WELL-POSEDNESS OF A DIFFUSIVE GYROKINETIC MODEL.

MAXIME HAURAY and ANNE NOURI

LATP, Aix-Marseille University, France

Abstract. We study a finite Larmor radius model used to describe the ions distribution function in the core of a tokamak plasma, that consists in a gyro-kinetic transport equation coupled with an electro-neutrality equation. Since the last equation does not provide enough regularity on the electric potential, we introduce a simple linear collision operator adapted to the finite Larmor radius approximation. We next study the two-dimensional dynamics in the direction perpendicular to the magnetic field. Thanks to the smoothing effects of the collision and the gyro-average operators, we prove the global existence of solutions, as well as short time uniqueness and stability.

Résumé. On étudie un modèle à rayon de Larmor fini décrivant la fonction de distribution des ions dans un plasma de coeur de tokamak. Il consiste en une équation de transport gyrocinétique couplée à une équation de quasi-neutralité. L'équation de quasi-neutralité donnant peu de régularité au potentiel électrique, on introduit un opérateur de collisions linéaire adapté. On étudie alors la dynamique du système dans la direction perpendiculaire au champ magnétique. L'effet régularisant des opérateurs de collisions et de gyro-moyenne permet de démontrer l'existence globale de solutions ainsi que leur unicité et stabilité locales en temps.

1 Introduction.

The model studied in this paper describes the density of ions in the core of a tokamak plasma. In such a highly magnetized plasma, the charged particles have a very fast motion of gyration around the magnetic lines, called the Larmor gyration. A good approximation is then to consider that the particles are uniformly distributed on gyro-circles, parametrized by their gyro-centers and Larmor radii r_L, that are proportionnal to the speed of rotation u. In what follows we will forget the physical constant of proportionality and take $r_L = u$. The models obtained in that new variables are kinetic in the direction parallel to the magnetic field lines, and fluid (precisely a superposition of fluid models) in the perpendicular direction. For a rigorous derivation of such models and a more complete discussion on their validity, we refer to [5] and our previous work [6], in which the derivation is performed from a Vlasov equation in the limit of a large magnetic field.

Such gyro-kinetic models are usually closed by an electro-neutrality equation. The derivation of the electroneutrality equation from the Euler-Poisson system has been performed in [3]. Existence of weak entropy solutions to the Euler-Poisson system has been proven in [4]. The electroneutrality equation provides few regularity to the electric field, so that the well-posedness of gyro-kinetic models is unknown, at least to our knowledge. In this article, we add a 'gyro-averaged' collision operator to the model and study the dynamics in the directions perpendicular to the field only.

Let us now describe the model precisely. The ion distribution function $f(t, x, u)$ in gyro-coordinates depends on the time t, the gyro-center position $x \in \mathbb{T}^2$ and the velocity of the fast Larmor rotation $u \in \mathbb{R}^+$. The electric potential Φ depends only on (t, x). They satisfy the following system of equation on $\Omega = \mathbb{T}^2 \times \mathbb{R}^+$,

$$\frac{\partial f}{\partial t} + (J_u^0 \nabla_x \Phi)^\perp \cdot \nabla_x f = \beta u \partial_u f + 2\beta f + \nu \left(\Delta_x f + \frac{1}{u} \partial_u (u \partial_u f) \right), \tag{1.1}$$

$$(\Phi - \Phi *_x H_T)(t, x) = T\left(\rho(t, x) - 1\right), \tag{1.2}$$

$$\rho(t, x) = \int (J_u^0 f(t, x, u) 2\pi u du), \tag{1.3}$$

$$f(0, x, v) = f_i(x, v), \qquad (x, u) \in \Omega, \tag{1.4}$$

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where β and ν are two positive constants, ρ is the density in physical space, T is the ion temperature,

$$J_u^0 h(x_g) = \frac{1}{2\pi}\int_0^{2\pi} h(x_g + ue^{i\varphi_c})\, d\varphi_c\,, \tag{1.5}$$

is the well known zeroth order Bessel operator [12] and

$$H_T(x) = \frac{e^{-\frac{|x|^2}{4T}}}{2\pi^{\frac{3}{2}}\sqrt{T}|x|}\,. \tag{1.6}$$

We also used the notation $b^\perp = (-b_2, b_1)$, for any vector $b = (b_1, b_2)$ of $\mathbb{R}^2$.

This model without the Fokker-Planck operator ($\nu = \beta = 0$) has been studied in a previous work [6] - to which we refer for an heuristic derivation of the electro-neutrality equation (1.2) - and is used by physicists for simulations, for instance in the Gysela code [7]. Here we just mention that (1.2) is obtained in a close to equilibrium setting, with an adiabatic hypothesis on the distribution of the electrons $n_e = n_0 exp(-\frac{e\Phi}{T_e}) \approx n_0\big(1 + \frac{e\Phi}{T_e}\big)$, and an hypothesis of adiabatic response of the ions on the gyro-circles which gives rise to the $\Phi * H_T$ term. As usual with the quasi-neutrality equation, there are no good a priori estimates on the regularity of $E = -\nabla\Phi$.

Remark that although equation (1.1) is derived from a Vlasov model (a rigourous derivation of a more general $3D$ model is performed for fixed field E in section 2), it is of 'fluid' nature. In fact there is no transport in the velocity variable u, and the position of the gyro-center is transported by the eletric drift $(J_u^0 E)^\perp$. Therefore the equation is similar to the $2D$ Navier-Stokes equation written in vorticity. More precisely, we have a family of fluid models depending on a parameter u, which are coupled thanks to diffusion in the u variable, and by the closure used for E described below.

Moreover, we prove in the following that thanks to the gyro-average operator J_u^0, the equation has almost the same regularity as the two-dimensional Navier-Stokes equation in vorticity. In fact, for a fixed $u > 0$, the force field $J_u^0 \nabla_x \Phi$ belongs naturally to H^1 if $f \in L^2$ with some weight, but unfortunately for small values of u the H^1 bound explodes. That is why we obtain results a little poorer as those known for the two-dimensional Navier-Stokes equation (which are global existence, uniqueness and stability) and prove only global existence, short time stability and uniqueness and, in the case $\beta = 0$ the global stability and uniqueness for small initial data.

To state our results properly, we will need the following definitions and notations :

- In the sequel, the letter C will represent a numerical constant, that may change from line to line. Unless it is mentioned, such constants are independent of anything.
- $L^2_u(\Omega) = L^2(\Omega, udxdu)$ is the space of square integrable functions with respect to the measure $udxdu$.
- We shall use various norms on $\mathbb{T}^2$ or on Ω. To avoid confusion, we will use the following convention. All the norms performed on the whole Ω will have their weights with respect to u as additional index. For instance $\|\cdot\|_{2\pi u}$, $\|\cdot\|_{H^1_{2\pi u(1+u^2)}}$. All the norms without any index are norms on $\mathbb{T}^2$ (in x) only, and thus usually depends on u.
- For any weight function $k : \mathbb{R}^+ \mapsto \mathbb{R}^+$, the norm $\|\cdot\|_{2,k}$ is defined for any function f on Ω by

$$\|f\|_{2,k} = \left(\int \|f(\cdot,u)\|_2 k(u)\, du\right)^{\frac{1}{2}}.$$

- The most useful weights will be $m(u) = 2\pi u(1+u^2)$ and $\tilde{m}(u) = 1 + u^2$.
- We change a little the duality used to define distributions in the following definition.

Definition 1.1. *Using distributions with the weight u means that duality is performed as*

$$\langle f, g\rangle_u = \int fg\, dx_g dv_{||}\, udu \quad or \quad \langle f, g\rangle_u = \int fg\, dx_g\, udu\,.$$

in the 3d or 2D cases.

This definition may seem a little artificial because the simple definition of derivative with respect to u is not valid. Instead,

$$\langle \partial_u f, g \rangle_u = -\langle f, \partial_u g \rangle_u - \langle \frac{f}{u}, g \rangle_u \,.$$

However, this weight respects the underlying physics (u is in fact the 1D norm of a 2D velocity variable) and has many advantages. For instance the operator $(1/u)\partial_u(u\partial_u)$ is self-adjoint with this weight.

Our precise results are the following. First we prove global existence under the hypothesis $\|f_i\|_{2,m} < +\infty$.

Theorem 1.1. *Let f_i satisfy $\|f_i\|_{2,m} < +\infty$. Then there exists at least a weak solution $f \in L^\infty(\mathbb{R}^+, L^2_u(\Omega)) \cap L^2(\mathbb{R}^+, H^1_u(\Omega))$ to (1.1)-(1.2) with initial condition f_i, which also satisfies all the a priori estimates of Lemma 3.8, 3.9, 3.10 if the hypotheses on the initial data are satisfied.*

Then we prove short time uniqueness and stability under the additional hypothesis $\|\nabla_x f\|_{2,m} < +\infty$. In the case $\beta = 0$ it also implies global uniqueness and stability for small initial data.

Theorem 1.2. *Let f_i satisfy*

$$\|f_i\|_{2,m} + \|\nabla_x f\|_{2,m} < +\infty \,.$$

Then the positive (or infinite) time $\tau^\star$ defined in Lemma 3.10 is such that the weak solution to (1.1)-(1.4) , defined in Theorem 1.1, is unique on $[0, \tau^\star]$.

Moreover, this solution is stable on $[0, \tau^\star]$ in the following sense. Assume that $(f^n)_{n\in\mathbb{N}}$ is a family of solutions given by theorem 1.1 with initial conditions f_i^n satisfying

$$\lim_{n\to+\infty} \|f_i^n - f_i\|_{2,m} = 0 \,, \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \|f_i^n\|_{L^2_m(L^4)} < +\infty \,.$$

Then

$$\lim_{n\to+\infty} \sup_{t\in[0,\tau^*]} \|f^n(t) - f(t)\|_{2,m} = 0 \,.$$

Remark 1.1. *The above theorems will remain globally true if a sufficiently regular source term s is added in the right-hand side of (1.1). The regularity required is :*

- *Existence result : $\int_0^T \|s\|_{2,m}^2 < +\infty$, for any $T > 0$.*
- *Uniqueness result : $\int_0^T \|\nabla_x s\|_{2,m}^2 < +\infty$, for any $T > 0$.*

The only difference is that it is no longer true that $\tau^* = +\infty$ for small initial conditions in the case $\beta = 0$. All the other conclusions are valid, and only require a simple adaptation of the following proof. That case with a source term is physically important since in tokamak plasmas, particles are injected in the core of the plasma.

This local result has some more consequence when relating it to the a-priori bounds of Lemma 3.8, satisfied by any solution in the sense of 1.1, that imply that $\|\nabla_x f\|_{2,m}$ is almost surely finite for any weak solution. Both theorems for instance imply that any solution is stable on a dense subset in time, and may explode only on a small subset (in some sense).

