Kinetic and Related Models
©American Institute of Mathematical Sciences
Volume 5, Number 4, December 2012
doi:10.3934/krm.2012.5.673
pp. 673–695

EXPONENTIAL STABILITY OF THE SOLUTIONS TO THE BOLTZMANN EQUATION FOR THE BENARD PROBLEM

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(Communicated by Tong Yang)

Abstract. We complete the result in [2] by showing the exponential decay of the perturbation of the laminar solution below the critical Rayleigh number and of the convective solutions above the critical Rayleigh number, in the kinetic framework.

1. **Introduction.** The arising of convective motions in a fluid between two thermal walls under the action of the gravity field g, when the bottom wall is hotter than the top wall, is one of the classical examples of bifurcation of a stationary solution in Fluid-Dynamics and is known as the "Benard problem". The bifurcation is driven by a parameter Ra, the Rayleigh number which is proportional to the product of the gravity and the temperature difference. It consists in the fact that, when the Rayleigh number Ra is below a critical value Ra_c, the incompressible Navier-Stokes-Fourier system (INSF) in an external gravity has only the conductive solution, characterized by vanishing velocity field and a linear temperature profile. Instead, when Ra crosses the threshold Ra_c convective solutions appear with non vanishing velocity field. With the increase of the Rayleigh number, a large variety of complex phenomena occur. Here we wish to restrict our attention to a small right neighborhood of Ra_c, where only the first bifurcation occurs and the laminar solution bifurcates: above Ra_c both the laminar and the two convective motions, corresponding to clockwise and anti-clockwise rotation, are stationary solutions, but only the last two are stable.

The analysis of the linear and non linear stability of the stationary solutions to the Benard problem, at the level of Fluid-Dynamics, has been performed in a vast literature ([4, 6, 8, 9, 10, 13, 11]). The same problem, in the framework of the Boltzmann equation, has been addressed in [1, 2], where the stationary solutions to the Boltzmann equation (1.1), both below and above the critical Rayleigh number,

2000 *Mathematics Subject Classification.* Primary: 82B40, 82C26; Secondary 76P05.
Key words and phrases. Boltzmann equation, hydrodynamical limit, Benard problem.

have been constructed and their asymptotic stability has been proved, without computing the rate of decay of the perturbation.

The aim of this paper is to complete the result in [2] by proving exponential decay rate of the perturbation. Unfortunately, the key spectral inequality we used in [2] is incorrect. Therefore, we begin with fixing this error by giving the correct inequality, then we modify consequently the proofs given in [2]. This requires a slight change of perspective. As already mentioned, the Rayleigh number is proportional to the product of the gravity times the temperature difference. Therefore, in order to achieve a supercritical Rayleigh number, either we consider a sufficiently small gravity and a corresponding temperature difference, or we fix a sufficiently small temperature difference and deal with a corresponding gravity. The former point of view is the one used in [2]. In this paper, due to the extra terms deriving from the corrected spectral inequality, we adopt, at least in two dimensions, the latter point of view, which requires minor modifications in several lemmas.

To be more specific, we state the main problem. We follow as closely as possible the notation of [2] to which we will also refer for many details which are just a repetition of the arguments given there.

We look for the solution to the initial-boundary-value problem for the Boltzmann equation with diffuse reflection at the boundary modelling two thermal walls the bottom one at temperature $T_- = 1$ and the top one at themperature $T_+ = 1-2\pi\varepsilon\lambda$:

$$\begin{aligned}
&\frac{\partial F}{\partial t} + \frac{1}{\varepsilon} v\cdot\nabla^\mu F - G\frac{\partial F}{\partial v_z} = \frac{1}{\varepsilon^2}Q(F,F),\\
&F(0,x,z,v) = F_0(x,z,v), \quad (x,z)\in[-\pi,\pi)\times(-\pi,\pi)\equiv\Omega,\ v\in\mathbb{R}^3, \qquad (1.1)\\
&F(t,x,\mp\pi,v) \ = M_\mp(v)\int_{w_z\lesseqgtr 0}|w_z|F(t,x,\mp\pi,w)dw,\ t>0,\ v_z\gtrless 0,\ x\in[-\pi,\pi),
\end{aligned}$$

where $\mu = \frac{h}{d}$ is the aspect ratio of the convective cell, $\nabla^\mu = (\mu\partial_x, \partial_z)$ and $v\cdot\nabla^\mu = \mu v_x\partial_x + v_z\partial_z$. Indeed, we have rescaled the variables z to make the width of the slab 2π and the variable x so that all the functions are periodic in x with fixed period 2π. Moreover,

$$F_0 \geq 0, \quad M_- = \frac{1}{2\pi}e^{-\frac{v^2}{2}}, \quad M_+(v) = \frac{1}{2\pi(1-2\pi\varepsilon\lambda)^2}e^{-\frac{v^2}{2(1-2\pi\varepsilon\lambda)}}.$$

The parameter $\varepsilon = \frac{\ell_0}{d}$ is the ratio between the mean free path and the width of the slab, T_+ and $T_- > T_+$ are the temperatures on the top and bottom plates, $G = \frac{1}{\varepsilon}\frac{dg}{2T_-}$ is the rescaled gravity field, $\lambda = \frac{1}{\varepsilon}\frac{T_- - T_+}{2\pi T_-}$ measures the rescaled temperature gradient. Moreover,

$$Q(f,g)(z,v,t) = \frac{1}{2}\int_{\mathbb{R}^3}dv_*\int_{S_2}d\omega B(\omega, v-v_*)\big\{f'_*g' + f'g'_* - f_*g - g_*f\big\}.$$

Here h', h'_*, h, h_* stand for $h(x,z,v',t), h(x,z,v'_*,t), h(x,z,v,t), h(x,z,v_*,t)$ respectively, $S_2 = \{\omega\in\mathbb{R}^3\,|\omega^2 = 1\}$, B is the differential cross section $2B(\omega,V) = |V\cdot\omega|$ corresponding to hard spheres, and v, v_* and v', v'_* are pre-collisional and post-collisional velocities or conversely. Note that the boundary conditions are chosen so that the impermeability condition

$$\int_{\mathbb{R}^3} dvFv_z = 0 \qquad (1.2)$$

is formally satisfied at the boundaries.

A comment is in order about the assumptions on collision cross section and boundary conditions: the method presented here can be probably extended to collision cross sections corresponding to hard potentials with Grad angular cutoff. This would require extra technical efforts and we prefered to restrict ourselves to the simplest case. It does not seem possible to include soft potentials with cutoff and non cutoff potentials in this treatment. About boundary conditions we remark that the Benard setup requires thermal walls that could also be modeled by a combination of diffuse reflection and elastic or reverse reflection. Unfortunately the boundary terms due to elastic or reverse reflection are too singular to be treated with our methods, hence we have to confine our analysis to the purely diffusive boundary conditions. Purely elastic reflection or reverse reflection are not considered because they do not model thermal walls.

