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ANTIPLANE SHEARING MOTIONS OF A VISCO-PLASTIC SOLID*

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Abstract. The authors consider antiplane shearing motions of an incompressible isotropic visco-plastic solid. The flow rule employed is a properly invariant generalization of Coulomb sliding friction and assumes a constant yield stress or threshold above which plastic flow occurs. In this model stresses above yield are possible; but when this condition obtains, the plastic flow rule forces the plastic strain to change so as to lower the stress levels in the material and dissipate energy. On the yield surface, the flow rule looks like the classical one for a rate independent elastic-perfectly plastic material when the velocity gradients are small enough but differs from the classical model for large gradients.

Key words. plastic waves, visco-plasticity, time-dependent problems

AMS subject classifications. 73E70, 73E60, 73E50

1. Introduction.

In this note we consider antiplane shearing motions of an incompressible isotropic visco-plastic solid. This work generalizes and compliments earlier work of Greenberg [1], [2], where he considered simple shearing flows for such materials. The flow rule we employ is a properly invariant generalization of Coulomb sliding friction and assumes a constant yield stress or threshold above which plastic flow occurs. As with most such theories, we assume a multiplicative decomposition of the deformation gradient into an elastic and plastic part, and we assume further that the deviatoric part of the Cauchy Stress tensor depends only on the elastic portion of the deformation gradient. For antiplane shearing motions this decomposition presents no precedence problems; i.e., does the elastic deformation precede the plastic or vice versa? One key feature of this model is that stresses above yield are possible. When this condition obtains, the plastic flow rule forces the plastic strain to change so as to lower the stress levels in the material and dissipate energy. The principal difficulty in formulating this model occurs when the stress is at yield. Motivated by results of Seidman [3], Utkin [4], and Filippov [5] on sliding modes induced by discontinuous vector fields, we are led to the flow rule advanced in (2.38). On the yield surface, this flow rule looks like the classical one for a rate independent elastic-perfectly plastic material when the velocity gradients are small enough but differs from the classical model for large gradients. This rule differentiates between loading and unloading and generates an energy identity which guarantees that uniqueness obtains for initial and initial-boundary value problems.

The organization of this paper is as follows. In §2 we develop the appropriate equations describing antiplane shearing flows in visco-plastic solids. Section 3 focuses on the uniqueness issue. Our basic estimate is that the energy associated with the difference between two solutions generated by the same data is nonincreasing. This estimate relies in an essential way on the definition of the plastic flow rule. In §4 we examine a one-dimensional signalling problem and discuss (1) the structure of this

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solution, and (2) a procedure to analytically obtain an approximate solution. We also compare this solution with what obtains for the more studied model of a rate independent elastic-perfectly plastic material where uniqueness fails. Section 5 deals with a numerical experiment for a two-dimensional signalling problem in the corner domain $r > 0$ and $\pi/2 < \theta < 2\pi$. Here the stresses are singular as one approaches the corner and care must be taken in the implementation of the boundary conditions.

We note that in the last several years there have been a number of other efforts aimed at capturing the essence of plastic flows. Antman and Szymczak [6], [7] have advanced a finite deformation theory of such materials which is similar in spirit to ours but differs in a number of essential ways. Their model is formally rate independent where ours is not but their model also requires a history dependent strain hardening mechanism. The predictions of the two theories are often qualitatively different; these differences arise since in their model the imposition of large loads tends to elevate the yield stress and create a temporally constant permanent plastic deformation, whereas in our model such loading would generate a constant plastic deformation rate and thus a plastic deformation which varies linearly in time. This may be seen by examining the solution constructed in §4. Other efforts on elasto-plastic modelling may be found in Coleman and Owen [8], Buhite and Owen [9], Coleman and Hodgdon [10], and Owen [11].

2. Model development. We say that a body is undergoing antiplane shear if material points $\xi = \xi_1\mathbf{e_1} + \xi_2\mathbf{e_2} + \xi_3\mathbf{e_3}$ move to $\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}$ with

$$x_1 = \xi_1, \quad x_2 = \xi_2, \quad \text{and } x_3 = \xi_3 + \phi(\xi_1, \xi_2, t) \tag{2.1}$$

under the action of a Cauchy stress tensor of the form

$$\begin{aligned} T = &-\pi(\mathbf{e_1}\otimes\mathbf{e_1} + \mathbf{e_2}\otimes\mathbf{e_2} + \mathbf{e_3}\otimes\mathbf{e_3}) \\ &+(S_{11}\mathbf{e_1}\otimes\mathbf{e_1} + S_{22}\mathbf{e_2}\otimes\mathbf{e_2} + S_{33}\mathbf{e_3}\otimes\mathbf{e_3}) \\ &+S_{31}(\mathbf{e_1}\otimes\mathbf{e_3} + \mathbf{e_3}\otimes\mathbf{e_1}) + S_{32}(\mathbf{e_2}\otimes\mathbf{e_3} + \mathbf{e_3}\otimes\mathbf{e_2}). \end{aligned} \tag{2.2^1}$$

Here, π is the hydrostatic pressure and S is the deviatoric stress tensor and satisfies

$$\operatorname{trace}(S) = S_{11} + S_{22} + S_{33} = 0. \tag{2.3}$$

Relative to the above basis, the matrix representation of the Cauchy stress is given by

$$\mathcal{J} = -\pi\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} S_{11} & 0 & S_{31} \\ 0 & S_{22} & S_{32} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}, \tag{2.4}$$

and relative to the same basis the deformation gradient is given by

$$\mathcal{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_{31} & F_{32} & 1 \end{pmatrix}, \tag{2.5}$$

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[1] $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the standard basis elements for R^3 and $\mathbf{e_i}\otimes\mathbf{e_j} = \mathbf{e_i}\mathbf{e_j}^\top$ are the standard basis elements for linear operators from R^3 to R^3.

where

$$F_{31} = \frac{\partial \phi}{\partial x_1} \quad \text{and} \quad F_{32} = \frac{\partial \phi}{\partial x_2}. \tag{2.6}$$

Noting that matrices

$$\mathcal{F}_{(a,b)} :\stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

satisfy the commutation relation

$$\mathcal{F}_{(a_1,b_1)}\mathcal{F}_{(a_2,b_2)} = \mathcal{F}_{(a_2,b_2)}\mathcal{F}_{(a_1,b_1)} = \mathcal{F}_{(a_1+a_2,b_1+b_2)}, \tag{2.7}$$

we feel justified in decomposing the deformation gradient $\mathcal{F}$ into its elastic and plastic parts $\mathcal{E}$ and $\mathcal{P}$ by

$$\mathcal{E} :\stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{P} :\stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{31} & p_{32} & 1 \end{pmatrix}, \tag{2.8}$$

where

$$\mathcal{F} = \mathcal{E}\mathcal{P} = \mathcal{P}\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{31}+p_{31} & e_{32}+p_{32} & 1 \end{pmatrix}. \tag{2.9}$$

For such antiplane shear flows one need not make any assumption about the precedence of the elastic and plastic parts of the flow.

Our basic constitutive assumption is that under a change of reference frame E transforms in the same way as F and that the deviatoric stress S is an isotropic, frame indifferent, trace free function of the elastic deformation gradient E.[2] The constraint that S is an isotropic, frame indifferent function of E implies that $\mathcal{S}$ must have the functional form

$$\mathcal{S} = \alpha I + \beta \mathcal{E}\mathcal{E}^\top + \gamma \mathcal{E}^{-\top}\mathcal{E}^{-1} \tag{2.10}$$

or

(2.11)[3]

$$\mathcal{S} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & e_{31} \\ 0 & 1 & e_{32} \\ e_{31} & e_{32} & 1+e_{31}^2+e_{32}^2 \end{pmatrix} + \gamma \begin{pmatrix} 1+e_{31}^2 & e_{31}e_{32} & -e_{31} \\ e_{31}e_{32} & 1+e_{32}^2 & -e_{32} \\ -e_{31} & -e_{32} & 1 \end{pmatrix},$$

where α, β, and γ are functions of the invariants of $\mathcal{E}\mathcal{E}^\top$, in this case the scalar $e_{31}^2 + e_{32}^2$. Equation (2.4) implies that $\mathcal{S}_{21} = \mathcal{S}_{12} = 0$ and this, in turn, implies that $\gamma \equiv 0$ while the condition that trace$\mathcal{S} = 0$ implies that $\alpha = -\beta(1 + ((e_{31}^2 + e_{32}^2)/3))$. Combining these identities with (2.11) yields

$$\mathcal{S} = \beta \begin{pmatrix} -\frac{1}{3}(e_{31}^2 + e_{32}^2) & 0 & e_{31} \\ 0 & -\frac{1}{3}(e_{31}^2 + e_{32}^2) & e_{32} \\ e_{31} & e_{32} & \frac{2}{3}(e_{31}^2 + e_{32}^2) \end{pmatrix}. \tag{2.12}$$

[2] E is the tensor whose matrix representation relative to the basis elements $\mathbf{e_i} \otimes \mathbf{e_j}$ is given by $(2.8)_1$.

[3] For details see Gurtin [12].