In the next section the diffusive operator of (1.1) is rigourously derived from a linear Vlasov-Fokker-Planck equation in the limit of a large magnetic field. In the third section, some useful lemmas are established, proving regularizing properties of the gyro-average operator, global preservation of some weighted norm of f, the short time preservation of the m-moment of $\nabla_x f$ by the system (1.1)-(1.2), and controlling the electric potential by the physical density. This allows to prove the global existence (Theorem 1.1) of solutions to the Cauchy problem in the fourth section and their short time uniqueness and stability (Theorem 1.2) in the fifth section. Finally some useful properties of the zeroth order Bessel function J^0 together with a version of the Sobolev embeddings on $\mathbb{T}^2$ are proven in the appendix.

2 Derivation of the gyro-Fokker-Planck operator

In this section, we rigorously justify the form of the Fokker-Planck operator appearing in the right-hand side of (1.1). The usual collision operator for plasmas is the nonlinear Landau operator originally introduced by Landau [9]. Because of its complexity, simplified collision operators have been introduced. An important physical litterature exists on the subject, also in the gyro-kinetic case (See [2] and the references therein). In this paper we choose the simplest possible operator, namely a linear Fokker-Planck operator. The reasons of this choice are:

- Its simplicity, that will allow to focus on the other difficulties of the model.
- The fact that physicists studying gyro-kinetic models for the core of the plasma mainly assume that the dynamics stays close to equilibrium, in which case a linear approximation of the collision operator is relevant.
- The aim of the paper is not a precise description of collisions. In fact, even if they exist in tokamaks, being needed to produce energy, their effect is small compared to the turbulent transport. However, we are interested by their regularizing effect, since the electro-neutrality equation (1.2) does not provide enough regularity to get a well-posed problem. This is a major difference to the Poisson equation setting.

We start from a simple model for a $3D$ plasma, i.e. a linear Vlasov-Fokker-Planck equation with an external electric field, an external uniform magnetic field and linear collision and drift terms, and obtain in the limit of large magnetic field a 3D (in position) equation analog to (1.1). In particular, we show that a usual linear Fokker-Planck term on the speed variables turns into an equation with diffusion terms both in space and Larmor radius variables in the limit.

Precisely, for any small parameter $\epsilon > 0$ we study the distribution $f_\epsilon(t, x, v)$ of ions submitted to an exterior electric field $E(t, x)$ (independent of ϵ) and an uniform magnetic field $B_\epsilon = (1/\epsilon, 0, 0)$. We also model collisions (with similar particles and the others species) by a simple linear Fokker-Planck operator. To avoid any problem with possible boundary collisions, which are really hard to take into account in gyro-kinetic theory, we assume that $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$, where $\mathbb{T}^3$ is the 3D torus. When the scale length of all the parameters are well chosen (in particular the length scale in the direction perpendicular to the magnetic field should be chosen of order ϵ times the length scale in the parallel direction, we refer to our previous work [6] for more details on the scaling), the Vlasov equation that f_ϵ satisfies is

$$\frac{\partial f_\epsilon}{\partial t} + v_{\parallel} \partial_{x_{\parallel}} f_\epsilon + E \cdot \nabla_v f_\epsilon + \frac{1}{\epsilon}(v_\perp \cdot \nabla_{x_\perp} f_\epsilon + v^\perp \cdot \nabla_{v_\perp} f_\epsilon) = \mathrm{div}_v(\beta v f_\epsilon) + \nu \Delta_v f_\epsilon \,, \tag{2.7}$$

where β, ν are two positive parameters, the subscript $\parallel$ (resp. $\perp$) denotes the projection on the direction parallel (resp. on the plane perpendicular) to B, and the superscript $\perp$ denotes the projection on the plane perpendicular to B composed with the rotation of angle $\pi/2$. In others words if $v = (v_1, v_2, v_3)$,

$$v_\perp = (v_1, v_2, 0), \quad v_{||} = (0, 0, v_3), \quad v^\perp = (-v_2, v_1, 0)\,.$$

The next results require the additional notation,

$$\tilde{J}_u^0 g(x_g, u, v_\parallel) = \frac{1}{2\pi} \int_0^{2\pi} g(x_g + ue^{i\varphi_c}, ue^{i(\varphi_c - \frac{\pi}{2})} + v_\parallel e_\parallel)\, d\varphi_c \,, \tag{2.8}$$

with the convention $e^{i\varphi} = (\cos\varphi, \sin\varphi, 0)$. This defines a gyro-average performed in phase space, that will be used as an initial layer to adapt the initial condition to the fast Larmor gyration.

Theorem 2.3. *Let $E \in L_t^\infty(L^2)$ and f_ϵ be a family of solutions to equation (2.7) with initial condition $f_i \in L^2$ satisfying $\sup_{t \le T} \|f_\epsilon(t)\|_2 \le \|f_i\|_2$ for any $T > 0$. Then the family $\bar{f}_\epsilon$ defined by*

$$\bar{f}_\epsilon(t, x_g, v) = f(t, x_g + v^\perp, v)$$

admits a subsequence that converges in the sense of distributions towards a function $\bar{f}$ depending only on $(t, x_g, u = |v|, v_{||})$ and solution to

$$\begin{aligned} \partial_t \bar{f} + v_\parallel \partial_{x_\parallel} \bar{f} + J_u^0 E_\parallel \partial_{v_\parallel} \bar{f} + (J_u^0 E)^\perp \cdot \nabla_{x_g} \bar{f} = \\ \beta(v_{||} \partial_{v_{||}} \bar{f} + u \partial_u \bar{f} + 3\bar{f}) + \nu \left(\Delta_{x_{g\perp}} \bar{f} + \frac{1}{u} \partial_u (u \partial_u \bar{f}) \right) \,, \end{aligned} \tag{2.9}$$

in the sense of distributions with the weight u, with the initial condition $\tilde{J}_u^0(f_i)$.

Remark 2.2. *The reason for the change of variables is that the $1/\epsilon$-term in equation (2.7) induces a very fast rotation in the perpendicular direction both in the x and v variables,*

$$v(t) = v^0 e^{it/\epsilon}\,, \qquad x(t) = x^0 + v^{0\perp} + v^0 e^{i(t/\epsilon - \pi/2)}\,.$$

But in the gyro-coordinates this fast rotation is simply described by a rotation in v,

$$v(t) = v^0 e^{it/\epsilon}\,, \qquad x_g(t) = x_g^0.$$

Remark 2.3. *The final diffusion appears in all dimensions except the $x_{g_{||}}$ one. It does not mean that there is no regularization in that direction. Indeed, the models have diffusion in $v_{||}$, which after some time provides regularity in the $x_{g_{||}}$ direction. This mechanism is well known for the Fokker-Planck equation (see for instance [1]). However, we are not able to prove this phenomenon in the non-linear setting because the electric field of the model lacks regularity. This is the reason why we will only study the 2D model.*

Proof of Theorem 2.3. We proved in a previous work [6] that, provided $f^0 \in L^2$ and $E \in L^1_{loc}(\mathbb{R}, L^2)$, a subsequence of f_ϵ solutions of (2.7) without the collision term converges towards a solution of (2.9) without the collision term. In order to simplify the presentation, we will neglect the electric field and the parallel translation terms. To obtain the result in full generality, the only thing to do is to add the argument given in our previous work to the one given below. For the same reason, we shall also not treat the problem of initial conditions.

So consider the above Vlasov Fokker-Planck equation without electric force field nor parallel translation,

$$\partial_t f + \frac{1}{\epsilon}(v_\perp \cdot \nabla_{x_\perp} f + v^\perp \cdot \nabla_{v_\perp} f) = \mathrm{div}_v(\beta v f) + \nu \Delta_v f\,. \tag{2.10}$$

The first step is to use the change of variables $(x, v) \to (x_g = x + v^\perp, v)$. Since

$$\begin{aligned} \nabla_v f &= \nabla_v \bar f - \nabla^\perp_{x_g} \bar f, \\ \Delta_v f &= \Delta_v \bar f + \Delta_{x_{g\perp}} \bar f - 2\nabla_v \cdot \nabla^\perp_{x_g} \bar f, \\ \nabla_v \cdot (v f) &= v \cdot \nabla_v \bar f + 3\bar f - v \cdot \nabla^\perp_{x_g} \bar f\,, \end{aligned}$$

equation (2.10) becomes

$$\partial_t \bar f_\epsilon + \frac{1}{\epsilon} v^\perp \cdot \nabla_v \bar f_\epsilon = -\beta\Big(v \cdot \nabla_v \bar f_\epsilon + 3\bar f_\epsilon - v \cdot \nabla^\perp_{x_g} \bar f_\epsilon\Big) + \nu\Big(\Delta_v \bar f_\epsilon + \Delta_{x_g} \bar f_\epsilon - 2\nabla_v \cdot \nabla_{x_g} \bar f_\epsilon\Big)\,. \tag{2.11}$$

By hypothesis $\bar f_\epsilon$ is bounded in $L^\infty_{loc}(\mathbb{R}, L^2_{x;v})$. Therefore, at least a subsequence of $(\bar f_\epsilon)$ converges weakly to some $\bar f \in L^\infty_{loc}(\mathbb{R}, L^2)$. Passing to the limit in (2.10), it holds that

$$v^\perp \cdot \nabla_v \bar f = 0\,,$$

since all the other terms are bounded. For $v = (ue^{i\varphi}, v_{||})$ where φ is the gyro-phase, the previous equality means that $\bar f$ is independent of the gyro-phase (in the sense of distribution and thus as an L^2 function).

Equation (2.10) tested against a smooth function g independent of the gyro-phase writes

$$\int \bar f_\epsilon \left(\partial_t g - \beta(v \cdot \nabla_v g - v \cdot \nabla^\perp_{x_g} g) - \nu(\Delta_v g + \Delta_{x_{g\perp}} g - 2\nabla^\perp_{x_g} \cdot \nabla_v g)\right)\, dx_g dv = 0\,. \tag{2.12}$$

We may also pass to the limit when ϵ tends to zero in this equation and obtain that the same equality holds for $\bar f_\epsilon$ replaced by $\bar f$, considered as a function defined on $\mathbb{T}^3 \times \mathbb{R}^3$.