We note that above definitions of the parameters correspond to the choice $\varepsilon = 2Kn\sqrt{\frac{6}{5\pi}}$ where Kn is the Knudsen number. We have also set the Mach number $Ma = \varepsilon\sqrt{\frac{6}{5}}$. With such a choice of the parameters, the Rayleigh number is given by (see for example [14])

$$Ra = 32G\lambda, \tag{1.3}$$

independent of ε. As mentioned before (see e.g. [4]), there is a critical value of Ra, denoted by Ra_c, such that the laminar solution to the hydrodynamic equations becomes linearly unstable. In the rest of this paper $\lambda > 0$ will be a fixed value, smaller than a suitable λ_0 that will be specified later, and G will be the control parameter of the bifurcation, which will occur when G crosses the threshold G_c such that $32\lambda G_c = Ra_c$. Moreover, we will use the notation $\delta = (G - G_c)G_c^{-1}$, and our analysis will hold either for $0 \le G \le G_c$ or for $\delta > 0$ sufficiently small. We stress that the smallness of the parameters λ and δ is independent of ε, so that the results we obtain are valid also in the limit $\varepsilon \to 0$, the hydrodynamic limit, with λ and δ small but fixed.

We now recall the Fluid-Dynamics results for the Benard problem relevant to our purposes. We refer to [6, 7, 8] for more details. The laminar solution to the INSF system is characterized by the temperature field $T_l = -\lambda\frac{z+\pi}{2\pi}$ and $u_l = 0$. We write the INSF system for the deviations from the laminar solution. They are:

$$\begin{aligned}
&\partial_t u + u\cdot\nabla^\mu u = \hat\eta\Delta^\mu u - \nabla p - e_z G\theta,\\
&\partial_t\theta + u\cdot\nabla^\mu\theta + \lambda u_z = \hat k\Delta^\mu\theta \qquad\qquad \text{in } \Omega = [-\pi,\pi)\times(-\pi,\pi)\\
&\nabla^\mu\cdot u = 0,
\end{aligned}\tag{1.4}$$

with u a vector in $\mathbb{R}^2$ whose components are u_x and u_z respectively, $u\cdot\nabla^\mu = \mu u_x\partial_x + u_z\partial_z$, $\Delta^\mu = \mu^2\partial_{xx} + \partial_{zz}$, e_z the unit vector in the positive z direction. p is the pressure of the incompressible fluid, θ is the deviation from the linear temperature profile, $\hat\eta$ is the kinematic viscosity and $\hat k$ is the heat conductivity multiplied by a factor $\frac{2}{5}$.

The INSF system (1.4) has to be solved with homogeneous boundary data:

$$u(t,x,\pm\pi) = 0, \quad \theta(t,x,\pm\pi) = 0, \quad x\in[-\pi,\pi), \tag{1.5}$$

and periodic boundary conditions in the variable x. The couple $h = (u,\theta)$ denotes the solution to the problem (1.4),(1.5). For $G \le G_c$, the laminar solution, $h = 0$ is the only steady solution and it is stable up to the critical Rayleigh number. Moreover, there is $\delta_1 > 0$ such that, if $G \in (G_c, G_c(1+\delta))$ for $\delta < \delta_1$, then there are two periodic roll solutions, h_s, with period which fixes the aspect ratio μ, rotating clockwise and anti-clockwise respectively, such that

$$h_s = \delta h_{\rm con} + \delta^2 h_R, \tag{1.6}$$

where $h_{\rm con}$ are the eigenvectors corresponding to the least eigenvalues of the linearization of the problem (1.4),(1.5) around the laminar solution $h = 0$. The remainder h_R is in a suitable Sobolev space $(H_k(\Omega))^3$ with its Sobolev norm bounded uniformly in δ: namely, there is a constant C such that, for any $\delta < \delta_1$,

$$\|h_R\|_{H_k(\Omega)} \le C. \tag{1.7}$$

Furthermore, there are n_0 and ζ_1 such that if $h_0 \in (H_k(\Omega))^3$ for k sufficiently large and has H_k-norm smaller than n_0, then the time dependent solution to the problem (1.4), (1.5) is such that

$$\|h(t)\|_{H_{k'}(\Omega)} \le C\delta \mathrm{e}^{-\zeta_1 t} \tag{1.8}$$

for any $k' < k$ (see Proposition 3.1).

A stationary solution to the problem (1.1) is constructed by means of a truncated expansion in ε with remainder, so that we have the representation

$$F_s = M + \varepsilon f_s + O(\varepsilon^2). \tag{1.9}$$

The first term of the expansion is the standard Maxwellian M

$$M(v) = \frac{1}{(2\pi)^{\frac32}} \mathrm{e}^{-\dfrac{|v|^2}{2}}; \tag{1.10}$$

the first order correction is given by

$$f_s = M\Big(\rho_s + u_s \cdot v + T_s \frac{|v|^2 - 3}{2}\Big) \tag{1.11}$$

where, for $G \le G_c$ we have $u_s = 0$, $T_s = T_l$ and $\rho_s = -(\lambda + G)z$ is computed by using the Boussinesq condition

$$\nabla(\rho + T) = G e_z. \tag{1.12}$$

When $G > G_c$ and $\delta < \delta_1$, u_s, T_s and ρ_s are computed in terms of h_s.

The higher order terms will be described later. Now we are in position to state the main theorem. In the statement we use the norm $\|\cdot\|_{2,2}$ which represents the L^2-norm on the phase space $\Omega \times \mathbb{R}^3$ with weight M^{-1}, and the norm $\|\cdot\|_{2,2,2}$, the L^2-norm on $\Omega \times \mathbb{R}^3 \times \mathbb{R}^+$ with weight M^{-1}, including also integration of all the positive times.

Theorem 1.1. *There are $\lambda_0 > 0$, $\delta_0 > 0$, $\varepsilon > 0$ such that, if $0 \le \lambda < \lambda_0$ and $G \in (0, G_c(1+\delta))$ with $\delta < \delta_0$, then there is a positive, locally unique, stationary solution F_s to the Boltzmann equation such that for any $\varepsilon < \varepsilon_0$,*

$$\|F_s - (M + \varepsilon f_s)\|_{2,2} \le c\varepsilon^2. \tag{1.13}$$

Furthermore, if the initial perturbation Φ_0^ε to the stationary solution is such that $F_s + M\Phi_0^\varepsilon$ is positive and satisfies the conditions (3.2), (3.3), and the hydrodynamic perturbation satisfies the smallness assumptions in Proposition 3.1, then the positive solution to the time dependent problem (1.1) exists, and there are $\bar\zeta > 0$ and c independent of ε such that, for any $\zeta \le \bar\zeta$,

$$\|(F - F_s)e^{\zeta t}\|_{2,2,2} \le c. \tag{1.14}$$

Section 2 is devoted to the construction of the stationary solution. In Section 3 we show the exponential decay of the perturbation.

2. **Stationary solution.** As discussed before, in this paper we want to show that a small perturbation of the stationary solution F_s to the problem (1.1) decays exponentially fast, as $t \to +\infty$. However, since the paper [2] contains an inconsistency in the construction of such a stationary solution, we need to review part of the proof of the main existence result for the stationary solutions.

We recall the notation adopted for the norms: the norm in the bulk is defined, for any $1 \le q \le +\infty$ as

$$\| f\|_{q,2} = \left(\int_{\mathbb{R}^3} dv M(v) \left(\int_\Omega dxdz |f(x,z,v)|^q\right)^{\frac{2}{q}}\right)^{\frac{1}{2}}.$$

The space of measurable functions on $\Omega \times \mathbb{R}^3$ with the above norm finite is denoted by $\tilde{L}^q$.

