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A large data existence result for the stationary Boltzmann equation in a cylindrical geometry

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Abstract. An L^1-existence theorem is proved for the nonlinear stationary Boltzmann equation with hard forces and no small velocity truncation—only the Grad angular cut-off—in a setting between two coaxial rotating cylinders when the indata are given on the cylinders.

1. Introduction

General L^1-solutions for stationary, fully nonlinear equations of Boltzmann type, have so far been obtained by weak compactness techniques. Examples are existence results far from equilibrium for the stationary Povzner equation in bounded domains of $\mathbf{R}^n$, as obtained in [3], [15], and general L^1-solutions of the stationary nonlinear Boltzmann equation in a slab, as studied in [2] and [4]. Also half-space problems for the stationary, nonlinear Boltzmann equation in the slab with given indata can sometimes be solved by such techniques; see [5] for a collision operator truncated for large velocities and for small values of the velocity component in the slab direction. For more complete references the reader is referred to the above cited papers.

For bounded domains in $\mathbf{R}^n$, a general existence result was obtained in [6] for the stationary Boltzmann equation under a supplementary truncation for small velocities. The removal of the small velocity cut-off for the nonlinear, stationary Boltzmann equation with large boundary data, remained an open problem in more than one space dimension. The present paper studies that problem in a particular $\mathbf{R}^2$ case, a two-roll configuration without any small velocity truncation, using a generalization of the techniques from the slab case. Also recall that the close to equilibrium $\mathbf{R}^n$-situation is better understood, since there more powerful techniques such as contraction mappings, can be used. In that case a number of existence results are published, see [11], [12], [13], [14], [17] and others. In particular, see [7] and [16] for the present two-roll problem close to equilibrium.

The set-up for the two-roll problem is as follows. Consider the stationary Boltzmann equation in the space Ω between two coaxial cylinders,

$$v\cdot\nabla_x f(x,v)=Q(f,f),\quad x\in\Omega,\ v\in\mathbf{R}^3, \tag{1.1}$$

for axially homogenous solutions f. We may then take $\Omega\subset\mathbf{R}^2$ as the annulus between two concentric circles of radii $r_A<r_B$. The nonnegative solution $f(x,v)$ represents the density of a rarefied gas with x the position and v the velocity. Solutions are here understood in the weak sense, which is somewhat stronger than the renormalized one or equivalently the mild, exponential, iterated integral form (cf. [9] and [1]). The operator Q is the nonlinear Boltzmann collision operator with angular cut-off,

$$Q(f,f)(v)=\int_{\mathrm{R}^3}\int_{S^2}B(v-v_*,\omega)(f(x,v')f(x,v'_*)-f(x,v)f(x,v_*))\,dv_*\,d\omega,$$

where $v'=v-(v-v_*,\omega)\omega$, $v'_*=v_*+(v-v_*,\omega)\omega$. The function B is the kernel of the classical nonlinear Boltzmann operator for hard forces,

$$|v-v_*|^\beta b(\theta)\quad\text{with } 0\le\beta<2,\ b\in L^1_+((0,2\pi)),\ b(\theta)\ge c>0 \text{ a.e.}$$

The solutions considered, are axially and rotationally uniform functions with respect to the space variables. Denoting by (r,θ,z) and (v_r,v_θ,v_z), respectively, the spatial coordinates and the velocity in cylindrical coordinates, the solutions are thus functions $f(r,v_r,v_\theta,v_z)$. As boundary conditions, functions f_b are given on the ingoing boundary $\partial\Omega^+$ at A and B, equal to $f_A>0$ and $f_B>0$ defined on $\{(r_A,v);v_r>0\}$ and $\{(r_B,v);v_r<0\}$, respectively. Solutions $f(r,v)$ to (1.1) are sought with profiles f_A and f_B on the inner and the outer cylinders, i.e.

$$f(r_A,v)=kf_A(v),\ v_r>0,\quad f(r_B,v)=kf_B(v),\ v_r<0, \tag{1.2}$$

for some positive constant k. The test functions φ are taken in $L^\infty(\bar\Omega\times\mathbf{R}^3)$ with $v\cdot\nabla_x\varphi\in L^\infty(\Omega\times\mathbf{R}^3)$, continuously differentiable along characteristics, with compact support in $\Omega\times\widetilde{\mathbf{R}}^3$, and vanishing on $\{(r_A,v);v_r<0\}\cup\{(r_B,v);v_r>0\}$. Here $\widetilde{\mathbf{R}}^3=\mathbf{R}^3\setminus\{v;v_r=0\}$.

The main result of this paper is the following result.

Theorem 1.1. *Suppose that*

$$\int_{v_r>0} v_r(1+|v|^2+\log^+ f_A(v))f_A(v)\,dv + \int_{v_r<0} |v_r|(1+|v|^2+\log^+ f_B(v))f_B(v)\,dv < \infty.$$

Then, for any $m>0$, *the equation* (1.1) *has a weak* L^1*-solution* f_m *satisfying* (1.2) *with* $k=k_m>0$, *and*

$$\int_{r_A}^{r_B}\int_{\mathbb{R}^3} f_m(r,v)r(1+|v|)^\beta\,dv\,dr = m. \tag{1.3}$$

The weak compactness arguments in the proof below do not provide continuity for the map $m\mapsto k_m$. Connected to this, the theorem does not state the existence of a solution with arbitrary indata. Instead a particular moment is fixed, leaving only the profile of the arbitrary indata free at the boundary.

Entropy related quantities are widely used to study kinetic equations and kinetic formulations of conservation laws. In the context of stationary kinetic problems, it is often the entropy dissipation term that provides the most useful control. That was the case in the Povzner and Boltzmann slab papers [2], [3], [4], where this term was an important tool to obtain existence results for (1.1) under (1.3) via weak L^1-compactness. In the present paper the same approach is generalized from the slab case to cylinders. Approximations of the problem at hand are first constructed in Section 2, similarly to those earlier papers. Starting from those approximations, Section 3 is devoted to taking the approximations into true solutions through a sequence of limit steps.

2. Approximations

Without loss of generality we can restrict the discussion to the case $m=1$. Denote by $f_*=f(x,v_*)$, $f'=f(x,v')$, and $f'_*=f(x,v'_*)$. Let $s>1/\varrho>0$, and let $\chi^s_\varrho(v,v_*,\omega)$ be a C^∞-function, such that $0\le\chi^s_\varrho\le 1$ is invariant with respect to the collision transformation $J(v,v_*,\omega)=(v',v'_*,-\omega)$, as well as to an exchange of v and v_*, and such that

$$\chi^s_\varrho(v,v_*,\omega)=1, \quad \text{if } |v_r|\ge s+\frac{1}{\varrho},\ |v_{*r}|\ge s+\frac{1}{\varrho},\ |v'_r|\ge s+\frac{1}{\varrho} \text{ and } |v'_{*r}|\ge s+\frac{1}{\varrho},$$

$$\chi^s_\varrho(v,v_*,\omega)=0, \quad \text{if } |v_r|\le s,\ \text{or } |v_{*r}|\le s,\ \text{or } |v'_r|\le s,\ \text{or } |v'_{*r}|\le s,$$

and define $\chi^s:=\chi^s_\infty$. Set $\bar\chi^s(v)=1$ if $|v_r|\ge s$ and $\bar\chi^s(v)=0$ otherwise. Denote by

$$Q^s(f,f)(r,v)=\int_{\mathbb{R}^3\times S^2}\chi^s(v,v_*,\omega)B(f'f'_*-ff_*)\,dv_*\,d\omega.$$