For the change of variable $v = (ue^{i\varphi}, v_{||})$,

$$\nabla_v g = (e^{i\varphi}\partial_u g + ie^{i\varphi}\partial_\varphi g, \partial_{v_{||}}).$$

Hence, for any function g independent of the gyrophase φ, it holds that

$$\begin{aligned} &\Delta_{v_g} g = \partial^2_{v_{||}} g + \frac{1}{u}\partial_u(u\partial_u g)\,, \\ &(\nabla^\perp_{x_g} \cdot \nabla_{v_g}) g = \nabla^\perp_{x_g} \cdot (e^{i\varphi}\partial_u g) = e^{i\varphi} \cdot \nabla^\perp_{x_g} \partial_u g, \\ &v \cdot \nabla_v g = v_{||}\partial_{v_{||}} g + u\partial_u g\,. \end{aligned}$$

The other terms appearing in (2.12) remain unchanged. Then,

$$\int \bar{f}\Big(\partial_t g - \beta(v_{||}\partial_{v_{||}} g + u\partial_u g - ue^{i\varphi}\cdot\nabla^{\perp}_{x_g} g) - \nu(\partial^2_{v_{||}} g + \frac{1}{u}\partial_u(u\partial_u g) \\ + \Delta_{x_{g\perp}} g - 2e^{i\varphi}\cdot\nabla^{\perp}_{x_g}\partial_u g)\Big)\, dx_g dv_{||} 2\pi u du d\varphi = 0\,. \tag{2.13}$$

Since $\bar{f}$ is independent of φ, performing the integration in φ first makes the term containing φ vanish. So the function $\bar{f}$ of the five variables $(x_g, u, v_{||})$ satisfies

$$\int \bar{f}\Big(\partial_t g - \beta(v_{||}\partial_{v_{||}} g + u\partial_u g) - \nu(\partial^2_{v_{||}} g + \frac{1}{u}\partial_u(u\partial_u g) + \Delta_{x_{g\perp}} g)\Big)\, dx_g dv_{||} u du = 0\,. \tag{2.14}$$

It exactly means that $\bar{f}$ satisfies the equation

$$\partial_t \bar{f} = \beta(v_{||}\partial_{v_{||}}\bar{f} + u\partial_u \bar{f} + 3\bar{f}) + \nu\left(\partial^2_{v_{||}}\bar{f} + \Delta_{x_{g\perp}}\bar{f} + \frac{1}{u}\partial_u(u\partial_u\bar{f})\right)\,, \tag{2.15}$$

in the sense of distributions with weight u. It is the equation (2.9) without parallel transport nor electric field. □

If we look at solutions of this equation invariant by translation in the direction of B, we exactly get the 2D-model announced in the introduction. Formally, if $\bar{f}$ is a solution of (2.9), then

$$f(t, x_\perp, u) = \int \bar{f}(t, x_\perp, x_{||}, u, v_{||})\, dv_{||} dx_{||}$$

is a solution of (1.1). Such an assumption on f is reinforced by experiments and numerical simulations, where it is observed that the distribution of ions is quite homogeneous in $x_{\|}$.

3 Some useful lemmas

We prove here some a priori estimates useful for the proof of our theorems. In order to simplify the proof of some of the following Lemmas, we sometimes use the following formulation of (1.1) with the genuine two-dimensional velocity variable. Denote by $\tilde{f}(t,x,\vec{u}) = f(t,x,|\vec{u}|)$, $\vec{u}\in\mathbb{R}^2$. It is solution (in the sense of distribution with usual duality) of the following equation with $2D$ in space and velocity variables,

$$\partial_t\tilde{f} - \nabla^{\perp}_x(J^0_{|\vec{u}|}\Phi)\cdot\nabla_x\tilde{f} = \nu(\Delta_x\tilde{f} + \Delta_{\vec{u}}\tilde{f}) + \beta(2\tilde{f} + \vec{u}\cdot\nabla_{\vec{u}}\tilde{f}). \tag{3.16}$$

Heuristically, a radial in $\vec{u}$ solution of equation (3.16) is a solution of (1.1). For instance we can state a precise Lemma in the case where Φ is fixed and smooth.

Lemma 3.4. *For a fixed smooth potential Φ, f is the unique solution of* (1.1) *with initial condition f_i if and only if $\tilde{f}$ is the unique solution of* (3.16) *with intial condition $\tilde{f}_i$.*

Proof of the Lemma 3.4 : The proof relies on the uniqueness of the solution to (3.16) (See [8]) and the conservation of the radial symmetry of the solution. ■

3.1 Regularizing properties of the gyro-average operator.

In this section, some regularizing property of the gyro-average operator are proven. They are based on the fact that $\hat{J}^0 \sim k^{-\frac{1}{2}}$ for large k (the precise bound are proved in the appendix A), which implies that J^0 maps H^s onto $H^{s+\frac{1}{2}}$. It is important since the formula (1.2) giving the gyro-averaged potential in terms of the distribution f involves two gyro-averages, and thus a gain of one derivative for the gyro-averaged potential w.r.t. f. However, the regularizing properties of J^0_u are bad for small u, which raises difficulties.

The first lemma of this section gives the regularity of the gyro-averaged potential in term of the potential Φ. The second one gives the regularity of the density ρ in terms of the distribution f. We will need the two following definitions before stating them.

Definition 3.2. *Let f be a measurable function defined on Ω. Denote by*

$$\|f\|_{L^2_m(H^s)} = \left(\int \|f(\cdot,u)\|^2_{H^s} m(u)\, dw \right)^{\frac{1}{2}}$$

the norm with the weight $m(u) = 2\pi u(1+u^2)$.

For any $U>0$, let F be a measurable function defined on $\Omega_U = \mathbb{T}^2 \times [0,U]$. Denote by

$$\|F\|_{H^1_U} = \left(\int_{\mathbb{T}^2} \int_0^U \left(|f|^2 + |\nabla_x f|^2 + |\partial_u f|^2 \right) 2\pi u\, du \right)^{\frac{1}{2}}.$$

The lemmas stating the regularity of Φ and ρ are the following.

Lemma 3.5. *For any $s \in \mathbb{R}$, $u>0$ and Φ with 0-mean, it holds that*

$$\begin{aligned} &i) \quad \|J^0_u \Phi\|_{H^s} \le \|\Phi\|_{H^s}, \\ &ii) \quad \|J^0_u \Phi\|_{H^{s+\frac{1}{2}}} \le \frac{1}{\sqrt{u}} \|\Phi\|_{H^s}, \\ &iii) \quad \|\partial_u J^0_u \Phi\|_{H^s} \le \frac{1}{\sqrt{u}} \|\Phi\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

As a consequence, for any $U>0$,

$$iv) \quad \|J^0_u \Phi\|_{H^1_U} \le 2\sqrt{\pi U} \|\Phi\|_{H^{\frac{1}{2}}}.$$

Lemma 3.6. *For any $s>0$, if $\int f\, 2\pi u\, dx du = 1$ and ρ is defined by (1.3), then*

$$\|\rho - 1\|_{H^{s+\frac{1}{2}}} \le 2^{\frac{1}{4}} \pi \|f\|_{L^2_m(H^s)}. \tag{3.17}$$

<u>Proof of Lemmas 3.5</u>.
Denote by $\hat{\Phi}(k)$ the $k-th$ Fourier coefficient of Φ. Then

$$\|J^0_u \Phi\|^2_{H^s} = \sum_{k=1}^{\infty} |J^0_u(k)|^2 |\Phi(k)|^2 \le \sum_{k=1}^{\infty} |\Phi(k)|^2 = \|\Phi\|^2_{H^s},$$

using the bound $\|\hat{J}^0\|_\infty \le 1$ proved in Lemma A.12. For the second inequality, use $ii)$ of Lemma A.12 in

$$\begin{aligned} \|J^0_u \Phi\|^2_{H^{s+\frac{1}{2}}} &= \sum_{k \neq 0} |\hat{J^0_u \Phi}(k,u)|^2 (1+|k|^2)^{s+\frac{1}{2}} \\ &= \sum_{k \neq 0} |\hat{\Phi}(k)|^2 |\hat{J}^0(|k|u)|^2 (1+|k|^2)^{s+\frac{1}{2}} \\ &\le \sum_{k \neq 0} |\hat{\Phi}(k)|^2 (1+|k|^2)^s \sqrt{\frac{1+|k|^2}{2|k|^2 u^2}} \\ &\le \frac{1}{u} \|\Phi\|^2_{H^s}. \end{aligned}$$

For the third estimate of Lemma 3.5, remark that

$$\left(\partial_u \hat{J^0_u \Phi} \right)(k) = \partial_u \left(\hat{J}^0(|k|u) \hat{\Phi}(k) \right) = |k| \hat{\Phi}(k) \hat{J^0}'(|k|u)$$

and use the bound $iii)$ of Lemma A.12 to get

$$|(\partial_u \hat{J^0_u \Phi})(k)| \quad \le \quad \sqrt{\frac{|k|}{u}} |\hat{\Phi}(k)|.$$

From this, we obtain

$$\|\partial_u(J^0_u\Phi)\|_{H^s} \quad \leq \quad \frac{1}{\sqrt{u}}\|\Phi\|_{H^{s+\frac{1}{2}}}.$$

The point *iv)* uses the previous inequalities. First remark that the norm $\|\ \|_{H^1_U}$ is also equal to

$$\|F\|_{H^1_U} = \left(\int_0^U \Big(\|\partial_u F(\cdot,u)\|^2_{L^2} + \|F(\cdot,u)\|^2_{H^1}\Big)2\pi u\,du\right)^{\frac{1}{2}}.$$

Using this formulation and *ii)*- *iii)* leads to

$$\begin{aligned}\|J^0_u\Phi\|^2_{H^1_U} &= \int_0^U \Big(\|\partial_u J^0\Phi\|^2_{L^2} + \|J^0\Phi\|^2_{H^1}\Big)2\pi u\,du \\ &\leq 2\pi\|\Phi\|^2_{H^{\frac{1}{2}}}\int_0^U \frac{1+1}{u}u\,du \leq 4\pi U\|\Phi\|^2_{H^{\frac{1}{2}}},\end{aligned}$$

which gives the desired result and ends the proof of Lemma 3.5. ■

Proof of Lemma 3.6.
Denote by $\hat\rho(k)$ the k-th Fourier term of ρ with respect to the space variable, i.e.

$$\hat\rho(k) \quad = \quad 2\pi\int J^0(|k|u)\hat f(k,u)u\,du.$$

By the Lemma A.12,

$$|\hat\rho(k)| \quad \leq \quad 2\pi\int\frac{|\hat f|(k,u)u}{(1+u^2|k|^2)^{1/4}}\,du.$$

It follows from the inequality below

$$\forall\, k\in\mathbb{Z}^*, \quad \frac{1+|k|^2}{1+u^2|k|^2} = \frac{1}{u^2}\frac{1+\frac{1}{|k|^2}}{1+\frac{1}{u^2|k|^2}} \leq \frac{2}{u^2}, \tag{3.18}$$

that for $k\neq 0$,

$$\begin{aligned}(1+|k|^2)^{\frac{2s+1}{4}}|\hat\rho(k)| &\leq 2^{\frac{5}{4}}\pi\int_0^\infty |\hat f|(k,u)(1+|k|^2)^{\frac{s}{2}}\sqrt{u}\,du \\ &\leq 2^{\frac{5}{4}}\pi\left(\int_0^\infty |\hat f|^2(k,u)(1+|k|^2)^s u(1+u^2)\,du\right)^{1/2}\left(\int_0^\infty\frac{du}{(1+u^2)}\right)^{1/2} \\ &= 2^{\frac{1}{4}}\pi\left(\int_0^\infty |\hat f|^2(k,u)(1+|k|^2)^s 2\pi u(1+u^2)\,du\right)^{1/2}.\end{aligned}$$

Hence, since $\hat\rho(0)=\int_{\mathbb{T}^2}\rho(x)\,dx = 1$ by mass conservation,

$$\begin{aligned}\|\rho-1\|_{H^{s+\frac{1}{2}}} &\leq 2^{\frac{1}{4}}\pi\sqrt{\sum_{k\neq 0}\left(\int_0^\infty |\hat f(k,u)|^2(1+|k|^2)^{\frac{s}{2}}2\pi u(1+u^2)\,du\right)} \\ &\leq 2^{\frac{1}{4}}\pi\left(\int_0^\infty \|f(u)\|^2_{H^s}2\pi u(1+u^2)\,du\right)^{1/2},\end{aligned}$$

and Lemma 3.6 is proved.

