SIAM J. MATH. ANAL.
Vol. 29, No. 1, pp. 30–47, January 1998

003

A CONDENSATION–EVAPORATION PROBLEM IN KINETIC THEORY*

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Abstract. A linear Boltzmann model is used for studying a condensation–evaporation problem in a bounded domain. First the time asymptotic limit is derived, which solves the associated stationary problem. Then the Milne problem is discussed for the boundary layer. Finally a fluid approximation is obtained in the small mean free path limit with initial and boundary layers of zeroth order.

Key words. boundary layer, condensation, evaporation, hydrodynamic limit, initial layer, Milne problem, time asymptotics

AMS subject classification. 58G20

PII. S0036141096301737

Introduction. The kinetic description of a rarefied gas can be given through the Boltzmann equation for the density function $f(t, x, v)$ of particles with velocity v at position x and time t. A coarser theory consists of describing the gas as a continuous fluid with local density $\rho(t, x)$, velocity $u(t, x)$, and temperature $T(t, x)$ satisfying the Euler or Navier–Stokes equations. In the limit of small mean free path, the fluid dynamic equations may be derived from the Boltzmann equation through either a Hilbert or Chapman–Enskog expansion; see, e.g., [2, 8, 9, 12]. However, the fluid dynamic limits fail near shocks and for general indata near spatial or temporal boundaries.

Among the many studies of the boundary layer structure let us mention the following. In [3], the steady nonlinear Boltzmann equation for a gas with zero bulk velocity between two plates at two different temperatures is solved for a small mean free path, using a Chapman–Enskog expansion between the two plates. Here the fluid part of the solution contains Fourier's law for heat conduction which can be made to satisfy different temperature values at the two plates. This is why the boundary layer terms only need to be of first order with respect to the mean free path. An analogous study also including the initial layer is performed in [16] for the linear semiconductor case where further references in the field may also be found. For more results in the area see also [5, 10, 13, 19].

The present paper addresses the added presence of condensation–evaporation on the boundary. In this context a formal analysis and numerical computations are carried out in [17, 18] for a rarefied gas with varying temperatures and condensation–evaporation on the boundaries. On the basis of the linearized Boltzmann equation for hard sphere molecules, zeroth-order boundary layer terms are needed for solving the problem. Our paper considers the same problem for a rarefied solute in a solvent gas, and with varying temperatures on the boundary. The linear Boltzmann equation is used as a model for the solute. We prove that a fluid approximation in the interior together with initial and boundary layer structures are available to describe the solute

*Received by the editors April 10, 1996; accepted for publication (in revised form) November 12, 1996.
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gas. Here the fluid approximation is derived from the boundary layer analysis. Indeed, like [17, 18] this boundary layer structure requires zeroth-order terms with respect to the mean free path.

In the first section an existence and uniqueness result for the initial boundary value problem with given indata in a bounded region is recalled. We then determine the solution to the stationary boundary value problem from the time asymptotics of the initial boundary value solution. The approach is designed for prospective future use in the nonlinear case. For another approach to the nonlinear stationary problem see [1]. Section 2 is devoted to the solution of the Milne problem. For indepth discussions and bibliography see [4, 6]. Depending on the sign of the normal velocity of the solvent gas, two kinds of solutions are of interest for the following boundary layer analysis. In the last section we perform in the slab case a fluid approximation with respect to the mean free path by splitting the solution into a zeroth-order initial layer term together with a stationary boundary value contribution having a fluid part with zeroth-order boundary layer terms and a first order remainder term.

1. The initial boundary value problem and its time asymptotic behavior. The linear Boltzmann equation models the interaction between a solvent gas and a solute gas. The solute gas is rarefied enough so that collisions with itself are negligible in comparison with collisions with the solvent gas. Both gases are located in a bounded convex domain $\Omega \subset \mathbb{R}^3$. The distribution function $f(t,x,v)$ of the solute gas satisfies the linear Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f = Q(f), \tag{1.1}$$

where

$$Q(f)(t,x,v) = \int B(\theta,w)(f'F'_* - fF_*)dv_* d\theta d\epsilon = Q^+(f) - \nu f.$$

Here

$$\begin{gathered} f' = f(t,x,v'), \quad F'_* = F(t,x,v'_*), \\ f = f(t,x,v), \quad F_* = F(t,x,v_*), \\ w = |v-v_*|, \quad v' = v - \frac{2}{1+\kappa}((v-v_*)\cdot e)e, \\ v'_* = v_* + \frac{2\kappa}{1+\kappa}((v-v_*)\cdot e)e, \quad e\in S^2. \end{gathered}$$

F is the solvent distribution function, assumed to be known, and κ is the ratio between the solute molecular mass m and the solvent molecular mass m_*.

Assuming that the collisions between the two gases are governed by a cut-off inverse power law interaction potential $U(\rho) = c\rho^{-k+1}, k > 2$ depending on the distance ρ of two colliding particles, the weight function B is $B(\theta,w) = w^\gamma b(\theta)$, $0 \le \theta < \frac{\pi}{2}$, $w > 0$ (cf. [7]), where $\gamma = \frac{k-5}{k-1}$ and b is a nonnegative L^1-function defined on $[0, \frac{\pi}{2}]$, with $\int_0^{\frac{\pi}{2}} b(\theta)d\theta > 0$. We assume hard interactions, i.e., $k > 5$ or $0 < \gamma < 1$. A principle of detailed balance only holds [14], when F is a Maxwellian,

$$F(v) = \left(\frac{2\pi T}{m_*}\right)^{-\frac{3}{2}} \exp\left(-m_*\frac{(v-U)^2}{2T}\right).$$

This is also assumed throughout the paper.

The collision frequency $\nu(v)$ is bounded from above and below by a positive multiple of $(1+|v|)^\gamma$. The choice of the bulk velocity $U=(u,0,0)\in\mathbb{R}^3$ in connection with the given boundary temperature follows from the boundary value problem for the solvent gas. The present study of the solute holds for any U and boundary temperature. The solute Maxwellian with the same bulk velocity U and temperature T is $M(v)=(\frac{2\pi T}{m})^{-\frac{3}{2}}\exp(-m\frac{(v-U)^2}{2T})$. It satisfies

$$F_*M=F'_*M'. \tag{1.2}$$

(1.1) is complemented with an initial condition

$$f(0,x,v)=f_i(x,v) \tag{1.3}$$

and given indata on the boundary

$$f(t,x,v)=f_b(x,v),\quad x\in\partial\Omega,\quad v\cdot n(x)>0. \tag{1.4}$$

Here $n(x)$ denotes the inward normal at x. Let $(\partial\Omega\times\mathbb{R}^3)^+$ and $(\partial\Omega\times\mathbb{R}^3)^-$ denote the sets of $(x,v)\in\partial\Omega\times\mathbb{R}^3$ such that $v\cdot n(x)>0$ and $v\cdot n(x)<0$, respectively.

For $\partial\Omega$ sufficiently smooth, say C^1, the existence and uniqueness approach of [15] can be used to prove the following theorem.