Lemma 2.1. *For any $0<s<1$, there are a function f^s and a real number $k^s>0$ which form a solution to*

$$sf^s+v\cdot\nabla_x f^s=Q^s(f^s,f^s),\quad (x,v)\in\Omega\times\mathbf{R}^3, \tag{2.1}$$

$$f^s(x,v)=k^s f_b(x,v),\quad (x,v)\in\partial\Omega^+, \tag{2.2}$$

$$\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f^s(x,v)\,dx\,dv=1. \tag{2.3}$$

Moreover,

$$0<k^s<c_0,\quad \int_{\Omega\times\mathrm{R}^3}|\tilde{v}|^2 f^s(x,v)\,dx\,dv\le c_1k^s, \tag{2.4}$$

and

$$\int_{\Omega\times\mathrm{R}^6\times S^2}\chi^s B(f^{s\prime}f_*^{s\prime}-f^sf_*^s)\log\frac{f^{s\prime}f_*^{s\prime}}{f^sf_*^s}\,dx\,dv\,dv_*\,d\omega\le c_2k^s, \tag{2.5}$$

where $\tilde{v}=(v_r,v_\theta)$, and c_0, c_1 and c_2 are positive constants independent of s.

Proof. Only the main lines of the proof are given, similar arguments being developed in [6].

Let $0<j,p,n,\mu\in\mathbf{N}$ and $\varrho\ge1$ be given, as well as a positive C^∞ regularization $\tilde{b}$ of b. Let K be the closed and convex subset of $L^1(\Omega\times\mathbf{R}^3)\times\mathbf{R}_+$, defined by

$$K=\left\{f\in L^1_+(\Omega\times\mathbf{R}^3)\,;\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f(x,v)\,dx\,dv=1,\right.$$
$$\left.f(x,v)=0\text{ for }|v_r|<s\right\}\times[0,c_3].$$

Here

$$c_3=\frac{e^{(1+8\pi^2n^3j\mu|\tilde{b}|_{L^1}/3)2r_B}}{\int_{\Omega\times\{v;|\tilde{v}|\ge1\}}\bar{\chi}^s(1+|v|)^\beta(f_b(x-s^+(x,v)v,v)\wedge j)\,dx\,dv},$$

f_b is the ingoing boundary value f_A, f_B, and

$$s^+(x,v):=\inf\{s\in\mathbf{R}_+\,;(x-sv,v)\in\partial\Omega^+\},\quad f_b\wedge j=\min\{f_b,j\}.$$

Similarly, take

$$s^-(x,v):=\inf\{s\in\mathbf{R}_+\,;(x+sv,v)\in\partial\Omega^-\},$$

where $\partial\Omega^-$ denotes the outgoing boundary.

Define the map T on K by

$$T(f,k)=\left(\frac{\bar{\chi}^s F}{\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta F(x,v)\,dx\,dv},\frac{1}{\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta F(x,v)\,dx\,dv}\right),$$

where F is the solution to

$$\begin{aligned}
sF+v\cdot\nabla_x F=\int_{\mathrm{R}^3\times S^2}\chi_\varrho^s\widetilde{\chi}^{pn}B_\mu\Bigg[&\frac{F}{1+\frac{kF}{j}}(x,v')\frac{f*\varphi_\varrho}{1+\frac{f*\varphi_\varrho}{j}}(x,v'_*)\\
&-F(x,v)\frac{f*\varphi_\varrho}{1+\frac{f*\varphi_\varrho}{j}}(x,v_*)\Bigg]\,dv_*\,d\omega,\quad (x,v)\in\Omega\times\mathbf{R}^3, \qquad (2.6)\\
F(x,v)=f_b(x,v)\wedge j,\quad (x,v)\in\partial\Omega^+.&
\end{aligned}$$

Here,

$$B_\mu(v,v_*,\omega)=\max\left\{\frac{1}{\mu},\min\{\mu,|v-v_*|^\beta\}\right\}\tilde{b}(\theta).$$

The function $\widetilde{\chi}^{pn}(v,v_*,\omega)$ is taken in C^∞, such that $0\le\widetilde{\chi}^{pn}\le1$, invariant with respect to the collision transformation $J(v,v_*,\omega)=(v',v'_*,-\omega)$, and invariant under an exchange of v and v_*. Moreover, it satisfies

$$\widetilde{\chi}^{pn}(v,v_*,\omega)=1,\quad \text{if } v^2+v_*^2\le\frac{n^2}{2},\ \frac{1}{p}\le\left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right| \text{ and } |v-v_*|\ge\frac{1}{p},$$

$$\widetilde{\chi}^{pn}(v,v_*,\omega)=0,\quad \text{if } v^2+v_*^2\ge n^2 \text{ or } \left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right|\le\frac{1}{2p} \text{ or } |v-v_*|\le\frac{1}{2p}.$$

The functions φ_ϱ are mollifiers in x defined by

$$\varphi_\varrho(x)=\varrho^2\varphi(\varrho x),\quad 0\le\varphi\in C_0^\infty(\mathbf{R}^2),\quad \varphi(x)=0 \text{ for } |x|\ge1,\quad \int_{\mathrm{R}^2}\varphi(x)\,dx=1.$$

The function T maps K into K. Indeed, from the exponential form of F, obtained by integration of (2.6) along characteristics,

$$\begin{aligned}
F(x,v)\ge f_b(x-s^+(x,v)v,v)e\wedge j\exp\Bigg(&-ss^+(x,v)\\
&-\int_{-s^+(x,v)}^0\int_{\mathrm{R}^3\times S^2}\chi_\varrho^s\widetilde{\chi}^{pn}B_\mu\frac{f*\varphi_\varrho}{1+\frac{f*\varphi_\varrho}{j}}(x+tv,v_*)\,dv_*\,d\omega\,dt\Bigg)
\end{aligned}$$

for $(x,v)\in\Omega\times\mathbf{R}^3$. Hence,

$$F(x,v)\geq (f_b(x-s^+(x,v)v,v)\wedge j)\exp\bigl(-\bigl(1+\tfrac{8}{3}\pi^2n^3j\mu|\tilde{b}|_{L^1}\bigr)2r_B\bigr),\quad x\in\Omega,\ |\tilde{v}|\geq 1.$$

And so,

$$\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta F(x,v)\,dx\,dv\geq\frac{1}{c_3}.$$