■

3.2 Control of the potential by the density.

Denote by L_T the operator that maps any function Φ on $\mathbb{T}^2$ with zero mean to $\frac{1}{T}(\Phi - \Phi *_x H_T)$ and by $H_0^s(\mathbb{T}^d)$ the space of H^s functions with zero mean. This section is devoted to a proof of the boundedness of L_T^{-1} from $H_0^s(\mathbb{T}^d)$ onto $H_0^s(\mathbb{T}^d)$. Recall that in a Fourier setting (See the Appendix of [6] for more details), the operator $H_T = I - TL_T$ is the multiplication by

$$\hat{H}_T(k) = \frac{2}{T}\int_0^{+\infty} J^0(ku)^2 e^{-u^2/T} u\, du.$$

The precise results are stated in the following lemma.

Lemma 3.7. *The Fourier multipliers $\hat{H}_T(k)$ satisfy,*

$$|1-\hat{H}(k)| \geq \frac{|k|^2 T}{4}\left(1 - e^{-\frac{1}{|k|^2 T}}\right), \quad k \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

As a consequence, the operator L_T^{-1} maps any H_0^s, $s \in \mathbb{R}$, into itself with norm

$$\|L_T^{-1}\|_{H_0^s} \leq c_T := \frac{4}{1-e^{-\frac{1}{T}}}. \tag{3.19}$$

Remark 3.4. *Lemma 3.7 shows that $\|L_T^{-1}\|$ is bounded for small T, and of order T for large T, the physical case of interest. The boundedness of the spatial domain is essential. When defined on the whole space $\mathbb{R}^2$ rather than on the torus, the operator L_T^{-1} is not bounded. Its norm explodes in the low frequency range.*

Proof of the Lemma 3.7 Two bounds on $J^0(l)$ are used, namely one of the bounds of Lemma A.12 for $l \geq 1$ and the following bound given by the Taylor expansion of J^0 near 0 for $l \leq 1$,

$$0 \leq (J^0(l))^2 \leq 1 - \frac{l^2}{4}, \quad \text{if } 0 \leq l \leq 1\,.$$

Consequently,

$$\begin{aligned}
|\hat{H}_T(k)| &\leq \frac{2}{T}\int_0^{\frac{1}{|k|}}\left(1-\frac{(|k|u)^2}{4}\right)e^{-u^2/T}u\,du + \frac{\sqrt{2}}{|k|T}\int_{\frac{1}{|k|}}^{\infty} e^{-\frac{u^2}{T}}\,du \\
&\leq 2\int_0^{w}\left(1-\frac{x^2}{4w^2}\right)e^{-x^2}x\,dx + \sqrt{2}w\int_w^{\infty} e^{-x^2}\,dx \\
&\leq 1 - \frac{3}{4}e^{-w^2} - \frac{1}{4w^2}(1-e^{-w^2}) + \sqrt{2}w\int_w^{\infty} e^{-x^2}\,dx,
\end{aligned}$$

where $w = (|k|\sqrt{T})^{-1}$. Now, using the bounds $2^{-\frac{1}{2}} < \frac{3}{4}$ and

$$w\int_w^{\infty} e^{-x^2}\,dx \leq \int_w^{\infty} xe^{-x^2}\,dx = \frac{e^{-w^2}}{2}\,,$$

it holds that

$$1 - |\hat{H}_T(k)| \geq \frac{1}{4w^2}(1-e^{-w^2}).$$

This is the first claim of lemma 3.7 The function of w in the right-hand side of the previous inequality on $|\hat{H}(k)|$ is decreasing and goes from $\frac{1}{4}$ at 0 to 0 at $+\infty$. Consequently its minimal value is obtained for large w i.e. for small $|k|$, namely $|k| = 1$. Precisely ,

$$1 - \sup_{k\neq 0}|\hat{H}_T(k)| \geq \frac{T}{4}\left(1-e^{-\frac{1}{T}}\right).$$

Since the Fourier representation of L_T^{-1} is the multiplication by $T(1-\hat{H}(k))^{-1}$ we obtain that in any H_0^s, $s \in \mathbb{R}$,

$$\|L_T^{-1}\|_{H_0^s} = \sup_{k\neq 0}\frac{T}{|1-\hat{H}(k)|} \leq \frac{4}{1-e^{-\frac{1}{T}}},$$

which is the desired result. ■

3.3 Propagation of L^2_m and $L^2_m(L^4)$ norms of f.

The two following lemmas will be useful in the sequel.

Lemma 3.8. *Assume that $\|f_i\|^2_{2,u} < +\infty$. Then, any solution of (1.1) and (1.4), for regular potential ϕ, satisfies*

$$\|f(t)\|^2_{2,u} + \nu \int_0^t e^{2\beta(t-s)} \|\nabla_{x,u} f(s)\|^2_{2,u}\, ds \le e^{2\beta t}\|f_i\|^2_{2,u}.$$

Assume moreover that $\|f_i\|_{2,m} < +\infty$. Then any solution f satisfies

$$\|f(t)\|^2_{2,m} + \nu \int_0^t \|\nabla_{x,\vec{u}} \tilde{f}(s)\|^2_{2,\tilde{m}}\, ds \le \|f_i\|^2_{2,m} + (2\nu+\beta)\frac{e^{2\beta t}-1}{\beta}\|f_i\|^2_{2,2\pi u}\,, \tag{3.20}$$

with the convention that $\frac{e^{2\beta t}-1}{\beta} = 2t$ if $\beta = 0$.

Lemma 3.9. *Assume $\|f_i\|_{L^2_m(L^4)} < +\infty$ and f is a solution of (1.1) with initial condition f_i with a regular potential ϕ. Then f satisfies*

$$\|f(t)\|_{L^2_m(L^4)} \le e^{(\beta+2\nu)t}\|f_i\|_{L^2_m(L^4)}. \tag{3.21}$$

Remark 3.5. *A more careful analysis will show that*

$$\|f(t)\|^2_{L^2_m(L^4)} \le \|f_i\|^2_{L^2_m(L^4)} + (2\nu+\beta)\frac{e^{2\beta t}-1}{\beta}\|f_i\|^2_{L^2_{2\pi u}(L^4)}\,,$$

but the simple estimate of Lemma 3.9 will be sufficient.

Proof of Lemma 3.8 Multiply equation (3.16) written in $4D$ by $\tilde{f}$. Using the notations

$$u = |\vec{u}|, \quad g(t,u) = \frac{1}{2}\|\tilde{f}(t,\cdot,\vec{u})\|^2_2, \tag{3.22}$$

and integrating in the x variable leads to

$$\partial_t g - \nu\Delta_{\vec{u}} g = -\|\nabla_{x,\vec{u}}\tilde{f}(t,\cdot,u)\|^2_2 + \beta(4g + \vec{u}\cdot\nabla_{\vec{u}} g). \tag{3.23}$$

Multiplying the previous equation by $k(\vec{u})$, where k is a smooth function on $\mathbb{R}^2$ with compact support and integrating it in the velocity variable $\vec{u}$ leads to

$$\partial_t\left(\int g(t,u)k(\vec{u})\,d\vec{u}\right) + \int \|\nabla_{x,\vec{u}}\tilde{f}(t,\cdot,u)\|^2_2 k(\vec{u})\,d\vec{u} = \int (\nu\Delta_{\vec{u}} k(\vec{u}) + 4\beta k(u) - \beta\,\mathrm{div}(k(\vec{u})\vec{u}))g(t,u)\,d\vec{u}.$$

By approximation, this is still true for functions k with unbounded supports. For $k(\vec{u}) = 1$,

$$\partial_t\left(e^{-2\beta t}\int g(t,u)\,d\vec{u}\right) + \nu e^{-2\beta t}\int \|\nabla_{x,\vec{u}}\tilde{f}(s,u)\|^2_2\,d\vec{u} \le 0.$$

Coming back to the 1D original quantities, it means that

$$\|f(t)\|^2_{2,u} + \nu\int_0^t e^{2\beta(t-s)}\|\nabla_{x,u} f(s)\|^2_{2,u}\,ds \le e^{2\beta t}\|f_i\|^2_{2,u}. \tag{3.24}$$

For $k(\vec{u}) = \tilde{m}(u)$, then $\Delta k = 4$ and

$$4\tilde{m}(u) - \mathrm{div}(\tilde{m}(u)\vec{u}) = 2\tilde{m}(u) - \tilde{m}'(u)u = 2\,.$$

Therefore,

$$\int g(t,u)\tilde{m}(u)\,d\vec{u} + \nu\int \|\nabla_{x,\vec{u}}\tilde{f}(t,\cdot,u)\|^2_2\tilde{m}(u)\,d\vec{u} \le \int g(0,u)\tilde{m}(u)\,d\vec{u} + 2(2\nu+\beta)\int_0^t\int g(s,u)\,d\vec{u}\,ds.$$

In other words,

$$\|f(t)\|_{2,m}^2 + \nu \int_0^t \|\nabla_{x,\vec u}\tilde f(s)\|_{2,\tilde m}^2\, ds \le \|f_i\|_{2,m}^2 + 2(2\nu+\beta)\int_0^t \|f(s)\|_{2,u}^2\, ds\,.$$

Using the bound of equation (3.24), we get

$$\|f(t)\|_{2,m}^2 + \nu \int_0^t \|\nabla_{x,\vec u}\tilde f(s)\|_{2,\tilde m}^2\, ds \le \|f_i\|_{2,m}^2 + (2\nu+\beta)\frac{e^{2\beta t}-1}{\beta}\|f_i\|_{2,u}^2\,.$$