In the sequel we shall assume that β is a positive constant. Equation (2.12) implies that we may regard the elements S_{31} and S_{32} as basic descriptors of our system. In terms of these $\mathcal{S}$ and $\mathcal{E}$ take the form

$$\mathcal{S} = \begin{pmatrix} -\frac{1}{3\beta}(S_{31}^2 + S_{32}^2) & 0 & S_{31} \\ 0 & -\frac{1}{3\beta}(S_{31}^2 + S_{32}^2) & S_{32} \\ S_{31} & S_{32} & \frac{2}{3\beta}(S_{31}^2 + S_{32}^2) \end{pmatrix} \tag{2.13}$$

and

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{S_{31}}{\beta} & \frac{S_{32}}{\beta} & 1 \end{pmatrix}. \tag{2.14}$$

We now turn to the equations of motion. Equation (2.1) implies that the Eulerian velocity field $\mathbf{u}$ is of the form

$$\mathbf{u} = u(x_1, x_2, t_1)\mathbf{e_3}, \tag{2.15}$$

where

$$u(x_1, x_2, t_1) = \frac{\partial \phi}{\partial t_1}(x_1, x_2, t_1); \tag{2.16}$$

and (2.16), when combined with (2.6), implies that

$$\frac{\partial F_{31}}{\partial t_1} - \frac{\partial u}{\partial x_1} = 0 \tag{2.17}$$

and

$$\frac{\partial F_{32}}{\partial t_1} - \frac{\partial u}{\partial x_2} = 0. \tag{2.18}$$

Additionally, (2.9) and (2.14) imply that

$$F_{31} = \frac{S_{31}}{\beta} + p_{31} \tag{2.19}$$

and

$$F_{32} = \frac{S_{32}}{\beta} + p_{32}. \tag{2.20}$$

Balance of momentum in the $\mathbf{e_1}$ and $\mathbf{e_2}$ directions implies that

$$\frac{\partial}{\partial x_1}\left(\pi + \frac{(S_{31}^2 + S_{32}^2)}{3\beta}\right) = \frac{\partial}{\partial x_2}\left(\pi + \frac{(S_{31}^2 + S_{32}^2)}{3\beta}\right) = 0 \tag{2.21}$$

or equivalently that

$$\pi = \pi_0(x_3, t) - \frac{(S_{31}^2 + S_{32}^2)}{3\beta}(x_1, x_2, t), \tag{2.22}$$

whereas balance of momentum in the $\mathbf{e}_3$ direction yields

$$\rho_0 \frac{\partial u}{\partial t_1} - \frac{\partial S_{31}}{\partial x_1} - \frac{\partial S_{32}}{\partial x_2} = -\frac{\partial \pi_0}{\partial x_3}. \tag{2.23}$$

Here, ρ_0 is the constant mass density of the material. Since $\partial\pi_0/\partial x_3$ depends on x_3 and t_1, whereas all quantities on the left-hand side of (2.23) depend only on x_1, x_2, and t_1, we conclude that for antiplane shearing flows $\partial\pi_0/\partial x_3$ is independent of x_3. In what follows we shall assume this quantity is zero.

We now turn our attention to "yield condition" and the flow rule for the plastic strain tensor $\mathcal{P}$ of $(2.8)_2$. We assume that yield is determined by whether the scalar $S_{31}^2 + S_{32}^2$ exceeds a threshold S_y^2 or not. This assumption relies on the special form of $\mathcal{S}$ (see (2.13)) and is equivalent to a yield criteria determined by the norm of $\mathcal{S}$, where

$$\|\mathcal{S}\|^2 :\stackrel{\text{def}}{=} \mathcal{S}_{ij}\mathcal{S}_{ij} = 2(S_{31}^2 + S_{32}^2) + \frac{2}{3\beta^2}(S_{31}^2 + S_{32}^2)^2 \tag{2.24}$$

or one based on the maximum shear stress

$$S_*^2 :\stackrel{\text{def}}{=} \max_{\{\mathbf{e}|\mathbf{e}\cdot\mathbf{e}=1\}} \|\mathcal{S}\mathbf{e} - (\mathcal{S}\mathbf{e}\cdot\mathbf{e})\mathbf{e}\|^2. \tag{2.25}$$

In the sequel we let H denote the Heaviside function

$$H(x) :\stackrel{\text{def}}{=} \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{2.26}$$

and define ψ_1 and ψ_2 by

$$\psi_1 = \frac{1}{2}\int_{-\infty}^{(S_{31}^2+S_{32}^2)} H(x - S_y^2)\,dx \tag{2.27}$$

and

$$\psi_2 = \int_{-\infty}^{\sqrt{S_{31}^2+S_{32}^2}} H(x - S_y)\,dx, \tag{2.28}$$

where $S_y > 0$ is the "yield stress."

We shall confine our attention to the Coulomb type sliding law

$$\frac{\partial p_{31}}{\partial t_1} = \frac{1}{\beta T_0}\frac{\partial \psi_1}{\partial S_{31}} = \frac{S_{31}}{\beta T_0} H(S_{31}^2 + S_{32}^2 - S_y^2) \tag{2.29}$$

and

$$\frac{\partial p_{32}}{\partial t_1} = \frac{1}{\beta T_0}\frac{\partial \psi_1}{\partial S_{32}} = \frac{S_{32}}{\beta T_0} H(S_{31}^2 + S_{32}^2 - S_y^2), \tag{2.30}$$

though much of what we say applies equally well to the flow rule

$$\frac{\partial p_{31}}{\partial t_1} = \frac{S_y}{\beta T_0}\frac{\partial \psi_2}{\partial S_{31}} = \frac{S_y S_{31}}{\beta T_0\sqrt{S_{31}^2 + S_{32}^2}} H\left(\sqrt{S_{31}^2 + S_{32}^2} - S_y\right) \tag{2.31}$$

and

$$\frac{\partial p_{32}}{\partial t_1} = \frac{S_y}{\beta T_0}\frac{\partial \psi_2}{\partial S_{32}} = \frac{S_y S_{32}}{\beta T_0\sqrt{S_{31}^2+S_{32}^2}} H\left(\sqrt{S_{31}^2+S_{32}^2} - S_y\right). \tag{2.32}$$

The constant β is the shear modulus in (2.12), S_y is the yield stress, and $T_0 > 0$ is a fixed relaxation time. The flow rule is defined for $S_{31}^2 + S_{32}^2 \neq S_y^2$ and the problem remains to define it on the yield surface.

We first note that if $S_{31}^2 + S_{32}^2 \neq S_y^2$, we can combine (2.17)–(2.20) and (2.29) and (2.30) to obtain the following system for S_{31}, S_{32}, and u:

$$\frac{1}{\beta}\frac{\partial S_{31}}{\partial t_1} - \frac{\partial u}{\partial x_1} = \frac{-S_{31}H(S_{31}^2+S_{32}^2-S_y^2)}{\beta T_0}, \tag{2.33}$$

$$\frac{1}{\beta}\frac{\partial S_{32}}{\partial t_1} - \frac{\partial u}{\partial x_2} = \frac{-S_{32}H(S_{31}^2+S_{32}^2-S_y^2)}{\beta T_0}, \tag{2.34}$$

and

$$\rho_0\frac{\partial u}{\partial t_1} - \frac{\partial S_{31}}{\partial x_1} - \frac{\partial S_{32}}{\partial x_2} = 0. \tag{2.35}$$

Equations (2.33) and (2.34) imply that for $S_{31}^2 + S_{32}^2 \neq S_y^2$,

$$\begin{aligned}\frac{\partial}{\partial t_1}(S_{31}^2+S_{32}^2) = 2\beta&\left(S_{31}\frac{\partial u}{\partial x_1} + S_{32}\frac{\partial u}{\partial x_2}\right)\\ &-\frac{2}{T_0}(S_{31}^2+S_{32}^2)H(S_{31}^2+S_{32}^2-S_y^2),\end{aligned} \tag{2.36}$$

and (2.36), together with the results of [3], [4], [5], motivates our extension of the flow rule on the yield surface $S_{31}^2 + S_{32}^2 = S_y^2$. We extend (2.29) and (2.30) to the yield surface $S_{31}^2 + S_{32}^2 = S_y^2$ by

$$\frac{\partial p_{31}}{\partial t_1} = \frac{\alpha S_{31}}{\beta T_0} \quad\text{and}\quad \frac{\partial p_{32}}{\partial t_1} = \frac{\alpha S_{32}}{\beta T_0}, \tag{2.37}$$

where

$$\alpha = \begin{cases} 1 \ \text{ if } S_{31}^2+S_{32}^2 = S_y^2 \ \text{ and } \ S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2} > \dfrac{S_y^2}{\beta T_0},^{4} \\ \beta T_0\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right)\Big/ S_y^2 \ \text{ if } S_{31}^2+S_{32}^2 = S_y^2 \ \text{ and} \\ 0 \le S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2} \le \dfrac{S_y^2}{\beta T_0}, \\ 0 \ \text{ if } S_{31}^2+S_{32}^2 = S_y^2 \ \text{ and } \ S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2} < 0. \end{cases} \tag{2.38}$$

[4] The relations (2.37) and (2.38) transform in a frame indifferent fashion.

In the sequel we shall confine our attention to the extended flow rule (2.29), (2.30), (2.37) and (2.38). The relevant equations are

$$\frac{1}{\beta}\frac{\partial S_{31}}{\partial t_1} - \frac{\partial u}{\partial x_1} = -\frac{\alpha S_{31}}{\beta T_0}, \tag{2.39}$$

$$\frac{1}{\beta}\frac{\partial S_{32}}{\partial t_1} - \frac{\partial u}{\partial x_2} = -\frac{\alpha S_{32}}{\beta T_0}, \tag{2.40}$$

$$\rho_0\frac{\partial u}{\partial t_1} - \frac{\partial S_{31}}{\partial x_1} - \frac{\partial S_{32}}{\partial x_2} = 0, \tag{2.41}$$

where now

(2.42)

$$\alpha = \begin{cases} 1 \quad \text{if } S_{31}^2 + S_{32}^2 > S_y^2, \\ 1 \quad \text{if } S_{31}^2 + S_{32}^2 = S_y^2 \quad \text{and} \quad \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) > 1, \\ \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) \quad \text{if } S_{31}^2 + S_{32}^2 = S_y^2 \quad \text{and} \\ 0 \le \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) \le 1, \\ 0 \quad \text{if } S_{31}^2 + S_{32}^2 = S_y^2 \quad \text{and} \quad \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) < 0, \\ 0 \quad \text{if } S_{31}^2 + S_{32}^2 < S_y^2, \end{cases}$$

and these are solved together with appropriate initial and boundary conditions. Having solved the above system for S_{31}, S_{32}, and u we recover the deformation gradients F_{31} and F_{32} by solving

$$\frac{\partial F_{31}}{\partial t_1} - \frac{\partial u}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial F_{32}}{\partial t_1} - \frac{\partial u}{\partial x_2} = 0 \tag{2.43}$$

together with appropriate initial conditions. The plastic strains p_{31} and p_{32} are then given by

$$p_{31} = F_{31} - \frac{S_{31}}{\beta} \quad \text{and} \quad p_{31} = F_{32} - \frac{S_{32}}{\beta}. \tag{2.44}$$

These equations should be contrasted with what obtains in the more commonly studied theory of rate independent elastic-perfectly plastic materials. In that theory (2.37), (2.39)–(2.41), (2.43) and (2.44) still hold but α is given by