The boundary norm is defined as

$$\| f \|_{q,2,\sim} = \sup_{\pm} \left(\int_{\mathbb{R}^3_\pm} dv |v_z| M(v) \left(\int_{[-\pi,\pi)} dx |f(x,\mp\pi,v)|^q\right)^{\frac{2}{q}}\right)^{\frac{1}{2}},$$

where $\mathbb{R}^3_\pm$ is the set of velocities such that $v_z \gtrless 0$. The set of functions on $[-\pi,\pi) \times \{-\pi\} \times \mathbb{R}^3_+ \cup [-\pi,\pi) \times \{\pi\} \times \mathbb{R}^3_-$ with bounded $\| \cdot \|_{2,2,\sim}$-norm is denoted by L^+.

The stationary solution F_s corresponding to the laminar and convective solutions to the INSF system will be constructed as follows: set

$$F_s = M(1+\Phi_s^\varepsilon).$$

Then

$$\Phi_s^\varepsilon(x,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi_s^{(n)}(x,z,v) + \varepsilon R_{s,\varepsilon}(x,z,v),$$

where $\Phi_s^{(1)} = f_s$ and $\Phi_s^{(j)}$, for $j > 1$ are constructed by means of a bulk-boundary layer expansion already discussed in [5, 1, 2]. Here we summarize the relevant properties of the $\Phi_s^{(n)}$'s in the following theorem taken from [2]:

Proposition 2.1. *The functions $\Phi_s^{(n)}$, $n = 1,\dots,5$ and $\psi_{n,\varepsilon}$ can be determined so as to satisfy the boundary conditions*

$$\begin{aligned}\Phi^{(n)}(x,\mp\pi,v) \quad &= \frac{M_\mp(v)}{M(v)} \int_{w_z \lesseqgtr 0} |w_z| M[\Phi^{(n)}(x,\mp\pi,w) - \psi_{n\varepsilon}(x,\mp\pi,w)]dw \\ &+\psi_{n,\varepsilon}(x,\mp\pi,w),\ t>0,\ v_z \gtrless 0,\end{aligned}$$

and the normalization condition $\int_{\mathbb{R}^3\times[-\pi,\pi]^2} dvdxdz \Phi^{(n)} = 0$, so that the asymptotic expansion in ε for the stationary problem (1.1), truncated to the order 5 is given by

$$F_s^{(exp)}(x,z,v) = M(v)\left(1 + \sum_{n=1}^{5} \varepsilon^5 \Phi^{(n)}(x,z,v)\right).$$

If $G \le G_c$ then the functions $\Phi^{(n)}$'s, corresponding to the laminar solution satisfy the conditions

$$\| \Phi^{(n)} \|_{2,2} \le C\lambda, \quad \| \Phi^{(n)} \|_{\infty,2} \le C\lambda\ , \quad n = 1,\dots,5, \tag{2.1}$$

for a suitable constant C. Moreover if $G \geq G_c$ and $\delta < \delta_1$, then the $\Phi^{(j)}$'s differ from those of the laminar solution by $O(\delta)$ and the inequalities (2.1) are replaced by

$$\| \Phi^{(n)} \|_{2,2} \leq C(\lambda + \delta), \quad \| \Phi^{(n)} \|_{\infty,2} \leq C(\lambda + \delta)\ , \quad n = 1, \dots, 5. \tag{2.2}$$

The functions $\psi_{n,\varepsilon}$ are such that $\|\psi_{n,\varepsilon}\|_{q,2,\sim}$, $q = 2, \infty$ are exponentially small as $\varepsilon \to 0$, and $\int_{\mathbb{R}^3} dv v_z M(v) \psi_{n,\varepsilon} = 0$.

The space where the remainder will be constructed is the following:

$$\mathcal{W}^{q,-} := \{ f : [-\pi, \pi]^2 \times \mathbb{R}^3 \to \mathbb{R} \,|\, \nu^{\frac{1}{2}} f \in \tilde{L}^q,\ \nu^{-\frac{1}{2}} Df \in \tilde{L}^q,\ \gamma^+ f \in L^+ \},$$

for $q = 2$ or $q = \infty$. Here, Df denotes first order derivatives of f and $\gamma^{\pm} f$ are the ingoing (resp. outgoing) trace operators defined as the restrictions of f to the ingoing (resp. outgoing) boundary, $[-\pi, \pi) \times \{-\pi\} \times \mathbb{R}^3_+ \cup [-\pi, \pi) \times \{\pi\} \times \mathbb{R}^3_-$ (resp. $[-\pi, \pi) \times \{-\pi\} \times \mathbb{R}^3_- \cup [-\pi, \pi) \times \{\pi\} \times \mathbb{R}^3_+$).

Before stating the main theorem of this section we recall the properties of the linearized Boltzmann operator L,

$$LR = 2M^{-1} Q(M, MR),$$

defined on a suitable dense subset of $H = L^2_M(\mathbb{R}^3)$, namely $L^2(\mathbb{R}^3)$ with weight M. The space H will be equipped with the inner product $(\,\cdot\,,\,\cdot\,)_H = (\,\cdot\,\sqrt{M}, \,\cdot\, \sqrt{M}\,)_{L^2(\mathbb{R}^3)}$.

The operator L has a non trivial null space. An orthonormal basis in the null space is given by the functions $\psi_0 = 1$, $\psi_1 = v_x$, $\psi_2 = v_y$, $\psi_3 = v_z$ and $\psi_4 = \frac{1}{\sqrt{6}}(|v|^2 - 3)$. The orthogonal projection on the null space of L is denoted by P. For the operator L the decomposition $L = -\nu I + K$ holds, where I is the identity, ν is a positive function of $|v|$ which, for hard sphere is such that $\nu \sim (1 + |v|)$ and K is a compact operator on H. Finally L is symmetric on H and the quadratic form associated to L is negative semi-definite in the sense that there is a positive constant C such that

$$-(f, Lf)_H \leq C((I - P)f, \nu(I - P)f)_H.$$

Now we state the main theorem of this section:

Theorem 2.1. *There are positive ε_0, δ_0 and λ_0 such that given $\lambda < \lambda_0$, $\delta < \delta_0$, for any $\varepsilon < \varepsilon_0$ there exists a stationary solution to (1.1) in the form*

$$F_s = F_s^{(exp)} + \varepsilon R_{s,\varepsilon},$$

with $R_{s,\varepsilon} \in \mathcal{W}^{2,-} \cap \mathcal{W}^{\infty,-}$. The remainder $R_{s,\varepsilon}$, simply denoted by R, solves the boundary value problem

$$\begin{aligned} & v \cdot \nabla^{\mu} R - \varepsilon G M^{-1} \frac{\partial (MR)}{\partial v_z} \\ & = \frac{1}{\varepsilon} LR + \sum_{n=1}^{5} \varepsilon^{n-1} J(\Phi^{(n)}, R) + J(R, R) + \varepsilon A, \end{aligned} \tag{2.3}$$

$$\begin{aligned} R(x, \mp\pi, v) = {} & \frac{M_{\mp}(v)}{M(v)} \int_{w_z \lessgtr 0} |w_z| M(w) \Big(R(x, \mp\pi, w) + \frac{1}{\varepsilon} \bar{\psi}_{\varepsilon}(x, \mp\pi, w) \Big) dw \\ - {} & \frac{1}{\varepsilon} \bar{\psi}_{\varepsilon}(x, \mp\pi, v), \qquad \textit{for } v_z \gtrless 0 \ \textit{ and } x \in [-\pi, \pi], \end{aligned} \tag{2.4}$$

where $\frac{1}{\varepsilon}\bar\psi_\varepsilon = -\sum_{n=1}^{5}\varepsilon^n\psi_{n,\varepsilon}$, $J(h,g) = \dfrac{2}{M}Q(Mh,Mg)$, *and* A *is a smooth function computed in terms of the* $\Phi_s^{(j)}$*'s, bounded in* $\|\cdot\|_{q,2}$, $q=2,\infty$, *and* $\int_{\mathbb{R}^3} dvM(v)A = 0$. *Moreover, the remainder satisfies the impermeability conditions*

$$\int_{\mathbb{R}^3} dvv_z R = 0$$

for $z=\pm\pi$.

The construction of the stationary solution is obtained by an iteration scheme where, in the equation for the iterate R^{n+1}, the non linear term is computed in terms of R^n. Therefore, the main step of the analysis is the study of the equation (2.3) where $A + J(R^n,R^n)$ is replaced by a known function.

As in [2], we introduce the operator L_J for fixed x,z as follows: for any f in the domain of L,

$$L_J f = Lf + \varepsilon NPf, \tag{2.5}$$

where, for a given $L^\infty_M(\Omega\times\mathbb{R}^3)$ function $\mathfrak{q}$, the operator N is defined as

$$Nf = J(\mathfrak{q}, f).$$

In the rest of this section we will use the function

$$\mathfrak{q} = \sum_{n=1}^{5}\varepsilon^{n-1}\Phi_s^{(n)}.$$

With this choice of the function $\mathfrak{q}$, we have

$$\|\mathfrak{q}\|_\infty \le C(\lambda+\delta+\varepsilon), \tag{2.6}$$

for some constant C. In next section there will be a different choice of $\mathfrak{q}$, and the above estimate will be consequently modified.

The operator L_J also has a non trivial null space $\mathtt{Kern}(L_J)$, which is spanned by the vectors $\bar\psi_j$, $j=0,\dots,4$ as proved in [2]. The vectors $\bar\psi_j$'s differ from the ψ_j's for terms of order ε:

$$\bar\psi_j = \psi_j - \varepsilon L^{-1}N\psi_j. \tag{2.7}$$

The operator P_J denotes the orthogonal projector on $\mathtt{Kern}(L_J)$.

We underline that L_J is not symmetric, so we will also consider the adjoint of L_J, denoted L_J^*. The null space of L_J^* coincides with the null space of L, $\mathtt{Kern}(L)$.

The difference $P_J - P$ is estimated as follows:

$$\|\nu^{\frac12}(P_J-P)f\|_{2,2} \le C\varepsilon\|\mathfrak{q}\|_\infty\|f\|_{2,2}, \tag{2.8}$$

for some constant C.

The following proposition replaces Proposition 2.1 in [2]:

Proposition 2.2. *There is* $\varepsilon_0>0$ *such that for* $\varepsilon<\varepsilon_0$ *there are positive constants* c_1 *and* c_2 *such that*

$$-(L_Jf,f)_{2,2} \ge c_1(\nu(I-P_J)f,(I-P_J)f)_{2,2} - c_2\varepsilon^2\|\mathfrak{q}\|^2_\infty\|\nu^{\frac12}P_Jf\|^2_{2,2}, \tag{2.9}$$

$$-(L_J^*f,f)_{2,2} \ge c_1(\nu(I-P)f,(I-P)f)_{2,2} - c_2\varepsilon^2\|\mathfrak{q}\|^2_\infty\|\nu^{\frac12}P_Jf\|^2_{2,2}. \tag{2.10}$$

Proof. By the decomposition $f=(I-P_J)f+P_Jf$, we have

$$-(f,L_Jf)_{2,2} = -((I-P_J)f, L_J(I-P_J)f)_{2,2} - (P_Jf, L_J(I-P_J)f)_{2,2}.$$

The first part is bounded from below as in Proposition 2.1 of [2] by $c(\nu(I-P)f,(I-P)f)_{2,2}$. For the second term, since $(Pf, L_J(I-P_J)f)_{2,2}=0$, we have

$$|(P_Jf, L_J(I-P_J)f)_{2,2}| = |((P_J-P)f, L_J(I-P_J)f)_{2,2}|$$
$$\leq C\varepsilon\|\mathfrak{q}\|_\infty\|f\|_{2,2}\|\nu^{\frac12}(I-P_J)f\|_{2,2}.$$

Then since $\|f\|_{2,2}=\|(I-P_J)f\|_{2,2}+\|P_Jf\|_{2,2}$, for any positive η,

$$\|f\|_{2,2}\|\nu^{\frac12}(I-P_J)f\|_{2,2}\leq \|\nu^{\frac12}(I-P_J)f\|^2_{2,2}(1+\frac{1}{\eta})+\frac{\eta}{4}\|P_Jf\|^2_{2,2}.$$

Thus we obtain (2.9) by choosing $\eta=\frac{3C}{c}\varepsilon\|\mathfrak{q}\|_\infty$. Consequently $c_1=\frac{c}{2}$ and $c_2=\frac{C^2}{c}$. □

The first consequence of the extra term appearing in Proposition 2.2 is in the Green inequality (2.15) of [2] which is modified as follows:

Proposition 2.3 (The Green Inequality)**.** *Consider the linear problem*

$$v\cdot\nabla^\mu f-\varepsilon GM^{-1}\frac{\partial(Mf)}{\partial v_z}=\frac{1}{\varepsilon}L_Jf+g, \tag{2.11}$$

with the prescribed inhomogeneous term g such that $\int Mg dv=0$ and prescribed incoming data

$$f(x,\pm\pi,v)=p(x,\pm\pi,v),\quad v_z\lesseqgtr 0. \tag{2.12}$$

Then, for any $\eta>0$,

$$\begin{aligned}&\|\ \gamma^-f\ \|^2_{2,2,\sim}+\frac{c}{2\varepsilon}\ \|\ \nu^{\frac12}(I-P_J)f\ \|^2_{2,2}\\ &\leq C\Big(\varepsilon\ \|\ \nu^{-\frac12}(I-P_J)g\ \|^2_{2,2}+(\eta+\varepsilon^2\|\mathfrak{q}\|^2_\infty)\ \|\ P_Jf\ \|^2_{2,2}\\ &+\frac{1}{\eta}\ \|\ P_Jg\ \|^2_{2,2}+\|p\|^2_{2,2,\sim}\Big).\end{aligned} \tag{2.13}$$

A similar inequality holds when L_J is replaced by L_J^*.

The proof is the same as in [2], taking into account the modified spectral gap inequality for L_J.