∎

Proof of Lemma 3.9 . In order to simplify the presentation, we will first perform the calculations without justifying every integration by parts and division. But once we obtain an a-priori result, we will explain the small adaptations needed to make it rigorous. First, we denote by

$$\alpha(t,\vec u) = \int |\tilde f(t,x,\vec u)|^4 dx = \|f(t,u)\|_4^4\,, \qquad \gamma(t,\vec u) = \int |\tilde f|^2 |\nabla_{\vec u}\tilde f|^2\, dx.$$

Multiplying equation (1.1) by $3\,\mathrm{sign}(f)|f|^3$ and integrating with respect to x leads to

$$\partial_t \alpha = -12\nu \int f^2 |(\nabla_x,\nabla_{\vec u})f|^2\, dx + \nu\Delta_{\vec u}\alpha + 8\beta\alpha + \beta\vec u\cdot\nabla_{\vec u}\alpha\,. \tag{3.25}$$

Hence, dividing by $2\sqrt{\alpha}$,

$$\partial_t\sqrt{\alpha} = \frac{\partial_t\alpha}{2\sqrt{\alpha}} \le -6\nu\frac{\gamma}{\sqrt{\alpha}} + \frac{\nu\Delta_{\vec u}\alpha}{2\sqrt{\alpha}} + 4\beta\sqrt{\alpha} + \beta\frac{\vec u\cdot\nabla_{\vec u}\alpha}{2\sqrt{\alpha}}.$$

Now, we multiply by $\tilde m(u)$ and integrate with respect to $\vec u$, so that

$$\partial_t\left(\int \sqrt{\alpha}\,\tilde m(u) d\vec u\right) \le -6\nu\int \frac{\gamma}{\sqrt{\alpha}}\,\tilde m(u)\, d\vec u + \frac{\nu}{2}\int \frac{\Delta_{\vec u}\alpha}{\sqrt{\alpha}}\tilde m(u) d\vec u + 4\beta\int\sqrt{\alpha}\,\tilde m(u)\, d\vec u + \beta\int \frac{\vec u\cdot\nabla_{\vec u}\alpha}{2\sqrt{\alpha}}\tilde m(u)\, d\vec u.$$

With the help of some integrations by parts, we get that

$$\begin{aligned}
\int \frac{\vec u\cdot\nabla_{\vec u}\alpha}{2\sqrt{\alpha}}\tilde m(u)\, d\vec u &= -2\int\sqrt{\alpha}(\tilde m(u)+u^2)\, d\vec u,\\
\int \frac{\Delta_{\vec u}\alpha}{\sqrt{\alpha}}\tilde m(u) d\vec u &= -2\int \frac{\vec u\cdot\nabla_{\vec u}\alpha}{\sqrt{\alpha}}\, d\vec u + \int \frac{|\nabla_{\vec u}\alpha|^2}{2\alpha^{\frac32}}\tilde m(u)\, d\vec u\\
&= 8\int\sqrt{\alpha}\, d\vec u + \int\frac{|\nabla_{\vec u}\alpha|^2}{2\alpha^{\frac32}}\tilde m(u)\, d\vec u\,.
\end{aligned}$$

Thanks to that, the previous inequality simplifies in

$$\partial_t\left(\int \sqrt{\alpha}\,\tilde m(u) d\vec u\right) \le -6\nu\int\frac{\gamma}{\sqrt{\alpha}}\,\tilde m(u)\, d\vec u + \frac{\nu}{4}\int\frac{|\nabla_{\vec u}\alpha|^2}{\alpha^{\frac32}}\tilde m(u)\, d\vec u + 2(\beta+2\nu)\int\sqrt{\alpha}\, d\vec u.$$

Next we can estimate $|\nabla_{\vec u}\alpha|$ in terms of γ. In fact by Hölder's inequality

$$\begin{aligned}
\nabla_{\vec u}\alpha &= \nabla_{\vec u}\left(\int f^4\, dx\right) = 4\int f^3\nabla_{\vec u} f\, dx,\\
|\nabla_{\vec u}\alpha|^2 &\le 16\left(\int f^4\, dx\right)\left(\int f^2|\nabla_{\vec u} f|^2\, dx\right) = 16\alpha\gamma.
\end{aligned}$$

Therefore the second term in the right hand side of the previous inequality is controlled up to a constant by the first one. We precisely get

$$\partial_t\left(\int \sqrt{\alpha}\,\tilde m(u) d\vec u\right) \le -2\nu\int\frac{\gamma}{\sqrt{\alpha}}\,\tilde m(u)\, d\vec u + 2(\beta+2\nu)\int\sqrt{\alpha}\, d\vec u. \tag{3.26}$$

From this we conclude easily.

In the previous calculation, we have not justified all the integrations by parts. To make the argument rigorous, a possibility is to choose a smooth function ξ_1 from $\mathbb{R}^+$ into $[0,1]$ such that $\xi_1(u)=1$ if $u\in[0,1]$ and $\xi_1(u)=0$ if $u\in[2,+\infty)$, and define for all $U>0$, $\xi_U(u)=\xi\left(\frac{u}{U}\right)$. Remark that $|U\xi_U'|_\infty\leq|\xi'|_\infty$ and $|U^2\xi_U''|_\infty\leq|\xi''|_\infty$. Then, we perform the previous calculation with the weight $\tilde{m}_U=\tilde{m}\xi_U$ and obtain a similar inequality as (3.26),

$$\partial_t\left(\int\sqrt{\alpha}\,\tilde{m}_U(u)d\vec{u}\right)\leq-2\nu\int\frac{\gamma}{\sqrt{\alpha}}\,\tilde{m}_U(u)\,d\vec{u}+\left[2(\beta+2\nu)+\frac{C}{U^2}\right]\int\sqrt{\alpha}\,d\vec{u}. \tag{3.27}$$

Hence $\|f(t)\|_{L^2_{\tilde{m}_U}(L^4)}\leq e^{(\beta+2\nu+\frac{C}{U^2})t}\|f_i\|_{L^2_{\tilde{m}_U}(L^4)}$, which gives the desired result letting U going to infinity.

The other point not rigorously justified is the division by $\sqrt{\alpha}$ that may be zero. However, since we have a diffusion equation, it may be proved that for $t>0$, $\alpha>0$ everywhere. Either we can use a family of smooth approximations of $\sqrt{\cdot}$. Or we can say that $\alpha+\epsilon$ satisfy (3.25) with an additional term that has the right sign, so that it will satisfy (3.27). We will obtain the desired inequality letting first ϵ go to zero then U go to infinity. It is well justified since the maximum principle applies there so that any solution with a non-negative initial condition remains non-negative.

3.4 Short time estimate of the m-moment of $\nabla_x f$.

The following lemma is central in the proof of the stability and uniqueness of the solution for short time.

Lemma 3.10. *Assume that f is a solution of the system (1.1)-(1.4) satisfying $\|\nabla_x f_i\|_{2,m}<+\infty$ initially. Then there exists a constant C^* and a time τ^* depending on $(T,\nu,\|\nabla_x f_i\|_{2,m},\|f_i\|_{2,u})$, such that*

$$\|\nabla_x f_i\|_{2,m}^2+\frac{\nu}{2}\int_0^{\tau^*}\|(\nabla_x,\partial_u)\nabla_x f\|_{2,m}^2\,dt\leq C^*.$$

Moreover, if $\beta=0$ and

$$\|\nabla_x f_i\|_{2,m}\|f_i\|_{2,u}\leq\frac{C\nu^2}{c_T^2},$$

then $\tau^*=+\infty$.

We also mention that the result is true if the definition of Φ in (1.2) is replaced by another definition which still satisfies the bound given in Lemma 3.6 and 3.5. Precise bounds from below of τ^* are given at the end of the proof.

Proof of Lemma 3.10 : We take the x-gradient of equation (3.16), written in $2D$ in $\vec{u}$ (with $u=|\vec{u}|$), and obtain

$$\partial_t\nabla_x\tilde{f}-\nabla_x^\perp(J_u^0\Phi)\nabla^2_{x,x}\tilde{f}=\beta(2\nabla_x\tilde{f}+\vec{u}\cdot\nabla_{\vec{u}}(\tilde{f}))+\nu\Delta_{x,\vec{u}}(\nabla_x\tilde{f})-\nabla_x(\nabla_x^\perp(J_u^0\Phi)\nabla_x\tilde{f}.$$

If we now multiply by ${}^t\nabla_x\tilde{f}$ on the left and integrate in x, the function h defined by $h(t,u)=\frac{1}{2}\int|\nabla_x\tilde{f}|^2\,dx$ satisfies,

$$\partial_t h(u)=\beta(4h(u)+\vec{u}\cdot\nabla_{\vec{u}}h(u))\nu\Delta_{\vec{u}}h(u)-\nu\|\nabla_{x,\vec{u}}\nabla_x\tilde{f}\|_2^2-\int{}^t\nabla_x\tilde{f}\,\nabla_x(\nabla_x^\perp J_u^0\Phi)\,\nabla_x\tilde{f}\,dx. \tag{3.28}$$

We may also multiply this equation by $\tilde{m}(u)=(1+u^2)$ and integrate it in $\vec{u}$. Hence

$$\frac{1}{2}\partial_t\|\nabla_x\tilde{f}\|_{2,\tilde{m}}^2+\nu\|\nabla_{x,\vec{u}}\nabla_x\tilde{f}\|_{2,\tilde{m}}^2=(4\nu+2\beta)\|\nabla_x\tilde{f}\|_{2,u^0}^2-\iint{}^t\nabla_x\tilde{f}\Big(\nabla_x(\nabla_x^\perp J_u^0\Phi)\Big)\nabla_x\tilde{f}\,\tilde{m}(u)\,dx\,d\vec{u}. \tag{3.29}$$

To go on, we need to understand a little better the matrix $M(t,x,u)=\nabla_x(\nabla_x^\perp J_u^0\Phi)$. First remember that $\Phi=L_T^{-1}(\rho-1)$, and then remark that from their definitions, J^0 and L^{-1} commute with derivation in x. Therefore $M=J_u^0L^{-1}(\nabla_x(\nabla_x^\perp\rho))$. Using bounds of Lemmas 3.6 and 3.7 we obtain that

$$\|M(t,u)\|_2=\|J_u^0L^{-1}(\nabla_x(\nabla_x^\perp\rho))\|_2\leq\frac{Cc_T}{\sqrt{u}}\|\nabla\rho\|_{H^{\frac{1}{2}}}\leq\frac{Cc_T}{\sqrt{u}}\|\nabla\tilde{f}\|_{L^2_{\tilde{m}}},\quad u>0.$$

So we may bound the last term of the right hand side of (3.29) by

$$\iint{}^t\nabla_x\tilde{f}\Big(\nabla_x(\nabla_x^\perp J_u^0\Phi)\Big)\nabla_x\tilde{f}\,\tilde{m}(u)\,dx\,d\vec{u}\leq\int\|M(t,u)\|_2\|\nabla_x\tilde{f}\|_4^2\tilde{m}(u)\,d\vec{u}$$