THEOREM 1.1. *If* $(1+|v|)^\gamma f_i$ *and* $(1+|v|)^\gamma f_b$ *belong to* $L^1(\Omega\times\mathbb{R}^3)$ *and* $L^1_{v\cdot n(x)}((\partial\Omega\times\mathbb{R}^3)^+)$, *respectively, then there exists a unique solution* f *of* (1.1)–(1.3–1.4) *with* $f(t)(1+|v|)^\gamma\in L^1(\Omega\times\mathbb{R}^3)$ *for* $t>0$. *Moreover,* f *is nonnegative whenever* f_i *and* f_b *are nonnegative.*

Let us next discuss the collisions and the collision operator in velocity space. The momentum and energy conservations imply

$$mv+m_*v_*=mv'+m_*v'_*,$$
$$m|v|^2+m_*|v_*|^2=m|v'|^2+m_*|v'_*|^2.$$

A transformation to the equal mass situation $m=m_*$ is given by

$$\tilde v=v-\frac{\alpha}{2}(v-v_*),\quad \tilde v_*=v_*-\frac{\alpha}{2}(v-v_*),$$

where $\alpha=\frac{m_*-m}{m_*+m}$. Hence

$$\tilde v+\tilde v_*=\tilde v'+\tilde v'_*,$$
$$|\tilde v|^2+|\tilde v_*|^2=|\tilde v'|^2+|\tilde v'_*|^2.$$

Denote by $\tilde f$, $\tilde Q^+(\tilde f)$, and $\tilde Q(\tilde f)$

$$\tilde f=f\sqrt{\frac{\nu}{M}},\quad \tilde Q^+(\tilde f)=\frac{1}{\sqrt{\nu M}}\,Q^+(f),\quad \tilde Q(\tilde f)=\tilde Q^+(\tilde f)-\tilde f.$$

By (1.2)

$$\tilde Q^+(\tilde f)=\int B\sqrt{\frac{F'_*F_*}{\nu'\nu}}\;\tilde f'dv*d\theta d\epsilon.$$

Let $(\,,)$ denote the scalar product in $L^2(\mathbb{R}^3)$.

LEMMA 1.2. *Every $\tilde{f} \in L^2(\mathbb{R}^3)$ can uniquely be written*

$$\tilde{f} = c_f\sqrt{\nu M} + \tilde{w}_f, \tag{1.5}$$

with $(\sqrt{\nu M},\, \tilde{w}_f) = 0$. Moreover

$$|(\tilde{Q}^+\tilde{w}_f, \tilde{w}_f)| \leq (1-\sigma)\|\tilde{w}_f\|^2_{L^2} \tag{1.6}$$

for some constant σ such that $0 < \sigma < 1$.

Proof. $\tilde{Q}^+$ satisfies Grad's conditions [12], so $\tilde{Q}^+$ is a compact operator in $L^q_s := L^q(\mathbb{R}^3, 1+|v|^s)$, $1 \leq q < \infty$, $s \in \mathbb{R}$. Moreover, $\tilde{Q}^+$ is symmetric in L^2. Hence its eigenvector spaces span L^2 and are finite dimensional for nonzero eigenvalues. Then

$$|(\tilde{Q}^+\tilde{f}, \tilde{f})| = \int B\sqrt{\frac{F'_*}{\nu'}}\ \tilde{f}'\ \sqrt{\frac{F_*}{\nu}}\ \tilde{f} \leq \int B\frac{F_*}{\nu}|\tilde{f}|^2 = \int |\tilde{f}|^2,$$

so $-\tilde{Q}$ is positive in L^2 and $\|\tilde{Q}^+\| \leq 1$. The $\tilde{Q}^+$-eigenvalue 1 is simple. Indeed, $\tilde{Q}^+\tilde{f} = \tilde{f}$ implies $(\tilde{Q}\tilde{f}, \tilde{f}) = 0$, which can be written

$$\int B\left(\sqrt{\frac{F'_*}{\nu'}}\ \tilde{f}' - \sqrt{\frac{F_*}{\nu}}\tilde{f}\right)^2 dv dv_*\ d\theta d\epsilon = 0.$$

Hence

$$\sqrt{\frac{F'_*}{\nu'}}\ \tilde{f}' = \sqrt{\frac{F_*}{\nu}}\ \tilde{f},$$

or $\frac{f'}{M'} = \frac{f}{M}$ by (1.2). It follows (see [14]) that $\tilde{f} = c\sqrt{\nu M}$, where c is a constant. Now -1 is not an eigenvalue of $\tilde{Q}^+$. Otherwise, $\tilde{Q}^+\tilde{f} = -\tilde{f}$ for some $\tilde{f}$ implies

$$\int B\left(\sqrt{\frac{F'}{\nu'}}\ \tilde{f}' + \sqrt{\frac{F_*}{\nu}}\ \tilde{f}\right)^2 dv dv_*\ d\theta d\epsilon = 0,$$

so $\frac{f'}{M'} = -\frac{f}{M}$. Varying v_* and the angular coordinate for v fixed gives that f has a constant sign. Hence $\tilde{f} = 0$. Since $\tilde{Q}^+$ is compact and symmetric, $\|\tilde{Q}^+\| \leq 1$, -1 is not an eigenvalue, and the eigenspace of 1 is $c\sqrt{\nu M}$, it follows that every $\tilde{f} \in L^2$ can be uniquely written as

$$\tilde{f} = c_f\sqrt{\nu M} + \tilde{w}_f, \quad \text{with } (\sqrt{\nu M}, \tilde{w}_f) = 0$$

and

$$|(\tilde{Q}^+\tilde{w}_f, \tilde{w}_f)| \leq (1-\sigma)\|\tilde{w}_f\|^2_{L^2}, \quad 0 < \sigma < 1. \qquad \square$$

Let us next describe the time asymptotics for the solution of the initial boundary value problem (1.1)–(1.3–1.4).

THEOREM 1.3. *Let f_i and f_b be functions belonging to $L^2_{\frac{1}{M}}(\Omega \times \mathbb{R}^3)$ and $L^2_{\frac{v\cdot n(x)}{M}}((\partial\Omega \times \mathbb{R}^3)^+)$. When t tends to infinity, the solution to the initial boundary value problem (1.1)–(1.3–1.4) converges in $L^1(\Omega \times \mathbb{R}^3)$ to the unique stationary solution g of the linear stationary Boltzmann equation*

$$v \cdot \nabla_x g = Q(g), \tag{1.7}$$

with $\tilde{g} \in L^2$, complemented with the boundary condition

$$g(x,v) = f_b(x,v), \quad (x,v) \in (\partial\Omega \times \mathbb{R}^3)^+. \tag{1.8}$$

Proof. Due to the linearity of (1.1), f can be split into the sum of the solution to (1.1) with initial condition f_i and zero boundary condition, and the solution to (1.1) with a zero initial condition and f_b boundary condition. Again by linearity it is enough to consider nonnegative initial and boundary values. Let us first prove that the first part tends to zero in $L^1_{x,v}$ when t tends to infinity. Let $d\alpha(x)$ denote the measure on the boundary $\partial\Omega$. The Green formula applied to (1.1), together with (1.6), implies

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} \frac{|\tilde{f}(t,x,v)|^2}{\nu(v)} dxdv \\
&+ \int_0^t \int_{(\partial\Omega\times\mathbb{R}^3)^-} \frac{|v\cdot n(x)|}{\nu(v)} |\tilde{f}(s,x,v)|^2 ds d\alpha(x) dv \\
+ \sigma \int_0^t \int_{\Omega\times\mathbb{R}^3} |\tilde{w}(s,x,v)|^2 dsdxdv \leq &\int_{\Omega\times\mathbb{R}^3} \frac{|\tilde{f}(t,x,v)|^2}{\nu(v)} dxdv \\
&+ \int_0^t \int_{(\partial\Omega\times\mathbb{R}^3)^-} \frac{|v\cdot n(x)|}{\nu(v)} |\tilde{f}(s,x,v)|^2 ds d\alpha(x) dv \\
- \int_0^t \int_\Omega (\tilde{Q}\tilde{w},\tilde{w})(s,x) dsdx = &\int_{\Omega\times\mathbb{R}^3} \frac{|\tilde{f}_i(x,v)|^2}{\nu(v)} dxdv.
\end{aligned} \tag{1.9}$$