(2.45)

$$\alpha = \begin{cases} \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) & \text{if } S_{31}^2 + S_{32}^2 = S_y^2 \quad \text{and} \\ 0 \leq \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right), & \\ 0 \quad \text{if } S_{31}^2 + S_{32}^2 = S_y^2 \quad \text{and} \quad \dfrac{\beta T_0}{S_y^2}\left(S_{31}\dfrac{\partial u}{\partial x_1} + S_{32}\dfrac{\partial u}{\partial x_2}\right) < 0, & \\ 0 \quad \text{if } S_{31}^2 + S_{32}^2 < S_y^2. & \end{cases}$$

The unboundedness of α on the yield surface $S_{31}^2 + S_{32}^2 = S_y^2$ presents difficulties not encountered in our model. In particular, across nonstationary shocks where F_{31}, F_{32}, u, S_{31}, and S_{32} experience jump discontinuities, we must admit jumps in the plastic strains p_{31} and p_{32}. The reason for this is that in the classical rate independent theory—α as in (2.45)—we must allow "dirac" type singularities in the terms $\alpha S_{31}/\beta T_0$ and $\alpha S_{32}/\beta T_0$ and therefore, we cannot conclude that

$$cn_1[p_{31}] = cn_2[p_{32}] = 0. \tag{2.46}$$

Here, c is the normal velocity of the shock wave and $\mathbf{n}=(n_1, n_2)$ is the unit normal to the shock. In our model α is bounded, no "dirac" type singularities arise in the terms $\alpha S_{31}/\beta T_0$ and $\alpha S_{32}/\beta T_0$, and thus (2.46) holds. This implies that with our model all nonstationary shocks satisfy $c^2 = 1$; that is, they propagate with the speed of elastic signals. With our model, the only surfaces across which the plastic strains can jump are stationary, i.e., $c = 0$. Such jumps are also allowed in the classical theory.

We conclude this section by writing down a dimensionless version (2.39)–(2.44). We let

$$\begin{gathered} x = \sqrt{\frac{\rho_0}{\beta}}\frac{x_1}{T_0}, \quad y = \sqrt{\frac{\rho_0}{\beta}}\frac{x_2}{T_0}, \quad t = \frac{t_1}{T_0} \\ v = \sqrt{\frac{\rho_0}{\beta}}u, \quad \tau_{31} = \frac{S_{31}}{\beta}, \quad \tau_{32} = \frac{S_{32}}{\beta}, \quad \text{and } \tau_y = \frac{S_y}{\beta} \end{gathered} \tag{2.47}$$

and observe that (2.39)–(2.42) transform to

$$\frac{\partial \tau_{31}}{\partial t} - \frac{\partial v}{\partial x} = -\hat{\alpha}\tau_{31}, \tag{2.48}$$

$$\frac{\partial \tau_{32}}{\partial t} - \frac{\partial v}{\partial y} = -\hat{\alpha}\tau_{32}, \tag{2.49}$$

$$\frac{\partial v}{\partial t} - \frac{\partial \tau_{31}}{\partial x} - \frac{\partial \tau_{32}}{\partial y} = 0, \tag{2.50}$$

where

(2.51)
$$\hat{\alpha} = \begin{cases} 1 \quad \text{if } \tau_{31}^2 + \tau_{32}^2 > \tau_y^2, \\ 1 \quad \text{if } \tau_{31}^2 + \tau_{32}^2 = \tau_y^2 \quad \text{and} \quad \dfrac{1}{\tau_y^2}\left(\tau_{31}\dfrac{\partial v}{\partial x_1} + \tau_{32}\dfrac{\partial v}{\partial x_2}\right) > 1, \\ \dfrac{1}{\tau_y^2}\left(\tau_{31}\dfrac{\partial v}{\partial x_1} + \tau_{32}\dfrac{\partial v}{\partial x_2}\right) \quad \text{if } \tau_{31}^2 + \tau_{32}^2 = \tau_y^2 \text{ and} \\ 0 \leq \dfrac{1}{\tau_y^2}\left(\tau_{31}\dfrac{\partial v}{\partial x_1} + \tau_{32}\dfrac{\partial v}{\partial x_2}\right) \leq 1, \\ 0 \quad \text{if } \tau_{31}^2 + \tau_{32}^2 = \tau_y^2 \quad \text{and} \quad \dfrac{1}{\tau_y^2}\left(\tau_{31}\dfrac{\partial v}{\partial x_1} + \tau_{32}\dfrac{\partial v}{\partial x_2}\right) < 0, \\ 0 \quad \text{if } \tau_{31}^2 + \tau_{32}^2 < \tau_y^2. \end{cases}$$

The transformed versions of (2.43) and (2.44) are

$$\frac{\partial F_{31}}{\partial t} - \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F_{32}}{\partial t} - \frac{\partial v}{\partial y} = 0 \tag{2.52}$$

and

$$p_{31} = F_{31} - \tau_{31} \quad \text{and} \quad p_{32} = F_{32} - \tau_{32}. \tag{2.53}$$

3. Uniqueness results. Our task in this section is to establish the following

THEOREM 3.1. *Let Ω be an open domain in R^2 with smooth boundary $\partial\Omega$. Then, there is at most one piecewise smooth,*[5] *$L^2_{\mathrm{loc}}(\Omega)$ solution $(\tau_{31}, \tau_{32}, v)$ to (2.48)–(2.51) satisfying*

$$\lim_{t\to 0^+} (\tau_{31}, \tau_{32}, v)(x, y, t) = (\tau_{31}^0, \tau_{32}^0, v^0)(x, y), \tag{3.1}$$

$$\lim_{(x,y)\in\Omega;(x,y)\to\partial\Omega_1} (n_1\tau_{31} + n_2\tau_{32})(x, y, t) = f_1(x, y, t), \tag{3.2}$$

$$\lim_{(x,y)\in\Omega;(x,y)\to\partial\Omega_2} v(x, y, t) = f_2(x, y, t). \tag{3.3}$$

Here $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_1 \cap \partial\Omega_2$ is at worst a finite collection of points, $\mathbf{n}=(n_1, n_2)$ is the unit exterior normal to $\partial\Omega_1$, and the f_i's are smooth functions in $L^2_{\mathrm{loc}}(\partial\Omega_i \times [0, \infty))$.

Proof. We first note that if $(\tau_{31}^b, \tau_{32}^b, v^b)$ and $(\tau_{31}^a, \tau_{32}^a, v^a)$ are two solutions to (2.48)–(2.51), then their differences satisfy

$$\frac{\partial}{\partial t}(\tau_{31}^b - \tau_{31}^a) - \frac{\partial}{\partial x}(v^b - v^a) = -(\hat{\alpha}^b\tau_{31}^b - \hat{\alpha}^a\tau_{31}^a), \tag{3.4}$$

$$\frac{\partial}{\partial t}(\tau_{32}^b - \tau_{32}^a) - \frac{\partial}{\partial y}(v^b - v^a) = -(\hat{\alpha}^b\tau_{32}^b - \hat{\alpha}^a\tau_{32}^a), \tag{3.5}$$

and

$$\frac{\partial}{\partial t}(v^b - v^a) - \frac{\partial}{\partial x}(\tau_{31}^b - \tau_{31}^a) - \frac{\partial}{\partial y}(\tau_{32}^b - \tau_{32}^a) = 0. \tag{3.6}$$

[5] This formulation admits shocks which propagate with normal velocity c satisfying $c^2 = 1$.

Here, $\hat{\alpha}^b$ and $\hat{\alpha}^a$ represent the bounded function $\hat{\alpha}$ defined in (2.51) evaluated at $(\tau_{31}^b, \tau_{32}^b, v^b)$ and $(\tau_{31}^a, \tau_{32}^a, v^a)$, respectively. The last three identities imply that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}&\left[(\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2+(v^b-v^a)^2\right]\\ &-\frac{\partial}{\partial x}\left[(\tau_{31}^b-\tau_{31}^a)(v^b-v^a)\right]-\frac{\partial}{\partial y}\left[(\tau_{32}^b-\tau_{32}^a)(v^b-v^a)\right]\\ &=-\left[(\tau_{31}^b-\tau_{31}^a)(\hat{\alpha}^b\tau_{31}^b-\hat{\alpha}^a\tau_{31}^a)+(\tau_{32}^b-\tau_{32}^a)(\hat{\alpha}^b\tau_{32}^b-\hat{\alpha}^a\tau_{32}^a)\right].\end{aligned} \tag{3.7}$$

We now claim that

$$p :\stackrel{\text{def}}{=} (\tau_{31}^b-\tau_{31}^a)(\hat{\alpha}^b\tau_{31}^b-\hat{\alpha}^a\tau_{31}^a)+(\tau_{32}^b-\tau_{32}^a)(\hat{\alpha}^b\tau_{32}^b-\hat{\alpha}^a\tau_{32}^a) \tag{3.8}$$

is nonnegative. In verifying this assertion there is no loss in generality in assuming that

$$0 \le \hat{\alpha}^a \le \hat{\alpha}^b \le 1. \tag{3.9}$$

We first note that p may be rewritten as

$$\begin{aligned}p = \hat{\alpha}^a &\left[(\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2\right]\\ &+(\hat{\alpha}^b-\hat{\alpha}^a)\left[(\tau_{31}^b)^2-\tau_{31}^a\tau_{31}^b+(\tau_{32}^b)^2-\tau_{32}^a\tau_{32}^b\right].\end{aligned} \tag{3.10}$$

If $\hat{\alpha}^a = 0$, then $(\tau_{31}^a)^2+(\tau_{32}^a)^2 \le \tau_y^2$ and $\tau_{31}^a\tau_{31}^b+\tau_{32}^a\tau_{32}^b \le \tau_y\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}$ and, therefore, (3.10) implies that

$$p \ge \hat{\alpha}^b\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}\left(\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}-\tau_y\right). \tag{3.11}$$

If $\hat{\alpha}^b = 0$, then (3.10) implies that $p = 0$, whereas if $0 < \hat{\alpha}^b \le 1$, (2.51) implies that $\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2} \ge \tau_y$, and (3.11) then yields $p \ge 0$. We now turn to the case where $0 < \hat{\alpha}^a \le \hat{\alpha}^b \le 1$. If $\hat{\alpha}^b = \hat{\alpha}^a$, the nonnegativity of p follows from (3.10), and thus to complete the verification that $p \ge 0$ it suffices to consider the case where $0 < \hat{\alpha}^a < \hat{\alpha}^b \le 1$. Here we know that $(\tau_{31}^a)^2+(\tau_{32}^a)^2 = \tau_y^2$ and $(\tau_{31}^b)^2+(\tau_{32}^b)^2 \ge \tau_y^2$. The former identity, along with (3.10) and $\tau_{31}^a\tau_{31}^b+\tau_{32}^a\tau_{32}^b \le \tau_y\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}$, implies that

$$\begin{aligned}p \ge \hat{\alpha}^a &\left[(\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2\right]\\ &+(\hat{\alpha}^b-\hat{\alpha}^a)\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}\left(\sqrt{(\tau_{31}^b)^2+(\tau_{32}^b)^2}-\tau_y\right),\end{aligned} \tag{3.12}$$

and (3.12), $0 < \hat{\alpha}^a < \hat{\alpha}^b \le 1$, and $(\tau_{31}^b)^2+(\tau_{32}^b)^2 \ge \tau_y^2$ complete the proof of the assertion that p is nonnegative.