We now use the Sobolev inequality $\|u\|_4^2 \leq C\|u\|_2 \|\nabla u\|_2$ (valid for u with zero mean, as it is not so common on the torus, we add the proof in the Appendix A), and get

$$\begin{aligned}\iint {}^t\nabla_x \tilde f \Big(\nabla_x(\nabla_x^\perp J_u^0 \Phi)\Big)\nabla_x \tilde f\, \tilde m(u)\, dx\, d\vec u \quad &\leq \quad Cc_T \|\nabla_x \tilde f\|_{2,\tilde m} \int \|\nabla_x \tilde f\|_2 \|\nabla^2_{x,x}\tilde f\|_2 \frac{\tilde m(u)}{\sqrt{u}}\, d\vec u \\ &\leq \quad Cc_T \|\nabla_x \tilde f\|_{2,\tilde m} \|\nabla^2_{x,x}\tilde f\|_{2,\tilde m} \|\nabla_x \tilde f\|_{2,\frac{\tilde m}{u}} . \end{aligned} \tag{3.30}$$

In order to get a bound on $\|\nabla_x \tilde f\|_{2,\frac{\tilde m}{u}}$, we choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that $\phi = 1$ on $[0,1]$, $\phi = 0$ on $[2,+\infty)$ and $\|\phi'\|_\infty \leq 2$. Then, we use

$$\|\nabla_x \tilde f\|_{2,\frac{\tilde m}{u}} = \|\nabla_x \tilde f\|_{2,u} + \|\nabla_x \tilde f\|_{2,\frac{\phi}{u}} + \|\nabla_x \tilde f\|_{2,\frac{1-\phi}{u}} .$$

First remark that $\|\nabla_x \tilde f\|^2_{2,u} \leq \|\nabla_x \tilde f\|_{2,u^0}\|\nabla_x \tilde f\|_{2,\tilde m}$, that $\|\nabla_x \tilde f\|_{2,\frac{1-\phi}{u}} \leq \|\nabla_x \tilde f\|_{2,u^0}$ by definition of ϕ and

$$\begin{aligned}\|\nabla_x \tilde f\|^2_{2,\frac{\phi}{u}} \quad &= \quad 2\int h(u)\frac{\phi(u)}{u} d\vec u = 2\int h(u)\phi(u)\,\mathrm{div}_{\vec u}\left(\frac{\vec u}{|u|}\right) d\vec u \\ &= \quad -\int \nabla_{\vec u}(|\nabla_x \tilde f|^2)\cdot \frac{\vec u}{|u|}\phi(u)\, dx d\vec u - 2\int h(u)\phi'(u)\, d\vec u \\ &\leq \quad 2\|\nabla_x \tilde f\|_{2,u^0}(\|\nabla_x \tilde f\|_{2,u^0} + \|\nabla^2_{x,\vec u}\tilde f\|_{2,u^0}) .\end{aligned}$$

With the help of the Poincaré inequality $\|\nabla_x \tilde f\|_2 \leq \|\nabla^2_{x,x}\tilde f\|_2$ we finally get

$$\begin{aligned}\|\nabla_x \tilde f\|^2_{2,\frac{\tilde m}{u}} \quad &\leq \quad C\|\nabla_x f\|_{2,u^0}\Big(\|\nabla_x \tilde f\|_{2,\tilde m} + \|\nabla_{\vec u}\nabla_x \tilde f\|_{2,\tilde m}\Big) \\ &\leq \quad C\|\nabla_x \tilde f\|_{2,u^0}\,\|\nabla_{x,\vec u}\nabla_x \tilde f\|_{2,\tilde m} .\end{aligned} \tag{3.31}$$

Therefore, using that into (3.30) we obtain

$$\frac{1}{2}\partial_t \|\nabla_x \tilde f\|^2_{2,\tilde m} + \nu\|\nabla_{x,\vec u}\nabla_x \tilde f\|^2_{2,\tilde m} \leq (4\nu + 2\beta)\|\nabla_x \tilde f\|^2_{2,u^0} + Cc_T\|\nabla_x \tilde f\|^{\frac12}_{2,u^0}\|\nabla_x \tilde f\|_{2,\tilde m}\|\nabla_{x,\vec u}\nabla_x \tilde f\|^{\frac32}_{2,\tilde m} . \tag{3.32}$$

Now, we need to eliminate the term involving $\|\nabla_{x,\vec u}\nabla_x \tilde f\|_{2,\tilde m}$ in the right-hand side, with the help of the $\|\nabla_{x,\vec u}\nabla_x \tilde f\|_{2,\tilde m}$ of the left-hand side. We will use the Young inequality $xy \leq \frac34 x^{\frac43} + \frac{y^4}{4}$. Taking into account the constants ν, c_T we get

$$\frac12 \partial_t \|\nabla_x \tilde f\|^2_{2,\tilde m} + \frac{\nu}{2}\|\nabla_{x,\vec u}\nabla_x \tilde f\|^2_{2,\tilde m} \leq (4\nu + 2\beta)\|\nabla_x \tilde f\|^2_{2,u^0} + \frac{Cc_T^4}{\nu^3}\|\nabla_x \tilde f\|^2_{2,u^0}\|\nabla_x \tilde f\|^4_{2,\tilde m} .$$

With the notations $h = \|\nabla_x \tilde f\|^2_{2,\tilde m}$, $a(t) = (4\nu + 2\beta)\|\nabla_x \tilde f\|^2_{2,u^0}$ and $\gamma = \frac{Cc_T^4}{\nu^3(4\nu+2\beta)}$ it gives

$$\frac12 \partial_t h \leq a(t)\left(1 + \gamma h^2\right) .$$

This implies that

$$\arctan\left(\gamma^{\frac12} h(t)\right) \leq \gamma^{\frac12}\int_0^t a(s)\, ds + \arctan\left(\gamma^{\frac12} h(0)\right) ,$$

or equivalently

$$h(t) \leq \frac{h(0) + \gamma^{-\frac12}\tan\left(\gamma^{\frac12}\int_0^t a(s)\, ds\right)}{1 - \gamma^{\frac12} h(0)\tan\left(\gamma^{\frac12}\int_0^t a(s)\, ds\right)} .$$

Thus the existence time τ^* satisfies

$$\gamma^{\frac12} h(0)\tan\left(\gamma^{\frac12}\int_0^{\tau^*} a(s)\, ds\right) \geq 1.$$

Using the definition of γ, the a-priori estimates of Lemma 3.8 and the inequality $\tan x \geq x$ we obtain the sufficient condition

$$e^{2\beta\tau^*}\|\nabla_x \tilde f_i\|^2_{2,\tilde m}\|f_i\|^2_{2,u} \geq \frac{C\nu^4}{c_T^4}, \tag{3.33}$$

or again for $\beta > 0$

$$\tau^* \geq -\frac{1}{\beta} \ln \left(\frac{Cc_T^2}{\nu^2} \|\nabla_x f_i\|_{2,m} \|f_i\|_{2,u^0} \right) .$$

In the case $\beta = 0$, (3.33) does not depend on τ. In that case, if

$$\|\nabla_x f_i\|_{2,m} \|f_i\|_{2,u} \leq \frac{C\nu^2}{c_T^2} ,$$

then $\tau^* = \infty$.

The case of physical interest is when ν and β are small and c_T large. In that case, we see that for very small ∇f_i, the existence time is quite long. ■

4 Existence of solutions.

In this section, we prove the existence theorem 1.1. The proof will use the following notation and a preliminary lemma. A priori estimates of the Lemma 3.8 on the solution (f, Φ) to 1.1-1.2 on $[0, T]$ lead to the definition of the set K of functions f such that

$$\|f(t)\|_{2,m} \leq \sqrt{M}, \quad a.a.t \in 0, T,$$

where

$$M = \|f_i\|_{2,m}^2 + (2\nu + \beta) \frac{e^{2\beta t} - 1}{\beta} \|f_i\|_{2,2\pi u}^2.$$

For each $n > 0$, we also introduce an approximation of the potential Φ_n defined for any $f \in L_m^2$ by

$$\Phi_n(t,x) := \sum_{|k| \leq n; k \neq 0} e^{ik\cdot x} \frac{1}{1 - \hat{H}(k)} \Big(\int 2\pi J_w^0 \hat{f}_n(t,k,w) w dw - 1 \Big). \tag{4.34}$$

Lemma 4.11. *For any $n \in \mathbb{N}^*$ and any $T > 0$, there is a unique f_n in $K \cap L^2(0,T; H_u^1(\Omega))$ solution to (1.1) with the potential Φ replaced by $\Phi_n = \Phi_n(f_n)$ and initial condition f_i. This solution satisfies all the a priori estimates of the previous section.*

Proof of Lemma 4.11. Let S be the map defined on K by $S(f) = g$, where g is the solution in $K \cap L^2(0,T; H_u^1(\Omega))$ to (1.1) with the potential $\Phi_n(f)$ and initial condition f_i. The existence and uniqueness of $S(F)$ follows from [10] Thm 4.1 p 257, since $\nabla \Phi_n$ is bounded in $L^\infty(0,T; H^3(\mathbb{T}^2))$ by $c_n M$ for some constant c_n. Then S maps K into K. Moreover, S is a contraction in $L^\infty(0,T; L_u^2(\Omega))$ for T small enough. Indeed, let $g_1 = S(f_1)$ (resp. $g_2 = S(f_2)$). By estimates very similar to the one performed in Lemma 3.6 it holds that

$$\|(\Phi_n(f_1) - \Phi_n(f_2))(t,\cdot)\|_{L^\infty(\mathbb{T}^2)} \leq \bar{c_n} \|(f_1 - f_2)(t,\cdot)\|_{2,m}, \quad t \geq 0,$$

for some constant $\bar{c_n}$. Substracting the equation satisfied by g_2 from the equation satisfied by g_1 and integrating over Ω leads to

$$\begin{aligned}
&\frac{e^{2(2\nu+\beta)t}}{2} \frac{d}{dt} \left(e^{-2(2\nu+\beta)t} \|g_1 - g_2\|_{2,m}^2 \right) \\
&\quad \leq -\nu \|(\nabla_x, \partial_u)(g_1 - g_2)\|_{2,m}^2 - \int g_2 \nabla^\perp (J_u^0(\Phi_n(f_2) - \Phi_n(f_1))) \cdot \nabla (g_1 - g_2)\, m(u)\, dx du \\
&\quad \leq -\nu \|(\nabla_x, \partial_u)(g_1 - g_2)\|_{2,m}^2 + \bar{c_n} \|(f_1 - f_2)\|_{2,m} \|\nabla_x (g_1 - g_2)\|_{2,m} \|g_2\|_{2,m} \\
&\quad \leq \frac{\bar{c_n}^2}{4\nu^2} \|(f_1 - f_2)\|_{2,m}^2 \|g_2\|_{2,m}^2 \leq \frac{\bar{c_n}^2 M}{4\nu^2} \|(f_1 - f_2)\|_{2,m}^2.
\end{aligned}$$

And so,

$$\|g_1 - g_2\|_{L^\infty(0,T; L_m^2)} \leq cTe^{2(\nu+\beta)T} \|f_1 - f_2\|_{L^\infty(0,T; L_m^2)}.$$

Hence there is a unique fixed point of the map S on $[0, T_1]$ for T_1 small enough. The bounds used for defining T_1 being independent of T_1, a unique solution of the problem can be determined globally in time by iteration.