By a monotone iteration scheme applied to (2.6), it is easy to see that T is well defined. As in [6], the map T is continuous and compact for the strong L^1 topology. Hence the Schauder fixed point theorem applies. A fixed point

$$(f,k_{s,j,n,p,\varrho,\mu}),\ \text{with}\ k_{s,j,n,p,\varrho,\mu}=\frac{1}{\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta F(x,v)\,dx\,dv},$$

satisfies

$$\begin{aligned}
sf+v\cdot\nabla_xf&=\int_{\mathrm{R}^3\times S^2}\chi_\varrho^s\widetilde{\chi}^{pn}B_\mu\Biggl(\frac{f}{1+\frac{f}{j}}(x,v')\frac{f*\varphi_\varrho}{1+\frac{f*\varphi_\varrho}{j}}(x,v'_*)\\
&\qquad -f(x,v)\frac{f*\varphi_\varrho}{1+\frac{f*\varphi_\varrho}{j}}(x,v_*)\Biggr)\,dv_*\,d\omega,\qquad (x,v)\in\Omega\times\mathbf{R}^3,\qquad (2.7)\\
f(x,v)&=k_{s,j,n,p,\varrho,\mu}f_b(x,v)\wedge j,\qquad (x,v)\in\partial\Omega^+,\\
&\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f(x,v)\,dx\,dv=1,
\end{aligned}$$

with $0<k_{s,j,n,p,\varrho,\mu}<c_3$.

Again following the proof in [6], a strong L^1 compactness argument can be used to pass to the limit in (2.7) when ϱ tends to infinity. It gives rise to a solution f of

$$\begin{aligned}
sf+v\cdot\nabla_xf&=\int_{\mathrm{R}^3\times S^2}\chi^s\widetilde{\chi}^{pn}B_\mu\Biggl(\frac{f}{1+\frac{f}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(x,v'_*)\\
&\qquad -f(x,v)\frac{f}{1+\frac{f}{j}}(x,v_*)\Biggr)\,dv_*\,d\omega,\qquad (x,v)\in\Omega\times\mathbf{R}^3,\qquad (2.8)
\end{aligned}$$

$$f(x,v)=k_{s,j,n,p,\mu}f_b(x,v)\wedge j,\qquad (x,v)\in\partial\Omega^+,\qquad (2.9)$$

$$\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f(x,v)\,dx\,dv=1.\qquad (2.10)$$

Here $0<k_{s,j,n,p,\mu}<c_3$, since the norm of F is bounded from above.

For γ a unit vector in the plane, let Ω_γ denote the line segment which is the orthogonal projection of Ω onto a line in $\mathbf{R}^2$ orthogonal to γ. For any $x\in\Omega$, denote by x_γ its orthogonal projection on Ω_γ. The length of Ω_γ is $|\Omega_\gamma|=2r_B$. It follows from (2.10) that

$$\int_{\Omega_\gamma}\int_{\{\tau;x_\gamma+\tau\gamma\in\Omega\}}\int_{\mathrm{R}^3}\bar{\chi}^s(1+|v_*|)^\beta f(x_\gamma+\tau\gamma,v_*)\,dv_*\,dx_\gamma\,d\tau=1.$$

Hence there is a subset $\widetilde{\Omega}_\gamma$ of Ω_γ with $|\widetilde{\Omega}^c_\gamma|<\frac{1}{2}|\Omega_\gamma|$ such that for $2\le|v|\le4$, $\tilde{v}/|\tilde{v}|=\gamma$, it holds that

$$\begin{aligned}&\int_{-s^+(x,v)}^0\int_{\mathrm{R}^3\times S^2}\chi^s\widetilde{\chi}^{pn}B_\mu\frac{f}{1+\dfrac{f}{j}}(x_\gamma+sv,v_*)\,dv_*\,d\omega\,ds\\&\qquad\le\int_{-s^+(x,\gamma)}^0\int_{\mathrm{R}^3\times S^2}\chi^s\widetilde{\chi}^{pn}B_\mu\frac{f}{1+\dfrac{f}{j}}(x_\gamma+\tau\gamma,v_*)\,dv_*\,d\omega\,d\tau<c|\tilde{b}|_{L^1},\quad x_\gamma\in\widetilde{\Omega}_\gamma.\end{aligned}$$

By the exponential form of (2.8), (2.9) and (2.10),

$$\begin{aligned}1&>\int_{\substack{|v_r|\ge s\\|\tilde{v}|>1\\2\le|v|\le4}}\int_{x\in\Omega}(1+|v|)^\beta f(x,v)\,dx\,dv\\&\ge k_{s,j,n,p,\mu}e^{-(2r_B+c|\tilde{b}|_{L^1})}\int_{\substack{|v_r|\ge s\\|\tilde{v}|>1\\2\le|v|\le4}}\int_{x_\gamma\in\widetilde{\Omega}_\gamma}(1+|v|)^\beta f_b(x-s^+(x,v)v,v)\wedge1\,dx\,dv.\end{aligned}$$

Hence the family $(k_{s,j,n,p,\mu})$ is bounded from above by a constant c_0, uniformly with respect to s, j, n, p, and μ. Denote the solution of (2.8) by f^j. Multiplying (2.8) by $1+\log(f^j/1+f^j/j)$, then integrating the resulting equation over $\Omega\times\mathbf{R}^3$, and using Green's formula, implies that

$$s\int_{\Omega\times\mathrm{R}^3}f^j(1+\log f^j)(x,v)\,dx\,dv\le c<\infty,$$

uniformly with respect to j. And so, as in the time-dependent case (cf. [8]), the weak L^1-limit f of f^j when j tends to infinity, satisfies

$$\begin{aligned}sf+v\cdot\nabla_xf&=\int_{\mathrm{R}^3\times S^2}\chi^s\widetilde{\chi}^{pn}B_\mu(f(x,v')f(x,v'_*)-f(x,v)f(x,v_*))\,dv_*\,d\omega,\\&\qquad(x,v)\in\Omega\times\mathbf{R}^3,\\f(x,v)&=k_{s,n,p,\mu}f_b(x,v),\quad(x,v)\in\partial\Omega^+,\\&\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f(x,v)\,dx\,dv=1,\end{aligned}\tag{2.11}$$

with $0<k_{s,n,p,\mu}\leq c_0$. Here $0<k_{s,n,p,\mu}$ is a consequence of Green's formula.

