An existence theorem for the multifluid Navier-Stokes problem.

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Abstract. Existence of a weak solution of the Navier-Stokes problem describing a multifluid flow is proved. The velocity fields associated to each fluid solves the Navier-Stokes equations in a time-dependent domain. Classical immiscibility conditions on the varying fluids interfaces are taken into account by a new formulation of the problem. This formulation was introduced in [3] and used in numerical computations. This paper follows a previous one where an existence theorem for Stokes multifluid problems was derived ([5]).

Introduction. Fluid interface computation has recently become a subject of interest because of its many industrial applications. In a previous paper ([5]), we studied the mathematical problem associated with coextrusion, i.e. extrusion of several polymers. In this case, the flows of the polymers could be modelled with Stokes equations. Here, no smallness assumption of the Reynolds numbers is introduced, so that the flow of N fluids filling time-dependent subdomains $\Omega_k(t), k = 1, ..., N$ of a fixed domain $\Omega \subset I\!R^D, D = 2$ or 3, is described in each subdomain by Navier-Stokes equations. Non-miscibility conditions at interfaces are shown to be equivalent to a transport equation on the whole domain for the viscosities and the concentrations. Transmission conditions at interfaces are the continuity of the velocity and of the normal component of the stress tensor. These conditions are obtained by variational considerations. The aim of this paper is to derive an existence result of weak solutions for the non-linear system of equations obtained by coupling this transport equation with the Navier-Stokes equations. A first difficulty arises because of insufficient smoothness of the velocity field for using classical result concerning transport equations. Therefore, using the concept of renormalized solutions introduced by R. J. DiPerna and P. L. Lions ([2]) is necessary. Following the classical proof of existence of the Navier-Stokes equations, the linearized multi-fluid Navier-Stokes system is first solved, and the problem of then passing to the limit in non-linear terms is overcome by properties induced by the renormalized solutions and a compactness result of Aubin's lemma type. Finally the existence proof of solutions for the coupled system is obtained by means of the Schauder fixed point theorem.

1 The multifluid Navier-Stokes problem

We consider N viscous fluids with viscosities

$$\eta_k, \quad 1 \leq k \leq N, \quad \eta_1 < \eta_2 < ... < \eta_N, \tag{1.1}$$

flowing in an open domain Ω of $I\!R^D$. The k^{th} fluid occupies at time t the open subdomain $\Omega_k(t)$. Let η be the globally defined viscosity such that $\eta(t,x) = \eta_k$ if the point $x \in \Omega$ is occupied at time t by the k^{th} fluid. We have

$$\Omega_k(t) := \overline{\{x \in \Omega; \eta(t,x) = \eta_k\}}^{\,\cdot} \tag{1.2}$$

The velocity u is globally defined on Ω by

$$u = u_k(t,x), x \in \Omega_k(t), k = 1, ..., N, \tag{1.3}$$

where $u_k(t,x)$ denotes the velocity of the k^{th} fluid for x belonging to $\Omega_k(t)$. In the following, letters without subscript, such as γ, denote functions defined on the whole domain Ω, and the same letters with subscripts, such as γ_k, denote the restriction of these functions to the subdomains $\Omega_k(t), k = 1, ..., N$.
The strain tensor ϵ is defined by

$$\epsilon(u_k) = \frac{1}{2}(\bigtriangledown u_k + \bigtriangledown u_k^t), x \in \Omega_k(t). \tag{1.4}$$

Denote ρ the globally defined density. The incompressibility of the fluids is expressed by

$$div(u_k) = 0, \quad \rho = \rho_k \in I\!R \;\; in \;\; \Omega_k(t). \tag{1.5}$$

where div denotes the divergence with respect to the space variables. The newtonian behaviour gives the expression of the stress tensor σ_k with respect to the viscosity, the strain tensor and the pressure p_k

$$\sigma_k = 2\eta_k \epsilon(u_k) - p_k Id. \tag{1.6}$$

Denote f the applied exterior force. The conservation of the mass and the fundamental law of mechanics are respectively

$$\rho_t + div(\rho u) = 0, \tag{1.7}$$

$$(\rho u)_t + div(\sigma) = f. \tag{1.8}$$

Denote $h_m(t)$ an interface between fluids k and l and define

$$\begin{aligned} \Pi &= [0,T] \times \Omega, \\ \Pi_k &= \{(t,x) \;\; s.t. \;\; 0 \leq t \leq T \;\; and \;\; x \in \Omega_k(t)\}, \\ H_m &= \{(t,x) \;\; s.t. \;\; 0 \leq t \leq T \;\; and \;\; x \in h_m(t)\}. \end{aligned} \tag{1.9}$$

Denote N the normal to Π_k at the boundary, $U = (1, u(t,x))$ and V and H the spaces

$$V = \{u \in (H^1(\Omega))^D, div(u) = 0, u_{/\partial\Omega} = 0\}. \tag{1.10}$$

$$H = \{u \in (L^2(\Omega))^D, divu = 0\}. \tag{1.11}$$

We first recall a trace property ([5]).