The fact that this unique solution satisfies the a-priori estimates of the next section is clear since these estimates only depend on the bound satisfied by Φ and not on its precise form. ■

Proof of Theorem 1.1
For any $T, U > 0$, the sequence (f_n) is compact in $L^2_{loc,u}([0,T]\times\Omega)$ (Recall that $\Omega_U = \mathbb{T}^2 \times [0,U]$). Indeed, it is bounded in $L^\infty(0,T;L^2_u(\Omega_U)) \cap L^2(0,T;H^1_u(\Omega_U))$. It follows from the interpolation theory that (f_n) is bounded in $L_u^{\frac{10}{3}}([0,T]\times\Omega_U)$. Together with the boundedness of $(\nabla_x\Phi_n(f_n))$ in $L^2_u([0,T]\times\Omega_U)$ (Lemma (3.5), this implies that $(\frac{\partial f_n}{\partial t})$ is bounded in $W_u^{-1,\frac{5}{4}}([0,T]\times\Omega_U)$. By the Aubin lemma [10], it holds that (f_n) is compact in $L^2_u([0,T]\times\Omega_U)$, so converges up to a subsequence to some function f in $L^2_u([0,T]\times\Omega_U)$.
It remains to pass to the limit when $n\to+\infty$ in the weak formulation satisfied by f_n. A weak form of (1.1)-(1.4) is that for every smooth test function α with compact support in $[0,T[\times\Omega$,

$$\int f_i(x,u)\alpha(0,x,u)\,u\,dxdu + \int_0^t\int f_n\Big(\frac{\partial\alpha}{\partial t} + \nabla_x^\perp(J_u^0\Phi_n(f_n))\cdot\nabla_x\alpha\Big)\,u\,dxduds$$
$$= \int_0^t\int\Big(u\nu\nabla_x f_n\cdot\nabla_x\alpha + \partial_u f_n\partial_u\alpha + \beta u^2 f_n\partial_u\alpha\Big)\,dxduds\,. \quad (4.35)$$

The passage to the limit in (4.35) when $n\to+\infty$ can be performed if

$$\lim_{n\to\infty}\int_0^t\int uf_n\nabla_x^\perp(J_u^0(\Phi_n(f_n))\cdot\nabla_x\alpha dxduds = \int_0^t\int uf\nabla_x^\perp(J_u^0(\Phi(f))\cdot\nabla_x\alpha dxduds.$$

This holds since (f_n) (resp. $(\nabla_x(J_u^0(\Phi_n)(f_n)))$ strongly (resp. weakly) converges to f (resp. $\nabla_x(J_u^0(\Phi(f)))$ in $L^2_u([0,T]\times\Omega_U)$ for any $U>0$. And since the f_n satisfy all the a priori bounds, the limit f also satisfies them. ■

5 Short time uniqueness and stability of the solution.

In this section we prove the short time uniqueness and stability theorem 1.2.

Proof of Theorem 1.2. Denote by f_1 (resp. f_2) a solution to (1.1) for the field Φ_1 (resp. Φ_2), by $\delta f = f_1 - f_2$ and by $\delta\Phi = J_u^0(\Phi_1-\Phi_2)$. Multiplying the equation satisfied by $(1+u^2)\delta f$ by δf and integrating w.r.t. (x,u) with the weight u leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta f\|^2_{2,m} &\leq -\nu\|(\nabla_x,\partial_u)\delta f\|^2_{2,m} + (4\nu+2\beta)\|\delta f\|^2_{2,u} + \int\delta f\nabla_x^\perp(\delta\Phi)\cdot\nabla_x f_2\,m(u)\,dxdu\\ &\leq -\nu\|(\nabla_x,\partial_u)\delta f\|^2_{2,m} + (4\nu+2\beta)\|\delta f\|^2_{2,u} + \|\delta f\nabla_x(\delta\Phi)\|_{2,um}\|\nabla_x f_2\|_{2,\frac{m}{u}}\,.\end{aligned}$$

To estimate $\|\delta f\nabla(\delta\Phi)\|_{2,u\,m}$, apply the inequality

$$ab \leq e^a - b + b\ln b\,, \qquad a,b>0,$$

to $(a,b) = \left(\left(\frac{\nabla_x\delta\Phi}{6\|\nabla_x^2\delta\Phi\|_2}\right)^2, \left(\frac{\delta f}{\|\delta f\|_2}\right)^2\right)$ for every nonnegative u and apply the Trüdinger inequality (See [11])

$$\int_{\mathbb{T}^2} e^{\left(\frac{\nabla_x\delta\Phi}{6\|\nabla_x^2\psi\|_2}\right)^2}\,dz \leq 2\,. \quad (5.36)$$

Therefore, using $\|\nabla_x^2\delta\Phi\|_2 \leq \frac{Cc_T}{\sqrt{u}}\|\nabla\delta f\|_{2,m}$ (Lemmas 3.6 and 3.5 and 3.7) and the Jensen inequality,

$$\begin{aligned}\|\delta f\nabla_x(\delta\Phi)\|_2^2 &= C\|\nabla^2\delta\Phi\|_2^2\|\delta f\|_2^2\int\left(\frac{\delta f}{\|\delta f\|_2}\right)^2\left(\frac{|\nabla_x\delta\Phi|}{6\|\nabla^2\delta\Phi\|_2}\right)^2 dx\\ &\leq C\|\nabla^2\delta\Phi\|_2^2\|\delta f\|_2^2\left(1+\int\frac{(\delta f)^2}{\|\delta f\|_2}\ln\left(\frac{(\delta f)^2}{\|\delta f\|_2^2}\right)dx\right)\\ &\leq \frac{Cc_T^2}{u}\|\nabla_x\delta f\|^2_{2,m}\|\delta f\|_2^2\left(1+\ln\left(\frac{\|\delta f\|^4}{\|\delta f\|_2^4}\right)\right).\end{aligned}$$

Integrating in u with the weight um, it holds using again Jensen inequality that

$$\begin{aligned}\|\delta f\nabla(\delta\Phi)\|_{2,um}^2 &\le 2Cc_T^2\|\nabla_x\delta f\|_{2,m}^2\|\delta f\|_{2,m}^2\int\frac{\|\delta f\|_2^2}{\|\delta f\|_{2,m}^2}\left(1+\ln\left(\frac{\|\delta f\|_4^2}{\|\delta f\|_2^2}\right)\right)m(u)\,du\\ &\le Cc_T^2\|\nabla_x\delta f\|_{2,m}^2\|\delta f\|_{2,m}^2\left(1+\ln\left(\frac{\|\delta f\|_{L_m^2(L^4)}^2}{\|\delta f\|_{2,m}^2}\right)\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac12\frac{d}{dt}\|\delta f\|_{2,m}^2 \le Cc_T\|\nabla_x\delta f\|_{2,m}\|\delta f\|_{2,m}&\sqrt{1+\ln\left(\frac{\|\delta f\|_{L_m^2(L^4)}^2}{\|\delta f\|_{2,m}^2}\right)}\|\nabla f_2\|_{2,\frac{m}{u}}\\ &-\nu\|(\nabla_x,\partial_u)\delta f\|_{2,m}^2+(4\nu+2\beta)\|\delta f\|_{2,m}^2\,,\\ \le \frac{Cc_T^2}{4\nu}\|\delta f\|_{2,m}^2\ln&\left(\frac{2e\|\delta f\|_{L_m^2(L^4)}^2}{\|\delta f\|_{2,m}}\right)\|\nabla f_2\|_{2,\frac{m}{u}}^2+(4\nu+2\beta)\|\delta f\|_{2,m}^2\end{aligned}$$

and finally using the inequality $\|f(t)\|_{L_m^2(L^4)}\le e^{(\beta+2\nu)t}\|f_i\|_{L_m^2(L^4)}$ from lemma 3.9,

$$\frac12\frac{d}{dt}\|\delta f\|_{2,m}^2\le\frac{Cc_T^2}{4\nu}\|\delta f\|_{2,m}^2\ln\left(\frac{r^2e^{2(\beta+2\nu)t}}{\|\delta f\|_{2,m}^2}\right)\|\nabla f_2\|_{2,\frac{m}{u}}^2+(4\nu+2\beta)\|\delta f\|_{2,m}^2\,,$$

where

$$r=e^{\frac12}\left(\|f_{1,i}\|_{L_m^2(L^4)}+\|f_{2,i}\|_{L_m^2(L^4)}\right).$$

Defining $s(t)=\frac{1}{r^2}\|\delta f\|_{2,m}^2e^{-2(\beta+2\nu)t}$, we get

$$\dot s(t)\le\frac{Cc_T^2}{4\nu}\|\nabla f_2\|_{2,\frac{m}{u}}^2\,s(t)\ln\frac{1}{s(t)}\,.$$

It follows from the Osgood lemma that

$$s(t)\le s(0)^{e^{-H(t)}}, \tag{5.37}$$

with $H(t)=\frac{Cc_T^2}{4\nu}\int_0^t\|\nabla f_2(s)\|_{2,\frac{m}{u}}^2\,ds$. We will show below that H is well defined on $[0,\tau^*]$, the interval of time defined in Lemma 3.10. Then

$$\|\delta f(t)\|_{2,m}\le e^{(\beta+2\nu)t}\,r^{1-e^{-H(t)}}\,\|\delta f(t)\|_{2,m}^{e^{-H(t)}}\,,$$

which implies the short time uniqueness and stability. Remark that the previous calculation does not use $\nabla_x f_1$ and this is why we do not need an assumption on this quantity in the stability result.

