

$$\ge \frac{1}{2}\left(\frac{f_n f_n^*}{1+\dfrac{f_n f_n^*}{j_n}} - \frac{\sqrt{f_n f_n^*}}{1+\dfrac{\sqrt{f_n f_n^*}}{j_n}}\right)\ln(f_n f_n^*)$$

$$= \frac{3}{8}\frac{f_n f_n^*}{1+\dfrac{f_n f_n^*}{j_n}}\ln(f_n f_n^*)$$

$$+ \frac{\sqrt{f_n f_n^*}}{2}\left(\frac{1}{4}\frac{1}{\dfrac{1}{\sqrt{f_n f_n^*}}+\dfrac{\sqrt{f_n f_n^*}}{j_n}} - \frac{1}{1+\dfrac{\sqrt{f_n f_n^*}}{j_n}}\right)\ln(f_n f_n^*),$$

where the last term is positive for n large. Hence for n large enough, and (v', v'^*) constructed from (x,v) in B_n and $(v_*, \vartheta, \varphi)$ in $E_n(x,v)$,

$$\left(\frac{f_n f_n^*}{1 + \frac{1}{j_n} f_n f_n^*} - \frac{f_n' {f_n'}^*}{1 + \frac{1}{j_n} f_n' {f_n'}^*} \right) \ln \frac{f_n f_n^*}{f_n' {f_n'}^*} > \frac{1}{4} f_n f_n^* \ln(f_n f_n^*).$$

Arguing as in the proof of (1.10), we can from here conclude that for n large

$$\int_{(x,v) \in B_n \setminus C_n} f_n(x,v)\, dx\, dv \geq \frac{c_4}{8},$$

where $B_n \setminus C_n$ is a set $\{(x,v) \in B_n;\ |E_n(x,v)| \leq c/\ln(n)\}$. So there is (x_n, v_n) in B_n such that

$$|E_n(x_n, v_n)| \leq \frac{c_8}{\ln(n)}. \tag{1.16}$$

Let F_n' and ${F_n'}^*$ be defined by

$$F_n' = \Big\{ v' : v' = v_n - (v_n - v_*\ , \omega)\omega,\ v_* \in Z_n,\ \vartheta \in \left[\frac{\pi}{8}\ , \frac{3\pi}{8}\right],$$
$$\varphi \in [0, 2\pi],\ (v_*, \vartheta, \varphi) \notin E_n(x_n, v_n) \Big\},$$

and

$${F_n'}^* = \Big\{ v_*' :\ v' = v_n + (v_n - v_*\ , \omega)\omega,\ v_* \in Z_n,\ \vartheta \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right],$$
$$\varphi \in [0, 2\pi],\ (v_*, \vartheta, \varphi) \notin E_n(x_n, v_n) \Big\}.$$

Then

$$f_n(x_n, v') f_n(x_n, v_*') \geq \sqrt{f_n(x_n, v_n) f_n(x_n, v_{n*})} \geq \sqrt{n c_9},$$

and $v' \in F_n'$, $v_*' \in {F_n'}^*$ for $\vartheta \in [\pi/8\ , 3\pi/8]$, $\varphi \in [0, 2\pi]$, $(v_*, \vartheta, \varphi) \in E_n(x_n, v_n)$. So for n large and such v_*, ϑ, φ, due to the geometry, either $|\{v' \in {F_n'}^*;\ f_n(x_n, v') \geq (n c_9)^{1/4}\}|$ or $|\{v_*' \in {F_n'}^*;\ f_n(x_n, v_*') \geq (n c_9)^{1/4}\}|$ is bounded from below for n large enough, say by $c_{10} > 0$. Let us denote this set by F_n. Then using assumption (H1) and the last remark, we obtain

$$(n c_8)^{1/4} c_{10} \leq \int_{F_n} f_n(x_n, v) dv \leq \frac{c_1}{p_1^2}.$$

This leads to a contradiction as n tends to ∞ and proves that μ belongs to $L^1([0,a] \times \mathbb{R}^3)$. □

Define $\|v\|$ by,

$$\|v\|^2 = \xi^2 \vee (\eta^2 + \zeta^2),$$

and for any $\varepsilon > 0$, define ψ_ε and $\bar{\chi}_\varepsilon$ by

$$\psi_\varepsilon(\xi) = \begin{cases} \varepsilon^{-2}\xi^2 & \text{for } \|v\| \le \varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\bar{\chi}_\varepsilon = \begin{cases} 1 & \text{for } \min(\|v\|\,,\,\|v_*\|\,,\,\|v'\|\,,\,\|v'_*\|) > \varepsilon, \\ \dfrac{\xi_j^2}{\varepsilon^2} & \text{for } \|v_j\| < \varepsilon \text{ and } v_j = v,\ v_*,\ v',\ v'_*. \end{cases} \tag{1.17}$$

Starting from $(1 \wedge n^2\xi^2)f_n$ for $\|v\| \le \varepsilon$ and from $\psi_\varepsilon f_n$ for $\|v\| \le \varepsilon$, and arguing as in the proof of Theorem 1.3, we may also prove the following result.