Lemma 1.1 *Assume that the bounded set Π_k is Lipschitz. Then there is a continuous trace mapping Γ_k from $V_1(\Pi_k)$ onto $V_{\frac{1}{2}}(\partial\Pi_k)$, where $V_1(\Pi_k)$ and $V_{\frac{1}{2}(\partial\Pi_k)}$ are defined by*

$$V_1(\Pi_k) = \{u_{/\Pi_k}; u \in L^\infty(0,T;(H^1(\Omega))^D)\}, \tag{1.12}$$

$$V_{\frac{1}{2}}(\partial\Pi_k) = \{v \circ \psi; v \in L^\infty(0,T;(H^{\frac{1}{2}}(\partial B_1))^D))\}, \tag{1.13}$$

with

$$B_1 = \{x \in I\!R^D; \mid x \mid < 1\}, \tag{1.14}$$

$$\psi \in Lip([0,T] \times \overline{B_1}; \overline{\Pi_k}), \psi \;\; isonetooneandonto. \tag{1.15}$$

For the time-dependent case, the non-miscibility condition is classically (see [1])

$$U.N = 0, (t, x) \in H_m. \tag{1.16}$$

We introduce a transport equation for the viscosity on the whole domain which can be proved to be equivalent to (1.14). Indeed we have

Lemma 1.2 *Assume that* $\Pi_k, k = 1, ..., N$, *are Lipschitz domains and* u *belongs to* $L^\infty(0, T; V)$. *Then the following conditions are equivalent*

$$(i) \;\; \Gamma_k(U_k).N = 0 = \Gamma_l(U_l).N on H_m, \tag{1.17}$$

$$(ii) \;\; \frac{\partial \eta}{\partial t} + u. \bigtriangledown \eta = 0 \;\; in \mathcal{D}'(\otimes). \tag{1.18}$$

For a proof of Lemma 1.2 we refer to [5]. Let us point out that equation (1.18) is equivalent to

$$\frac{\partial \eta}{\partial t} + div(\eta u) = 0, \tag{1.19}$$

since u is divergence free.
For the Navier-Stokes multifluid problem to be well posed, it remains to recall the classical transmission conditions. They are expressed by the continuity of the velocity u and of the normal component of the stress tensor at any interface $h_m(t)$,

$$u_k(t, x) = u_l(t, x), x \in h_m(t), \tag{1.20}$$

$$\Sigma_k(t, x).N = \Sigma_l(t, x).N, x \in h_m(t), \tag{1.21}$$

if Σ denotes the tensor $diag(1, \sigma)$. The vector N is the normal to $h_m(t)$ at point x and time t. These conditions correspond to those obtained in multimaterial elasticity. Let us remark that if (1.20) is interpreted for almost every t as a trace equality in $H^{\frac{1}{2}}(h_m(t))$, it is satisfied as soon as u belongs to $L^2(0, T; (H^1(\Omega))^D)$. Finally, (1.21) is obtained the following way: multiply (1.7) by a compactly supported in $[0, T] \times \Omega$ test function and integrate over $[0, T] \times \Omega$. Then multiply (1.7) by compactly supported in $[0, T] \times \overline{\Omega_k(t)}$ test functions, integrate over $[0, T] \times \overline{\Omega_k(t)}$ and substract in the first equality. It comes

$$(\rho_k - \rho_l)U.(\Gamma(U).N) + [\Sigma.N] = 0. \tag{1.22}$$

Then the kinematic condition (1.16) implies (1.21). By this way we have proved that solving the multifluid Navier-Stokes problem (1.4)-(1.8), supplemented with the transport equation (1.18), implies the usual kinematic and transmission conditions. For this problem, we state

Theorem 1.1 *Let* $(\rho_0, \eta_0) \in \{(\rho_k, \eta_k), k = 1, ..., N\}$ *a.e., with* $\rho_k > 0, 0 < \eta_1 < \eta_2 < ... < \eta_N$ *and* $u_0 \in L^2(\Omega)$ *be such that* $div(u_0) = 0$*. Then there is at least a solution* (ρ, η, u) *in* $(L^\infty((0,T) \times \Omega))^2 \times L^2(0,T;V)$ *of the problem*

$$\partial_t \rho + div(\rho u) = 0 \quad in \quad \mathcal{D}'((\imath, \mathcal{T}) \times \otimes), \tag{1.23}$$

$$\partial_t \eta + div(\eta u) = 0 \quad in \quad \mathcal{D}'((\imath, \mathcal{T}) \times \otimes), \tag{1.24}$$

$$\partial_t(\rho u) + div(\rho u \otimes u) - div(\eta \epsilon(u)) = f \quad in \quad L^1(0,T;V'), \tag{1.25}$$

$$\rho(t = 0) = \rho_0, \eta(t = 0) = \eta_0, u(t = 0) = u_0. \tag{1.26}$$

Moreover

$$(\rho, \eta) \in \{(\rho_k, \eta_k), k = 1, ..., N\} \quad a.e., u \in L^\infty(0,T; L^2(\Omega)), \tag{1.27}$$

and there is $p \in L^1(0,T;L^2(\Omega))$ *such that*

$$\partial_t(\rho u) + div(\rho u \otimes u) - div(\eta \epsilon(u)) - \bigtriangledown p = f \quad in \quad \mathcal{D}'((\imath, \mathcal{T}) \times \otimes). \tag{1.28}$$

The remainder of the paper is devoted to the proof of this theorem.

Remark 1.1 *Following the same strategy as in [5], we could replace the homogeneous boundary condition for u by*

$$u_{/\partial\Omega} = u_0, u_0 = U_{0/\partial\Omega}, U_0 \in C^1(\overline{\Omega}), div(U_0) = 0, \tag{1.29}$$

and obtain the same result as in Theorem 4.1.