For any $(x_0, y_0) \in I\!R^2$, $r_0 > 0$, $T > 0$, and $0 \le t \le T$ we let

$$C(x_0, y_0, r_0, t) :\stackrel{\text{def}}{=} \{(x,y) | (x-x_0)^2+(y-y_0)^2 < (r_0+T-t)^2\}. \tag{3.13}$$

The identity (3.7) implies that if $(\tau_{31}^b, \tau_{32}^b, v^b)$ and $(\tau_{31}^a, \tau_{32}^a, v^a)$ are two solutions of (2.48)-(2.51) taking on the same data (3.1)–(3.3), then

$$\frac{1}{2}\int_{C(x_0,y_0,r_0,t)\cap\Omega}((\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2+(v^b-v^a)^2)\,dx\,dy \tag{3.14}$$

$$+\int_0^T \left(\int_{C(x_0,y_0,r_0,t)\cap\Omega} p(x,y,t)\,dx\,dy \right) dt$$
$$+\int_0^T \left(\int_{\partial C(x_0,y_0,r_0,T)\cap\Omega} \left(\frac{1}{2}((\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2+(v^b-v^a)^2) \right.\right.$$
$$\left.\left. -(v^b-v^a)\left(\frac{(x-x_0)(\tau_{31}^b-\tau_{31}^a)}{(r_0+T-t)} + \frac{(y-y_0)(\tau_{32}^b-\tau_{32}^a)}{(r_0+T-t)} \right)\right) ds \right) dt$$
$$= 0.$$

Here,

$$\partial C(x_0,y_0,r_0,t) = \{(x,y)|(x-x_0)^2+(y-y_0)^2 = (r_0+T-t)^2\}. \tag{3.15}$$

The vector $((x-x_0)/(r_0+T-t), (y-y_0)/(r_0+T-t))$ is the unit exterior normal to $\partial C(x_0,y_0,r_0,t)$, and ds is arc length along $\partial C(x_0,y_0,r_0,t)$. Since

$$\begin{aligned} &-(v^b-v^a)\left(\frac{(x-x_0)(\tau_{31}^b-\tau_{31}^a)}{(r_0+T-t)} + \frac{(y-y_0)(\tau_{32}^b-\tau_{32}^a)}{(r_0+T-t)} \right) \\ &\geq -|v^b-v^a|\sqrt{(\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2} \\ &\geq -\frac{1}{2}((\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2+(v^b-v^a)^2) \end{aligned} \tag{3.16}$$

and since $p \geq 0$, we see that all three integrals in (3.15) are nonnegative and their sum is zero. From this we obtain

$$\int_{C(x_0,y_0,r_0,T)\cap\Omega} ((\tau_{31}^b-\tau_{31}^a)^2+(\tau_{32}^b-\tau_{32}^a)^2+(v^b-v^a)^2)dxdy = 0, \tag{3.17}$$

which is the desired uniqueness result.

4. A signalling problem. In this section we consider an elementary one-dimensional signalling problem for the normalized system (2.48)–(2.53). The solution is of the form

$$(\tau_{31},\tau_{32},v) = (\tau(x,t),0,v(x,t)), \qquad 0<x<\infty, \tag{4.1}$$

where τ and v satisfy

$$\frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} = -\hat{\alpha}\tau, \qquad 0<x<\infty, \tag{4.2}$$

$$\frac{\partial v}{\partial t} - \frac{\partial \tau}{\partial x} = 0, \qquad 0<x<\infty, \tag{4.3}$$

and

$$\hat{\alpha} = \begin{cases} 1 & \text{if } \tau^2 > \tau_y^2, \\ 1 & \text{if } \tau^2 = \tau_y^2 \quad \text{and} \quad \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x} > 1, \\ \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x}, & \text{if } \tau^2 = \tau_y^2 \quad \text{and} \quad 0 \leq \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x} \leq 1, \\ 0 & \text{if } \tau^2 = \tau_y^2 \quad \text{and} \quad \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x} < 0, \\ 0 & \text{if } \tau^2 < \tau_y^2, \end{cases} \tag{4.4}$$

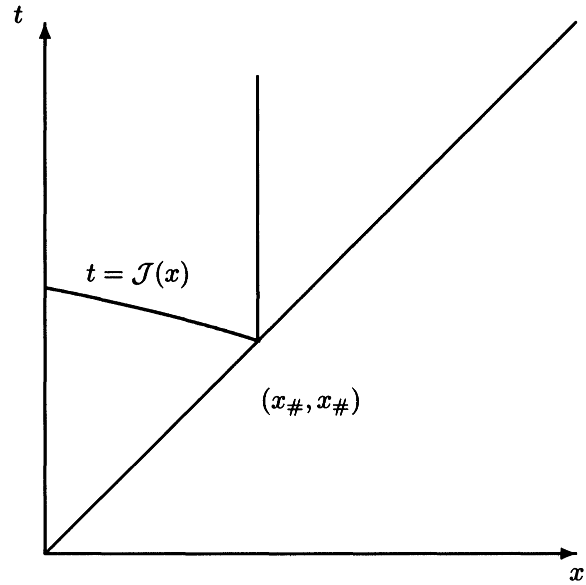

FIG. 1

and the initial and boundary conditions

$$(\tau, v)(x,0) = (0,0), \qquad 0 < x < \infty \tag{4.5}$$

and

$$v(0,t) = -\tau_0, \quad \text{where } \tau_0 > \tau_y. \tag{4.6}$$

We note that the results of the previous section guarantee there is at most one solution to the above problem.

In the region $0 \le t < x$, we have $(\tau, v) \equiv (0,0)$. Moreover, $\tau + v$ is continuous across the curve $t = x$ and thus satisfies $\tau_-(t,t) + v_-(t,t) \equiv 0$. The difficult part of the problem is to show there is a curve $t = \mathcal{J}(x)$, $0 < x < x_\#$, with $-1 < d\mathcal{J}/dx \le 0$, such that in the region $x < t < \mathcal{J}(x)$ with $0 < x < x_\#$, τ and v satisfy

$$\tau > \tau_y, \tag{4.7}$$

$$\frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} = -\tau \quad \text{and} \quad \frac{\partial v}{\partial t} - \frac{\partial \tau}{\partial x} = 0, \tag{4.8}$$

the boundary condition (4.6) and $\tau_-(t,t) + v_-(t,t) = 0$. On the curve $t = \mathcal{J}(x)$ we have $\lim_{\epsilon \to 0^+} \tau(x, \mathcal{J}(x) - \epsilon) = \tau_y$ and $\hat{v}(x) \stackrel{\text{def}}{:=} \lim_{\epsilon \to 0^+} v(x, \mathcal{J}(x) - \epsilon)$ satisfies $0 \le d\hat{v}/dx \le \tau_y$. In the region $\mathcal{J}(x) < t$ and $0 < x < x_\#$ we have $\tau(x,t) = \tau_y$ and $v(x,t) = \hat{v}(x)$, whereas in $x_\# \le x < t$, $\tau \equiv \tau_y$ and $v(x,t) = \hat{v}(x_\#) = -\tau_y$ (see Fig. 1).

The existence of a curve $t = \mathcal{J}(x)$ with the desired properties may be established by converting the system (4.6), (4.8), and $\tau_-(t,t) + v_-(t,t) = 0$ to integral equations for τ and v in $x < t$, verifying that for $0 < t - x \ll 1$ the stress satisfies $\tau > \tau_y$, and finally by obtaining qualitative information on the level line $t = \mathcal{J}(x)$ defined by $\lim_{\epsilon \to 0^+} \tau(x, \mathcal{J}(x) - \epsilon) = \tau_y$.

Rather than perusing that approach we shall show how to obtain simple approximate solutions satisfying (4.6)–(4.8) and $\tau_-(t,t)+v_-(t,t)=0$ as well as approximations to the level line $t=\mathcal{J}(x)$.