The Fourier transform with respect to the variable x, $\mathcal{F}_xf$ (sometimes just $\mathcal{F}f$ for brevity) is defined as follows: for any $\xi\in\mathbb{Z}$,

$$\mathcal{F}_xf(\xi)=\frac{1}{2\pi}\int_{[-\pi,\pi]}dxe^{-i\xi x}f(x).$$

For $(x,z)\in[-\pi,\pi]^2$, $\hat f(\xi_x,\xi_z)=(\mathcal{F}_x\mathcal{F}_zf)(\xi_x,\xi_z)$. For f function of x, z and v, $<f>$ is the zero Fourier coefficient of f:

$$<f>:=\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}f(x,z,v)dxdz,\quad a.a.\ v\in\mathbb{R}^3 \tag{2.14}$$

and $\tilde f:=f-<f>$.

In the rest of this paper, constants which, independently of the parameter ε, can be made sufficiently small for the purposes of the proofs, will generically be denoted η.

The statement of Lemma 2.1 in [2] holds provided that $\|\mathfrak{q}\|_\infty$ is sufficiently small:

Lemma 2.1. *Let $\varphi(x,z,v)$ be solution to*

$$v\cdot\nabla^{\mu}\varphi-\varepsilon GM^{-1}\frac{\partial(M\varphi)}{\partial v_z}=\frac{1}{\varepsilon}L_J^*\varphi+g, \tag{2.15}$$

periodic in x of period 2π, and with zero ingoing boundary values at $z=-\pi,\pi$. Then, if $\lambda+\delta+\epsilon$ is sufficiently small, it results:

$$\begin{aligned}\parallel\nu^{\frac12}(I-P)\varphi\parallel_{2,2}\quad&\le C\Big(\varepsilon\parallel\nu^{-\frac12}(I-P)g\parallel_{2,2}\\&+\parallel Pg\parallel_{2,2}+\eta\varepsilon\parallel<P\varphi>\parallel_2\Big),\qquad(2.16)\\\parallel\widetilde{P\varphi}\parallel_{2,2}\quad&\le C\Big(\parallel\nu^{-\frac12}(I-P)g\parallel_{2,2}+\frac1\varepsilon\parallel Pg\parallel_{2,2}\\&+\eta\parallel<P\varphi>\parallel_2\Big).\qquad(2.17)\end{aligned}$$

The statement of Lemma 2.1 is still true, if we replace the operator L_J^* with the operator L_J and the operator P with P_J.

Proof. Lemma 2.1 is proved as in [2]. Equation (2.21) in [2] provides a bound for $\|P\varphi\|_{2,2}^2$ in terms of $\varepsilon^{-2}\|\nu^{\frac12}(I-P)\varphi\|_{2,2}^2$. This, by the Green inequality, gives a term $\|\mathfrak{q}\|_\infty^2\|P\varphi\|_{2,2}^2$ in the right hand side, which can be absorbed in the left hand side provided that $\|\mathfrak{q}\|_\infty^2$ is sufficiently small. This is true, by (2.6), provided that $\lambda+\delta+\epsilon$ is sufficiently small. ☐

Put $H(R)=\sum_{n=1}^5\varepsilon^{n-1}J(\Phi_s^{(n)},R)$ and decompose H in accordance with the operator L_J. Set $H_1(\,\cdot\,)=H(\,\cdot\,)-J(q,P\,\cdot)=J(\mathfrak{q},f)-J(\mathfrak{q},Pf)$. We notice that $H_1(\cdot)$ is of order zero in ε and only depends on the non-hydrodynamic projection $(I-P)$.

At the stage $n+1$ of the iterative procedure we need to compute the remainder R^{n+1}, still for brevity denoted by R. We decompose it into two parts R_1 and R_2, solutions of two different equations. The part R_1 is periodic in x and solves the boundary value problem

$$\begin{aligned}&v\cdot\nabla^{\mu}R_1-\varepsilon GM^{-1}\frac{\partial(MR_1)}{\partial v_z}=\frac1\varepsilon L_JR_1+H_1(R_1)+g,\qquad(2.18)\\&R_1(x,\mp\pi,v)=-\frac1\varepsilon\bar\psi(x,\mp\pi,v),\quad v_z\gtrless0,\end{aligned}$$

where the incoming data are prescribed and the inhomogeneous term g includes A and the non linear term computed at the previous step. The part R_2 is discussed later. An existence proof for this problem can be obtained by the method of [12] . The nonhydrodynamic part of R_1 is estimated along the same lines of the proof of Lemma 2.1:

$$\begin{aligned}&\frac1\varepsilon\parallel\gamma^-R_1\parallel_{2,2,\sim}^2+\frac{c}{2\varepsilon^2}\parallel\nu^{\frac12}(I-P_J)R_1\parallel_{2,2}^2\\&\le C\Big(\parallel\nu^{-\frac12}(I-P_J)g\parallel_{2,2}^2+\frac{\eta+\lambda+\delta+\epsilon}{2\varepsilon}\parallel P_JR_1\parallel_{2,2}^2\\&+\frac{1}{2\eta\varepsilon}\parallel P_Jg\parallel_{2,2}^2+\frac1{\varepsilon^3}\parallel\bar\psi\parallel_{2,2,\sim}^2\Big),\end{aligned}$$

for small $\eta>0$, by using the inequality

$$|(R_1,H_1(R_1))_{2,2}|\le C(\parallel\nu^{\frac12}(I-P_J)R_1\parallel_{2,2}^2+\varepsilon^2\|\mathfrak{q}\|_\infty^2\parallel P_JR_1\parallel_{2,2}^2).$$

The duality technique used in [2] can still be applied to estimate $P_J R_1$. The term $H_1(R_1)$ is treated as a perturbation, after dealing in next lemma with the system without it.

Lemma 2.2. *Set $h := P_J R_1$. Then there are $\delta_0 > 0$, $\lambda_0 > 0$ such that for $0 < \delta < \delta_0$ and $0 < \lambda < \lambda_0$, and for any $G \in [0, G_c(1+\delta)]$,*

$$\| h \|_{2,2}^2 \leq C(\| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon^2} \| P_J g \|_{2,2}^2 + \frac{1}{\varepsilon^3} \| \bar{\psi} \|_{2,2,\sim}^2).$$

Proof of Lemma 2.2. We do not repeat the proof given in [2]. We only remind that it is based on the joint analysis of the boundary value problem for R_1 and the "dual" problem for φ, solution to

$$v \cdot \nabla^\mu \varphi - \varepsilon G M^{-1} \frac{\partial (M\varphi)}{\partial v_z} = \frac{1}{\varepsilon} L_J^* \varphi + h,$$

with zero ingoing boundary values at $z = -\pi, \pi$ and φ a 2π-periodic function in x.

By using Lemma 2.1, one is then left with a $< P\varphi >$-term which is the projection of $< \varphi >$. Now $< \varphi >$ is the average over the variable x, and thus satisfies a one dimensional equation similar to eq. (3.5) in [1]. By using the argument of Lemma 3.4 in [1], we obtain

$$\| < P\varphi > \|_2 \leq c \, \| < P\varphi >_x \|_{2,2} \leq \frac{c}{\varepsilon} \| h \|_{2,2} + \eta \| \varphi \|_{2,2},$$

where $< \cdot >_x$ denotes the average on x. □

Remark. We note that in [2], instead of Lemma 3.4 in [1], we used the arguments of Lemma 3.5 in the same paper, which require G small but permit any value of λ. In the present setup the use of Lemma 3.4 in [1] allows us to use $G \in [0, G_c(1+\delta)]$ provided that λ is sufficiently small. This is the only point where the condition G small was used in [2].