2 The linearized Navier-Stokes problem.

In this section, we are concerned with the following problem

$$Find \quad u \in L^2(0,T;V) \quad suchthat \tag{2.1}$$

$$\begin{cases} \partial_t(\rho u) + div(\rho v \otimes u) - div(\eta \epsilon(u)) + \bigtriangledown p = f, \\ u(t = 0) = u_0(x). \end{cases} \tag{2.2}$$

ρ, v and η are supposed to satisfy

$$v \in L^2(0,T;V), \tag{2.3}$$

$$(\rho,\eta) \in L^\infty(\Omega), \rho_M \geq \rho \geq \rho_m > 0, \eta_M \geq \eta \geq \eta_m > 0, \tag{2.4}$$

$$\partial_t \rho + div(\rho v) = 0, \tag{2.5}$$

$$\partial_t \eta + div(\eta v) = 0. \tag{2.6}$$

The data f and u_0 are such that

$$f \in L^2(0,T,V'), u_0 \in L^2(H). \tag{2.7}$$

Ω is a Lipschitz bounded set, which is a sufficient condition for

$$\mathcal{V} = \{\phi \in \mathcal{D}(\otimes)\int.\sqcup.\lceil\rangle\sqsubseteq(\phi) = \prime\} \tag{2.8}$$

to be dense in V. Then we have

Proposition 2.1 *Assuming (2.3)-(2.7), the problem (2.1)-(2.2) has at least one weak solution which belongs to $L^\infty(0,T;H)$ and satisfies the energy estimate*

$$\frac{1}{2} < \rho u^2(t,.) > + \int_0^t [< \eta\epsilon(u) : \epsilon(u)(s,.) > - \ll f, u \gg (s,.)]ds \\ \leq \frac{1}{2}\int_\Omega \rho_0(x) u_0^2(x)dx. \tag{2.9}$$

Moreover, if $N = 2$, (2.9) is an equality and the solution is unique.

Recall the definition of a weak solution of (2.1)-(2.2).

Definition 2.1 *A weak solution of (2.1)-(2.2) is $u \in L^2(0,T;V)$, solution of the variational problem*

$$\frac{d}{dt} < \rho(t,.)u(t,.)w > + < div(\rho(t,.)v(t,.) \otimes u(t,.))w > \\ + < \eta(t,.)\epsilon(u(t,.)) : \epsilon(w) > = \ll f, w \gg, \quad foreverywinV. \tag{2.10}$$

Here, $<>$ denotes the integral over Ω with respect to x, and $\ll\gg$ the duality between V' and V. Let us remark that, thanks to (2.5), (2.10) is equivalent to

$$< \rho(t,.)\frac{d}{dt}u(t,.)w > + < (\rho(t,.)v(t,.).D_x u(t,.)).w > \\ + < \eta(t,.)\epsilon(u(t,.))\epsilon(w) > = \ll f(t,.), w \gg . \tag{2.11}$$

The proof of Proposition 2.1 is based on the construction of an approximate solution by the Galerkin method. Since V is separable, there is a free and total family in V, $\{w_n\}_{n=1,...,\infty}$. Denote $V_n = \mathrm{Spann}\{w_k; k = 1, ..., n\}$. Then we have

Proposition 2.2 *The problem*
Find $u_n \in H^1(0,T;V_n)$ *such that*

$$\begin{aligned} u_n(0) = u_n^0 \in V_n, \\ < \rho(t,.)\frac{du_n}{dt}w_k > + < (\rho(t,.)v(t,.).D_x u_n(t,.)).w_k > \\ + < \eta(t,.)\epsilon(u_n(t,.))\epsilon(w_k) > = \ll f(t,.), w_k \gg, k = 1, ..., n, \end{aligned} \tag{2.12}$$

has a unique solution, which satisfies

$$\begin{aligned} \frac{1}{2} < \rho(t,.)u_n^2(t,.) > + \int_0^t < \eta\epsilon(u_n) : \epsilon(u_n) > (s,.)ds \\ - \int_0^t \ll f, u_n \gg (s)ds = \frac{1}{2} < \rho(t=0,.)u_n^0(.)^2 > . \end{aligned} \tag{2.13}$$

We first state

Lemma 2.1 *The following estimate holds.*
For any $\rho \in L^\infty((0,T)\times\Omega), v(t,.) \in L^2((0,T);V), w_1 \in V, w_2 \in V,$

$$< \rho(t,.)v(t,.).D_x w_1.w_2 > \in L^2(0,T). \tag{2.14}$$

Proof.

$$\begin{aligned} \| < \rho(t,.)v(t,.).D_x w_1.w_2 > \|_{L^2} \\ \leq \| \rho \|_{L^\infty} \| v \|_{L^2(0,T;L^4(\Omega))} \| D_x w_1 \|_{L^2} \| w_2 \|_{L^4} \\ \leq c \| \rho \|_{L^\infty} \| v \|_{L^2(0,T;L^4(\Omega))} \| w_1 \|_V \| w_2 \|_V, \end{aligned} \tag{2.15}$$

from Sobolev's imbedding.
Proof of Propositions 2.1 and 2.2.
We follow the strategy of proof of [6] for the existence of solutions of Navier-Stokes equations with constant viscosities and concentrations. Let us consider the problem (2.12)-(2.13). In view of (2.4), ρ is positive, so that the matrix $(< \rho w_l.w_k >)_{l,k=1,...,n}$ is a definite positive symmetric matrix, hence invertible. Then, in view of Lemma 2.1, we can write (2.12) as a finite system of ODE's with $L^2(0,T)$-coefficients. This linear differential system

together with the initial conditions given by (2.12) defines uniquely u_n on the whole interval $[0,T]$. Moreover, $u_n(t,.)$ belonging to V_n, (2.13) implies

$$\frac{1}{2}\{< \rho(t,.)\frac{d}{dt} \mid u_n(t,.) \mid^2> + < \rho(t,.)v(t,.).D_x(\mid u_n(t,.) \mid^2) >\}$$
$$+ < \eta(t,.)\epsilon(u_n(t,.)) : \epsilon(u_n(t,.)) >=\ll f(t,.), u_n(t,.) \gg . \quad (2.16)$$