Corollary 1.4. *Assume that* $-1 < \beta < 0$. *Let* (f_n) *verify* (H1), (H3), *and*

$$\sup_n \int_0^a \int_{\|v\| \ge \varepsilon} (1 \wedge n^2\xi^2)(1+|v|)^\beta f_n(x,v)\,dx\,dv < \infty.$$

Let each f_n *be a solution of* (0.1) *under* (K2) *with* $\chi_{\alpha\varepsilon}$ *which verifies* $(H2)$ *for* $D_n(f_n,f_n)$ *instead of* $D(f_n,f_n)$, *where* $\chi_{\alpha\varepsilon} = \chi_\alpha$ *when* $\bar{\chi}_\varepsilon = 1$, *and* $\chi_{\alpha\varepsilon} = \bar{\chi}_\varepsilon$ *otherwise, and*

$$D_n(f,f) = \int \chi_{\alpha\varepsilon} B(v-v_*,\omega)(ff^* - f'f'^*)\ln\frac{ff^*}{f'f'^*},$$

and $\alpha = 1/n$. *Then* $(\bar{\chi}_{\alpha\varepsilon} f_n)$ *is weakly relatively compact in* $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$. *Here* $\bar{\chi}_{\alpha\varepsilon} = 1 \wedge n^2\xi^2$ *when* $\bar{\chi}_\varepsilon = 1$ *and* $\bar{\chi}_{\alpha\varepsilon} = \bar{\chi}_\varepsilon$ *otherwise.*

Finally, for any sequence (f_n) of solutions (0.1), satisfying (1.12)–(1.13), the following partial converse of Lemma 1.1 holds.

Lemma 1.5. *Assume that*

$$\sup_n \int f_n(1+|v|)^\beta\,dx\,dv < \infty.$$

Then for any bounded measurable subset $Z \subset \mathbb{R}^3$, $|Z| > 0$, *where* $f_0 > 0$ *for* $\xi > 0$ *and* $f_a > 0$ *for* $\xi < 0$, *and any* $\varepsilon > 0$, *there is a constant* $c > 0$ *and for every* $n \in \mathbb{N}$, *there is* $Z_n \subset Z$ *with* $|Z_n| > |Z|(1-\varepsilon)$, *such that*

$$f_n(x,v) > c, \quad \text{a. e. } x \in [0,a],\ v \in Z_n.$$

Proof. Given Z, let $\tilde{c} \gg \sup_{v\in Z}|v|$. Then

$$\sup_{v\in Z}\ \sup_{n\in\mathbb{N}} \int_0^a \int_{|v_*|\geq \tilde{c}} |v-v_*|^\beta f_n(x,v_*)\,dx\,dv_* < \infty. \tag{1.18}$$

Further, for $\beta \geq 0$,

$$\sup_{v\in Z}\ \sup_{n\in\mathbb{N}} \int_0^a \int_{|v_*|\leq \tilde{c}} |v-v_*|^\beta f_n(x,v_*)\,dx\,dv_* < \infty. \tag{1.19}$$

Finally, for $-3<\beta<0$ since $\int_{v\in Z}|v-v_*|^\beta\,dv$ is uniformly bounded with respect to v_* such that $|v_*|\leq \tilde{c}$,

$$\sup_{n\in\mathbb{N}} \int_{v\in Z}\int_0^a \int_{|v_*|\leq \tilde{c}} |v-v_*|^\beta f_n(x,v_*)\,dv_*\,dx\,dv < \infty.$$

And so, given Z, there is a constant $\bar{c}>0$ and $Z_n \subset Z$ such that $|Z_n| > (1-\varepsilon)|Z|$ and

$$\sup_{n\in\mathbb{N}}\ \sup_{v\in Z_n} \int_0^a \int_{|v_*|\leq \tilde{c}} |v-v_*|^\beta f_n(x,v_*)\,dx\,dv_* < \bar{c}. \tag{1.20}$$

Since f_n is a solution of (0.1), it satisfies

$$\xi\frac{\partial f_n}{\partial x} + \nu(f_n)f_n = Q^+(f_n,f_n),$$

in weak form, where

$$\nu(f)(x,v) = \int_{(v_*,\omega)\in\mathbb{R}^3\times S^2} B(v-v_*\,,\,\omega)f(x,v_*)\,dv_*\,d\omega, \tag{1.21}$$

and

$$Q^+(f,f)(x,v) = \int_{(v_*,\omega)\in\mathbb{R}^3\times S^2} B(v-v_*\,,\,\omega)f(x,v')f(x,v'_*)\,dv_*\,d\omega.$$