We note that for each integer $N\geq 1$ the system (4.8) has solutions

$$v_N = -\tau_0 + \sum_{k=1}^{N} \lambda_{(k)}(t)x^{2k-1} \tag{4.9}$$

and

$$\tau_N = \lambda_0(t) + \sum_{k=1}^{N} \dot{\lambda}_{(k)} x^{2k}/2k \tag{4.10[6]}$$

where the coefficients satisfy

$$\dot{\lambda}_0 + \lambda_0 = \lambda_1, \tag{4.11}$$

$$\ddot{\lambda}_k + \dot{\lambda}_k = 2k(2k+1)\lambda_{k+1}, \qquad 1\leq k\leq N-1, \tag{4.12}$$

and

$$\ddot{\lambda}_N + \dot{\lambda}_N = 0. \tag{4.13}$$

These solutions satisfy the boundary condition $v_N(0^+,t)=-\tau_0$ and have $2N+1$ free parameters which are determined by insisting that the equation

$$\tau_N(t,t)+v_N(t,t)=0 \tag{4.14}$$

is satisfied to $O(t^{2N})$ as $t\to 0^+$. The approximate curve $t=\mathcal{J}_N(x)$ is subsequently determined by solving $\tau_N(x,\mathcal{J}_N(x))=\tau_y$. An easy calculation shows that $\mathcal{J}_N(x)=O((\tau_0-\tau_y)/\tau_0)$ and $d\mathcal{J}_N/dx<0$ which guarantees that the number $x_{\#}^N$ defined by $\mathcal{J}_N(x_{\#}^N)=x_{\#}^N$ is $O((\tau_0-\tau_y)/\tau_0)$, and thus on the boundary $x=t$, $\tau_N(t,t)+v_N(t,t)$ is at worst $O((\tau_0-\tau_y)/\tau_0)^{2N+1}$ for $0\leq t\leq x_{\#}^N$. We continue the approximate solutions to the rest of the region described by Fig. 1 via the extensions procedure used for the exact solution; that is, for $0<t<x$,

$$(\tau_N,v_N)\equiv(0,0), \tag{4.15}$$

for $\mathcal{J}_N(x)<t$ and $0<x<x_{\#}^N$

$$v_N(x,t)=v_N(x,\mathcal{J}_N(x)) \quad\text{and}\quad \tau_N(x,t)=\tau_y, \tag{4.16}$$

and for $x_{\#}\leq x<t$,

$$v_N(x,t)=v_N(x_{\#},\mathcal{J}_N(x_{\#})) \quad\text{and}\quad \tau_N(x,t)=\tau_y. \tag{4.17}$$

We are then guaranteed that the error made in failing to meet the boundary condition $\tau_N(t,t)+v_N(t,t)=0$ is at worst $O((\tau_0-\tau_y)/\tau_0)^{2N+1}$ for all $t>0$. We shall present the details of this procedure for the case $N=1$.

In this case,

$$v_1 = -\tau_0 + (\lambda_{1,0}+\lambda_{1,1}e^{-t})x \tag{4.18}$$

[6] Here $\cdot$ denotes differentiation with respect to t.

and

$$\tau_1 = (\lambda_{1,0} + \lambda_{1,1} t e^{-t} + \lambda_{0,1} e^{-t}) - \frac{\lambda_{1,1} e^{-t} x^2}{2}, \tag{4.19}$$

and the insistence that $\tau_1(t,t) + v_1(t,t) = O(t^3)$ as $t \to 0^+$ implies that

$$\begin{aligned} \lambda_{1,0} + \lambda_{0,1} &= \tau_0 \\ \lambda_{1,0} - \lambda_{0,1} + 2\lambda_{1,1} &= 0 \\ \lambda_{0,1} - 5\lambda_{1,1} &= 0 \end{aligned} \tag{4.20}$$

and hence that

$$v_1 = \tau_0 \left(-1 + \frac{(3 + 5e^{-t})x}{8} \right) \tag{4.21}$$

and

$$\tau_1 = \frac{\tau_0}{8} \left(3 + t e^{-t} + 5e^{-t} - \frac{e^{-t} x^2}{2} \right). \tag{4.22}$$

The approximate curve $t = \mathcal{J}_1(x)$ is obtained by solving $\tau_1(x, \mathcal{J}_1(x)) = \tau_y$ or equivalently the equation

$$\left(3 + \mathcal{J}_1 e^{-\mathcal{J}_1} + 5e^{-\mathcal{J}_1} - \frac{x^2 e^{-\mathcal{J}_1}}{2} \right) = \frac{8\tau_y}{\tau_0}. \tag{4.23}$$

The fact that $0 < \tau_y/\tau_0 < 1$ guarantees the unique solvability of this equation for $0 \le x \ll 1$ and that $\mathcal{J}_1(0) = O(2((\tau_0 - \tau_y)/\tau_0))$. A quick calculation also shows that

$$\frac{d\mathcal{J}_1}{dx} = \frac{-2x}{(8 + 2\mathcal{J}_1 - x^2)} < 0. \tag{4.24}$$

The number $x^1_{\#}$, where $\mathcal{J}_1(x^1_{\#}) = x^1_{\#}$ satisfies

$$\left(3 + x^1_{\#} e^{-x^1_{\#}} + 5e^{-x^1_{\#}} - \frac{(x^1_{\#})^2 e^{-x^1_{\#}}}{2} \right) = \frac{8\tau_y}{\tau_0}, \tag{4.25}$$

and for $0 < \tau_0 - \tau_y$ small enough we are guaranteed that $x^1_{\#} = O((\tau_0 - \tau_y)/\tau_0)$. This estimate, when combined with (4.24), implies that $-1 < d\mathcal{J}_1/dx$ for $0 \le x \le x^1_{\#}$.

Our final task is to show that the function

$$\hat{v}_1(x) :\stackrel{\text{def}}{=} \tau_0 \left(-1 + \frac{(3 + 5e^{-\mathcal{J}_1(x)})x}{8} \right) \tag{4.26}$$

satisfies

$$0 \le \frac{d\hat{v}_1}{dx}(x) \le \tau_y, \qquad 0 \le x \le x^1_{\#}. \tag{4.27}$$

The defining relation (4.26) implies that

$$\frac{d\hat{v}_1}{dx}(x) = \tau_0 \left(\frac{3 + 5e^{-\mathcal{J}_1(x)}}{8} \right) - \frac{5\tau_0 e^{-\mathcal{J}_1(x)}}{8} x \mathcal{J}_1'(x), \tag{4.28}$$

and this relationship, when combined with (4.23) and (4.24), implies that

$$\frac{d\hat{v}_1}{dx}(x) = \tau_y + \frac{\tau_0 e^{-\mathcal{J}_1}((x^2 - 2\mathcal{J}_1)(8 + 2\mathcal{J}_1 - x^2) + 20x^2)}{16(8 + 2\mathcal{J}_1 - x^2)}. \tag{4.29}$$

The fact that $\mathcal{J}_1(x) \geq x^1_{\#}$ for $0 \leq x \leq x^1_{\#} = O((\tau_0 - \tau_y)/\tau_0)$ implies that the second term in (4.29) is negative and this provides the desired upper bound for $d\hat{v}_1/dx$. The desired lower bound is an immediate consequence of (4.28) and the bounds for $d\mathcal{J}_1/dx$.

We conclude this section by contrasting the above solution with what obtains if we replace our flow rule—α given by (4.4)—with the one generated by (2.31) and (2.32) and also by the flow rule associated with a rate independent elastic-perfectly plastic material. In the former case, (4.2) is replaced by

$$\frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} = -\alpha\tau_y, \tag{4.30}$$

and (4.4) is unchanged.

In the region $0 \leq t < x$ we have $(\tau, v) \equiv (0,0)$, and $\tau + v$ is continuous across $x = t$. For $0 \leq x \leq t \leq 2(\tau_0 - \tau_y)/\tau_y$ we have

$$v = -\tau_0 + \frac{\tau_y x}{2} \quad \text{and} \quad \tau = \tau_0 - \frac{\tau_y t}{2}; \tag{4.31}$$

for $0 \leq x \leq 2(\tau_0 - \tau_y)/\tau_y$ and $t \geq 2(\tau_0 - \tau_y)/\tau_y$ we have

$$v = -\tau_0 + \frac{\tau_y x}{2} \quad \text{and} \quad \tau = \tau_y, \tag{4.32}$$

and finally for $2(\tau_0 - \tau_y)/\tau_y \leq x < t$ we have

$$v = -\tau_y \quad \text{and} \quad \tau = \tau_y. \tag{4.33}$$

With this flow rule the curve $t = \mathcal{J}(\cdot)$ is the constant function $\mathcal{J}(x) = 2(\tau_0 - \tau_y)/\tau_y$, $0 \leq x \leq 2(\tau_0 - \tau_y)/\tau_y$. Equations (2.52) and (2.53), the initial conditions $(F_{31}, p_{31})(x, 0) = (0, 0)$ for $x > 0$, and (4.31)–(4.33) allow us to determine (F_{31}, p_{31}). The result is

$$(F_{31}, p_{31}) = \begin{cases} (0,0), \quad 0 \leq t < x, \\ (\tau_0 + \tau_y(\frac{t}{2} - x), \tau_y(t - x)), \qquad 0 \leq x < t < \frac{2(\tau_0 - \tau_y)}{\tau_y}, \\ (\tau_0 + \tau_y(\frac{t}{2} - x), \tau_{01} - \tau_y + \tau_y(\frac{t}{2} - x)), \frac{2(\tau_0 - \tau_y)}{\tau_y} \leq t \quad \text{and} \\ 0 \leq x < \frac{2(\tau_0 - \tau_y)}{\tau_y}, \\ (\tau_y, 0), \frac{2(\tau_0 - \tau_y)}{\tau_y} < x < t. \end{cases} \tag{4.34}$$

It is worth noting that the above solution is unique. This can be established using the arguments of §3 directly on the system (4.30) and (4.3)–(4.6).

We now examine the signaling problem for a rate independent elastic-perfectly plastic material. Equations (4.1)–(4.3) and (4.5) and (4.6) still hold, except now $\hat{\alpha}$ is given by

$$\hat{\alpha} = \begin{cases} 0 & \text{if } \tau^2 < \tau_y^2 \\ 0 & \text{if } \tau^2 = \tau_y^2 \quad \text{and} \quad \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x} < 0, \\ \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x} & \text{if } \tau^2 = \tau_y^2 \quad \text{and} \quad 0 \leq \frac{\tau}{\tau_y^2}\frac{\partial v}{\partial x}. \end{cases} \tag{4.35}$$

We also have

$$\frac{\partial F_{31}}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial p_{31}}{\partial t} = \hat{\alpha}\tau, \quad \text{and} \quad F_{31} = \tau + p_{31}, \tag{4.36}$$

and these satisfy the initial conditions

$$(F_{31}, p_{31})(x, 0) = (0, 0), x > 0. \tag{4.37}$$

We seek solutions with structure similar to that obtained for the previous two models. Specifically, a shock curve $t = \hat{t}(x)$ such that in the region $0 < t < \hat{t}(x)$,

$$(F_{31}, p_{31}, \tau, v) = (0, 0, 0, 0), \tag{4.38}$$

and in the region $t > \hat{t}(x)$ the shear stress τ is at yield, i.e.,

$$\tau(x, t) = \tau_y, \qquad \hat{t}(x) < t. \tag{4.39}$$

We interpret (4.3) and $(4.36)_1$ as conservation laws, and this, together with (4.38) and (4.39), implies that on $t = \hat{t}(x)$,

$$v^-(x, \hat{t}(x)) + \tau_y \frac{d\hat{t}}{dx} = 0 \tag{4.40}$$

and

$$F_{31}^-(x, \hat{t}(x)) + v^-(x, \hat{t}(x)) \frac{d\hat{t}}{dx} = 0. \tag{4.41}$$