Integrating by parts $< \rho(t,.)v(t,.).D_x(\mid u_n(t,.) \mid^2) >$ and using (2.5) in (2.16), we obtain

$$\frac{1}{2} < \rho(t,.) \mid u_n(t,.) \mid^2> + \int_0^t < \eta(s,.)\epsilon(u_n(s,.)) : \epsilon(u_n(s,.)) > ds$$
$$= \int_0^t \ll f(t,.), u_n(t,.) \gg + \frac{1}{2} < \rho(0,.) \mid u_n^0(.) \mid^2>, t \in [0,T].(2.17)$$

Then, using Korn and Cauchy-Schwartz inequalities in (2.17), as well as the bounds from below of ρ and η given in (2.4), we obtain

$$\| u_n \|^2_{L^2(\Omega)} (t) + \| u_n \|^2_{L^2(0,T;V)} \leq c(\| f \|_{L^2(0,T;V')} + \| u_0 \|_{L^2(\Omega)}),(2.18)$$

where c is independent of n. Therefore, up to subsequences,

$$u_n \rightharpoonup u in L^2(0,T;V) weak, and u \in L^\infty(0,T;H), \quad (2.19)$$

$$\epsilon(u_n) \rightharpoonup \epsilon(u) in L^2((0,T) \times \Omega) weak. \quad (2.20)$$

The inequality (2.9) of Proposition 2.1 follows from (2.16), (2.18), (2.19), the convexity of $u \rightarrow < \eta\epsilon(u) : \epsilon(u) >$ and the convergence of (u_n^0) to u_0 in L^2. Then, in order to prove that u is a weak solution of the problem (2.1)-(2.2), we pass to the limit in (2.13) and obtain

$$< \rho\dot{u}w_k > + < (\rho v.D_x u).w_k > + < \eta\epsilon(u) : \epsilon(w_k) >$$
$$=\ll f, w_k \gg, \quad k \geq 1. \quad (2.21)$$

But $\rho v.D_x u$ is bounded in $L^1(0,T;V')$. Indeed for every smooth function $\phi = \phi(x)$,

$$\mid \int_\Omega (\rho v.D_x u).\phi dx \mid$$
$$\leq \rho_M \| v \|_{L^2(0,T;L^4(\Omega))} \| D_x u \|_{L^2((0,T)\times\Omega))} \| \phi \|_{L^2(0,T;L^4(\Omega))}, \quad (2.22)$$

and thanks to Sobolev's imbeddings,

$$\mid \int_\Omega (\rho v.D_x u).\phi dx \mid \leq c \| v \|_{L^2(0,T;V)} \| D_x u \|_{L^2(0,T;V)} \| \phi \|_{L^2(0,T;V)} .(2.23)$$

Then, since $\{w_k\}_{k\geq 1}$ is a total family,

$$< \rho\dot{u}.w > + < (\rho v.D_x u).w > + < \eta\epsilon(u) : \epsilon(w) > = \ll f, w \gg, \qquad w \in V, \quad (2.24)$$

is a consequence of (2.20). This proves that u is a weak solution of (2.1)-(2.2).

Let us now prove that (2.9) is an equality for $N = 2$, because of more regularity of $\rho v.D_x u$. The uniqueness of the weak solution will be a direct consequence of this equality. Let u be a weak solution of (2.1)-(2.2). First, prove that $\int_0^t < (\rho v.D_x u).u > ds$ is well defined. For any smooth function,

$$| \int_0^t < (\rho v.D_x u).u > ds | \leq \| \rho \|_{L^\infty((0,T)\times\Omega))} \| v \|_{L^2(0,T;L^4(\Omega))} \| u \|_{L^2(0,T;L^4(\Omega))} \| u \|_{L^2(0,T;V)} \quad (2.25)$$

But for $N = 2$, ([6]),

$$\| u(t,.) \|_{L^4(\Omega)} \leq c \| u(t,.) \|^{\frac{1}{2}}_{L^2(\Omega)} \| u(t,.) \|^{\frac{1}{2}}_V . \quad (2.26)$$

Therefore

$$\| u \|^2_{L^2(0,T;L^4(\Omega))} \leq c \| u \|_{L^\infty(0,T;L^2(\Omega))} \| u \|_{L^2(0,T;V)}, \quad (2.27)$$

and similarly for v. By a density argument, we obtain from (2.23),

$$\frac{1}{2} < \rho u^2 > (t) + \int_0^t < (\rho v.D_x u).u > (s)ds + \int_0^t [< \eta\epsilon(u) : \epsilon(u) > - \ll f, u \gg](s)ds = \frac{1}{2} < \rho u^2 > (0). \quad (2.28)$$

For smooth functions ϕ ([6]),

$$< (\rho v.D_x \phi)\phi > = 0, \quad (2.29)$$

so once again a density argument implies, u belonging to $L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega))$, that

$$< (\rho v.D_x u).u > = 0, \quad (2.30)$$

which leads to the result.