The final estimates for R_1 then follow as in [2]:

Lemma 2.3. *If R_1 is a solution to the system (2.18), then, under the same conditions on the parameters as before,*

$$\begin{aligned}
\| \nu^{\frac{1}{2}} R_1 \|_{2,2} \quad &\leq c\Big(\| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2} + \frac{1}{\varepsilon} \| P_J g \|_{2,2} + \varepsilon^{-\frac{3}{2}} \| \bar{\psi} \|_{2,2,\sim} \Big),\\
\| \nu^{\frac{1}{2}} R_1 \|_{\infty,2} \quad &\leq c\Big(\frac{1}{\varepsilon} \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2} + \frac{1}{\varepsilon^2} \| P_J g \|_{2,2} + \varepsilon \| \nu^{-\frac{1}{2}} g \|_{\infty,2}\\
&\quad + \varepsilon^{-\frac{5}{2}} \| \bar{\psi} \|_{2,2,\sim} \Big).
\end{aligned}$$

Now we discuss R_2. It is solution to the following boundary value problem:

$$v \cdot \nabla^\mu R_2 - \varepsilon G M^{-1} \frac{\partial (M R_2)}{\partial v_z} = \frac{1}{\varepsilon} L_J R_2 + H_1(R_2), \tag{2.19}$$

$$R_2(x, \mp\pi, v) = f^-(x, \mp\pi, v) + \frac{M_\mp(v)}{M(v)} \int_{w_z \lessgtr 0} R_2(x, \mp\pi, w) |w_z| M dw,$$
$$v_z \gtrless 0, \tag{2.20}$$

where

$$f^-(x, \mp\pi, v) = \frac{M_\mp(v)}{M(v)} \int_{w_z \lessgtr 0} \Big(R_1(x, \mp\pi, w) + \frac{1}{\varepsilon} \bar{\psi}(x, \mp\pi, w) \Big) |w_z| M dw, \; v_z \gtrless 0.$$

In order to estimate R_2, one can use the arguments given in [2], Lemmas 2.4, 2.5. Indeed the only modifications arise from the extra term in the Green inequality and they are managed by using the smallness of $\mathfrak{q}$ given in (2.6). One thus gets the final estimates for R_2 given in the following

Lemma 2.4. *A solution to the R_2-problem satisfies*

$$\begin{aligned}
&\| \nu^{\frac{1}{2}}(I-P_J)R_2 \|_{2,2}^2 \leq c\Big(\varepsilon \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon} \| P_J g \|_{2,2}^2 \\
&\qquad + \frac{1}{\varepsilon^2} \| \bar{\psi} \|_{2,2,\sim}^2 \Big), \\
&\| P_J R_2 \|_{2,2}^2 \leq c\Big(\frac{1}{\varepsilon} \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon^3} \| P_J g \|_{2,2}^2 + \frac{1}{\varepsilon^4} \| \bar{\psi} \|_{2,2,\sim}^2 \Big), \\
&\| \nu^{\frac{1}{2}} R_2 \|_{\infty,2}^2 \leq c(\frac{1}{\varepsilon^3} \| \nu^{-\frac{1}{2}}(I-P_J)g \|_{2,2}^2 + \frac{1}{\varepsilon^5} \| P_J g \|_{2,2}^2 + \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|_{\infty,2}^2 \\
&\qquad + \frac{1}{\varepsilon^6} \| \bar{\psi} \|_{2,\sim}^2).
\end{aligned}$$

The linear estimates of Lemmas 2.2 and 2.4 are sufficient to prove the existence of the solution to the equation for the remainder. This is Theorem 2.2 in [2], which we restate here:

Theorem 2.2. *There are positive λ_0, δ_0 and ε_0 such that, if $\lambda < \lambda_0$, $\delta < \delta_0$, $\varepsilon < \varepsilon_0$ and $G \in [0, G_c(1+\delta)]$, then there exists a solution R in $L^2_M([-\pi,\pi]^2 \times \mathbb{R}^3)$ to the rest term problem*

$$\begin{aligned}
&v \cdot \nabla^\mu R - \varepsilon G M^{-1} \frac{\partial (MR)}{\partial v_z} = \frac{1}{\varepsilon} LR + \frac{1}{2} J(R,R) + H(R) + \varepsilon A, \qquad (2.21)\\
&R(x,\mp\pi,v) = \int_{w_z \lesseqgtr 0} (R(x,\mp\pi,w) + \frac{1}{\varepsilon}\bar{\psi}(x,\mp\pi,w))|w_z| M_- dw \\
&\qquad - \frac{1}{\varepsilon}\bar{\psi}(x,\mp\pi,v), \quad v_z \gtrless 0.
\end{aligned}$$

3. **Initial boundary value problem.** We now study the initial boundary value problem (1.1) for an initial datum F_0 suitably close to the stationary solution. Indeed, we introduce the perturbation $\Phi = M^{-1}(F - F_s)$. The equation for the perturbation Φ is:

$$\begin{aligned}
&\frac{\partial \Phi^\varepsilon}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla^\mu \Phi^\varepsilon - \frac{G}{M} \frac{\partial (M\Phi^\varepsilon)}{\partial v_z} = \frac{1}{\varepsilon^2}\Big(L\Phi^\varepsilon + \frac{1}{2}J(\Phi^\varepsilon,\Phi^\varepsilon) + J(\Phi^\varepsilon_s,\Phi^\varepsilon)\Big), \qquad (3.1)\\
&\Phi^\varepsilon(0,x,z,v) = \zeta_0(x,z,v), \quad (x,z) \in (-\pi,\pi)^2,\ v \in \mathbb{R}^3,\\
&\Phi^\varepsilon(t,x,\pm\pi,v) = \frac{M_\pm}{M} \int_{w_z \gtreqless 0} |w_z| M \Phi^\varepsilon(t,x,\pm\pi,w) dw,\ v_z \lesseqgtr 0,\ t>0,\ x \in [-\pi,\pi].
\end{aligned}$$

The initial conditions for $M^{-1}(F(0,x,z,v) - F_s(x,z,v)) = \Phi^\varepsilon_0(x,z,v)$ are given with the initial datum Φ_0 specified as follows:

$$\Phi^\varepsilon_0(x,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(0,x,z,v) + \varepsilon^5 p_5 \qquad (3.2)$$

where $\Phi^{(n)}(0,x,z,v)$ is the n-th term of the expansion introduced in the next paragraph, computed at time $t=0$, and the ε-dependent contribution p_5 is arbitrary but for having total mass $\int dvdxdzM(v)p_5(x,z,v) = 0$ and

$$\| p_5 \|_{\infty,2} := \sup_{\varepsilon>0} \left(\int_{\mathbb{R}^3} dv \left| \sup_{(x,z)\in[-\pi,\pi]^2} p_5(x,z,v) \right|^2 M \right)^{\frac{1}{2}} < c, \tag{3.3}$$

for some constant c.