Given $s>0$, write $f^{n,p,\mu}$ for f in (2.11) to stress the parameter dependence. Multiplying (2.11) by $1+v^2$ and by $\log f^{n,p,\mu}$, then integrating both resulting equations over $\Omega\times\mathbf{R}^3$ and using Green's formula implies that

$$s\int_{\Omega\times\mathrm{R}^3}(1+v^2+\log f^{n,p,\mu})f^{n,p,\mu}(x,v)\,dx\,dv<\infty,$$

uniformly with respect to n, p, μ, and $\tilde{b}$. And so, when $\tilde{b}$ tends to b, n and p tend to infinity, and μ tends to zero, the weak limit f^s of $f^{n,p,\mu}$ satisfies

$$sf^s+v\cdot\nabla_x f^s=\int_{\mathrm{R}^3\times S^2}\chi^s B(f^{s'}f_*^{s'}-f^sf_*^s)\,dv_*\,d\omega,\quad (x,v)\in\Omega\times\mathbf{R}^3,$$

$$f^s(x,v)=k^sf_b(x,v),\qquad (x,v)\in\partial\Omega^+, \tag{2.12}$$

$$\int_{\Omega\times\mathrm{R}^3}\bar{\chi}^s(1+|v|)^\beta f^s(x,v)\,dx\,dv=1, \tag{2.13}$$

with $k^s\leq c_0$. Moreover, $k^s>0$ and

$$\int_{\Omega\times\mathrm{R}^3}|\tilde{v}|^2f^s(x,v)\,dx\,dv\leq c_1,$$

for some $c_1>0$, uniformly with respect to s. Indeed, multiplying (2.12) by $1+v^2$ and integrating it over $\Omega\times\mathbf{R}^3$ leads to

$$\begin{aligned}s\int_{\Omega\times\mathrm{R}^3}(1+v^2)f^s(x,v)\,dx\,dv+\int_{\partial\Omega^-}|v\cdot n(x)|(1+v^2)f^s(x,v)\,dx\,dv\\ =k^s\int_{\partial\Omega^+}v\cdot n(x)(1+v^2)f_b(x,v)\,dx\,dv\\ \leq c_0\int_{\partial\Omega^+}v\cdot n(x)(1+v^2)f_b(x,v)\,dx\,dv.\end{aligned} \tag{2.14}$$

It follows from (2.13) and the left-hand side equality in (2.14) that $k^s>0$ for $s>0$. Then, denote by (v_x,v_y,v_z) the three components of the velocity v in cartesian coordinates with (v_x,v_y) parallel to Ω. Multiply (2.12) by v_x and integrate it over $\Omega_a\times\mathbf{R}^3$, where Ω_a is the part of Ω with $x_1<a$. Set $S_a:=\{x\in\Omega;x_1=a\}$ and $\partial\Omega_a:=\partial\Omega\cap\bar{\Omega}_a$. This gives

$$\begin{aligned}s\int_{\Omega_a\times\mathrm{R}^3}v_xf^s(x,v)\,dx\,dv+\int_{S_a\times\mathrm{R}^3}v_x^2f^s(a,x_2,x_3,v)\,dx_2\,dx_3\,dv\\ -\int_{\partial\Omega_a\times\mathrm{R}^3}v_xv\cdot n(x)f^s(x,v)\,dx\,dv=0.\end{aligned} \tag{2.15}$$

Integrating (2.15) over $[-r_B, r_B]$, leads to

$$\int_{\Omega\times\mathrm{R}^3} v_x^2 f^s(x,v)\,dx\,dv \le 2r_B s\int_{\Omega\times\mathrm{R}^3}(1+v^2)f^s(x,v)\,dx\,dv + \int_{-r_B}^{r_B}\int_{\partial\Omega_a\times\mathrm{R}^3} v_x v\cdot n(x) f^s(x,v)\,dx\,dv\,da < c_1' k^s,$$

by (2.14). Analogously, $\int_{\Omega\times\mathrm{R}^3} v_y^2 f^s(x,v)\,dx\,dv$ is bounded from above, uniformly with respect to s. And so, the boundedness in (2.4) follows. Finally, Green's formula for $f^s\log f^s$ implies that, for some $c_2>0$,

$$\int_{\Omega\times\mathrm{R}^6\times S^2} \chi^s B(f^{s'}f_*^{s'} - f^s f_*^s)\log\frac{f^{s'}f_*^{s'}}{f^s f_*^s}\,dx\,dv\,dv_*\,d\omega \le c_2 k^s, \tag{2.16}$$

uniformly with respect to s. This ends the proof of Lemma 2.1. □

Lemma 2.2. *There is a constant c_3 such that*

$$\int_{\mathrm{R}^3} v_r^2 f^s(r,v)\,dv \le c_3 k^s, \quad \text{a.e. } r\in(r_A, r_B).$$

Proof. Multiplying (2.1) by $1+v^2$ and integrating over $\Omega\times\mathbf{R}^3$ leads to

$$\int_{v\cdot n<0} |v\cdot n|(1+v^2) f(r_A,v)\,dv + \int_{v\cdot n>0} v\cdot n(1+v^2)f(r_B,v)\,dv \le ck^s. \tag{2.17}$$

In cylindrical coordinates, f^s is a solution to

$$sf^s + v_r\frac{\partial f^s}{\partial r} + \frac{1}{r}\left(v_\theta^2\frac{\partial f^s}{\partial v_r} - v_\theta v_r\frac{\partial f^s}{\partial v_\theta}\right) = Q^s(f^s, f^s). \tag{2.18}$$

Multiplying (2.18) by v_r and integrating over $(r_A, r)\times(0,2\pi)\times\mathbf{R}^3$, gives by (2.4), (2.14), and (2.17) that

$$\begin{aligned}\int_{\mathrm{R}^3} rv_r^2 f^s(r,v)\,dv &\le \int_{\mathrm{R}^3} r_A v_r^2 f(r_A,v)\,dv - s\int_{r_A}^{r}\int_{\mathrm{R}^3} tv_r f^s(t,v)\,dv\,dt \\ &\quad + \int_{r_A}^{r}\int_{\mathrm{R}^3} v_\theta^2 f^s(t,v)\,dv\,dt \\ &\le ck^s + cs\int_{r_A}^{r_B}\int_{\mathrm{R}^3}(1+v^2)f^s(t,v)\,dv\,t\,dt \\ &\le c_3 k^s\end{aligned}$$

for some constant c_3. □

Lemma 2.3. *Let* α *be the angle of aperture of the cone starting at* $(r_B, 0)$ (*in cartesian coordinates*), *with axis* Ox (*where* O *denotes the origin*), *and tangent to the circle of radius* r_A. *For any* x *in* Ω, *denote by* $\widetilde{C}_x$ *the cone with axis* Ox, *summit on the outer cylinder, and tangent to the inner cylinder. Denote by* C_x *the homothetical cone with* $\frac{1}{2}\alpha$ *as angle of aperture. Then for any* $\delta>0$, *there is a constant* $c_\delta>0$ *such that*

$$f^s(x,v) \geq c_\delta k^s f_A(v), \quad a.e.\ x\in\Omega,\ v_r>0,\ v\in C_x,\ \delta\leq|v|\leq\frac{1}{\delta},$$
$$f^s(x,v) \geq c_\delta k^s f_B(v), \quad a.e.\ x\in\Omega,\ v_r<0,\ v\in C_x,\ \delta\leq|v|\leq\frac{1}{\delta}.$$

Proof. It follows from (2.1) written in exponential form with the collision frequency ν, that

$$f^s(x,v)\geq ck_s f_b(x-s^+(x,v)v,v)e^{-(s/\delta)2r_B-\int_{-s^+(x,v)}^0 \nu(f^s)(x+tv,v)\,dt}, \quad \delta\leq|v|\leq\frac{1}{\delta}.$$