Then the following exponential formulas also hold:

$$f_n(x,v) \geq f_0(v)e^{-\int_0^x (1/\xi)\nu(f_n)(y,v)\,dy} + \int_0^x \frac{1}{\xi}Q^+(f_n,f_n)(y,v)e^{\int_x^y (1/\xi)\nu(f_n)(z,v)\,dz}\,dy,$$
$$x\in[0,a],\ v\in Z,\ \xi>0,$$

and

$$f_n(x,v) \geq f_a(v) e^{-\int_a^x (1/\xi)\nu(f_n)(y,v)\,dy} + \int_a^x \frac{1}{\xi} Q^+(f_n,f_n)(y,v) e^{\int_x^y (1/\xi)\nu(f_n)(x,v)\,dz}\,dy,$$

$$x \in [0,a],\ v \in Z,\ \xi < 0.$$

Hence,

$$f_n(x,v) \geq f_0(v) e^{-\int_0^x (1/\xi)\nu(f_n)(y,v)\,dy},$$

$$x \in [0,a],\ v \in Z,\ \xi > 0,$$

and

$$f_n(x,v) \geq f_\alpha(v) e^{-\int_a^x (1/\xi)\nu(f_n)(y,v)\,dy},$$

$$x \in [0,a],\ v \in Z,\ \xi < 0,$$

which leads to the result, given (1.18), (1.19), (1.20), (1.21) and the assumption of positivity of f_0 (resp. f_a) on the component of Z where ξ is positive (resp. negative). □

Remark. The previous lemma also holds for diffuse reflection boundary conditions, where the total inflow

$$\int_{\xi>0} \xi f_n(0,v)\,dv + \int_{\xi>0} |\xi| f_n(a,v)\,dv$$

equals a given positive constant.

2. The stationary Boltzmann equation in the slab, with given boundary indata and reduced collision rate for small ξ. In this section we assume (K1) with $-1 < \beta < 0$, and (K2) with (1.15). Our aim is to prove the existence of a solution of the boundary value problem

$$\xi \frac{\partial f}{\partial x} = Q_\alpha(f,f), \quad x \in]0,a[, \quad v = (\xi,\eta,\zeta) \in \mathbb{R}^3, \tag{2.1}$$

$$f(0,v) = f_0(v), \quad \xi > 0, \tag{2.2}$$

$$f(a,v) = f_a(v), \quad \xi < 0. \tag{2.3}$$

To describe the weak form of this problem, we consider as admissible, test functions φ which are bounded and continuous, such that $(1/\xi)(\partial\varphi/\partial x)$ is continuous and such that φ is Lipschitz continuous with respect to v (with a Lipschitz constant not depending on x) and with compact support. In addition, we require that

$$\varphi(0,v) = 0 \quad \text{if } \xi < 0,$$

$$\varphi(a,v) = 0 \quad \text{if } \xi > 0.$$

We define f as a weak solution of (2.1)–(2.3) if

$$\int_0^a \int \xi f(x,v) \frac{\partial}{\partial x} \varphi(x,v)\, dv\, dx - \int_{\xi>0} \xi f_0(v) \varphi(0,v)\, dv + \int_{\xi<0} \xi f_a(v) \varphi(a,v)\, dv$$

$$= \int_0^a \int_v \int_{v*} \int_{S^2} [\varphi(x,v') - \varphi(x,v)] B_\alpha(v,v_*,\omega) f f^*\, d\omega\, dv_*\, dv\, dx,$$

for all admissible test functions. This concept of weak solution is equivalent to the usual mild and exponential ones (cf [5]).

Theorem 2.1. *Assume $f_0 > 0$ for $\xi > 0$ and $f_a > 0$ for $\xi < 0$. Then the stationary problem* (2.1)–(2.3) *has a weak solution which is weakly continuous in x, in the sense that $\int \varphi(x,v)\xi^2 f(x,v)\, dv$ is continuous in x for each test function φ.*

Proof. An approximate sequence (f_n) of solutions of

$$\xi \frac{\partial f_n}{\partial x} = Q_n(f_n, f_n),$$

will first be constructed in $C([0,a]; L^\infty_+(\mathbb{R}^3)$. The operator Q_n is obtained from Q_α by the following substitutions. B_α is replaced by

$$B_\alpha(v,v_*,\omega) = B_\alpha(v,v_*,\omega)\chi_n(v,v_*,v',v'_*),$$

where χ_n is defined by (1.14). The products ff^* and $f'f'^*$ in $Q_\alpha(f,f)$ are respectively replaced by

$$\frac{ff^*}{1+\frac{ff^*}{n}} \quad \text{and} \quad \frac{f'f'^*}{1+\frac{f'f'^*}{n}}.$$

The corresponding problem (2.1)–(2.3) with the boundary conditions $f_0 \wedge n$, $f_a \wedge n$ has an L^∞ solution. This can be proved by the type of fixed point argument used in [3], but here in an L^∞ context. Let X be $C([0,a];\ L^\infty_+(\mathbb{R}^3))$. For f in X we define

$$\rho_\varphi(f) = \sup_{x\in[0,a]} \left| \int_{\mathbb{R}^3} \varphi(v) f(x,v)\, dv \right|, \quad \varphi \in L^1(\mathbb{R}^3_v).$$