3 The fixed point procedure.

First we recall a result of DiPerna-Lions ([2]) concerning renormalized weak solutions. v and ρ_0 belonging to $L^2(0,T;V)$ and $L^\infty(\Omega)$ respectively, we consider the problem :
Find ρ in $L^\infty(0,T;\Omega)$ such that

$$\partial_t \rho + div(\rho v) = 0, \quad \rho(t=0,x) = \rho_0(x). \tag{3.1}$$

We have

Proposition 3.3 *If $v \in L^2(0,T;V)$, then there is a unique weak solution ρ in $L^\infty((0,T)\times\Omega)$, in the following sense*

$$\int_0^{+\infty}\int_\Omega \rho(\partial_t\phi + v.\nabla_x\phi)dxdt = \int_\Omega \rho_0(x)\phi(0,x)dx, \phi \in \mathcal{D}(I\!R^{\mathcal{N}+\infty}). \tag{3.2}$$

Moreover this solution is a renormalized solution, i.e. $\beta(\rho)$ is a weak solution associated to the data $\beta(\rho_0)$ for any $\beta \in C^1(I\!R)$. Furthermore if the data satisfies $\rho_0 \in \{\rho_m, ..., \rho_M\}$ a.e., then $\rho \in \{\rho_m, ..., \rho_M\}$ a.e.

For a proof of Proposition 3.3, we refer to [2] and [5]. Let us only point out that the problem (3.1) has not to be completed by boundary conditions because $v_{/\partial\Omega} = 0$.
Describe the fixed point procedure. For every v_n in $L^2(0,T;V_n)$, we solve

$$\partial_t \rho + div(\rho v_n) = 0, \rho(t=0) = \rho_0, \tag{3.3}$$

$$\partial_t \eta + div(\eta v_n) = 0, \eta(t=0) = \eta_0, \tag{3.4}$$

where $\rho_0 \in \{\rho_m, ..., \rho_M\}$ a.e. and $\eta_0 \in \{\eta_m, ..., \eta_M\}$ a.e. Then we solve (2.12)-(2.13), with $v = v_n$ and obtain a solution, denoted u_n. Let us define the map τ by $\tau v_n = u_n$.

Proposition 3.4 *The map τ has at least a fixed point.*

Proof of Proposition 3.4.
Thanks to Korn and Poincaré's inequalities,

$$< \eta\epsilon(w) : \epsilon(w) > \ \geq \frac{1}{4}\eta_m c(\Omega) \parallel w \parallel_V^2, w \in V, \tag{3.5}$$

where $c(\Omega)$ is a Poincaré's constant. Then, with the help of (2.14),

$$\parallel u_n \parallel_{L^2(0,T;V)} \leq c(\parallel f \parallel_{L^2(0,T;V')} + \parallel u_n^0 \parallel_{L^2(\Omega)}), \tag{3.6}$$

where c is a constant depending on ρ_m, ρ_M, η_m and Ω. Therefore, if

$$R = c(\| f \|_{L^2(0,T;V')} + \| u_n^0 \|_{L^2(\Omega)}), \tag{3.7}$$

τ maps the ball

$$B_R^n = \{u \in L^2(0,T;V_n) s.t. \| u \|_{L^2(0,T;V)} \leq R\} \tag{3.8}$$

in itself. To use Schauder fixed point theorem, it remains to prove that τ is a continuous and compact map for the topology induced by $L^2(0,T;V)$.
Let us prove the continuity of τ. Let $(v_n^p(t,.))_{p\in I\!N}$ be a sequence of $L^2(0,T;V_n)$ converging to some $v_n(t,.)$. Up to a subsequence, the associated solutions of (3.3)-(3.4) $(\rho^p)_{p\in I\!N}$ and $(\eta^p)_{p\in I\!N}$ respectively converge in $L^\infty((0,T)\times\Omega)$ weak star to ρ and η. Passing to the limit in the formulation (3.2) proves that ρ and η are the unique solutions of (3.3)-(3.4) associated to v_n. Therefore the whole sequence (ρ_p, η_p) weakly converges to (ρ, η). On the other hand, $(\rho^p), (\eta^p), \rho$ and η are renormalized solutions of transport equations. It follows that the choice of $\beta(t) = t^2$ gives

$$\| \rho^p \|_{L^2((0,T)\times\Omega)} = T \| \rho_0 \|_{L^2(\Omega)} = \| \rho \|_{L^2((0,T)\times\Omega)}, \tag{3.9}$$

$$\| \eta^p \|_{L^2((0,T)\times\Omega)} = T \| \eta_0 \|_{L^2(\Omega)} = \| \eta \|_{L^2((0,T)\times\Omega)} . \tag{3.10}$$