Then,

$$\begin{aligned}\int_{-s^+(x,v)}^0 \nu(f^s)(x+tv,v)\,dt &\leq c\int_{-s^+(x,v)}^0\int_{\mathbb{R}^3}|v-v_*|^\beta f^s(x+tv,v_*)\,dv_*\,dt\\ &\leq c_\delta\int_{-s^+(x,v)}^0\int_{\mathbb{R}^3}(1+|v_*|)^\beta f^s(x+tv,v_*)\,dv_*\,dt\\ &\leq c_\delta\int_{-s^+(x,\omega)}^0\int_{\mathbb{R}^3}(1+|v_*|)^\beta f^s(x+s\omega,v_*)\,dv_*\,ds,\end{aligned}$$

where $\omega=v/|v|$. And so, by the change of variables $s\mapsto r=|x+s\omega|$, with Jacobian $|Ds/Dr|=|x+s\omega|/|(\omega,x+s\omega)|$ uniformly bounded from above by the definition of the cone C_x,

$$\int_{-s^+(x,v)}^0 \nu(f^s)(x+tv,v)\,dt\leq c_\delta\int_{r_A}^{r_B}\int_{\mathbb{R}^3}(1+|v_*|)^\beta f^s(r,v_*)\,dv_*\,r\,dr\leq c_\delta. \quad \square$$

3. Passage to the limit

For proving the existence Theorem 1.1, it remains to pass to the limit in (2.1)–(2.3) when $s\to 0$.

Lemma 3.1.

$$\sup_{0<s<1} k^s = k_0 < \infty.$$

Proof. It follows from Lemma 2.3 that

$$f^s(r,v) \geq ck^s f_A(v), \quad \text{a.e. } r\in(r_A,r_B),\ v_r>\tfrac{1}{2},\ v\in C,\ \tfrac{1}{2}\leq|v|\leq 2,$$

so that

$$1=\int_{r_A}^{r_B}\int_{\mathbb{R}^3}\bar{\chi}^s(1+|v|)^\beta f^s(r,v)\,dv\,dr\geq ck^s,$$

and

$$k^s\leq k_0:=\frac{1}{c}. \quad \square$$

Lemma 3.2.

$$\liminf_{s\to 0} k^s>0.$$

Proof. We shall prove Lemma 3.2 by contradiction. If $\liminf_{s\to 0} k^s=0$, then there is a sequence $(s_j)_{j=1}^{\infty}$ tending to zero when $j\to\infty$, such that $k_j:=k^{s_j}$ tends to zero when $j\to\infty$. Fix $\varepsilon\ll 1$. Prove that for j large enough, $f^j:=f^{s_j}$ and $\bar{\chi}^j:=\bar{\chi}^{s_j}$ satisfy

$$\int_{r_A}^{r_B}\int_{\mathbb{R}^3}\bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\,dr<5\varepsilon,$$

contradicting

$$\int_{r_A}^{r_B}\int_{\mathbb{R}^3}\bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\,dr=1.$$

By Lemma 2.2, given $c'>0$ there is $c>0$ such that

$$\int_{c'|v_r|>\sqrt{v_\theta^2+v_z^2}} v^2 f^j(r,v)\,dv\leq c\int_{\mathbb{R}^3} v_r^2 f^j(r,v)\,dv\leq ck_j. \tag{3.1}$$

Let us next prove that for $\lambda\gg 10$,

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}>\lambda \\ s<|v_r|<\sqrt{v_\theta^2+v_z^2}/10}} (1+|v|)^\beta f^j(r,v)\,dv\,dr<2\varepsilon, \tag{3.2}$$

by splitting the integral into two pieces,

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}>\lambda \\ \lambda/10>|v_r|>s}}(1+|v|)^\beta f^j(r,v)\,dv\,dr,$$

and

$$\int_{\sqrt{v_\theta^2+v_z^2}>10|v_r|>\lambda}(1+|v|)^\beta f^j(r,v)\,dv\,dr.$$

For each of these two pieces, construct j-dependent v_*-set $V_*\subset C$ and ω-set $\Gamma\subset S^2$, with measures bounded from below by a positive constant, such that

$$|v-v_*|\geq c|v|,\ |v'_r|\geq c|v|,\ |v'_{*r}|\geq c|v|,\quad v_*\in V_*,\ \omega\in\Gamma.$$

Hence, for any $L>1$,

$$\begin{aligned}(1+|v|)^\beta f^j(r,v) &\leq c(1+|v|)^\beta f^j(r,v)\frac{f^j(r,v_*)}{k_j}\\ &\leq cL(|v'_r|^\beta+|v'_{*r}|^\beta)f^j(r,v')\frac{f^j(r,v'_*)}{k_j}\\ &\quad+\frac{c\chi^s}{k_j\log L}|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}.\end{aligned}$$

And so, using (3.1)

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}>\lambda\\ s<|v_r|<\sqrt{v_\theta^2+v_z^2}/10}}(1+|v|)^\beta f^j(r,v)\,dv\,dr\leq\frac{cL}{\lambda^2}+\frac{c}{\log L}<\varepsilon,$$

and

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}>\lambda\\ s<|v_r|<\lambda/10}}(1+|v|)^\beta f^j(r,v)\,dv\,dr\leq\frac{cL}{\lambda^2}+\frac{c}{\log L}<\varepsilon,$$

by first choosing L large enough, and then λ large enough. Similarly

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}\leq\lambda\\ s<|v_r|<\sqrt{v_\theta^2+v_z^2}/10}}(1+|v|)^\beta f^j(r,v)\,dv\,dr<cLk_j+\frac{c}{\log L}<\varepsilon,$$

by also choosing j large enough. This completes the proof of Lemma 3.2. □

Lemma 3.3. *The family* $(\bar\chi^s f^s)$ *is weakly precompact in* $L^1((r_A,r_B)\times\mathbf{R}^3)$.

Proof. First,

$$\int_{|v_r|\geq s}\bar\chi^s f^s(r,v)r\,dr\,dv\leq 1.$$

It follows from the proof of (3.1) and (3.2), that it is enough for $\delta>0$ to consider $|v|\leq 1/\delta$. There, let us prove the equiintegrability of $(\bar\chi^s f^s)$ by contradiction. Suppose that for some $\varepsilon>0$, there is a sequence $(f^j)_{j=1}^\infty$ from the family (f^s), and a sequence of subsets A_j of $\{(r,v);|v_r|>s_j$ and $|v|<1/\delta\}$ such that $|A_j|<1/j$ and $\int_{A_j}\bar{\chi}^j f^j(r,v)\,dv\,dr>\varepsilon$. Consequently,

$$\int_{\substack{(r,v)\in A_j\\ f^j(r,v)>\varepsilon j/2}} f^j(r,v)\,dv\,dr>\frac{\varepsilon}{2}.$$

For $(r,v)\in A_j$ such that $f^j(r,v)>\frac{1}{2}\varepsilon j$, choose $v_*\in C$ with $10/\delta<|v_*|<11/\delta$. In the v_*-volume thus created, there is a subvolume V_* of measure uniformly bounded from below by a positive constant, such that for v_* in this subvolume, there is a set $\Gamma\subset S^2$ again of positive measure uniformly bounded from below, such that for $(v_*,\omega)\in V_*\times\Gamma$ and using Lemma 2.3, v' and v'_* satisfy

$$|v'_r|>\frac{1}{\delta},\quad |v'_{*r}|>\frac{1}{\delta},\quad f^j(r,v')<c,\quad f^j(r,v'_*)<c,$$
$$f^j(r,v)\le c(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*)),$$

and

$$\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}>cj.$$

Moreover, $|v-v_*|>1/\delta$. And so,

$$\begin{aligned} f^j(r,v)<{}&\frac{c\bar{\chi}^j}{\log j}|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}. \end{aligned}\tag{3.3}$$