Here, $(v^-, F_{31}^-)(x, \hat{t}(x)) = \lim_{\epsilon \to 0^+}(v, F_{31})(x - \epsilon, \hat{t}(x))$. The identity (4.39) also implies that in $t > \hat{t}(x)$ the velocity v is a function of x only. Near $x = 0$ we choose

$$v(x, t) = -\tau_0 + \lambda x, \lambda > 0. \tag{4.42}$$

With this choice we obtain

$$p_{31} = \lambda(t - \hat{t}(x)) + p_-(x) \tag{4.43}$$

and

$$F_{31} = \tau_y + \lambda(t - \hat{t}(x)) + p_-(x). \tag{4.44}$$

Equation (4.40), together with $\hat{t}(0) = 0$, then yields

$$\hat{t}(x) = \frac{\tau_0^2 - (\tau_0 - \lambda x)^2}{2\lambda\tau_y}, \tag{4.45}$$

and (4.41), (4.44), and (4.45) imply that

$$p_-(x) = \frac{(\tau_0 - \lambda x)^2 - \tau_y^2}{\tau_y}. \tag{4.46}$$

We now let

$$x_\# = \frac{\tau_0 - \tau_y}{\lambda} \tag{4.47}$$

and note that

$$p_-(x) > 0, \qquad 0 \le x < x_\#, \tag{4.48}$$

$$p_-(x_\#) = 0, \tag{4.49}$$

and

$$\frac{d\hat{t}}{dx}(x_\#) = 1. \tag{4.50}$$

In the region $(\tau_0^2 - (\tau_0 - \lambda x)^2)/2\lambda\tau_y < t$ and $0 \le x < x_\# = (\tau_0 - \tau_y)/\lambda$ our solution is given by

$$F_{31} = \tau_y + \lambda\left(t + \frac{\tau_0^2 - (\tau_0 - \lambda x)^2}{2\lambda\tau_y}\right), \tag{4.51}$$

$$p_{31} = \lambda\left(t + \frac{\tau_0^2 - (\tau_0 - \lambda x)^2}{2\lambda\tau_y}\right), \tag{4.52}$$

$$v = -\tau_0 + \lambda x, \tag{4.53}$$

$$\tau = \tau_y. \tag{4.54}$$

The shock curve is continued to $x > x_\#$ by

$$\hat{t}(x) = \frac{\tau_0^2 - \tau_y^2}{2\lambda\tau_y} + \left(x - \frac{\tau_0 - \tau_y}{\lambda}\right) \tag{4.55}$$

and in the region $((\tau_0^2 - \tau_y^2)/2\lambda\tau_y) + (x - ((\tau_0 - \tau_y)/\lambda)) < t$ and $(\tau_0 - \tau_y)/\lambda = x_\# < x$,

$$F_{31} = \tau_y, \quad p_{31} = 0, \quad v = -\tau_y, \quad \text{and } \tau = \tau_y. \tag{4.56}$$

The line $x = x_\# = (\tau_0 - \tau_y)/\lambda$ is a stationary contact discontinuity and across it p_{31} jumps while the other fields are continuous. The interesting fact about the signaling problem for this model is the lack of uniqueness of solutions; we have a compatible solution for every $\lambda > 0$. This observation points out one of the weaknesses of the classical model.

5. Computational experiments. In this section we present the results of a computational experiment performed on the normalized system (2.48)–(2.52) when the pressure gradient is zero. The results reported deal with a two-dimensional generalization of the signalling problem of the previous section.

The experiment deals with the system (2.48)–(2.51) solved in the region $r > 0$ and $\pi/2 < \theta < 2\pi$, where $r = \sqrt{x^2 + y^2}$. At time $t = 0$ we assume that

$$(\tau_{31}, \tau_{32}, v) = (0, 0, 0) \tag{5.1}$$

for $r > 0$ and $\pi/2 < \theta < 2\pi$, and for $t > 0$ we assume that

$$v\left(r, \frac{\pi}{2}^+\right) = v(r, 2\pi^-) = \tau_0, \qquad r > 0, \tag{5.2}$$

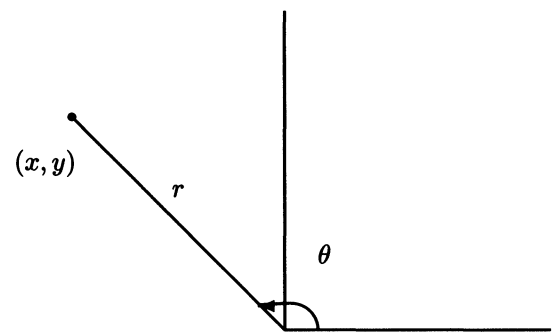

FIG. 2

where $\tau_0 > \tau_y$, and again $\tau_y > 0$ is the yield stress.

The elastic version of this problem, namely, the system

$$\frac{\partial \tau_{31}}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{5.3}$$

$$\frac{\partial \tau_{32}}{\partial t} - \frac{\partial v}{\partial y} = 0, \tag{5.4}$$

$$\frac{\partial v}{\partial t} - \frac{\partial \tau_{31}}{\partial x} - \frac{\partial \tau_{32}}{\partial y} = 0, \tag{5.5}$$

together with (5.1) and (5.2), was considered by Keller and Blank [13]. They obtained exact solutions to this and a number of other problems with self similar structure. Relevant to us here is the singular nature of $\tau_{31}^2 + \tau_{32}^2$ as $r \to 0^+$. Their results demonstrate that

$$\tau_{31}^2 + \tau_{32}^2 = O\left(\frac{t}{r}\right)^{2/3}, \qquad r \to 0^+. \tag{5.6}$$

This singular behavior also obtains for the plastic flow problem and forces us to treat the boundary conditions in our numerical simulation carefully. Our integration scheme for (2.48)–(2.51) is based on a symmetrized operator splitting algorithm for the governing differential equations. At time $t = nh$, $n = 0, 1, 2, \ldots$, our approximate solution consists of lattice data

$$(\tau_{31}, \tau_{32}, v)^n_{(k,m)} = (\tau_{31}, \tau_{32}, v)\left(\frac{(2k-1)}{2}h, \frac{(2m-1)}{2}h, nh\right). \tag{5.7}$$

For the problem under consideration the boundaries are not part of the computational lattice but are offset from it by a distance of $h/2$. The computational lattice is

$$\mathcal{S} = \{(k,m) \mid k \le 0 \text{ and } m = 0, \pm 1, \pm 2, \ldots\} \cup \{(k,m) \mid k \ge 1 \text{ and } m \le 0\}. \tag{5.8}$$

To update the data (5.7) we successively solve

$$\frac{\partial \tau_{31}}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial \tau_{32}}{\partial t} = 0, \quad \text{and} \quad \frac{\partial v}{\partial t} - \frac{\partial \tau_{31}}{\partial x} = 0, \quad 0 \le t \le h, \tag{5.9}$$

$$\frac{\partial \tau_{31}}{\partial t} = 0, \quad \frac{\partial \tau_{32}}{\partial t} - \frac{\partial v}{\partial y} = 0, \quad \text{and} \quad \frac{\partial v}{\partial t} - \frac{\partial \tau_{32}}{\partial y} = 0, \quad 0 \le t \le h, \tag{5.10}$$

and

$$\frac{\partial \tau_{31}}{\partial t} = -\hat{\alpha}\tau_{31}, \quad \frac{\partial \tau_{32}}{\partial t} = -\hat{\alpha}\tau_{32}, \quad \text{and} \quad \frac{\partial v}{\partial t} = 0, \quad 0 \le t \le h, \tag{5.11}$$

where of course $\hat{\alpha}$ is defined in (2.51). For (5.9) we use the approximate solution defined by (5.7) as initial data and let $(\tau_{31}^1, \tau_{32}^1, v^1)_{(k,m)}$ denote the value of this solution at $t = h$ on the lattice $\mathcal{S}$. We then solve (5.10) using the $(\tau_{31}^1, \tau_{32}^1, v^1)_{(k,m)}$ as initial data and let $(\tau_{31}^2, \tau_{32}^2, v^2)_{(k,m)}$ denote value of the solution at $t = h$ on $\mathcal{S}$. Finally, we solve (5.11) with $(\tau_{31}^2, \tau_{32}^2, v^2)_{(k,m)}$ as initial data and let $(\tau_{31}^3, \tau_{32}^3, v^3)_{(k,m)}$ denote the value of this solution at $t = h$ on $\mathcal{S}$.

We then repeat the process solving (5.10) first with the data (5.7), and, we let $(\tau_{31}^4, \tau_{32}^4, v^4)_{(k,m)}$ denote the lattice update at $t = h$. We then solve (5.9) using $(\tau_{31}^4, \tau_{32}^4, v^4)_{(k,m)}$ as initial data and let $(\tau_{31}^5, \tau_{32}^5, v^5)_{(k,m)}$ denote the lattice update. Finally we solve (5.11) with data $(\tau_{31}^5, \tau_{32}^5, v^5)_{(k,m)}$ and let $(\tau_{31}^6, \tau_{32}^6, v^6)_{(k,m)}$ denote the lattice update at $t = h$. The desired approximate solution $(\tau_{31}, \tau_{32}, v)_{(k,m)}^{(n+1)}$ is then obtained by averaging $(\tau_{31}^3, \tau_{32}^3, v^3)_{(k,m)}$ and $(\tau_{31}^6, \tau_{32}^6, v^6)_{(k,m)}$; that is,

$$(\tau_{31}, \tau_{32}, v)_{(k,m)}^{(n+1)} = \tfrac{1}{2}(\tau_{31}^3 + \tau_{31}^6, \tau_{32}^3 + \tau_{32}^6, v^3 + v^6)_{(k,m)}. \tag{5.12}$$

Of course, all of the intermediate updates are solved subject to the boundary conditions of the original problem. Here these boundary conditions manifest themselves as reflection conditions at those lattice points that are a distance $h/2$ away form the actual boundary. Formal accuracy could be maintained if we used either $(\tau_{31}^3, \tau_{32}^3, v^3)_{(k,m)}$ or $(\tau_{31}^6, \tau_{32}^6, v^6)_{(k,m)}$ for the updated approximate solution but either of these updates alone would, over time, tend to introduce asymmetries into the approximates not present in the actual solution. These asymmetries are removed with the algorithm employed.