We write also the time dependent solution in terms of a truncated expansion in ε,

$$\Phi^\varepsilon(t,x,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(t,x,z,v) + \varepsilon Y(t,x,z,v), \quad (x,z) \in \Omega,\ v \in \mathbb{R}^3,\ t > 0. \tag{3.4}$$

The first term of the expansion in ε is

$$\Phi^{(1)} = \rho + u \cdot v + \theta \frac{|v|^2 - 3}{2}\ ,$$

where the fields $(u(t,x,z), \theta(t,x,z))$ are solutions of the hydrodynamic equations for the perturbation, while $\rho(t,x,z)$ is determined by the Boussinesq condition (1.12). The hydrodynamic initial data are chosen as follows: let (u_0,θ_0) be an initial perturbation of the convective solution (u_s,θ_s) sufficiently small to ensure that the solution to (1.4), denoted here $(\tilde u(t,x,z), \tilde\theta(t,x,z)) = (u_s(x,z)+u(t,x,z), \theta_s(x,z)+\theta(t,x,z))$, exists globally in time and converges exponentially to (u_s,θ_s) as $t \to +\infty$, as stated in (1.8).

The construction of the time dependent solution is based, as the stationary solution, on an expansion which starts with the solution to the hydrodynamic equations. We need the following proposition on the stability of the hydrodynamic solution, whose proof is referred to the literature [6, 7, 8, 9, 10, 11, 13]:

Proposition 3.1. *For $\delta < \delta_1$, let (u,θ) be the periodic solution of the following equation for the perturbation*

$$\begin{aligned}
&\partial_t u + u_s \cdot \nabla^\mu u + u \cdot \nabla^\mu u_s + u \cdot \nabla^\mu u = \hat\eta \Delta^\mu u - \nabla^\mu p - e_z G\theta,\\
&\partial_t \theta + u_s \cdot \nabla^\mu \theta + u \cdot \nabla^\mu \theta_s + \lambda u_z = \frac{5}{2} \hat k \Delta^\mu \theta,\\
&\nabla^\mu \cdot u = 0,\\
&u(x,z,0) = u_0(x,z), \quad \theta(x,z,0) = \theta_0(x,z), \quad (x,z) \in [-\pi,\pi]\times[-\pi,\pi],\\
&u(x,-\pi,t) = u(x,\pi,t) = \theta(x,-\pi,t) = \theta(x,\pi,t) = 0, \quad x \in [-\pi,\pi], \quad t > 0.
\end{aligned}$$

If $(u_0,\theta_0) \in (H_k)^3$, for k sufficiently large, and $\| u_0 \|_{H_k} + \| \theta_0 \|_{H_k} < n_0$, for n_0 small enough, then there is $\zeta_1 > 0$ such that $(u,\theta)(x,z,t)$ is in $(H_k)^3$ for any $t > 0$ and $\lim_{t\to\infty} (\mathrm{e}^{\zeta_1 t} u, \mathrm{e}^{\zeta_1 t}\theta) = 0$ in $(H_{k'})^3$, for any $k' < k$.

The terms of the expansion $\Phi^{(n)}$, $n = 1,\dots,5$ are constructed by means of an Hilbert type expansion in the bulk, corrected by a boundary layer expansion designed to restore the correct boundary conditions. For the construction of the expansion we refer to [5]. To state next proposition, we need the norms

$$\| f \|_{2t,2,2} = \Big(\int_0^t \int_\Omega \int_{\mathbb{R}^3} |f(s,x,z,v)|^2 M(v) ds dx dz dv \Big)^{\frac{1}{2}},$$

$$\| f \|_{\infty,\infty,2} = \sup_{t>0} \Big(\int_{\mathbb{R}^3} \sup_{(x,z)\in\Omega} |f(t,x,z,v)|^2 M(v) dv \Big)^{\frac{1}{2}}\ .$$

Moreover $\| f \|_{2,2,2}$ is the norm $\| f \|_{2t,2,2}$ with $t = +\infty$. We also use the boundary norms

$$\begin{aligned}
\| f \|_{2t,2,\sim} \qquad &= \Big(\int_0^t \int_{-\pi}^{\pi} \int_{v_z>0} v_z M(v) \mid f(s,x,-\pi,v) \mid^2 dvdxds \Big)^{\frac{1}{2}} \\
&+\Big(\int_0^t \int_{-\pi}^{\pi} \int_{v_z<0} \mid v_z \mid M(v) \mid f(s,x,\pi,v) \mid^2 dvdxds \Big)^{\frac{1}{2}}, \\
\| f \|_{\infty,2,\sim} \qquad &= \Big(\sup_{t>0} \int_{-\pi}^{\pi} \int_{v_z>0} v_z M(v) \mid f(t,x,-\pi,v) \mid^2 dxdv \Big)^{\frac{1}{2}} \\
&+\Big(\sup_{t>0} \int_{-\pi}^{\pi} \int_{v_z<0} \mid v_z \mid M(v) \mid f(t,x,\pi,v) \mid^2 dxdv \Big)^{\frac{1}{2}},
\end{aligned}$$

and, as before, $\| \cdot \|_{2,2,\sim}$ corresponds to $t = +\infty$.

The estimates we need on the terms of the expansion $\Phi^{(n)}$ are summarized in the following proposition, whose proof can be readily obtained along the lines of [5, 1]:

Proposition 3.2. *Assume that at time zero, for some suitably large k,*

$$\| u_0 \|_{H_k} + \| \theta_0 \|_{H_k} < n_0. \tag{3.5}$$

Then for $\delta < \delta_1$ and for n_0 of Proposition 3.1, it is possible to determine the functions $\Phi^{(n)}$, $n = 1, \dots, 5$ and the boundary functions $\psi_{n,\varepsilon}$, $n = 2, \dots, 5$ in the asymptotic expansion so that the following boundary conditions are satisfied:

$$\begin{aligned}
&\Phi^{(n)}(t,x,\mp\pi,v) = \frac{M_{\mp}(v)}{M(v)} \int_{w_z \lesseqgtr 0} |w_z| M \big[\Phi^{(n)}(t,x,\mp\pi,w) - \psi_{n,\varepsilon}(t,x,\mp\pi,w) \big] dw \\
&+\psi_{n,\varepsilon}(t,x,\mp\pi,v), \;\; t > 0, \;\; v_z \gtrless 0, t > 0.
\end{aligned}$$

The $\Phi^{(n)}$ satisfy the zero mass condition

$$\int_{\mathbb{R}^3 \times \Omega_\mu} M \Phi^{(n)} dvdzdx = 0, \qquad t \in \mathbb{R}^+.$$

Moreover, there are constants C and C_1 such that for $n = 1, \dots, 5$,

$$\| e^{\zeta t} \Phi^{(n)} \|_{2,2,2} < C \Big(\| u_0 \|_{H_k} + \| \theta_0 \|_{H_k} \Big),$$

$$\| e^{\zeta t} \Phi^{(n)} \|_{\infty,\infty,2} < C \Big(\| u_0 \|_{H_k} + \| \theta_0 \|_{H_k} \Big),$$

and, for $n = 2 \dots, 5$,

$$\| e^{\zeta t} \psi_{n,\varepsilon} \|_{2,2,\sim} < C \mathrm{e}^{-C_1 \varepsilon^{-1}}, \quad \| e^{\zeta t} \psi_{n,\varepsilon} \|_{\infty,2,\sim} < C \mathrm{e}^{-C_1 \varepsilon^{-1}} \;,$$

for any $0 \le \zeta < \zeta_1$, with ζ_1 the decay rate of the hydrodynamic equation given in Proposition 3.1.