It follows that $\int_{\Omega\times\mathbb{R}^3} \frac{1}{\nu(v)} |\tilde{f}(t,x,v)|^2 dxdv$ decreases with time. Moreover, there is a sequence t_j tending to infinity and a function $\tilde{f}^\infty$ such that $\tilde{f}(t_j+t)$ tends weakly to $\tilde{f}^\infty$ in $L^2_{\frac{1}{\nu}}$, and $\int_0^t \int_{\Omega\times\mathbb{R}^3} |\tilde{w}(t_j+t,x,v)|^2 \, dtdxdv$ and $\int_0^1 \int_{(\partial\Omega\times\mathbb{R}^3)^-} \frac{|v\cdot n(x)|}{\nu(v)} |\tilde{f}(t_j + t,x,v)|^2 \, dt d\alpha(x) dv$ tend to zero when j tends to infinity. The function f^∞ is a weak solution to the equation (1.1) with zero boundary condition and $\tilde{w}_{f^\infty} = 0$. It follows that $f^\infty = c^\infty M$ for some constant c^∞. The null boundary conditions imply that $c^\infty = 0$. Hence $\tilde{f}(t)$ weakly converges to zero in $L^2_{\frac{1}{v}}$. Since

$$\|f(t)\|_{L^1_{x,v}} = \int_\Omega \int_{\mathbb{R}^3} \sqrt{\frac{M(v)}{\nu(v)}} \, \tilde{f}(t,x,v) dxdv,$$

$f(t)$ tends to zero strongly in $L^1_{x,v}$, when t tends to infinity.

Let us prove that the solution to the initial boundary value problem (1.1)–(1.3–1.4) with null initial condition and boundary condition f_b tends to a stationary solution g to (1.7–1.8). In view of possible future applications we prefer not to give a proof based on the existence of stationary solutions being known but instead to deduce their existence from the long time behavior. By translation invariance in time, the solution at time $t+s$ is the sum of the solution at time t and the contribution at time s carried forward with zero boundary values to $t+s$. So $f(t,x,v)$ is increasing with time and converges pointwise in x, v to a measurable function f^∞, when t tends to infinity. Let us prove that $\tilde{f}^\infty$ belongs to L^2. For any set $\Gamma \subset \Omega\times\mathbb{R}^3$, multiplying (1.1) by $\tilde{f}$ and using Green's formula leads to

$$\begin{aligned}
\int_{\Gamma^c} \frac{1}{\nu(v)} (|\tilde{f}(t+s)|^2 - |\tilde{f}(t)|^2) dxdv + \int_\Gamma \frac{1}{\nu(v)} |\tilde{f}(t+s)|^2 dxdv \\
+\sigma \int_t^{t+s} \int_{\Omega\times\mathbb{R}^3} |\tilde{w}(\tau,x,v)|^2 d\tau dxdv
\end{aligned}$$

$$+\int_t^{t+s}\int_{(\partial\Omega\times\mathbb{R}^3)^-}\frac{|v\cdot n(x)|}{\nu(v)}|\tilde{f}(\tau,x,v)|^2 d\tau d\alpha(x)dv$$

$$\leq \int_\Gamma \frac{1}{\nu(v)}|\tilde{f}(t)|^2 dxdv + sc, \tag{1.10}$$

where

$$c := \int_{(\partial\Omega\times\mathbb{R}^3)^+}\frac{v\cdot n(x)}{\nu(v)}\ |\tilde{f}_b(x,v)|^2 d\alpha(x)dv.$$

Let $\Gamma_{s\epsilon}\subset\Omega\times\mathbb{R}^3$ be the set of (y,v) such that $|v|\leq\frac{1}{\epsilon}$ and the characteristic starting at (t,y,v), namely, $\{(t+\tau,y+\tau v,v);\ \tau\geq 0\}$, reaches $(\partial\Omega\times\mathbb{R}^3)^-$ at a time smaller than $t+s$. Then from the exponential form of the equation

$$\int_t^{t+s}\int_{(\partial\Omega\times\mathbb{R}^3)^-}\frac{|v\cdot n(x)|}{\nu(v)}\ |\tilde{f}(\tau,x,v)|^2 d\tau d\alpha(x)dv$$

$$\geq\ c(s,\epsilon)\int_{\Gamma_{s\epsilon}}\ \frac{1}{\nu(v)}\ |\tilde{f}(t,x,v)|^2 dxdv$$

for some $c(s,\epsilon)\in(0,1)$. Hence by (1.10)

$$\int_{\Gamma_{s\epsilon}}\frac{1}{\nu}|\tilde{f}(t+s)|^2 dxdv\leq(1-c(s,\epsilon))\int_{\Gamma_{s\epsilon}}\frac{1}{\nu}|\tilde{f}(t)|^2 dxdv+sc.$$

It follows that

$$\sup_{t>0}\int_{\Gamma_{s\epsilon}}\frac{1}{\nu(v)}\ |\tilde{f}(t,x,v)|^2 dxdv$$

is finite. Then by (1.10)

$$\sup_{t>0}\int_0^{\frac{3}{2}}\int_{\Omega\times\mathbb{R}^3}|\tilde{w}(t+s,x,v)|^2 dsdxdv$$

is finite. Hence, by the previous two lines,

$$\sup_{t>0}\int_0^{\frac{3}{2}}\int_{\Gamma_{s\epsilon}}|c_f(t+s,x)|^2 M(v)dsdxdv$$

is bounded. Since Ω is bounded and convex, it follows that (for ϵ small)

$$\inf_{x\in\Omega}\int_{(x,v)\in\Gamma_{\frac{1}{2}\epsilon}}\nu(v)M(v)dv>c\int_{\mathbb{R}^3}\nu(v)M(v)dv.$$

This implies that

$$\sup_{t>0}\int_{\frac{1}{2}}^{\frac{3}{2}}\int_{\Omega\times\mathbb{R}^3}|c_f(t+s,x)|^2\nu(v)M(v)dsdxdv<\infty.$$

And so

$$\sup_{t>0}\int_0^1\int_{\Omega\times\mathbb{R}^3}|c_f(t+s,x)|^2\nu(v)M(v)dsdxdv$$

is bounded. Finally

$$\sup_{t>0} \int_0^1 \int_{\Omega\times\mathbb{R}^3} |\tilde{f}(t+s,x,v)|^2 ds dx dv$$

and (since $\tilde{f}$ is an increasing function of time)

$$\sup_{t>0} \int_{\Omega\times\mathbb{R}^3} |\tilde{f}(t,x,v)|^2 dx dv$$

are bounded. Hence $\tilde{f}^\infty$ belongs to $L^2(\Omega\times\mathbb{R}^3)$. Moreover, f^∞ solves the stationary problem (1.7–1.8), and $\tilde{f}(t)$ tends to $\tilde{f}^\infty$ in $L^2(\Omega\times\mathbb{R}^3)$, when t tends to infinity. Finally the solution of the stationary problem is unique in the class of functions g such that $\tilde{g}\in L^2$. Indeed, let us prove that if a function g such that $\tilde{g}\in L^2$ satisfies

$$v\cdot\nabla_x g = Q(g), \tag{1.11}$$

$$g(x,v)=0, \quad (x,v)\in(\partial\Omega\times\mathbb{R}^3)^+, \tag{1.12}$$

then $g=0$. We notice that

$$\int Q^+(g)\mathrm{sign}(g)dv - \int \nu|g|dv \le \int Q^+(|g|)dv - \int \nu|g|dv = 0.$$