It remains to prove that H is bounded on $[0,\tau^*]$. In fact, using the inequality (3.31) proved during the proof of Lemma 3.10, but rewritten in the 2D+1D setting, we get

$$\|\nabla f_2\|_{2,\frac{m}{u}}^2\le C\|\nabla_x f\|_{2,2\pi u}\,\|\nabla_{x,\vec u}\nabla_x f\|_{2,m}\,.$$

If we integrate that inequality with respect to time, we obtain

$$\begin{aligned}\int_0^{\tau^*}\|\nabla f(t)\|_{2,\frac{m}{u}}^2\,dt &\le 2\int_0^{\tau^*}\|\nabla_{x,\vec u}\nabla_x f(t)\|_{2,2\pi u}\|\nabla_x f(t)\|_{2,m}\\ &\le \sup_{t\le\tau^*}\|\nabla_x f(t)\|_{2,m}\sqrt{\tau^*}\left(\int_0^{\tau^*}\|\nabla_{x,\vec u}^2 f(t)\|_{2,m}\,dt\right)^{\frac12}\\ &\le C^*\sqrt{\frac{\tau^*}{\nu}}\,.\end{aligned}$$

■

A Appendix

A.1 Bounds on the Fourier transform of the Bessel operator of zero order

In the following, we use the same notation $J^0 = J_1^0$ for the zero order Bessel operator and its symbol in Fourier. Indeed, in the Fourier space, J_1^0 that appears in the definition of the gyroaverage of the electric field, is the multiplication by J^0. Some properties of the function J^0 are given in [12]. In this appendix, some bounds on J^0 and its first derivative are proven.

Lemma A.12. *J^0 satisfies the following estimates for all $k \in \mathbb{R}$,*

$$
\begin{aligned}
&i) \qquad |J^0(k)| \leq \min\left(1, \frac{1}{2^{1/4}\sqrt{k}}\right), \\
&ii) \qquad |J^0(k)| \leq (1+k^2)^{-\frac{1}{4}}, \\
&iii) \qquad |(J^0)'(k)| \leq \min\left(1, \sqrt{\frac{2}{\pi k}}\right), \\
&iv) \qquad |(J^0)'(k)| \leq (1+k^2)^{-\frac{1}{4}}.
\end{aligned}
$$

Proof of Lemma A.12

First Inequality : The bound $|J^0(k)| \leq 1$ is clear from the definition of J^0,

$$
J^0(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\cos\theta}\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(k\cos\theta)\, d\theta\,. \tag{A.38}
$$

The bound by $(\sqrt{2}k)^{-\frac{1}{2}}$ is obtained as follows. J^0 is solution of the ordinary differential equation

$$
k^2(J^0)'' + k(J^0)' + k^2 J^0 = 0\,, \quad J^0(0) = 1\,,\ (J^0)'(0) = 0\,. \tag{A.39}
$$

The new unknown $u = \sqrt{k}J^0$ is solution to

$$
u'' + \left(1 + \frac{1}{4k^2}\right) u = 0\,. \tag{A.40}
$$

There are no exact initial conditions for u. However,

$$
u(k) \underset{k\to 0^+}{=} \sqrt{k}[1 + O(k^2)]\,, \quad u'(k) \underset{k\to 0^+}{=} \frac{1}{2\sqrt{k}}[1 + O(k^2)].
$$

The second equation (A.40) admits the k-dependent energy,

$$
H(k) = H(k, u, u') = \frac{u'^2}{2} + \frac{u^2}{2}\left(1 + \frac{1}{4k^2}\right),
$$

that satisfies

$$
H(k) - H(k_0) = -\int_{k_0}^{k} \frac{u^2(l)}{4l^3}\, dl.
$$

It follows from the behaviour of u near 0 that

$$
H(k) \underset{k\to 0^+}{=} \frac{1}{4k} + O(k)\,.
$$

Moreover the series expansion of J^0 near $k = 0$,

$$
J^0(k) = \sum_{j=0}^{\infty} (-1)^j \frac{k^{2j}}{2^{2j}(j!)^2}
$$

and its alternating character if $k \leq 2$ imply that $u^2(k) \geq k - \frac{k^3}{2}$ (valid for $k \leq \sqrt{2}$). Using, the inequality and the behavior of H near 0, we get for $0 < k_0 \leq k \leq \sqrt{2}$ that

$$
\begin{aligned}
H(k) &\leq \frac{1}{4k_0} + O(k_0) - \int_{k_0}^{k}\left(\frac{1}{4l^2} - \frac{1}{8}\right) dl, \\
H(k) &\leq \frac{1}{4k} + \frac{k}{8}\,,
\end{aligned} \tag{A.41}
$$

since the first line is satisfied for any $k_0 > 0$. Therefore,

$$u^2(k) \quad \leq \quad k\frac{k^2+2}{4k^2+1}, \quad k \leq \sqrt{2}\,.$$

A simple calculation shows that the function appearing in the right-hand side is increasing in k, so that

$$u^2(k) \leq \frac{1}{\sqrt{2}}, \quad k \in [0, \sqrt{2}]\,.$$

For $k \geq \sqrt{2}$, simply remark that H is decreasing and that from (A.41)

$$u^2(k) \leq 2H(k) \leq 2H(\sqrt{2}) \leq \frac{1}{\sqrt{2}}.$$

In any case, $u^2(k) \leq 2^{-\frac{1}{2}}$, which gives the desired inequality.

Second inequality : It is a consequence of the first, for $k \geq 1$. For $k \leq 1$, it may be obtained from a comparison of the power series expansions of J^0 and $(1+k)^{-1/4}$ around the origin. We get

$$J^0(k) \leq 1 - \frac{k^2}{4} + \frac{k^4}{64} \leq 1 - \frac{k^2}{4} + \frac{5k^4}{32} - \frac{15k^6}{128} \leq (1+k^2)^{-1/4}.$$

Third inequality : Taking the derivative of J^0 in the definition (A.38),

$$(J^0)'(k) = \frac{i}{2\pi}\int_0^{2\pi} \cos\theta e^{ik\cos\theta}\, d\theta = -\frac{1}{\pi}\int_0^{\pi} \cos\theta \sin(k\cos\theta)\, d\theta\,,$$

from which it is clear that $|(J^0)'(k)| \leq 1$ for all k. Next we transform the previous integral in

$$\begin{aligned}(J^0)'(k) \quad &= \quad -\frac{2}{\pi}\int_0^1 \frac{\alpha\sin(k\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha\,,\\ &= \quad \sum_{i=0}^{j-1}(-1)^j \int_{h_i}^{h_{i+1}} \frac{|\sin(k\alpha)|}{\sqrt{1-\alpha^2}}\alpha\, d\alpha := \sum_{i=0}^{j-1}(-1)^j s_j\,,\end{aligned}$$

where $(h_i)_{1\leq i\leq j}$ are the points where $\sin(k\theta)$ vanishes and 1,

$$h_0 = 0 < h_1 = \frac{\pi}{k} < h_2 = \frac{2\pi}{k} < \ldots < h_{j-1} = \frac{(j-1)\pi}{k} < h_j = 1.$$

The previous sum has alternating signs, the larger terms occuring for large i. Its terms are with increasing absolute values, except for the last one which is incomplete and may be smaller than the next to last term. However,

$$-s_1 \leq s_0 - s_1 \leq \sum_{i=0}^{j}(-1)^j s_j \leq s_0 - s_1 + s_2 \leq s_0,$$

so that

$$\begin{aligned}|(J^0)'(k)| \quad &\leq \quad \max(s_0, s_1) \leq \frac{2}{\pi}\int_{1-\pi/k}^{1} \frac{\alpha d\alpha}{\sqrt{1-\alpha^2}}\\ &\leq \quad \frac{1}{\pi}\sqrt{\frac{2\pi}{k} - \frac{\pi^2}{k^2}} \leq \sqrt{\frac{2}{\pi k}}, \quad k \geq \pi.\end{aligned}$$

This ends the proof of the third inequality.
The proof of *iv)* is similar to the proof of *ii)*, since $\sqrt{\frac{2}{\pi}} < 2^{-\frac{1}{4}}$. ■

A.2 A Sobolev inequality on the torus

On the whole space of dimension two $\mathbb{R}^2$, the Sobolev inequality $\|u\|_4^2 \le \|u\|_2\|\nabla u\|_2$ holds. On a smooth bounded domain Ω, we are more familiar with the inequality $\|u\|_4 \le C\|u\|_{H^1(\Omega)}$. However, the more precise version $\|u\|_4^2 \le \|u\|_2\|u\|_{H^1(\Omega)}$ can still be obtained using extension operator. As we need this precise version in our article, we give a short proof of it on the torus.

Lemma A.13. *Let u be a function on $\mathbb{T}^2$ such that $\|u\|_2 + \|\nabla u\|_2 < +\infty$. Then*

$$\|u\|_4^4 \le \|u\|_2^2 \left(\|u\|_2 + \left\|\frac{\partial u}{\partial x}\right\|_2\right)\left(\|u\|_2 + \left\|\frac{\partial u}{\partial y}\right\|_2\right).$$

and if u is of average zero, we can combine this inequality with a Poincaré inequality to obtain

$$\|u\|_4^2 \le 3\|u\|_2\|\nabla u\|_2$$

Proof of Lemma A.13 We can see u as a periodic function on $\mathbb{R}^2$. For $y \in [0,1]$ and $-1 \le x' \le x < x'+1 \le 1$, we have

$$\begin{aligned} u(x,y)^2 &= u(x',y)^2 + 2\int_{x'}^{x} u(t,y)\frac{\partial u}{\partial x}(t,y)\,dt, \\ u(x,y)^2 &= u(x'+1,y)^2 - 2\int_{x}^{x'+1} u(t,y)\frac{\partial u}{\partial x}(t,y)\,dt, \\ u(x,y)^2 &\le u(x',y)^2 + \int_0^1 |u(t,y)|\left|\frac{\partial u}{\partial x}(t,y)\right|\,dt\,. \end{aligned}$$

The last line is obtained from the mean of the two first. As it is true for all x', we may average the last inequality on all $x' \in [0,1]$ and obtain

$$u(x,y)^2 \le \int_0^1 |u(t,y)|\left(|u(t,y)| + \left|\frac{\partial u}{\partial x}(t,y)\right|\right)\,dt := F(y)\,,$$

where F is defined by the right-hand side and depends only of y. Remark that

$$\int_0^1 F(y)\,dy \le \|u\|_2^2 + \|u\|_2\left\|\frac{\partial u}{\partial x}\right\|_2.$$

Very similarly, we obtain

$$u(x,y)^2 \le \int_0^1 |u(x,t)|\left(|u(x,t)| + \left|\frac{\partial u}{\partial y}(x,t)\right|\right)\,dt := G(x)\,,$$

and

$$\int_0^1 G(x)\,dx \le \|u\|_2^2 + \|u\|_2\left\|\frac{\partial u}{\partial x}\right\|_2.$$

Eventually, we write

$$\int_{\mathbb{T}^2} u^4(x,y)\,dxdy \le \int_{\mathbb{T}^2} G(x)F(y)\,dxdy = \int G(x)\,dx \int F(y)\,dy,$$

and obtain the desired inequality thanks to the bound on the L^1 norm of F and G. The Poincaré inequality may be proved in a very similar way. ■

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