The results of our experiment are shown in Figs. 3–7. Each snapshot shows two different representations of the velocity field and the total shear stress, namely the quantity $\sqrt{\tau_{31}^2 + \tau_{32}^2}$. This simulation was run with $h = 1/50$, $\tau_y = 1$, and $\tau_0 = 1.3$. The contours on the velocity plots are spaced 0.1 apart and run from $v = 0$ to $v = 1.3$. The stress contours run from 1 to 3.2 in increments of 0.2. In these snapshots one sees not only the plane wave solutions of the previous section but also the effect of the corner singularity which are confined to the region $0 \le r \le t$ and $\pi/2 < \theta < 2\pi$.

For comparison we have run the elastic version of this problem with the same boundary conditions and same values of h, τ_y, and τ_0. These results are shown in Figs. 8–12.

It should be noted that for both problems the velocity fields satisfy the additional condition

$$\lim_{r \to t^+} v(r, \theta, t) = 0, \qquad \frac{\pi}{2} < \theta < 2\pi \tag{5.13}$$

and that our numerical solutions meet this consistency condition automatically.

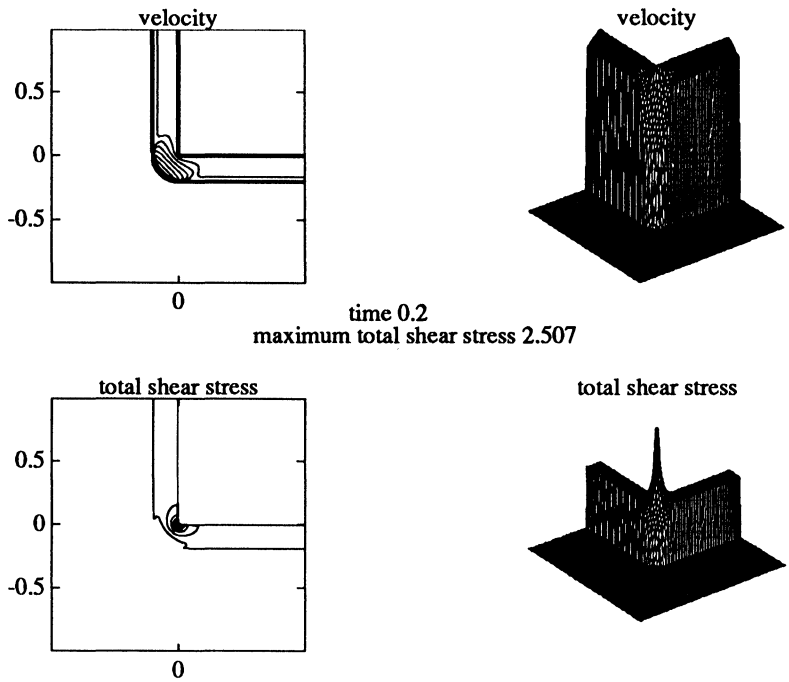
velocity
0.5
0
-0.5
0
velocity
time 0.2
maximum total shear stress 2.507
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 3

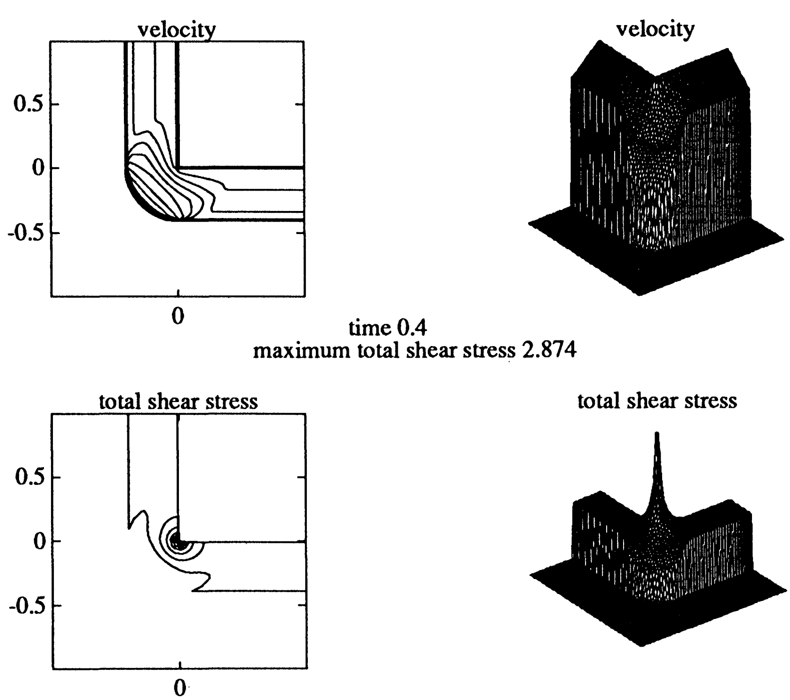
velocity
0.5
0
-0.5
0
velocity
time 0.4
maximum total shear stress 2.874
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 4

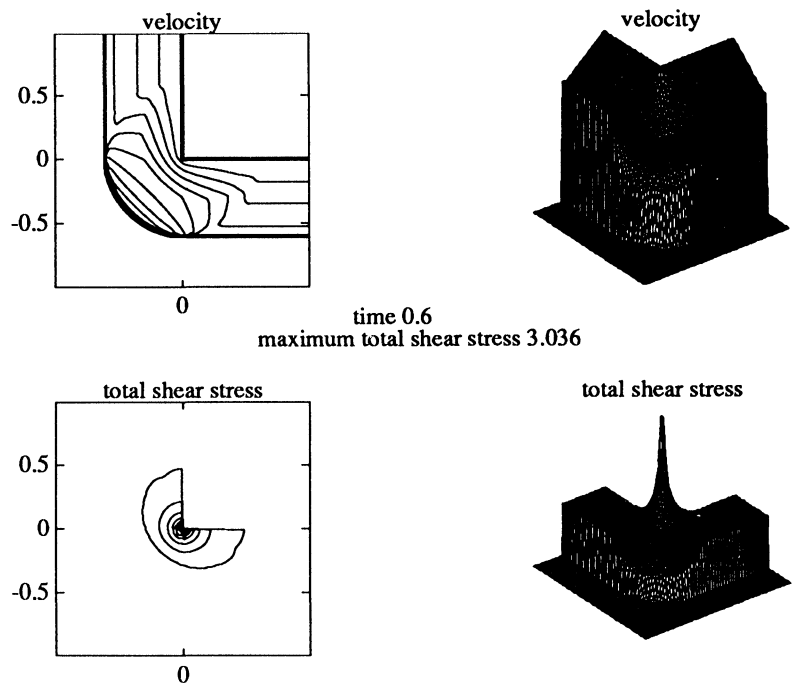
velocity
0.5
0
-0.5
0
velocity
time 0.6
maximum total shear stress 3.036
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 5

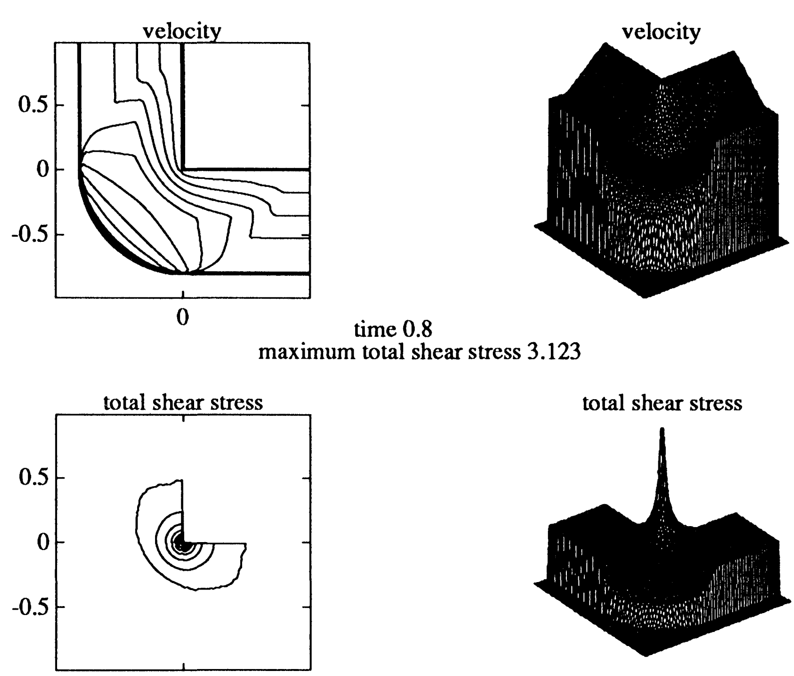
velocity
0.5
0
-0.5
0
velocity
time 0.8
maximum total shear stress 3.123
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 6

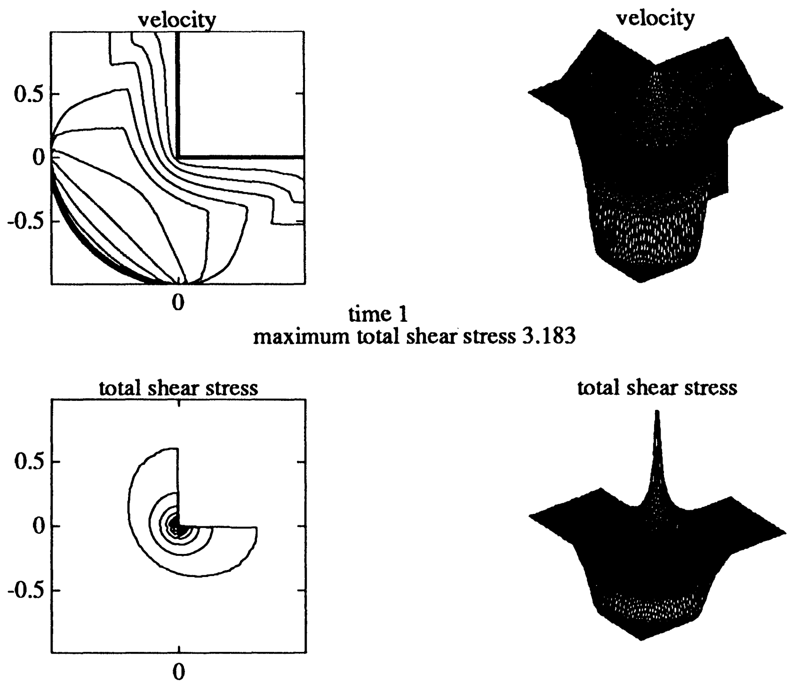
velocity
0.5
0
-0.5
0
velocity
time 1
maximum total shear stress 3.183
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 7