So, multiplying (1.11) by sign(g) and integrating implies that

$$\int_{(\partial\Omega\times\mathbb{R}^3)^-} v\cdot n|g| = 0.$$

Hence

$$g(x,v)=0, \quad (x,v)\in\partial\Omega\times\mathbb{R}^3. \tag{1.13}$$

$\tilde{g}$ belongs to L^2 and can be expressed by (1.5) as

$$\tilde{g} = c\sqrt{M\nu}+\tilde{w}. \tag{1.14}$$

It satisfies

$$\frac{1}{\nu(v)}\, v\cdot\nabla_x\tilde{g} = \tilde{Q}\tilde{w}. \tag{1.15}$$

Integrating (1.15) with respect to x and v using (1.6) implies by (1.13) that $\tilde{w}$ is equal to zero. Then $\tilde{g}=0$ follows from (1.11–1.12). □

2. The Milne problem. Write the velocity as $v=(\xi,v')$ with ξ the velocity component in the x-direction and v' the orthogonal velocity component. We consider the Milne problem

$$\frac{\xi}{\nu}\,\partial_x\tilde{f} = \tilde{Q}\tilde{f}, \quad x>0, \quad v\in\mathbb{R}^3, \tag{2.1}$$

$$\tilde{f}(0,v) = \tilde{\varphi}(v), \quad \xi>0. \tag{2.2}$$

THEOREM 2.1. *Let $\tilde{\varphi}\in L^2_{\frac{\xi}{\nu}}(\mathbb{R}_+\times\mathbb{R}^2)$. There is a solution to* (2.1–2.2) *in the set $\{\tilde{f};\exists c_\infty\in\mathbb{R},\ \tilde{f}-c_\infty\sqrt{\nu M}\in L^2(\mathbb{R}_+\times\mathbb{R}^3)\}$, which satisfies $\int\xi f(x,v)dv = c_\infty u$ for all $x\ge 0$. For $u<0$, this holds with $c_\infty=0$, i.e., $\int\xi f(x,v)dv=0$ for all $x\ge0$.*

Proof. There is—by the approach of Theorem 1.3—a unique solution $\tilde{f}^a \in L^2([0,a]\times\mathbb{R}^3)$ of

$$\frac{\xi}{\nu}\,\partial_x \tilde{f}^a = \tilde{Q}\tilde{f}, \quad x\in[0,a], \quad v\in\mathbb{R}^3,$$

together with (2.2) and boundary conditions at $x=a$ suitable for our purpose. For $u\geq 0$, we take

$$\tilde{f}^a(a,\xi,v') = \tilde{f}^a(a,-\xi+2u,v'), \quad \xi<0, \tag{2.3}$$

whereas for $u<0$,

$$\tilde{f}^a(a,v) = \frac{\sqrt{M(v)\nu(v)}}{\int_{\xi<0}|\xi|M(v)dv}\int_{\xi>0}\xi f^a(a,v)dv, \quad \xi<0. \tag{2.4}$$

Remark. The boundary condition (2.3) can only be used for $u>0$. A desired nonnegativity (2.9) would not be obtained from the boundary condition (2.4) for $u>0$.

Clearly $\int \xi f^a(x,v)dv$ is constant in both cases, moreover equal to zero for $u\leq 0$. Denote by uc_a this constant and bound it for $u>0$ from above and below. First

$$uc_a = \int \xi f^a(0,v)dv \leq \int_{\xi>0}\xi\varphi(v)dv. \tag{2.5}$$

Let $\tilde{f}^a(x,v) = c^a(x)\sqrt{\nu(v)M(v)} + \tilde{w}^a(x,v)$ be the decomposition of $\tilde{f}^a$ from section 1. By orthogonality

$$-(\tilde{Q}\tilde{f}^a,\tilde{f}^a) = -(\tilde{Q}\tilde{w}^a,\tilde{w}^a) \geq \sigma(\tilde{w}^a,\tilde{w}^a). \tag{2.6}$$

Multiplying (2.1) by $\tilde{f}^a$ and integrating over $\mathbb{R}^3_v$ leads to

$$\partial_x \int \frac{\xi}{\nu}\,|\tilde{f}^a|^2(x,v)dv = 2(\tilde{Q}\tilde{f}^a,\tilde{f}^a) \leq -2\sigma\|\tilde{w}^a\|^2 \leq 0. \tag{2.7}$$

Hence for $u\geq 0$,

$$\begin{aligned}\int \frac{\xi}{\nu}\,|\tilde{f}^a|^2(x,v)dv &\geq \int \frac{\xi}{\nu}\,|\tilde{f}^a|^2(a,v)dv \\ &\geq \int_{\xi<0}\frac{\xi}{\nu}\,|\tilde{f}^a|^2(a,v)dv + \int_{\xi>2u}\frac{\xi}{\nu}\,|\tilde{f}^a|^2(a,v)dv \\ &= 2u\int_{\xi>2u}\frac{1}{\nu}\,|\tilde{f}^a|^2(a,v)dv \geq 0,\end{aligned} \tag{2.8}$$

whereas for $u<0$,

$$\begin{aligned}\int \frac{\xi}{\nu}\,|\tilde{f}^a|^2(x,v)dv &\geq \int \frac{\xi}{\nu}\,|\tilde{f}^a|^2(a,v)dv \\ &\geq \left(1-\frac{\int_{\xi>0}\xi M}{\int_{\xi<0}|\xi|M}\right)\int_{\xi>0}\frac{\xi}{\nu}\,|\tilde{f}^a|^2(a,v)dv \geq 0.\end{aligned} \tag{2.9}$$

Indeed, using (2.4)

$$\int_{\xi<0} \frac{|\xi|}{\nu(v)} |\tilde{f}^a|^2(a,v)dv = \frac{1}{\int_{\xi<0}|\xi|M} \left(\int_{\xi>0} \xi f^a(a,v)dv\right)^2$$

$$\leq \frac{\int_{\xi>0}\xi M}{\int_{\xi<0}|\xi|M} \int_{\xi>0} \frac{\xi}{\nu(v)} |\tilde{f}^a|^2(a,v)dv.$$

But $\frac{\int_{\xi>0}\xi M(v)dv}{\int_{\xi<0}|\xi|M(v)dv} < 1$ for $u<0$ and so (2.9) follows. Finally by (2.8)

$$uc_a \geq \int_{\xi<0} \xi f^a(0,v)dv$$

$$= -\int_{\xi<0} |\xi| \sqrt{\frac{M(v)}{\nu(v)}} \tilde{f}^a(0,v)dv$$

$$\geq -\left(\int_{\xi<0}|\xi|M(v)dv \int_{\xi<0} \frac{|\xi|}{\nu} |\tilde{f}^a|^2(0,v)dv\right)^{\frac{1}{2}}$$

$$\geq -\left(\int_{\xi<0}|\xi|M(v)dv \int_{\xi>0} \frac{\xi}{\nu(v)} \tilde{\varphi}^2(v)dv\right)^{\frac{1}{2}}. \tag{2.10}$$