It means that the weak convergence of (ρ^p) and η^p in $L^2((0,T)\times\Omega)$ to ρ and η respectively is indeed a strong convergence. Since $(u_n^p)_{p\in I\!N}$ belongs to B_R^n, there is a subsequence, still denoted $(u_n^p)_{p\in I\!N}$, such that

$$u_n^p \rightharpoonup u_n \quad in \;\; L^2(0,T;V) \;\; weak. \tag{3.11}$$

More, $\rho^p v_n^p$ converges to ρv in $L^2((0,T)\times\Omega)$. Indeed

$$\begin{aligned} \| \rho^p v_n^p - \rho v \|_{L^2((0,T)\times\Omega)} & \\ & \leq \| \rho^p (v_n^p - v) \|_{L^2((0,T)\times\Omega)} + \| (\rho^p - \rho) v \|_{L^2((0,T)\times\Omega)} \\ & \leq \rho_M \| v_n^p - v \|_{L^2((0,T)\times\Omega)} \\ & \quad + \| \rho^p - \rho \|_{L^2(0,T;L^4(\Omega))} \| v \|_{L^2(0,T;L^4(\Omega))}, \end{aligned} \tag{3.12}$$

and $v \in L^2(0,T;L^4(\Omega))$ since $H^1(\Omega) \subset L^4(\Omega), (N \leq 3)$ and ρ^p converges to ρ in $L^2(0,T;L^4(\Omega))$, because it is bounded in $L^\infty((0,T)\times\Omega)$ and converges in $L^2((0,T)\times\Omega)$. Then, expressing

$$u_n^p = \sum_{k=1}^{n} \alpha_k^{n,p}(t) w_k(x) \tag{3.13}$$

we have, up to a subsequence,

$$\alpha_k^{n,p} \rightharpoonup \alpha_k in \ \ L^2(0,T) weak \tag{3.14}$$

and

$$< (\rho^p v_n^p . D_x u_n^p).w_k >= \sum_{l=1}^{n} \alpha_l^{n,p}(t) < (\rho^p v_n^p . D_x w_l).w_k > \tag{3.15}$$

tends to

$$\sum_{l=1}^{n} \alpha_l(t) < (\rho v_n . D_x w_l).w_k > in \mathcal{D}'(\prime, \mathcal{T}), \tag{3.16}$$

because of the strong convergence of $< (\rho^p v_n^p . D_x w_l).w_k >$ in $L^2(0,T)$. Analogous arguments prove that

$$< \eta^p \epsilon(u_n^p) : \epsilon(w_k) > \to < \eta \epsilon(u_n) : \epsilon(w_k) > in \mathcal{D}'(\prime, \mathcal{T}). \tag{3.17}$$

Let us prove that $((\alpha_k^{n,p}))_{p \in I\!N}$ is uniformly bounded in $(H^1(0,T))^n$. Denote M_n^p the matrix

$$(M_n^p)^{-1} = (< \rho^p(t,.) w_k w_l >)_{(k,l) \in [1,n]^2}. \tag{3.18}$$

$((M_n^p))_{p \in I\!N}$ is uniformly bounded in $\mathcal{M}_\backslash(\L^\infty(\prime, \mathcal{T}))$. Indeed

$$\rho^p(t,.) \geq \rho_m > 0, \tag{3.19}$$

and

$$\| (M_n^p)^{-1} \|_{\mathcal{M}_\backslash(\mathcal{L}^\infty(\prime,\mathcal{T}))} \leq \frac{1}{\rho_m}. \tag{3.20}$$

On the other hand, the matrix

$$A_n^p = (< (\rho^p v_n^p D_x w_l) w_k + \eta^p \epsilon(w_l) : \epsilon(w_k) >)_{(k,l) \in [1,n]^2}, \tag{3.21}$$

and the vector

$$B_n^p = (\ll f, w_k \gg)_{k \in [1,n]} \tag{3.22}$$

are uniformly bounded in $L^2(0,T)$. If $\alpha^p = (\alpha_k^p)_{k \in [1,n]}$, then (2.12) reduces to

$$\dot{\alpha^p} = -(M_n^p)^{-1} A_n^p \alpha^p + (M_n^p)^{-1} B_n^p. \tag{3.23}$$

Knowing that $(\alpha^p)_{p\in\mathbb{N}}$ is uniformly bounded in $(L^2(0,T))^n$, (3.23) and the uniform boundedness of $((M_n^p)^{-1}), (A_n^p)$ and (B_n^p) prove that $(\alpha^p)_{p\in\mathbb{N}}$ is uniformly bounded in $(W^{1,1}(0,T))^n$. Sobolev imbedding of $W^{1,1}(0,T)$ in $C^0(0,T)$ implies that $(\alpha^p)_{p\in\mathbb{N}}$ is uniformly bounded in $(C^0(0,T))^n$. Coming back to (3.23), we finally obtain that $(\alpha^p)_{p\in\mathbb{N}}$ is uniformly bounded in $(H^1(0,T))^n$. Then

$$< \rho^p(t,.)\dot{u}_n^p w_k > \rightarrow < \rho(t,.)\dot{u_n} w_k > in \mathcal{D}'(\prime,\mathcal{T}). \tag{3.24}$$

It follows from (3.16), (3.17) and (3.24) that u_n is the unique solution of (2.12), so that the whole sequence $(u_n^p)_{p\in\mathbb{N}}$ weakly converges to u_n in $H^1(0,T;V_n)$ and strongly converges to u_n in $L^2(0,T;V_n)$, since V_n is finite dimensional. This ends the proof of the continuity of τ. Finally τ is a compact map, as a consequence of the uniform bound of u_n in $H^1(0,T;V_n)$.