Define the mapping $T(\tau)$ on X by $g = T(\tau)(f)$ if

$$g(0,v) = f_0(v) \wedge n \quad \text{for } \xi > 0,$$

$$g(a,v) = f_a(v) \wedge n \quad \text{for } \xi < 0,$$

$$\xi \frac{\partial g}{\partial x} = 0, \quad |\xi| \le (\ln n)^{-1/8} \quad \text{or} \quad |\xi| \ge (\ln n)^{1/16},$$

$$\xi \frac{\partial g}{\partial x} + \tau g \int_{\mathbb{R}^3} \chi_n \frac{f^*}{1 + \frac{1}{n} f f^*} \, dv_* = Q_n(f,f) + \tau f \int_{\mathbb{R}^3} \chi_n \frac{f^*}{1 + \frac{1}{n} f f^*} \, dv_*,$$

$$(\ln n)^{-1/8} < |\xi| < (\ln n)^{1/16}.$$

The right hand side of the last equation is positive when f is positive, if τ is larger than $|S^2| c(n)$, where $c(n)$ is a bound of B_n from above. Choosing such a τ, $T(\tau)$ maps X into a set

$$B_R(0) = \{f \in X;\ \|f\|_\infty \le R\} \subset X.$$

Here R depends on n. In particular, $T(\tau)$ maps the convex and closed set $B_R(0)$ into itself. Evidently $T(\tau)$ is continuous and $T(\tau)$ is relatively compact in X with respect to the topology defined by the family of semi norms (ρ_φ). Hence, by Schaefer's fixed point theorem, $T(\tau)$ has a fixed point in $B_R(0)$. We denote the fixed point by f_n. Clearly f_n belongs to $C([0,a];\ L_+^\infty(\mathbb{R}^3))$ and satisfies

$$\xi \frac{\partial f_n}{\partial x} = Q_n(f_n, f_n) \tag{2.5}$$

in weak L^1-sense. Moreover,

$$\sup_n \sup_{x \in [0,a]} \int_{\mathbb{R}^3} \xi^2 f_n(x,v) \, dv < C(f_0, f_\alpha), \tag{2.6}$$

where

$$C(f_0, f_a) = \int_{\xi > 0} \xi (1 + \xi + |v|^2) f_0(v) \, dv + \int_{\xi < 0} |\xi| (1 + |v|)^2 f_a(v) \, dv.$$

Indeed,

$$\int \xi^2 f_n(x,v) \, dv = \int \xi^2 f_n(0,v) \, dv$$

$$= \int_{\xi > 0} \xi^2 f_0(v) \, dv + \int_{\xi < 0} \xi^2 f_n(0,v) \, dv,$$

and

$$\int_{\xi>0}\xi^2 f_0(v)\,dv+\int_{\xi<0}\xi^2 f_n(0,v)\,dv \le \int_{\xi>0}\xi^2 f_0(v)\,dv+\int_{\xi<0;|\xi|\le 1}|\xi| f_n(0,v)\,dv + \int_{\xi<0;|\xi|>1}|\xi|\;|v|^2 f_n(0,v)\,dv \le C(f_0,f_a)$$

since

$$\int_{\xi<0}|\xi| f_n(0,v)\,dv+\int_{\xi>0}\xi f_n(a,v)\,dv=\int_{\xi>0}\xi f_0(v)dv+\int_{\xi<0}|\xi| f_a(v)\,dv$$

and

$$\int_{\xi<0}|\xi|\;|v|^2 f_n(0,v)\,dv+\int_{\xi>0}\xi|v|^2 f_n(a,v)\,dv = \int_{\xi>0}\xi|v|^2 f_0(v)\,dv+\int_{\xi<0}|\xi|\;|v|^2 f_a(v)\,dv.$$

It follows essentially by approximation from (2.5) that

$$\int \xi f_n(a,v)\ln f_n(a,v)\,dv-\int \xi f_n(0,v)\ln f_n(0,v)\,dv = \int_0^a\int_{\mathbb{R}^3}(Q_n f_n)(x,v)\ln f_n(x,v)\,dv\,dx. \tag{2.7}$$

And so

$$\sup_n \tilde{D}_n(f_n,f_n)<\infty, \tag{2.8}$$

where $\tilde{D}_n(f,f)$ is defined by

$$\tilde{D}_n(f,f) = \int_{\mathbb{R}^3}\int_{S^2} B_n(v,v_*,\omega)\left[\frac{f'f'^*}{1+\frac{f'f'^*}{n}}-\frac{ff^*}{1+\frac{f'f'^*}{n}}\right]\ln\frac{f'f'^*}{ff^*}\,d\omega\,dv_*\,dv\,dx.$$