In the case $u=0$ the theorem can from here be derived using, e.g., [2] or [16]. So let us only detail the case when $u \neq 0$. First $\tilde{w}^a$ is bounded in $L^2([0,a]\times\mathbb{R}^3)$ uniformly with respect to a. Indeed by (2.8), (2.9)

$$\sigma \int_0^a \int |\tilde{w}^a|^2(x,v)dxdv$$

$$\leq -\int_0^a (\tilde{Q}\tilde{f}^a,\tilde{f}^a) = -\int_0^a\int \frac{\xi}{\nu(v)} \tilde{f}^a\partial_x\tilde{f}^a dxdv$$

$$= \frac{1}{2}\left(\int \frac{\xi}{\nu(v)} |\tilde{f}^a|^2(0,v)dv - \int \frac{\xi}{\nu} |\tilde{f}^a|^2(a,v)dv\right) \tag{2.11}$$

$$\leq \frac{1}{2}\int_{\xi>0} \frac{\xi}{\nu(v)} \tilde{\varphi}^2(v)dv.$$

Since $f^a(x,v) = c^a(x)M(v) + w^a(x,v)$,

$$|c_a - c^a(x)| = \frac{1}{|u|} \left|\int \xi w^a(x,v)dv\right|$$

$$\leq \frac{1}{|u|}\left(\int \xi^2 \frac{M(v)}{\nu(v)} dv \int |\tilde{w}^a|^2(x,v)dv\right)^{\frac{1}{2}}$$

so that

$$\int_0^a \int \left|\tilde{f}^a(x,v) - c_a\sqrt{M(v)\nu(v)}\right|^2 dxdv \leq c\int_0^a\int |\tilde{w}^a|^2(x,v)dxdv. \tag{2.12}$$

By (2.5), (2.11), and (2.12), there exist a sequence (a_j) tending to infinity, a number c_∞, and a function $\tilde{f}$ such that c_{a_j} tends to c_∞ and $\tilde{f}^{a_j} - c_{a_j}\sqrt{\nu M}$ converges weakly in L^2 to $\tilde f - c_\infty\sqrt{\nu M}$. One can then check that $\tilde f$ is a solution to the Milne problem (2.1–2.2) with the desired properties. ∎

For the boundary layer study in section 3, some decay of $\tilde g := \tilde f - c_\infty\sqrt{\nu M}$ is needed.

PROPOSITION 2.2. *Assume that*

$$\sup_{v\in\mathbb{R}^3_+} |\tilde\varphi(v)|(1+|v|)^s < \infty, \quad s\in\mathbb{R}_+. \tag{2.13}$$

Then for $s\in\mathbb{R}_+$, $\int \frac{|\xi|}{\nu(v)}|\tilde g(x,v)|^2 dv \le cx^{-s}, \quad x>0.$

This result can essentially be found in [11]. For the convenience of the reader we give their proof with the differences introduced by the nonzero bulk velocity of the Maxwellian. The proof is based on the entropy method introduced by Bardos, Santos, and Sentis [2], and uses the following decay properties of $\tilde g$, pointwise in v and integral in x.

LEMMA 2.3. *Under* (2.13) *for* $s\in\mathbb{R}_+$,

$$\sup_{x>0, v\in\mathbb{R}^3} (1+|v|)^s|\tilde g(x,v)| \le c_1, \tag{2.14}$$

$$\int_0^\infty\int_{\mathbb{R}^3} x^s|\tilde g(x,v)|^2 dxdv \le c_2. \tag{2.15}$$

The constants c_1, c_2 *depend on* φ *and* s.

Proof of Proposition 2.2. Write

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\xi|}{\nu}\,|\tilde g(x,v)|^2 dv &\le \int_{|\xi|\le r} \frac{|\xi|}{\nu}\,|\tilde g(x,v)|^2 dv\\ +\int_{|\xi|>r,|v|\le\rho} \frac{|\xi|}{\nu}\,|\tilde g(x,v)|^2 dv &+ \int_{|v|\ge\rho} \frac{|\xi|}{\nu}\,|\tilde g(x,v)|^2 dv\\ &:= a+b+c.\end{aligned}$$

By (2.14), $a(r)$ and $c(\rho)$ satisfy

$$a(r) \le cr\int_{|\xi|\le r}(1+|v|)^{-2s-\gamma}dv \le cr \quad \text{for } s>\frac{3}{2}-\frac{\gamma}{2}, \tag{2.16}$$

$$c(\rho) \le c\int_{|v|\ge\rho}(1+|v|)^{-2s+1-\gamma}dv \le c\rho^{-1} \quad \text{for } s>\frac{5}{2}-\frac{\gamma}{2}. \tag{2.17}$$

Evidently

$$b(r,\rho) \le cr^{-1}\rho^\gamma \int_{\mathbb{R}^3} \left|\frac{\xi\tilde g(x,v)}{\nu(v)}\right|^2 dv.$$

Now

$$\frac{\xi}{\nu}\,(\tilde g(y,v)-\tilde g(x,v)) = \int_x^y \tilde Q(\tilde w_g)(z,v)dz,$$

and so by (2.15)

$$\begin{aligned}\int_{|\xi|>r,|v|\le\rho} \left|\frac{\xi}{\nu}\,(\tilde g(y,v)-\tilde g(x,v))\right|^2 dv &\le c\left(\int_x^y\left(\int_{\mathbb{R}^3} |\tilde w_g(z,v)|^2dv\right)^{\frac12} dz\right)^2\\ &\le\ cx^{-s+1}\int_x^y\int_{\mathbb{R}^3} z^s|\tilde g(z,v)|^2dvdz \le cx^{-s+1}.\end{aligned}$$

Since $\tilde{g} \in L^2(\mathbb{R}_+ \times \mathbb{R}^3)$, a sequence $y_j \to \infty$ can be chosen so that $\lim_{j\to\infty} \tilde{g}(y_j, .\,) = 0$ in $L^2(\mathbb{R}^3)$. It follows that

$$\int_{|\xi|>r, |v|\leq\rho} \left| \frac{\xi}{\nu}\, \tilde{g}(x,v) \right|^2 dv \leq cx^{-s+1}$$

and so

$$b(r,\rho) \leq cr^{-1}\rho^{\gamma}x^{-s+1}.$$

The choice $r = x^{-\frac{s}{3}}$, $\rho = x^{\frac{s}{3}}$ in (2.16–2.17) gives the desired result. □

Proof of (2.14). By [11, Prop. 4.3]

$$\sup_{x>0} \|\tilde{g}(x,.\,)\|_{L^2(\mathbb{R}^3)} \leq c, \tag{2.18}$$

where c depends on $\tilde{\varphi}$ in the $L^2 \cap L^\infty$ sense. Also

$$\tilde{Q}^+(\tilde{g})(x,v) = \int_{\mathbb{R}^3} k(v,v_1)\tilde{g}(x,v_1)dv_1 \tag{2.19}$$

with

$$|k(v,v_1)| \leq (1+|v|+|v_1|)^{-1+\gamma}(1+|v_1|)^{-\frac{\gamma}{2}}\phi(v,v_1)$$

and

$$\int_{\mathbb{R}^3} \phi^2(v,v_1)dv_1 \leq c(1+|v|)^{-1-\gamma}.$$

Hence

$$(1+|v|)^{\frac{3}{2}-\frac{\gamma}{2}}|\tilde{Q}^+\tilde{g}(x,v)| \leq c\|\tilde{g}(x,.\,)\|_{L^2(\mathbb{R}^3)}.$$