Integrate (3.3) over $X:=\{(r,v,v_*,\omega);(r,v)\in A_j,\, v_*\in V_*$ and $\omega\in\Gamma\}$. Hence,

$$\begin{aligned} c\frac{\varepsilon}{2}&<\frac{c}{\log j}\int_X \chi^j|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}\,dr\,dv\,dv_*\,d\omega\\ &\le\frac{c}{\log j}\int_{\Omega\times\mathbb{R}^3\times S^2}\chi^j|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}\,dr\,dv\,dv_*\,d\omega\\ &\le\frac{c}{\log j}. \end{aligned}$$

This leads to a contradiction when $j\to\infty$. $\square$

By Lemma 3.3 there is a sequence $(f^j)_{j=1}^{\infty}$ from the family (f^s) and a function f, such that $\lim_{j\to\infty}\bar{\chi}^j f^j = f$ weakly in $L^1((r_A, r_B)\times\mathbf{R}^3)$. It follows from Lemma 2.2 and the proof of (3.1) and (3.2), that the limit f satisfies the moment condition (2.3). So Theorem 1.1 holds, if $\int_{r_A}^{r_B}\int_{\mathrm{R}^3} Q^{j\pm}(f^j, f^j)\varphi(r,v)\,dv\,dr$ have the limits $\int_{r_A}^{r_B}\int_{\mathrm{R}^3} Q^{\pm}(f,f)\varphi(r,v)\,dv\,dr$. For this we first prove the following four lemmas.

Lemma 3.4.

$$\lim_{\varepsilon\to 0}\sup_{\substack{S\subset(r_A,r_B)\\ |S|<\varepsilon}}\int_{S\times\mathrm{R}^3}\bar{\chi}^j(1+|v|)^{\beta}f^j(r,v)\,dv\,dr = 0,$$

uniformly with respect to j.

Proof. Analogously to the proof of Lemma 3.2, for each $(r,v)\in S\times\mathbf{R}^3$, determine subsets V_* of C and Γ of S^2 of positive measures, such that for each $(v_*,\omega)\in V_*\times\Gamma$,

$$|v-v_*|\geq c(1+|v|),\quad |v'_r|\geq c|v|,\quad |v'_r|\geq c,\quad |v'_{r*}|\geq c|v|,\quad |v'_{r*}|\geq c,$$

and for any $L>1$,

(3.4)

$$\begin{aligned}\bar{\chi}^j(1+|v|)^{\beta}f^j(r,v) &\leq cL(|v'_r|^{\beta}+|v'_{*r}|^{\beta})f^j(r,v')f^j(r,v'_*)\\ &\quad+\frac{c\bar{\chi}^j}{\log L}|v-v_*|^{\beta}b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}.\end{aligned}$$

So by Lemma 2.2

$$\int_{S\times\mathrm{R}^3}\bar{\chi}^j(1+|v|)^{\beta}f^j(r,v)\,dv\,dr\leq cL|S|+\frac{c}{\log L}.$$

The result of Lemma 3.4 follows, by first choosing L large enough, and then $|S|$ small enough. □

Lemma 3.5. *Given $\eta>0$, there is an integer j_0 such that for $j>j_0$ and outside of a j-dependent set in r of measure smaller than η,*

$$\lim_{\lambda\to\infty}\int_{\sqrt{v_\theta^2+v_z^2}>\lambda}\bar{\chi}^j(1+|v|)^{\beta}f^j(r,v)\,dv = 0$$

uniformly with respect to r and j.

Proof. For each (r,v) with $\sqrt{v_\theta^2+v_z^2}>\lambda$, choose v_* in a subset of C of measure uniformly bounded from below by a positive constant, so that for a subset of $\omega\in S^2$ of uniformly positive measure,

$$c|v'_r|>|v'|>\bar{c}\sqrt{v_\theta^2+v_z^2}\,,\qquad f^j(r,v')\le 1,$$
$$c|v'_{*r}|>|v'_*|>\bar{c}\sqrt{v_\theta^2+v_z^2}\,,\quad f^j(r,v'_*)\le 1,$$

and for any $L>1$,

$$\begin{aligned}\bar{\chi}^j(1+|v|)^\beta f^j(r,v) &\le c\bar{\chi}^j(1+|v|)^\beta f^j(r,v)f^j(r,v_*)\\ &\le cL|v'_r|^\beta f^j(r,v')f^j(r,v'_*)\\ &\quad+\frac{c\bar{\chi}^j}{\log L}|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}.\end{aligned}$$

It follows from (2.5) that uniformly in j,

$$\begin{aligned}\int_{\mathbf{R}^6\times S^2}&\chi^j|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\qquad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}\,dv\,dv_*\,d\omega\le c_\eta\end{aligned}$$

outside of a set $S'_j\subset(r_A,r_B)$ of measure η. Hence,

$$\int_{\sqrt{v_\theta^2+v_z^2}>\lambda}(1+|v|)^\beta f^j(r,v)\,dv\le\frac{cL}{\lambda^{2-\beta}}+\frac{c_\eta}{\log L},\qquad x\in S'^c_j.$$

The result of Lemma 3.5 follows, by first choosing L large enough, and then λ large enough. □

Lemma 3.6. *Given $\lambda>0$ and $\varepsilon>0$, there is an integer j_0 such that for $j>j_0$ and outside of a j-dependent set in r of measure smaller than ε,*

$$\lim_{i\to\infty}\int_{\substack{\sqrt{v_\theta^2+v_z^2}\le\lambda\\ |v_r|<1/i}}\bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv=0$$

uniformly with respect to r and j.