Because of the χ_α-factor, the estimates (1.18), (1.20) hold in the present case. Then the conclusion of Lemma 15, i.e. (H3), holds. Using this together with

(2.6)–(2.8), we conclude by Theorem 1.3 that a subsequence of $\xi^2 f_n$ weakly converges to some $\xi^2 f$ in $L^1_{\mathrm{loc}}([0,a]\times\mathbb{R}^3)$. For the collision term it is enough to discuss the convergence of the integral of

$$[\varphi(x,v')-\varphi(x,v)]\chi_n\chi_\alpha|v-v_*|^\beta f_n(x,v)\,f_n(x,v_*)$$

over $[0,a]\times\Omega_v\times\mathbb{R}^3$ for any C^1_0 test function φ and any bounded measurable set Ω_v in $\mathbb{R}^3$. Because of the $|v-v_*|^\beta$ factor with $\beta<0$, it is even enough to consider the weak compactness in $L^1([0,a]\times\Omega_v\times\{v_*;\ |v_*|\le\lambda\})$ for $\lambda>0$. But there $[\varphi(x,v')-\varphi(x,v)]|v-v_*|^\beta$ is bounded because $-1<\beta$. Also $\sup\chi_n\chi_\alpha/(\xi^2\xi_*^2)<\infty$, and so in $L^1([0,a]\times\Omega_v\times\{v_*;\ |v_*|\le\lambda\})$, the weak compactness of $\chi_n\chi_\alpha f_n(x,v)f_n(x,v_*)$ follows from the weak compactness of $\xi_*^2 f_n(x,v_*)$ and from the fact that

$$\sup_{n,x}\int\xi^2 f_n(x,v)\,dv<\infty.$$

Hence we can pick a L^1-weakly converging subsequence of $[\varphi(x,v')-\varphi(x,v)]\chi_n\chi_\alpha$ $|v-v_*|^\beta f_n(x,v)f_n(x,v_*)$. By choosing suitable factorized test functions, it is easy to see that the limit equals $[\varphi(x,v')-\varphi(x,v)]\chi_\alpha\ |v-v_*|^\beta f_n(x,v)f_n(x,v_*)$. Also,

$$[\varphi(x,v')-\varphi(x,v)]|v-v_*|^\beta\chi_n\chi_\alpha\frac{f_nf_n^*}{1+\dfrac{f_nf_n^*}{n}}$$

has the same limit. This is so because the sequence $\chi_n\chi_\alpha f_nf_n^*$ is uniformly bounded in L^1, and it is uniformly integrable. Hence, given ε, there is a constant c such that $\int_{\chi_n\chi_\alpha f_nf_n^*>c}\chi_n\chi_\alpha f_nf_n^*\,dv<\varepsilon$ for all n, and for $\delta>0$,

$$\chi_n\chi_\alpha\frac{f_nf_n^*}{1+\dfrac{c}{\delta n}}$$

weakly converges to $\chi_\alpha ff^*$. Evidently the other terms of f_n in (2.4) also converge to the corresponding terms of f when n tends to ∞. Moreover, the continuity of $\int\varphi(x,v)\xi^2 f(x,v)\,dv$ with respect to x can be deduced from (2.4).

Remark. Theorem 2.1 holds also when the condition of a given incoming flow at $x=a$ is replaced by reflexion at $x=a$, as well as for the case of a diffusive boundary condition at $x=0$ and reflexion at $x=a$. It also holds (with some technical complications in the proof) in the case when

$$f_0\ge0,\quad f_a\ge0,\quad \int_{\xi>0}f_0(v)\xi\,dv+\int_{\xi<0}f_a(v)|\xi|\,dv>0.$$

3. Diffuse reflection boundary conditions. In this section we shall keep the Boltzmann equation (2.1) with $-1 < \beta < 0$ in (K1) and (K2) with the reduced collision rate in the slab direction (1.15), but change the boundary conditions to diffuse reflection

$$f(0,v) = M_0(v) \int_{\xi'<0} |\xi'| f(0,v')\, dv', \quad \xi > 0, \tag{3.1}$$

$$f(a,v) = M_a(v) \int_{\xi'>0} \xi' f(a,v')\, dv', \quad \xi < 0. \tag{3.2}$$

Here, for $j \in \{0,a\}$, M_j is a (normalized) Maxwellian

$$M_j(v) = \frac{\vartheta_j^2}{2\pi} \exp\left(-\frac{\vartheta_j}{2}|v|^2\right), \tag{3.3}$$

$\vartheta_j > 0$ being the inverse boundary temperature. This implies that the inflow is equal to the outflow at each endpoint separately. We follow a suggestion by N. Maslova and consider the case of fixed total inflow (or outflow)

$$\int_{\xi>0} \xi f(0,v)\, dv + \int_{\xi<0} |\xi| f(a,v)\, dv = 1. \tag{3.4}$$

Theorem 3.1. *The stationary problem* (2.1), (3.1)–(3.4) *has a weak solution, which is weakly continuous in x in the sense that $\int \varphi(x,v)\xi^2 f(x,v)\, dv$ is continuous with respect to x for each test function φ.*