The exponential form of (2.1–2.2) gives

$$\begin{aligned}(1+|v|)^{\frac{3}{2}-\frac{\gamma}{2}}|\tilde{g}(x,v)| \leq |\tilde{\varphi}(v)|(1+|v|)^{\frac{3}{2}-\frac{\gamma}{2}}\chi_{\xi>0} \\ + c\sup_{x>0}\|\tilde{g}(x,.\,)\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{2.20}$$

Here $\chi_{\xi>0}$ is the characteristic function of the set $\{v \in \mathbb{R}^3;\ \xi > 0\}$. By (2.18) the right-hand side is finite. Also

$$\int_{\mathbb{R}^3} (1+|v|)^{s+1}(1+|v_1|)^{-s}k(v,v_1)dv_1 < \infty, \quad s \in \mathbb{R}_+.$$

Using this together with (2.20), a direct estimate in the exponential form of (2.1–2.2) gives (2.14). □

Proof of 2.15. By (2.11), which also holds for $\tilde{w}_g$, there is a sequence $y_j \to \infty$ such that

$$\int |\tilde{w}_g(y_j,v)|^2 dv \to 0. \tag{2.21}$$

It follows from Theorem 2.1 that $\int_{\mathbb{R}^3} \xi g(x,v)dv = 0$, $x \in \mathbb{R}_+$, and so the orthogonal decomposition $\tilde{g}(x,v) = c^\infty(x)\sqrt{\nu(v)M(v)} + \tilde{w}_g(x,v)$ gives

$$\begin{aligned} |uc^\infty(x)| &= \left| \int \xi w_g(v)dv \right| \\ &\leq \left(\frac{\xi^2 M(v)}{\nu(v)} dv \int |\tilde{w}_g(x,v)|^2 dv \right)^{\frac{1}{2}}. \end{aligned} \tag{2.22}$$

In particular

$$\lim_{j\to\infty} c^\infty(y_j) = 0. \tag{2.23}$$

Now the proof is based on a study of the entropy flux

$$H(x) = \int \frac{\xi}{\nu} |\tilde{g}(x,v)|^2 dv.$$

Using the orthogonal decomposition of $\tilde{g}$ and splitting the domain of integration we get

$$\begin{aligned} \lim_{j\to\infty} H(y_j) \leq \lim_{j\to\infty} Cc(y_j)^2 &+ \lim_{j\to\infty} C \int_{\mathbb{R}^3} \tilde{w}_g(y_j,v)^2 dv \\ &+ \lim_{j\to\infty} C \int_{|v|\geq\rho} \frac{\xi}{\nu(v)} \tilde{g}(y_j,v)^2 dv. \end{aligned}$$

By (2.21) and (2.23) the first two of these limits are zero. By (2.14) the third one is bounded by

$$c \int_{|v|\geq\rho} \frac{|\xi|}{\nu(v)} (1+|v|)^{-5} dv \leq \frac{c}{\rho}.$$

It follows that $\lim_{j\to\infty} H(y_j) = 0$. A multiplication of (2.1) by $\tilde{g}$ and v-integration show that $H(x)$ is nonincreasing. And so

$$0 \leq H(x) \leq H(0) \leq \int_{\xi>0} \frac{\xi}{\nu} \tilde{\varphi}(v)^2 dv. \tag{2.24}$$

Since $\tilde{g} \in L^2(\mathbb{R}_+ \times \mathbb{R}^3)$, it is enough for (2.15) to consider

$$\int_1^\infty \int_{\mathbb{R}^3} x^s \tilde{g}(x,v)^2 dx dv.$$

A multiplication of (2.1) by $x^s\tilde{g}$ and integration gives

$$\begin{aligned} H(y)y^s + \int_1^y \left(x^s \int_{\mathbb{R}^3} \tilde{w}_g(x,v)^2 dv - sx^{s-1}H(x) \right) dx \\ \leq H(1) \leq \int_{\xi>0} \frac{\xi}{\nu} \tilde{\varphi}(v)^2 dv. \end{aligned} \tag{2.25}$$

The positivity of $H(y)$ implies that

$$\int_1^y \left(x^s \int_{\mathbb{R}^3} \tilde{w}_g(x,v)^2 dv - sx^{s-1}H(x) \right) dx \leq H(1).$$

Now

$$\int_{|v|\leq\rho} \frac{|\xi|}{\nu}\,\tilde{g}(x,v)^2dv \leq c\rho^{1-\gamma}\|\tilde{g}(x,.)\|^2_{L^2(\mathbb{R}^3)},$$

and by (2.14), for any $\lambda \in \mathbb{R}_+$,

$$\int_{|v|\geq\rho} \frac{|\xi|}{\nu}\,\tilde{g}(x,v)^2dv \leq c_\lambda\rho^{-\lambda}.$$

This together with (2.22) and (2.25) implies for some $\alpha > 0$ that

$$\int_1^y x^s \left(\int \tilde{g}(x,v)^2dv\right)\left(\alpha - \frac{c\rho^{1-\gamma}}{x}\right)dx \leq H(1) + c_\lambda \int_1^y \rho^{-\lambda} s x^{s-1}dx.$$

The choice $\rho(x) = (\frac{\alpha x}{2c})^{\frac{1}{1-\gamma}}, \lambda > s(1-\gamma)$ yields

$$\int_1^\infty x^s \int \tilde{g}(x,v)^2dvdx \leq c_s. \qquad \square$$

3. The fluid approximation with initial and boundary layers for nonzero bulk velocity. Introduce the mean free path $\epsilon > 0$ and take $u > 0$. This section considers the slab problem

$$\partial_t f_\epsilon + \frac{1}{\epsilon^2}\xi\partial_x f_\epsilon = \frac{1}{\epsilon^3}Q(f_\epsilon), \quad t>0, \quad x\in(0,1), \quad v\in\mathbb{R}^3, \tag{3.1}$$

together with the initial condition

$$f_\epsilon(0,x,v) = f_i(x,v), \quad x\in(0,1), \quad v\in\mathbb{R}^3, \tag{3.2}$$

and the boundary conditions

$$f_\epsilon(t,0,v) = f_0(v), t>0, \xi>0; \qquad f_\epsilon(t,1,v) = f_1(v), \quad t>0, \quad \xi<0. \tag{3.3}$$

After an initial layer, the unique solution satisfies the stationary problem

$$\xi\partial_x g_\epsilon = \frac{1}{\epsilon}Q(g_\epsilon), \quad x\in(0,1), \quad v\in\mathbb{R}^3, \tag{3.4}$$

together with the boundary conditions

$$g_\epsilon(0,v) = f_0(v), \quad \xi>0; \quad g_\epsilon(1,v) = f_1(v), \quad \xi<0, \tag{3.5}$$

if one disregards the error term from the initial layer. Moreover, g_ϵ can be split into a fluid part cM in the interior of the domain together with boundary layers and with the error term tending to zero strongly in L^1, when ϵ tends to zero.