4 Proof of the main theorem.

It consists in passing to the limit when $n \rightarrow +\infty$. Thanks to Proposition 3.4, there is a solution (u_n,ρ^n,η^n) in $H^1(0,T;V_n)\times(L^\infty((0,T)\times\Omega))^2$, of the following problem :

$$\partial_t\rho^n + div(\rho^n u_n) = 0, \tag{4.1}$$

$$\partial_t\eta^n + div(\eta^n u_n) = 0, \tag{4.2}$$

$$\begin{aligned} &< \rho^n(t,.)\dot{u_n}(t,.).w_k > + < (\rho^n(t,.)u_n(t,.).D_x u_n(t,.)).w_k > \\ &\quad + < \eta^n\epsilon(u_n) : \epsilon(w_k) > = \ll f, w_k \gg, \quad k = 1,...,n, \end{aligned} \tag{4.3}$$

$$\rho^n(t=0) = \rho_0, \quad \eta^n(t=0) = \eta_0, \quad u_n(t=0) = u_n^0. \tag{4.4}$$

This solution satisfies the uniform estimates

$$\| u_n \|_{L^2(0,T;V)} \leq c(\| f \|_{L^2(0,T;V')} + \| u_n^0 \|_{L^2(\Omega)}) \leq c, \tag{4.5}$$

$$\rho^n \in \{\rho_m,...,\rho_M\} \quad a.e, \tag{4.6}$$

$$\eta^n \in \{\eta_m,...,\eta_M\} \quad a.e.. \tag{4.7}$$

Then, up to subsequences,

$$\rho^n \rightharpoonup \rho in \;\; L^\infty((0,T)\times\Omega) \;\; weakstar, \tag{4.8}$$

$$\eta^n \rightharpoonup \eta in \ \ L^\infty((0,T)\times\Omega) \ \ weakstar, \tag{4.9}$$

$$u_n \rightharpoonup u in \ \ L^2(0,T;V) \ \ weak. \tag{4.10}$$

We first pass to the limit in (4.1)-(4.2) with the help of the following lemma.

Lemma 4.1 *Let* $(\rho^n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *be such that*

$$\rho^n \rightharpoonup \rho in \ \ L^\infty((0,T)\times\Omega) \ \ weakstar, \tag{4.11}$$

$$u_n \rightharpoonup u \ \ in \ \ L^2(0,T;H^1_0(\Omega)) \ \ weak, \tag{4.12}$$

$$(\partial_t\rho^n)_{n\in\mathbb{N}} is bounded in \ \ L^2(0,T;H^{-1}(\Omega)). \tag{4.13}$$

Then $\rho^n u_n$ *tends to* ρu *in* $\mathcal{D}'((\prime,\mathcal{T})\times\otimes)$.

For a proof of Lemma 4.1, see [5].
Therefore

$$\partial_t\rho + div(\rho u) = 0, \quad \rho(t=0) = \rho_0, \tag{4.14}$$

$$\partial_t\eta + div(\eta u) = 0, \quad \eta(t=0) = \eta_0. \tag{4.15}$$

Hence the $L^2((0,T)\times\Omega)$ norms of $(\rho^p)_{p\in\mathbb{N}}$ and ρ are equal, as well as those of $(\eta^p)_{p\in\mathbb{N}}$ and η. This leads to the strong convergence of $(\rho^p)_{p\in\mathbb{N}}$ and $(\eta^p)_{p\in\mathbb{N}}$ to ρ and η respectively in $L^2((0,T)\times\Omega)$. Then (4.3) is also

$$< \{\partial_t(\rho^n u_n) + div(\rho^n u_n \otimes u_n)\}w_k > + < \eta^n \epsilon(u_n) : \epsilon(w_k) > \\ = \ll f, w_k \gg . \tag{4.16}$$

Multiply (4.16) by $\alpha(t) \in \mathcal{D}'(\mathbb{R})$ and integrate it over $\mathbb{R}_+$. After integrating by parts, we obtain

$$\int_0^{+\infty} \{< \rho^n u_n \partial_t(\alpha w_k) > + < (\rho^n u_n \otimes u_n).D_x(\alpha w_k) > \\ - < \eta^n \epsilon(u_n) : \alpha\epsilon(w_k) > + \ll f, \alpha w_k \gg\} ds \\ = \alpha(0) < \rho_0 u_n^0 w_k > . \tag{4.17}$$

But thanks to the weak convergence of $(u_n)_{n\in\mathbb{N}}$ and $(\epsilon(u_n))_{n\in\mathbb{N}}$ to u and $\epsilon(u)$ in $L^2((0,T)\times\Omega)$, and the strong convergence of $(\rho^n)_{n\in\mathbb{N}}$ and $(\eta^n)_{n\in\mathbb{N}}$ in $L^2((0,T)\times\Omega)$,

$$\rho^n u_n \to \rho u in \ \ \mathcal{D}'((\prime,\mathcal{T})\times\otimes), \tag{4.18}$$

and

$$\eta^n \epsilon(u_n) \rightarrow \eta\epsilon(u) in \;\; \mathcal{D}'((\prime, \mathcal{T}) \times \otimes). \tag{4.19}$$

Let us now study the limit of $\rho^n u_n \otimes u_n$. First, for $N \leq 4$,

$$\begin{aligned} \| \rho^n u_n \otimes u_n \|_{L^2(\Omega)} &\leq \| \rho^n \|_{L^\infty(\Omega)} \| u_n \|^2_{L^4(\Omega)} \\ &\leq \| \rho^n \|_{L^\infty(\Omega)} \| u_n \|^2_V . \end{aligned} \tag{4.20}$$