Proof. We start as in the proof of Theorem 2.1, by solving the truncated and mollified equation (2.5) with the boundary conditions (3.1)–(3.4). For $\tau \geq |S^2|c(n)$, where $c(n)$ is a bound of B_n from above, define

$$\tilde{T}(\tau) : B_R(0) \times [0,1] \to B_R(0) \times [0,1]$$

by $(g,\tilde{\vartheta}) = \tilde{T}(\tau)(f,\vartheta)$ if

$$g(0,v) = \vartheta M_0(v), \quad \text{for } \xi > 0,$$
$$g(a,v) = (1-\vartheta) M_a, \quad \text{for } \xi < 0,$$
$$\xi \frac{\partial g}{\partial x} = 0, \quad \text{for } |\xi| \leq (\ln n)^{-1/8} \text{ or } |\xi| \geq (\ln n)^{1/16},$$

$$\xi\frac{\partial g}{\partial x}+\tau g\int_{\mathbb{R}^3}\chi_n\frac{f^*}{1+\dfrac{1}{n}ff^*}\,dv_*=Q_n(f,f)+\tau f\int_{\mathbb{R}^3}\chi_n\frac{f^*}{1+\dfrac{1}{n}ff^*}\,dv_*,$$

$$\text{for } (\ln n)^{-1/8}<|\xi|<(\ln n)^{1/16},$$

and

$$\tilde{\vartheta}=\int_{\xi<0}|\xi|g(0,v)\,dv\left(\int_{\xi<0}|\xi|g(0,v)\,dv+\int_{\xi>0}\xi\vartheta g(a,v)\,dv\right)^{-1}.$$

Using Schaefer's fixed point theorem on the mapping $\tilde{T}$ and arguing as in Section 2, it follows that the problem (2.5), (3.1)–(3.4) has a solution. Letting n tend to ∞, a solution of (2.1), (3.1)–(3.4) with the desired continuity properties can be obtained as in the proof of Theorem 2.1.

Remark. Theorem 3.1 also holds if M_0 and M_a are replaced by nonnegative functions φ_0 and φ_α, satisfying (1.12)–(1.13) and

$$\int_{\xi>0}\xi\varphi_0(v)\,dv=1,\quad \int_{\xi<0}|\xi|\varphi_a(v)\,dv=1.$$

4 The slab problem with solution and data only depending on ξ and $\eta^2+\zeta^2$. In this section, we discuss the Boltzmann equation (0.1) with $-1<\beta<0$ in (K1), when the density f and the boundary data only depend on ξ and $\eta^2+\zeta^2$. We consider the case of given indata (2.2)–(2.3). In (K2), χ_α of (1.15) is replaced by $\bar{\chi}_\varepsilon$ of (1.17).

Theorem 4.1. *Assume $f_0>0$ for $\xi>0$ and $f_a>0$ for $\xi<0$, together with the cylindrical symmetry above. Then the stationary equation* (0.1) *with given inflow* (2.2)–(2.3) *has a weak solution, which satisfies*

$$\int_{\|v\|>\varepsilon}(1+|v|)^\beta f(x,v)\,dx\,dv<\infty,\qquad \sup_{0\le x\le\alpha}\int\xi^2 f(x,v)\,dx\,dv<\infty.$$

Moreover, it is weakly continuous in x in the sense that $\int\varphi(x,v)\psi_\varepsilon(v)\xi^2 f(x,v)\,dv$ is continuous in x for each test function φ.

Proof. We first construct approximate solutions. For this purpose, let χ_α be given by (1.15) and take $\alpha=1/n$. Let f_n be a solution of Theorem 2.1 with the kernel of Q equal to $\chi_{\alpha\varepsilon}B(v-v_*\,,\omega)$, where $\chi_{\alpha\varepsilon}$ is defined in Corollary 1.4. Let f_n^j be an approximation of f_n satisfying (2.6). The convex function

$$J(t,s)=\left(\frac{t}{1+\dfrac{t}{n}}-\frac{s}{1+\dfrac{s}{n}}\right)\ln\left(\frac{t}{s}\right)$$

is, for fixed (t,s), increasing as a function of n. And so, $\tilde{D}_m(f,f) \leq \tilde{D}_n(f,f)$ for $m \leq n$. Hence, using (2.6) and (2.7),

$$\sup_m \sup_{j\geq m} \tilde{D}_m(f_n^j, f_n^j) < \bar{C}(f_0, f_a), \tag{4.1}$$

where $\bar{C}(f_0,f_a)$ only depends on f_0 and f_a. But any convex function which is lower semi-continuous for the strong topology of L^1 is also lower semi-continuous for the weak topology of L^1, and $(f_n^j f_n^{j*})$ is weakly converging in L^1 to $f_n f_n^*$ when j tends to ∞. It follows that (H2) holds by (4.1), i.e.,

$$\sup_n D_n(f_n, f_n) < \bar{C}(f_0, f_a).$$

For proving (H3), it is enough, as in the proof of Lemma 1.5, to prove that for $\lambda > 0$,

$$\sup_n \int_{|v|\leq\lambda} \int |v - v_*|^\beta f_n(x, v_*) \chi_{\alpha\varepsilon} b(\vartheta)\, dx\, dv\, dv_* d\omega < \infty.$$