THEOREM 3.1. *Let f_i, f_0, f_1 be given with $\tilde{f}_i \in L^2_{\frac{1}{\nu}}((0,1)\times\mathbb{R}^3)$, $\tilde{f}_0 \in L^2_{\frac{\xi}{\nu}}(\mathbb{R}^\xi_+\times\mathbb{R}^2)$, $\tilde{f}_1 \in L^2_{\frac{|\xi|}{\nu}}(\mathbb{R}^\xi_-\times\mathbb{R}^2)$. Denote by f_ϵ and g_ϵ the unique solutions of* (3.1–3.3)*, respectively,* (3.4–3.5) *with these given initial and boundary values. Then for $t > 0$*

$$\lim_{\epsilon\to 0} f_\epsilon(t,.) - g_\epsilon(.) = 0$$

strongly in $L^1((0,1)\times\mathbb{R}^3)$.

THEOREM 3.2. *Under the same hypotheses there are a constant c, boundary layer terms $l_\epsilon(x,v) = l^0(\frac{x}{\epsilon}, v)$, and $r_\epsilon(x,v) = r^0(\frac{x-1}{\epsilon}, v)$, with $\tilde{l}^0$ and $\tilde{r}^0$, respectively, belonging to $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and $L^2(\mathbb{R}_- \times \mathbb{R}^3)$ such that*

$$g_\epsilon = cM + l_\epsilon + r_\epsilon + S_\epsilon. \tag{3.6}$$

Here the terms $\tilde{l}^0$ and $\tilde{r}^0$ have the decay properties of Proposition 2.2, *and the remainder term S_ϵ tends to* 0 *in $L^1((0,1) \times \mathbb{R}^3_v)$, when ϵ tends to* 0.

The proof of Theorem 3.1 is based on the following lemma.

LEMMA 3.3. *Let f_i be given with $0 \le \tilde{f}_i \in L^2_{\frac{1}{\nu}}((0,1) \times \mathbb{R}^3)$. Denote by $f_\epsilon(t,x,v)$ the solution of* (3.1–3.3) *with f_i as initial value and boundary values $f_0 = f_1 = 0$. For $s > 0$, $f_\epsilon(s,.,.)$ converges strongly in $L^1((0,1) \times \mathbb{R}^3)$ to zero, when ϵ tends to zero.*

Proof. After scaling $t \to \frac{t}{\epsilon^2}$, the solution (still denoted f_ϵ) satisfies

$$(\partial_t + \xi\partial_x) f_\epsilon = \frac{1}{\epsilon}\, Q(f_\epsilon), \quad t \in \mathbb{R}_+, \quad x \in (0,1), \quad v \in \mathbb{R}^3,$$
$$f_\epsilon(0,.) = f_i(.),$$
$$f_\epsilon(t,0,v) = 0, \quad t > 0, \quad \xi > 0, \quad f_\epsilon(t,1,v) = 0, \quad t > 0, \quad \xi < 0.$$

Green's formula implies that mass and entropy

$$\int_0^1 \int_{\mathbb{R}^3} f_\epsilon(t,x,v)dxdv, \qquad \int_0^1 \int_{\mathbb{R}^3} \frac{\tilde{f}_\epsilon(t,x,v)^2}{\nu(v)} dxdv$$

are decreasing with time. Suppose that the lemma does not hold. Then for some $s > 0$, there is a sequence (ϵ_j) with $\lim_{j\to\infty} \epsilon_j = 0$ such that

$$\inf_j \int_0^1 \int_{\mathbb{R}^3} f_{\epsilon_j}(t_j,x,v)dxdv > 0.$$

Here $t_j = \frac{s}{\epsilon_j^2}$. The lemma follows if for a subsequence of (t_j) (still denoted (t_j)) there is a sequence (t'_j) with $0 \le t'_j \le t_j$ such that

$$\lim_{j\to\infty} \int_0^1 \int_{\mathbb{R}^3} f_{\epsilon_j}(t'_j,x,v)dxdv = 0.$$

With $\tilde{f}_{\epsilon_j} := \tilde{f}_j, \tilde{w}_{\epsilon_j} := \tilde{w}_j$, (1.9) gives

$$\int_0^1 \int_{\mathbb{R}^3} \frac{\tilde{f}_j(t_j,x,v)^2}{\nu(v)}\, dxdv + \frac{\sigma}{\epsilon_j} \int_0^{t_j} \int_0^1 \int_{\mathbb{R}^3} \tilde{w}_j(\tau,x,v)^2 d\tau dxdv$$
$$\le \int_0^1 \int_{\mathbb{R}^3} \frac{\tilde{f}_i(x,v)^2}{\nu(v)}\, dxdv := \sigma c_1.$$

If each of the $s\epsilon_j^{-\frac{3}{2}}$ intervals $[l\epsilon_j^{-\frac{1}{2}}, (l+1)\epsilon_j^{-\frac{1}{2}}]$ of $[0,t_j]$ has

$$\frac{1}{\epsilon_j} \int_{l\epsilon_j^{-\frac{1}{2}}}^{(l+1)\epsilon_j^{-\frac{1}{2}}} \int_0^1 \int_{\mathbb{R}^3} \tilde{w}_j(\tau,x,v)^2 d\tau dxdv > \epsilon_j^{\frac{3}{2}}\ \frac{c_1}{s},$$

then

$$\frac{\sigma}{\epsilon_j}\int_0^{t_j}\int_0^1\int_{\mathbb{R}^3}\tilde{w}_j(\tau,x,v)^2 d\tau dx dv > c_1\sigma.$$

This contradiction implies that for some interval $I_j \subset [0,t_j]$ and of length $\epsilon_j^{-\frac{1}{2}}$

$$\frac{1}{\epsilon_j}\int_{I_j}\int_0^1\int_{\mathbb{R}^3}\tilde{w}_j(\tau,x,v)^2 d\tau dx dv \le e_j^{\frac{3}{2}}\ \frac{c_1}{s}.$$

In particular

$$\lim_{j\to\infty}\epsilon_j^{-2}\int_{I_j}\int_0^1\int_{\mathbb{R}^3}\tilde{w}_j(\tau,x,v)^2 d\tau dx dv = 0.$$

With $I_j = (t'_j, t''_j)$ it follows that for $t \ge 0$ (and some subsequence of the j's)

$$\tilde{f}_j(t'_j + t, x, v) \rightharpoonup \tilde{f}_\infty(t,x,v)$$

weakly in $L^2_{\frac{1}{\nu}}((0,1)\times\mathbb{R}^3)$. Here

$$\int_0^1\int_{\mathbb{R}^3}\frac{\tilde{f}_\infty(t,x,v)^2}{\nu(v)}\ dx dv \le \sigma c_1.$$

By the equicontinuity in t, it is enough to prove the above weak L^2-convergence for rational t's. Using (1.9) we have for t fixed and j large enough that

$$\int_0^1\int_{\mathbb{R}^3}\frac{1}{\nu(v)}\ f_j(t'_j + t, x, v)^2 dx dv \le \sigma c_1.$$

So a subsequence of $\tilde{f}_j(t'_j + t)$ converges weakly when $j \to 0$. We conclude with a Cantor diagonalization argument.