Proof. Given $0<\eta^2<\eta$, r and j, either

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}<\lambda\\ |v_r|<1/i}}\bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\le\eta^2,$$

or

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}<\lambda\\ |v_r|<1/i}} \bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv>\eta^2.$$

In the latter case,

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}<\lambda\\ |v_r|<1/i\\ f^j(r,v)\le\eta^2 i/4\pi\lambda^{2+\beta}}} \bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\le\frac{\eta^2}{2},$$

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}<\lambda\\ |v_r|<1/i\\ f^j(r,v)\ge\eta^2 i/4\pi\lambda^{2+\beta}}} \bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\ge\frac{\eta^2}{2}.$$

For each (r,v) such that $f^j(r,v)\ge\eta^2 i/4\pi\lambda^{2+\beta}$, using (2.3) and Lemma 2.3, consider subsets $V_*\subset C$ and $\Gamma\subset S^2$ of measure uniformly bounded from below, such that

$$\begin{gathered}|v-v_*|\ge c|v|,\quad f^j(r,v_*)\ge c,\quad |v'_r|\ge 1,\quad f^j(r,v')\le c',\\ |v'_{*r}|\ge 1,\quad f^j(r,v'_*)\le c',\quad v_*\in V_*,\quad \omega\in\Gamma.\end{gathered}$$

Hence, for such r, v, v_*, and ω, and for i large enough,

$$\begin{aligned}\bar{\chi}^j(1+|v|)^\beta f^j(r,v)&\le c\bar{\chi}^j(1+|v|)^\beta f^j(r,v)f^j(r,v_*)\\ &\le\frac{c\bar{\chi}^j}{\log i}|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\\ &\quad\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}.\end{aligned}$$

It follows from (2.5) that there is a constant c'' such that

$$\int_{\mathbb{R}^6\times S^2}\chi^j|v-v_*|^\beta b(\theta)(f^j(r,v)f^j(r,v_*)-f^j(r,v')f^j(r,v'_*))\times\log\frac{f^j(r,v)f^j(r,v_*)}{f^j(r,v')f^j(r,v'_*)}\,dv\,dv_*\,d\omega$$

is bounded by c'', uniformly with respect to j, outside a j-dependent subset $S'_j\subset(r_A,r_B)$ of measure ε. Hence,

$$\int_{\substack{\sqrt{v_\theta^2+v_z^2}\le\lambda\\ |v_r|<1/i}} \bar{\chi}^j(1+|v|)^\beta f^j(r,v)\,dv\le\frac{c'c}{\log i}+2\eta<3\eta,\quad r\in S_j'^{\,c},$$

for i large enough. □

Lemma 3.7. *The sequence of loss terms*

$$Q^{j-}(f^j,f^j):=f^j\int_{\mathbb{R}^3\times S^2}\chi^{s_j}|v-v_*|^\beta b(\theta)f^j(r,v_*)\,dv_*\,d\omega$$

is weakly compact in

$$L^1\left(\left\{(r,v)\in(r_A,r_B)\times\mathbf{R}^3\,;|v_r|>\delta \text{ and } |v|<\frac{1}{\delta}\right\}\right).$$

Proof. It follows from (2.3) and Lemma 2.2 that

$$\int_{\substack{(r,v)\in(r_A,r_B)\times\mathbb{R}^3\\|v_r|>\delta\\|v|<1/\delta}}|v-v_*|^\beta b(\theta)f^j(r,v)f^j(r,v_*)\,dv\,dv_*\,dr \\ \leq c\int_{r_A}^{r_B}\int_{\mathbb{R}^3}(1+|v_*|)^\beta f^j(r,v_*)\,dv_*\,dr=c.$$

It remains to prove that, for any sequence of sets $(S_j)_{j=1}^\infty$ with

$$S_j\subset\{(r,v)\in(r_A,r_B)\times\mathbf{R}^3\,;|v_r|>\delta \text{ and } |v|<1/\delta\}$$

and $|S_j|<1/j$,

$$\lim_{j\to\infty}\int_{S_j}Q^{j-}(f^j,f^j)(r,v)\,dv\,dr=0. \tag{3.5}$$

First, for any sequence of sets $R_j\subset(r_A,r_B)$ such that $\lim_{j\to\infty}|R_j|=0$, it holds that

$$\lim_{j\to\infty}\int_{R_j}\int_{\substack{|v_r|>\delta\\|v|<1/\delta}}Q^{j-}(f^j,f^j)(r,v)\,dv\,dr=0. \tag{3.6}$$

Indeed, by Lemma 2.2

$$\int_{R_j}\int_{\substack{|v_r|>\delta\\|v|<1/\delta}}Q^{j-}(f^j,f^j)(r,v)\,dv\,dr\leq c\int_{R_j}\left(\left(\int_{\mathbb{R}^3}v_r^2f^j(r,v)\,dv\right)\right. \\ \left.\times\int_{\mathbb{R}^3}\bar\chi^j(1+|v_*|)^\beta f^j(r,v_*)\,dv_*\right)dr \\ \leq ck_0\int_{R_j}\int_{\mathbb{R}^3}\bar\chi^j(1+|v_*|)^\beta f^j(r,v_*)\,dv_*\,dr,$$

which tends to zero when $j\to\infty$ by Lemma 3.4. Then, let

$$X_j := \left\{ r \in (r_A, r_B)\,;\, \text{meas}\{v\,;\,(r,v)\in S_j\} > \frac{1}{\sqrt{j}} \right\}, \quad S'_j := \{(r,v)\in S_j\,;\, r\notin X_j\}.$$

Since $|X_j|<1/\sqrt{j}$,

$$\lim_{j\to\infty} \int_{X_j} \int_{\mathbf{R}^3} Q^{j-}(f^j, f^j)(r,v)\,dv\,dr = 0. \tag{3.7}$$

Given $\Delta>0$,

$$\int_{S'_j} \int_{|v_{*r}|>\Delta} \bar{\chi}^j(v)(1+|v|)^\beta f^j(r,v) f^j(r,v_*)\,dv_*\,dv\,dr \le c_\Delta k_0 \int_{S'_j} \bar{\chi}^j(v)(1+|v|)^\beta f^j(r,v)\,dv\,dr, \tag{3.8}$$

which tends to zero when $j\to\infty$, by Lemma 3.3. This also holds for

$$\int_{S'_j} \int_{|v_{*r}|>\Delta} \bar{\chi}^j(v)(1+|v_*|)^\beta f^j(r,v) f^j(r,v_*)\,dv_*\,dv\,dr \tag{3.9}$$

by a similar argument. By Lemmas 3.5 and 3.6,

$$\int_{|v_{*r}|<\Delta} \bar{\chi}^j(v_*)(1+|v_*|)^\beta f^j(r,v_*)\,dv_*$$

tends to zero when $\Delta\to 0$, uniformly with respect to r outside a j-dependent small set, that is taken care of by Lemmas 2.2 and 3.4. Hence,

$$\lim_{\Delta\to 0} \left(\limsup_{j\to\infty} \int_{S_j} \int_{|v_{*r}|<\Delta} \bar{\chi}^j(v)(1+|v|)^\beta \bar{\chi}^j(v_*)(1+|v_*|)^\beta \times f^j(r,v) f^j(r,v_*)\,dv_*\,dv\,dr \right) = 0. \tag{3.10}$$

But (3.5) follows from (3.6)–(3.10), and so the lemma holds. □

End of the proof of Theorem 1.1. It is a consequence of the weak compactness of $Q^{j-}(f^j,f^j)$ and the inequality (2.5) that $(Q^{j+}(f^j,f^j))$ is also weakly compact in any $L^1(\{(r,v);|v_r|>\delta \text{ and } |v|<1/\delta\})$. This implies a (subsequence) limit when $j\to\infty$ in the weak form of equation (2.1) for any test function φ with compact support and vanishing on $\{(r,v);|v_r|<\delta\}$ for some $\delta>0$. Let us prove that

$$\lim_{j\to\infty} \int_{\mathbf{R}^3} \int_{r_A}^{r_B} \varphi Q^{j-}(f^j,f^j)(r,v) r\,dr\,dv = \int_{\mathbf{R}^3} \int_{r_A}^{r_B} \varphi Q^-(f,f)(r,v) r\,dr\,dv. \tag{3.11}$$