Given $0 < \varepsilon_1 < \varepsilon$, under the present hypotheses this evidently holds with respect to the domain of integration

$$\{v_*;\ |\xi_*| \geq \varepsilon_1\} \cup \{v_*;\ |\xi_*| < \varepsilon_1,\ \eta_*^2 + \zeta_*^2 \leq \varepsilon\}. \tag{4.2}$$

So it remains for some $\varepsilon_1 > 0$ to consider the complement Ω of (4.2) in $\mathbb{R}^3$. But

$$\int_{v_*\in\Omega} |v - v_*|^\beta f_n(x, v_*) \chi_{\alpha\varepsilon} b(\vartheta)\, dx\, dv_*\, d\omega$$

$$\leq \left(a \int_0^a dx \left(\int_\Omega |v - v_*|^\beta f_n(x, v_*) \chi_{\alpha\varepsilon} b(\vartheta)\, dv_*\, d\omega \right)^2 \right)^{1/2}.$$

It is enough to consider the case when $|\xi| > 2\varepsilon_1$. Then for $|\xi| \leq \varepsilon_1$ and $|v| \leq \lambda$, $|v - v_*|^\beta$ behaves like $(1 + |v_*|)^\beta$ for $\eta_*^2 + \zeta_*^2 \geq \varepsilon^2$. So it is enough to prove that

$$\sup_n \int_0^a dx \left(\int_\Omega \psi_n(\xi_*)(1 + |v_*|)^\beta f_n(x, v_*)\, dv_* \right)^2 < \infty,$$

where $\psi_n(\xi_*) = \min(1, n^2\xi_*^2)$. For convenience, let $\varepsilon_1 \ll \varepsilon$. Let the variables (η, ζ) and (η_*, ζ_*) be in two opposite quadrants of $\mathbb{R}^2$ with $\eta^2 + \zeta^2 > \varepsilon$, $\eta_*^2 + \zeta_*^2 > \varepsilon$. Take $|\xi| \leq \varepsilon_1$ and $|\xi_*| \leq \varepsilon_1$, $\vartheta \in [\pi/8\ ,\ 3\pi/8]$ and φ in a 'uniformly large' part of $[0,2\pi]$ so that $|\xi'| \geq \varepsilon/2$ and $|\xi'_*| \geq \varepsilon/2$. Multiplying (1.2) by $b(\vartheta)|v-v_*|^\beta \chi_{\alpha\varepsilon}$ leads to

$$b(\vartheta)|v-v_*|^\beta \chi_{\alpha\varepsilon} f_n f_n^* \leq b(\vartheta)|v-v_*|^\beta \chi_{\alpha\varepsilon}\left(2f'_n f'^*_n + (f_n f_n^* - f'^*_n)\ln\frac{f_n f_n^*}{f'_n f'^*_n}\right).$$

Integrate this last inequality over Ω_q, which is the set of the v, v_*, ϑ and φ variables defined just above, and $x \in [0,a]$. The right hand side is bounded by a finite constant independent of n, because of (H1) and (H2). Hence the left hand side is also bounded. But for the v, v_* above, when $|v-v_*| > \varepsilon$, then

$$|v-v_*|^\beta \geq c(1+\max(|v|,|v_*|))^\beta,$$

and so

$$\begin{aligned}
&\sup_n \int_0^a \left(\int_\Omega \psi_n(\xi) f_n (1+|v|)^\beta dv\right)^2 dx \\
&< c \sup_n \int_{\Omega_q} b(\vartheta)|v-v_*|^\beta \chi_{\alpha\varepsilon} f_n(x,v) f_n(x,v_*)\, d\vartheta\, dv\, dv_*\, dx \\
&\leq c \sup_n \int_{\Omega_q} b(\vartheta)|v-v_*|^\beta \chi_{\alpha\varepsilon}\left(2f'_n f'^*_n + (f_n f_n^* - f'_n f'^*_n)\ln\frac{f_n f_n^*}{f'_n f'^*_n}\right) \\
&< \infty.
\end{aligned}$$

This completes the proof of (H3), and also proves that

$$\sup_n \int_0^a \int_{\|v\|>\varepsilon} (1 \wedge n^2\xi^2)(1+|v|)^\beta f_n(x,v)\, dx\, dv < \infty. \tag{4.3}$$