Also for a.e. $t > 0$,

$$\tilde{w}_j(t'_j + t, x, v) \to 0$$

strongly in $L^2((0,1)\times\mathbb{R}^3)$, and so

$$\tilde{f}_\infty(t,x,v) = c_\infty(t,x,)\sqrt{\nu(v)M(v)}.$$

But $\tilde{f}_\infty$ satisfies

$$(\partial_t + \xi\partial_x)\tilde{f}_\infty = 0,\quad t>0,\quad x\in(0,1),\quad v\in\mathbb{R}^3,$$
$$\tilde{f}_\infty(t,0,v) = 0,\quad t>0,\quad \xi>0;\quad \tilde{f}_\infty(t,1,v)=0,\quad t>0,\quad \xi<0,$$

and so $\tilde{f}_\infty \equiv 0$. In particular $\lim_{j\to\infty}\tilde{f}_j(t'_j,.,.) = 0$ weakly in $L^2_{\frac{1}{\nu}}((0,1)\times\mathbb{R}^3)$. It follows that

$$0 \le \lim_{j\to\infty}\int_0^1\int_{\mathbb{R}^3} f_j(t_j,x,v)dx dv$$
$$\le \lim_{j\to\infty}\int_0^1\int_{\mathbb{R}^3} f_j(t'_j,x,v)dx dv = \lim_{j\to\infty}\int_0^1\int_{\mathbb{R}^3}\sqrt{\frac{M(v)}{\nu(v)}}\ \tilde{f}_j(t'_j,x,v)dx dv$$
$$= 0.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 3.1. The function $f_\epsilon - g_\epsilon$ satisfies (3.1–3.3) with initial value $f_i - g_i$ and boundary value zero. By linearity it is enough to prove the theorem when the boundary values are zero and $f_i - g_i \geq 0$, and this case is contained in Lemma 3.3. □

Proof of Theorem 3.2. Essentially by section 1, there is a unique solution $g_\epsilon(x,v)$ with $\tilde{g}_\epsilon \in L^2$ to

$$\xi \partial_x g_\epsilon = \frac{1}{\epsilon}\, Q(g_\epsilon),$$

$$g_\epsilon(0,v) = f_0(v), \quad \xi > 0,$$

$$g_\epsilon(1,v) = f_1(v), \quad \xi < 0.$$

From the results on the Milne problem in Theorem 2.1, there is a constant c such that

$$\frac{\xi}{\nu(v)}\, \partial_x \tilde{q}(x,v) = \tilde{Q}\tilde{q}(x,v), \quad x > 0, \quad v \in \mathbb{R}^3,$$

$$\tilde{q}(0,v) = \tilde{f}_0(v), \quad \xi > 0$$

has a solution $\tilde{q} = c\sqrt{\nu M} + \tilde{l}$, with $\tilde{l} \in L^2(\mathbb{R}^+ \times \mathbb{R}^3_v)$. Define l^0 by

$$l^0(y,v) = \sqrt{\frac{M(v)}{\nu(v)}}\, \tilde{l}(y,v).$$

Also by Theorem 2.1 the Milne problem

$$\frac{\xi}{\nu(v)}\, \tilde{r}_x(x,v) = \tilde{Q}\tilde{r}(x,v), \quad x < 0, \quad v \in \mathbb{R}^3,$$

$$\tilde{r}(0,v) = \tilde{f}_1(v) - c\sqrt{\nu(v)M(v)}, \quad \xi < 0$$

has a solution $\tilde{r} \in L^2(\mathbb{R}_- \times \mathbb{R}^3_v)$. Indeed for $u > 0$, looking for a solution defined in $\mathbb{R}_-$ corresponds to considering $u < 0$ in the $\mathbb{R}_+$ situation. Define r^0 by

$$r^0(y,v) = \sqrt{\frac{M(v)}{\nu(v)}}\, \tilde{r}(y,v).$$

$S_\epsilon := g_\epsilon - cM - l_\epsilon - r_\epsilon$ satisfies

$$\xi \partial_x S_\epsilon = \frac{1}{\epsilon}\, Q(S_\epsilon), \quad x \in (0,1), \quad v \in \mathbb{R}^3, \tag{3.7}$$

$$S_\epsilon(0,v) = -r^0\left(-\frac{1}{\epsilon}, v\right), \quad \xi > 0,$$

$$S_\epsilon(1,v) = -l^0\left(\frac{1}{\epsilon}, v\right), \quad \xi < 0.$$

Introduce as above the orthogonal decomposition

$$\tilde{S}_\epsilon(x,v) = c^\epsilon(x)\sqrt{\nu M} + \tilde{w}_\epsilon.$$

It follows from (3.4), Green's formula, and (1.6) that

$$\begin{aligned}\int_{\xi<0}\frac{|\xi|}{\nu}\,|\tilde{S}_\epsilon(0,v)|^2dv+\int_{\xi>0}\frac{\xi}{\nu}\,|\tilde{S}_\epsilon(1,v)|^2dv\\ +\frac{\sigma}{\epsilon}\int_0^1\int_{\mathbb{R}^3}|\tilde{w}_\epsilon(x,v)|^2dxdv\\ \le\int_{\xi>0}\frac{\xi}{\nu}\left|\tilde{r}^0\left(-\frac{1}{\epsilon},v\right)\right|^2dv+\int_{\xi<0}\frac{|\xi|}{\nu}\left|\tilde{l}^0\left(\frac{1}{\epsilon},v\right)\right|^2dv.\end{aligned}\tag{3.8}$$

By Proposition 2.2 the right-hand side tends superalgebraically to zero, when ϵ tends to zero. By (3.4)

$$uc_\epsilon=\int\xi S_\epsilon(x,v)dv$$

is independent of x. Multiplying (3.7) with $\text{sign}S_\epsilon$ and integrating we get

$$\begin{aligned}\int_{\xi<0}|\xi|\,|S_\epsilon(0,v)|dv+\int_{\xi>0}\xi|S_\epsilon(1,v)|dv\\ \le\int_{\xi>0}\xi\left|r^0\left(-\frac{1}{\epsilon},v\right)\right|dv+\int_{\xi<0}\left|\xi l^0\left(\frac{1}{\epsilon},v\right)\right|dv.\end{aligned}$$

Thus

$$\begin{aligned}|c_\epsilon|\le\frac{1}{u}\int|\xi|S_\epsilon(0,v)dv\\ \le\frac{c}{u}\left(\int_{\xi>0}\frac{\xi}{\nu}\left|\tilde{r}^0\left(-\frac{1}{\epsilon},v\right)\right|^2dv+\int_{\xi<0}\frac{|\xi|}{\nu}\left|\tilde{l}^0\left(\frac{1}{\epsilon},v\right)\right|^2dv\right),\end{aligned}\tag{3.9}$$

which tends to zero superalgebraically, when ϵ tends to zero. As in (2.12)

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^3}\left|\tilde{S}_\epsilon(x,v)-c_\epsilon\sqrt{\nu(v)M(v)}\,\right|^2dxdv\\ \le c\int_0^1\int_{\mathbb{R}^3}|\tilde{w}_\epsilon(x,v)|^2dxdv.\end{aligned}\tag{3.10}$$

By (3.5–3.7)

$$\int_0^1\int_{\mathbb{R}^3}|\tilde{S}_\epsilon(x,v)|^2dxdv$$

tends to zero superalgebraically, when ϵ tends to zero. □

Remark. The evaporation at $x=0$ determines the (fluid dynamic) mass flux term cM through the boundary layer analysis. At the condensation boundary $x=1$ this term is removed from the boundary layer correction.

Remark. It follows from this proof that the solution of the Milne problem in Theorem 2.1 is unique. It also follows that the convergence to zero of the error term in Theorem 3.2 is superalgebraic.

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