Therefore $(\rho^n u_n \otimes u_n)_{n \in I\!N}$ is uniformly bounded in $L^1(0,T;L^2(\Omega))$. In order to obtain a bound on $\partial_t(\rho_n u_n)$, we specifically choose the free and total family $\{w_k\}$ as an orthonormal basis of $L^2(\Omega)$ as well as an orthogonal basis of $H^1(\Omega)$. Recall that $(w_k)_{k \in I\!N}$ moreover satisfies

$$div(w_k) = 0. \tag{4.21}$$

For instance $(w_k)_{k \in I\!N}$ is the orthogonal sequence in $H^1(\Omega)$ of eigenvectors of the following Stokes problems

$$-\nu \Delta w_k + \bigtriangledown p_k = \lambda_k w_k, div(w_k) = 0, w_k \in H^1_0(\Omega), p_k \in L^2_{loc}(\Omega), \tag{4.22}$$

where $(\lambda_k)_{k \geq 1}$ are the eigenvalues of the self-adjoint Stokes operator, which has a compact resolvent.
On the other hand, (4.16) implies that

$$\partial_t(\rho^n u_n) = \Pi_n(div(\rho^n u_n \otimes u_n) - div(\eta^n \epsilon(u_n)) - f), \tag{4.23}$$

where Π_n is the L^2-projection on V_n defined by

$$v = \Pi_n(z) \;\; iff \;\; v = \sum_{k=1}^{n} \frac{< z.w_k >}{< w_k.w_k >} w_k. \tag{4.24}$$

Let us remark that Π_n is also the H^1-projection on V_n since w_k is also orthogonal in $H^1(\Omega)$. We extend the operator Π_n on $H^{-1}(\Omega)$ by

$$< \Pi_n z, \phi > = < z, \Pi_n \phi >, z \in H^{-1}(\Omega), \phi \in H^1_0(\Omega)). \tag{4.25}$$

Then we have the following

Lemma 4.2 $(\Pi_n)_{n \in I\!N}$ *is uniformly bounded in* $\mathcal{L}(\mathcal{H}^{-\infty}(\otimes))$.

Proof of Lemma 4.2.
By definition of Π_n,

$$\begin{aligned}\| \Pi_n \|_{\mathcal{L}(\mathcal{H}^{-\infty}(\otimes))} &= \sup_{\|z\|_{H^{-1}(\Omega)=1}} \| \Pi_n z \|_{L^2(H^{-1})} \\ &= \sup_{\Phi\in H_0^1(\Omega)} \sup_{\|z\|_{H^{-1}(\Omega)=1}} |< \Pi_n z, \phi >| \\ &= \sup_{\Phi\in H_0^1(\Omega)} \sup_{\|z\|_{H^{-1}(\Omega)=1}} |< z, \Pi_n \phi >| \\ &\leq \sup_{\Phi\in H_0^1(\Omega)} \| \Pi_n \phi \|_{H^1(\Omega)} \leq c.\end{aligned}$$

Thanks to Lemma 4.2 and the uniform boundedness of $div(\rho^n u_n \otimes u_n) - div(\eta^n \epsilon(u_n)) - f$ in $L^1(0,T;H^{-1}(\Omega))$, we obtain that

$$(\partial_t(\rho^n u_n)) \;\; isuniformlyboundedinL^1(0,T;H^{-1}(\Omega)). \tag{4.26}$$

(4.27) and the boundedness of $(\rho^n u_n)_{n\in I\!N}$ in $L^2((0,T)\times\Omega)$ imply that ([4])

$$(\rho^n u_n) \;\; belongstoacompactsetofL^2(0,T;H^{-1}(\Omega)). \tag{4.27}$$

Then, up to a subsequence,

$$\rho^n u_n \to \rho u \;\; in \;\; L^2(0,T;H^{-1}(\Omega)), \tag{4.28}$$

which states, with the help of the weak convergence of u_n to u in $L^2(0,T;H_0^1(\Omega))$ that

$$\rho^n u_n \otimes u_n \to \rho u \otimes u \;\; in \;\; \mathcal{D}'((\prime,\mathcal{T})\times\otimes). \tag{4.29}$$

Given (4.30), (4.18) and (4.19), we can pass to the limit in (4.16). Since the family $\{w_k\}$ is total in V, we finally obtain that (4.16) holds for every w in V, which ends the proof of the existence result stated in Theorem 1.1. In order to complete the proof of Theorem 1.1, it remains to prove that $(\rho,\eta) \in \{(\rho_k,\eta_k), k=1,...,N\}$ a.e.. We already know that $\rho \in \{\rho_k, k=1,...,N\}$ a.e. and $\eta \in \{\eta_k, k=1,...,N\}$ a.e, but it remains to establish that ρ and η have the same k^{th} value together. Denote

$$\Delta\rho = max \mid \rho_k - \rho_l \mid, \Delta\eta = max \mid \eta_k - \eta_l \mid, \tag{4.30}$$

and choose $\lambda > 0$ such that

$$\lambda\Delta\rho < \Delta\eta. \tag{4.31}$$

Then $\lambda\rho + \eta$ is also a solution of the transport equation (4.31). Hence

$$\lambda\rho + \eta \in \{\lambda\rho_k + \eta_k\} \quad a.e.. \tag{4.32}$$

It follows that

$$(\rho, \eta) \in \{(\rho_k, \eta_l) \quad s.t. \quad \lambda\rho_k + \eta_l = \lambda\rho_p + \eta_p\}. \tag{4.33}$$

Then

$$\mid \eta_p - \eta_l \mid = \lambda \mid \rho_p - \rho_k \mid \leq \lambda\Delta\rho < \Delta\eta, \tag{4.34}$$

which implies $\eta_p = \eta_l$, and then $\rho_p = \rho_k$, i.e $(\rho_k, \eta_l) = (\rho_p, \eta_p)$.

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