By Corollary 1.4 there is a subsequence of $\bar{\chi}_{\alpha\varepsilon} f_n$, where $\bar{\chi}_{\alpha\varepsilon}$ is defined in Corollary 1.4, weakly converging in $L^1_{\text{loc}}([0,a]\times\mathbb{R}^3)$ to some $f\psi_\varepsilon$. From here, the convergence of all terms in the weak equation for f_n to the corresponding term for f is immediate except for the collision term. For the collision term the arguments of the proof of Theorem 2.1 prove for $\delta > 0$ the correct convergence with respect to the domain of integration where $|\xi| \geq \delta$ and $|\xi_*| \geq \delta$. It remains to prove that the collision term of (2.4), with kernel $\chi_{\alpha\varepsilon} B$, integrated over the complement, tends to zero uniformly in n when δ tends to zero.

Consider first the domain of integration with $|\xi| \leq \delta$ and $|\xi_*| \leq \delta$. In the set where $|v-v_*| < \delta_1$, it holds for some $\lambda > 0$ independent of $\delta_1 < \varepsilon$ but depending on $\text{supp}(\varphi)$, that $\|v\| \leq \lambda$, $\|v_*\| \leq \lambda$ when $\varphi(x,v') \neq 0$ or $\varphi(x,v) \neq 0$. Moreover, for $|v-v_*| < \delta_1$,

$$|\varphi(x,v') - \varphi(x,v)|\ |v-v_*|^\beta \leq c\delta_1^{1+\beta}.$$

By (4.3), for $n \in \mathbb{N}$,

$$\int_{\varepsilon<\|v\|\leq\lambda} dx \left(\int (1 \wedge n^2\xi^2) f_n(x,v)\, dv \right)^2 < c,$$

$$\sup_{0\leq x\leq a} \int_{\|v\|\leq\varepsilon} \chi_\varepsilon(v) f_n(x,v)\, dv < c.$$

Hence the collision term of (2.4) for f_n, integrated over the set where $|v - v_*| < \delta_1$ is $O(\delta_1^{1+\beta})$ uniformly in n. Next

$$|v - v_*| \gg \sup_{\tilde{v}\in \mathrm{supp}(\varphi)} |\tilde{v}|,$$

in the set where $|v| \gg \mathrm{supp}\{|\tilde{v}|;\ \tilde{v} \in \mathrm{supp}(\varphi)\}$, when $\varphi(x,v') \neq 0$. Uniformly over such v and v_*, for $\pi/8 < \vartheta < 3\pi/8$ and φ in a suitable subinterval of $[0,2\pi]$ of length, say $\pi/8$, the quantities $|\xi'|$ and $|\xi'_*|$ are of the same magnitude as $|v - v_*|$. Now (cf. [2])

$$(4.4) \quad b(\vartheta)\chi_{\alpha\varepsilon}|v - v_*|^\beta f_n f_n^*$$
$$\leq b(\vartheta)\chi_{\alpha\varepsilon}|v - v_*|^\beta \left(\frac{k}{\xi'^2 {\xi'_*}^2} \xi'^2 f'_n {\xi'_*}^2 {f'}^*_n + \frac{1}{\ln k}[f_n f_n^* - f'_n {f'}^*_n] \ln \frac{f_n f_n^*}{f'_n {f'}^*_n} \right).$$

So choosing k large and then λ large, the collision term integrated over the set where $|v| > \lambda$ can be made arbitrarily small uniformly in n and in $\delta < \varepsilon$. It remains for $|\xi| \leq \delta$ and $|\xi_*| \leq \delta$ to discuss the collision term integrated over the set where $|v| \leq \lambda$, $|v - v_*| > \lambda_1$. Using (4.4), uniformly in n and in $\delta < \varepsilon$, the integral over the set where $|v_*| > \lambda_*$ tends to zero whern $\lambda_* \to \infty$. It remains to consider for arbitrary λ and λ_*, the collision term integrated over the set where $|v - v_*| > \delta_1$, $|v| \leq \lambda$, $|v_*| \leq \lambda_*$, $|\xi| \leq \delta$, $|\xi_*| \leq \delta$, when $\delta \to 0$. Again using (4.4) together with (H1) and the compactness obtained in Corollary 1.4, uniformly in n, the collision term integrated over this set tends to zero when $\delta \to 0$.

Finally the domain of integration where $|\xi| \leq \delta$ and $|\xi_*| > \delta$ or conversely, can be treated as the previous one $|\xi| \vee |\xi_*| \leq \delta$ with some slight changes in the arguments due to the different geometry. The above analysis of the collision term also shows that the continuity of $\int \varphi(x,v)\psi_\varepsilon(v) f(x,v)dv$ with respect to x follows from an obvious estimate of the collision term with respect to the domain of integration with $|\xi| \geq \delta$, $|\xi_*| \geq \delta$, from x to $x+h$, and which tends to zero when $h \to 0$.

Remark. Theorem 4.1 also holds for diffuse reflection boundary conditions as in Section 3.

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