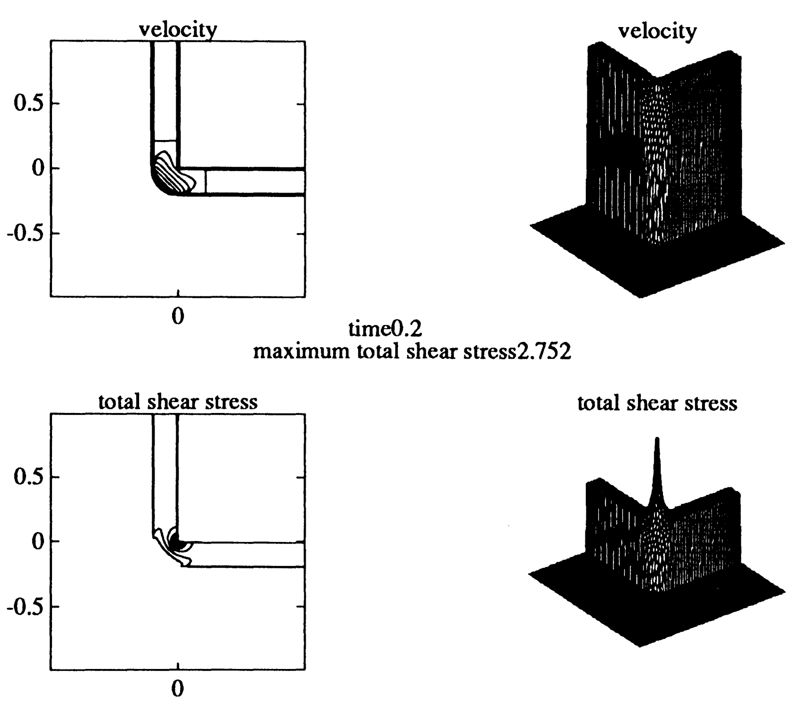
velocity
0.5
0
-0.5
0
velocity
time0.2
maximum total shear stress2.752
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 8

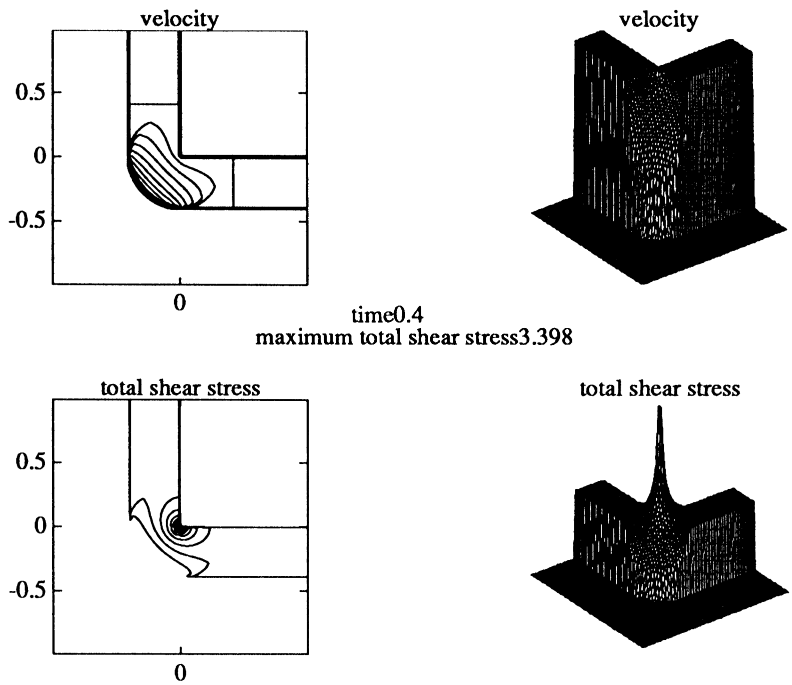
velocity
0.5
0
-0.5
0
velocity
time0.4
maximum total shear stress3.398
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 9

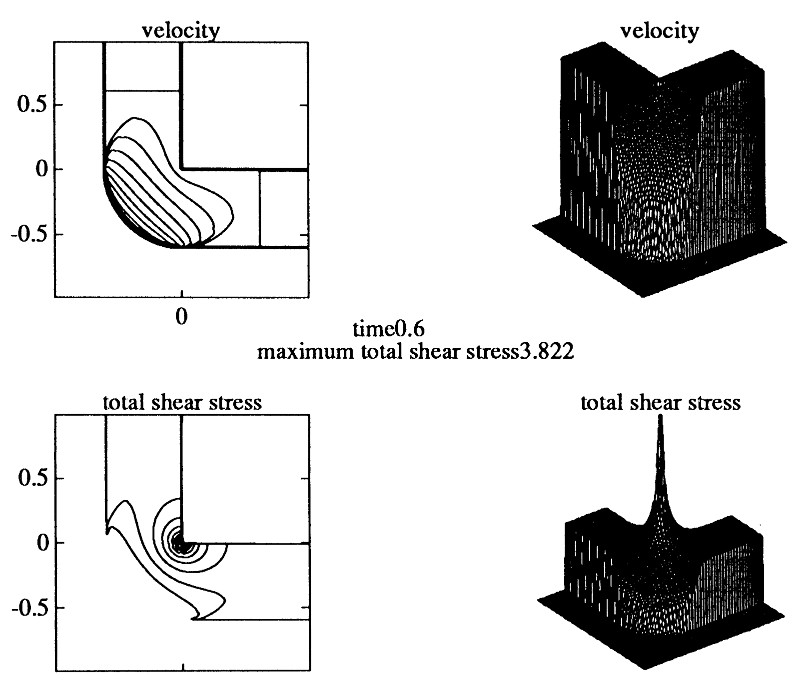
velocity
0.5
0
-0.5
0
velocity
time0.6
maximum total shear stress3.822
total shear stress
0.5
0
-0.5
0
total shear stress

FIG. 10

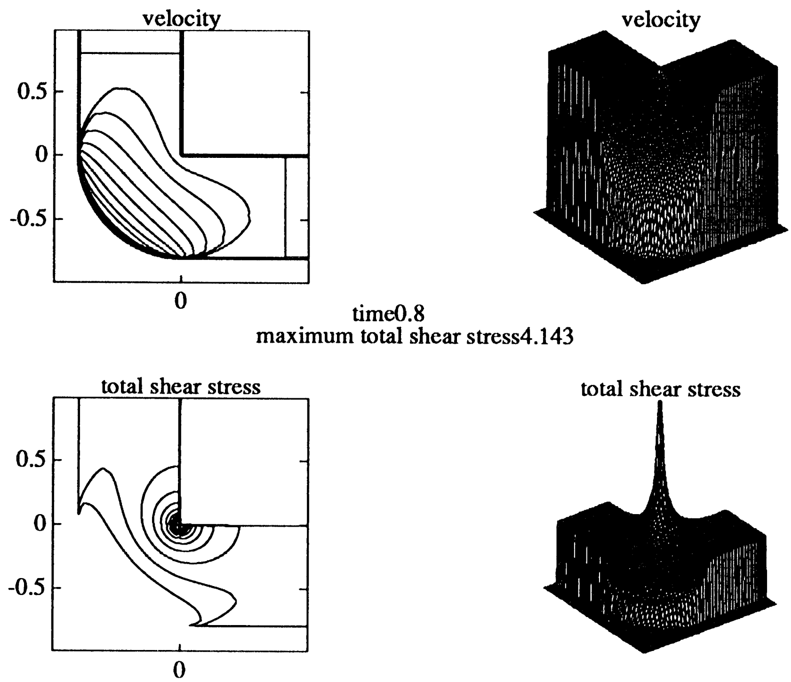
velocity
velocity
0.5
0
-0.5
0
time0.8
maximum total shear stress4.143
total shear stress
total shear stress
0.5
0
-0.5
0

FIG. 11

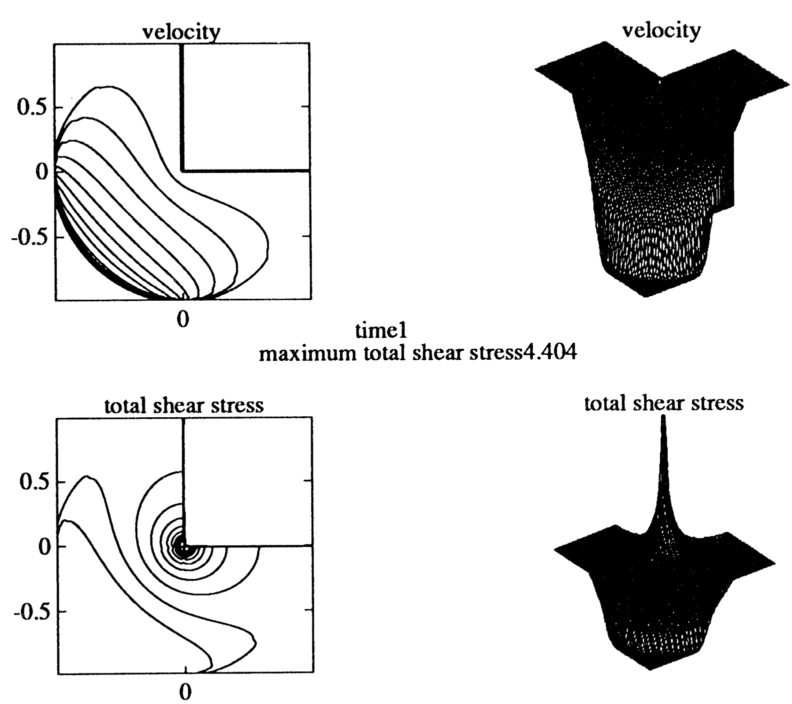
velocity
velocity
0.5
0
-0.5
0
time1
maximum total shear stress4.404
total shear stress
total shear stress
0.5
0
-0.5
0

FIG. 12

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