First,

$$\int_{|v_*|>V_*} B\varphi(f^j(r,v)f^j(r,v_*)\chi^j+f(r,v)f(r,v_*))r\,dr\,dv\,dv_*\,d\omega$$

can be made arbitrarily small for V_* large enough, since by Lemma 2.2,

$$\int_{|v_*|>V_*} B\varphi(f^j(r,v)f^j(r,v_*)\chi^j+f(r,v)f(r,v_*))r\,dr\,dv\,dv_*\,d\omega \\ \leq c_\delta \int_{|v_*|>V_*} \bar{\chi}^j(v_*)(1+|v_*|)^\beta(f^j(r,v_*)+f(r,v_*))\,dv_*\,dr,$$

which by the proof of (3.1) and (3.2) tends to zero when $V_*\to\infty$. For V_* fixed, let $\{v_*;|v_*|\leq V_*\}$ be covered by $\bigcup_i B_i^n$, where $B_i^n:=\{v_*;|v_*-w_i^n|\leq 1/n\}$. Using the averaging lemma (see [10]) and a diagonal process, $\int_{\mathbf{R}^3}|v-w_i^n|^\beta\varphi f^j(r,v)\,dv$ converges a.e., hence for each n, and outside of an arbitrarily small set $R\subset(r_A,r_B)$, uniformly with respect to i. Consequently,

$$\sum_i \int_{R^c}\left(\int_{\mathbf{R}^3}|v-w_i^n|^\beta\varphi f^j(r,v)\,dv\right)\left(\int_{B_i^n}\bar{\chi}^j(v_*)f^j(r,v_*)\,dv_*\right)r\,dr \\ \to \sum_i \int_{R^c}\left(\int_{\mathbf{R}^3}|v-w_i^n|^\beta\varphi f(r,v)\,dv\right)\left(\int_{B_i^n} f(r,v_*)\,dv_*\right)r\,dr\,dv, \quad j\to\infty.$$

Using Lemma 2.2,

$$\sum_i \int_{R^c}\int_{B_i^n}\left(\int_{\mathbf{R}^3}(|v-v_*|^\beta-|v-w_i^n|^\beta)\varphi f^j(r,v)\,dv\right)f^j(r,v_*)r\,dr\,dv_*\to 0,$$

when $n\to\infty$. It follows that

$$\int_{R\times\mathbf{R}^3}\varphi Q^-(f^j,f^j)r\,dr\,dv-\int_{R\times\mathbf{R}^3}\varphi Q^-(f,f)r\,dr\,dv\to 0, \quad \text{when } j\to\infty.$$

It remains to prove that

$$\lim_{j\to\infty}\int_{\mathbf{R}^3}\int_{r_A}^{r_B}\varphi Q^+(f^j,f^j)(r,v)r\,dr\,dv=\int_{\mathbf{R}^3}\int_{r_A}^{r_B}\varphi Q^+(f,f)(r,v)r\,dr\,dv. \tag{3.12}$$

For $R>0$, let f^R=weak-$L^1\lim_{j\to\infty}(f^j 1_{f^j<R})$. Split

$$\int_{\mathbf{R}^3}\int_{r_A}^{r_B}\varphi(Q^+(f^j,f^j)-Q^+(f,f))(r,v)r\,dr\,dv$$

into $I_1+I_2+I_3+I_4+I_5$, where

$$
\begin{aligned}
I_1 &:= \int_{\mathbb{R}^3}\int_{r_A}^{r_B} \varphi(Q^+(f^R,f)-Q^+(f,f))r\,dr\,dv,\\
I_2 &:= \int_{f^j(r,v')>R} \chi^j\varphi(r,v)Bf^j(r,v')f^j(r,v'_*)r\,dr\,dv\,dv_*\,d\omega,\\
I_3 &:= \int_{\substack{f^j(r,v')<R\\ |v_*|>V}} \chi^j\varphi(r,v)Bf^j(r,v')f^j(r,v'_*)r\,dr\,dv\,dv_*\,d\omega,\\
I_4 &:= \int_{v^2+v_*^2>V^2+1/\delta^2} B\varphi(r,v)f^R(r,v')f(r,v'_*)r\,dr\,dv\,dv_*\,d\omega,\\
I_5 &:= \int_{\substack{f^j(r,v')<R\\ v^2+v_*^2<V^2+1/\delta^2}} \chi^jB\varphi(r,v)f^j(r,v')f^j(r,v'_*)r\,dr\,dv\,dv_*\,d\omega\\
&\quad -\int_{v^2+v_*^2<V^2+1/\delta^2} B\varphi(r,v)f^R(r,v')f(r,v'_*)r\,dr\,dv\,dv_*\,d\omega.
\end{aligned}
$$

Let $\varepsilon>0$ be fixed. By the monotone convergence theorem,

$$|I_1|\le\varepsilon,\quad R>R_1, \tag{3.13}$$

for some $R_1>0$. Again arguing as in the proof of (3.1), (3.2) and (3.11), we get

$$|I_2|\le\varepsilon,\quad R>R_2, \tag{3.14}$$

for some $R_2>0$. Comparing the gain term with the loss term and the entropy production term, it holds that for $K>1$,

$$
\begin{aligned}
|I_3| &\le K\int_{|v_*|>V}\varphi(r,v)Bf^j(r,v)f^j(r,v_*)r\,dr\,dv\,dv_*\,d\omega+\frac{c}{\log K}\\
&\le c_\delta K\int_{|v_*|>V}\bar\chi^j(v_*)(1+|v_*|)^\beta f^j(r,v_*)r\,dr\,dv_*+\frac{c}{\log K}\le\varepsilon,
\end{aligned} \tag{3.15}
$$

for K, and then V large enough. Then,

$$|I_4|\le\varepsilon \tag{3.16}$$

for V large enough, by the integrability of $(r,v,v_*,\omega)\mapsto B\varphi(r,v)f(r,v')f(r,v'_*)$. Notice that f^R is also the weak* L^∞-limit of $f^j1_{f^j<R}$. Moreover, it follows from the averaging lemma that

$$\int_{v_*^2\le V^2+(1/\delta^2)-v^2}\chi^jB\varphi(r,v')f^j(r,v_*)\,dv_*\,d\omega$$

strongly converges in $L^1((r_A,r_B)\times\{v;|v|<R\})$ to

$$\int_{v_*^2\leq V^2+(1/\delta^2)-v^2} B\varphi(r,v')f(r,v_*)\,dv_*\,d\omega.$$

Hence $\lim_{j\to\infty} I_5=0$. And so,

$$\lim_{j\to\infty}\int_{\mathrm{R}^3}\int_{r_A}^{r_B}\varphi(Q^+(f^j,f^j)-Q^+(f,f))(r,v)r\,dr\,dv=0,$$

by choosing R and V large enough so that (3.13)–(3.16) hold. The limits (3.11), (3.12) are thus proved, which completes the proof of Theorem 1.1. □

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