The remainder Y satisfies the following initial boundary value problem:

$$\partial_t Y + \frac{1}{\varepsilon} v \cdot \nabla^\mu Y - G M^{-1} \frac{\partial (MY)}{\partial v_z} = \frac{1}{\varepsilon^2} LY + \frac{1}{2\varepsilon} J(Y,Y) + \frac{1}{\varepsilon} H(Y) + A, \tag{3.6}$$

$$\begin{aligned}
&Y(0,x,z,v) = Y_0(x,z,v) = \varepsilon^4 p_5(x,z,v), \\
&Y(t,x,\mp\pi,v) = \frac{M_{\mp}}{M} \int_{w_z \lesseqgtr 0} (Y(t,x,\mp\pi,w) + \frac{\psi}{\varepsilon}(t,x,\mp\pi,w)) |w_z| M dw \\
&\qquad - \frac{\psi}{\varepsilon}(t,x,\mp\pi,v), \qquad x \in [-\pi,\pi], \quad t > 0, \; v_z > 0,
\end{aligned}$$

where $\psi(t,x,\pm\pi,v)=\sum_{n=1}^{5}\varepsilon^n\psi_{n,\varepsilon}(t,x,\pm\pi,v)$. We have set

$$H(Y)=J(\varepsilon^{-1}\Phi_s^\varepsilon+\bar\Phi,Y),\quad \bar\Phi=\sum_{n=1}^{5}\varepsilon^{n-1}\Phi^{(n)}, \tag{3.7}$$

where we recall that Φ_s^ε is the full stationary solution constructed in Section 2, and $\Phi^{(n)}$ are the terms of the time dependent expansion.

The inhomogeneous term A is such that

$$\int_\Omega dxdz\int_{\mathbb{R}^3}dvA=0.$$

The expression for A is given in [5]. We omit it because we only use the following estimate for A,

Proposition 3.3. *There are $C>0$ and $C_1>0$ such that for any $0\le\zeta<\zeta_1$,*

$$\|e^{\zeta t}A\|_{2,2,2}+\|e^{\zeta t}A\|_{\infty,\infty,2}<C\varepsilon^3, \tag{3.8}$$

and

$$\|e^{\zeta t}\psi\|_{2,2,\sim}+\|e^{\zeta t}\psi\|_{\infty,2,\sim}<C\mathrm{e}^{-C_1\varepsilon^{-1}}.$$

The main result of this section is the stability result:

Theorem 3.1. *There are $\lambda_0>0$, $\delta_0>0$, $\varepsilon_0>0$ (possibly smaller than those introduced in Section 2), n_0 and $\zeta>0$ such that, if $\lambda<\lambda_0$, $\delta<\delta_0$, $G\in[0,G_c(1+\delta))$, $0<\varepsilon<\varepsilon_0$,*

$$\|\, u_0\,\|_{H_k}+\|\,\theta_0\,\|_{H_k}<n_0, \tag{3.9}$$

and p_5 satisfies (3.3), then the solution to the initial boundary value problem (3.1) exists and has the following decay property: there is a constant C independent of ε such that

$$\|\mathrm{e}^{\frac{1}{2}\zeta t}\Phi^\varepsilon\|_{2,2,2}<C\varepsilon^{\frac{7}{2}}. \tag{3.10}$$

In order to prove Theorem 3.1 the strategy we have discussed before consists in writing Φ^ε, as $\Phi^\varepsilon=\bar\Phi+\varepsilon Y$. Since the terms of the expansion are estimated by means of Proposition 3.2, we only need to estimate the remainder term Y. Again by Proposition 3.2, $\bar\Phi$ decays to zero exponentially in t. Therefore, to prove (3.10), we need to show that also the remainder Y decays exponentially. For this purpose, let us fix a positive $\zeta<\zeta_1$ and put $R=\mathrm{e}^{\zeta t}Y$. Then, R is solution of

$$\partial_tR-\zeta R+\frac{1}{\varepsilon}v\cdot\nabla^\mu R-GM^{-1}\frac{\partial(MR)}{\partial v_z} \tag{3.11}$$

$$=\frac{1}{\varepsilon^2}LR+\frac{1}{2\varepsilon}\mathrm{e}^{-\zeta t}J(R,R)+\frac{1}{\varepsilon}H(R))+\bar A, \tag{3.12}$$

$$R(0,x,z,v)=R_0(x,z,v)=\varepsilon^4p_5(x,z,v),$$

$$R(t,x,\mp\pi,v)=\frac{M_\mp}{M}\int_{w_z\lesseqgtr0}(R(t,x,\mp\pi,w)+\frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,w))|w_z|Mdw$$

$$-\frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,v),\qquad x\in[-\pi,\pi],\quad t>0,\ v_z>0.$$

Here $\bar A=\mathrm{e}^{\zeta t}A$ and $\bar\psi(t,x,\pm\pi,v)=\mathrm{e}^{\zeta t}\psi(t,x,\pm\pi,v)$. The estimates of Proposition 3.3 imply that for any $\zeta<\zeta_1$

$$\|\bar A\|_{2,2,2}+\|\bar A\|_{\infty,\infty,2}<C\varepsilon^3, \tag{3.13}$$

and

$$\|\bar{\psi}\|_{2,2,\sim} + \|\bar{\psi}\|_{\infty,2,\sim} < C\mathrm{e}^{-C_1\varepsilon^{-1}}.$$

We follow closely the approach in [2]. Therefore we will just recall the main theorems proved there, which are valid also in the present situation, and give explicitly the proofs when modifications are needed.

We decompose the operator H as in Section 2:

$$H(R) = J(\mathfrak{q}, PR) + H_1(R), \quad \mathfrak{q} = \varepsilon^{-1}\Phi^\varepsilon + \bar{\Phi}.$$

Then we define $L_J = L + \varepsilon J(\mathfrak{q}, PY)$. Note that Proposition 2.2 holds for the newly defined L_J and that, under the assumptions of Theorem 3.1, inequality (2.6) is replaced by

$$\|\mathfrak{q}\|_{2,2,2} \le C(\lambda + \delta + \varepsilon + n_0), \quad \|\mathfrak{q}\|_{\infty,\infty,2} \le C(\lambda + \delta + \varepsilon + n_0). \tag{3.14}$$

We notice that $H_1(R)$ is of order zero in ε, and only depends on the nonhydrodynamic part $(I - P)R$. To solve the equation for R we shall use an iteration procedure based on the decomposition of R in the sum $R_1 + R_2$, where R_1 and R_2 are solutions of two different problems. R_1 solves a problem with prescribed incoming data and prescribed inhomogeneous term, while R_2 solves a problem with diffusive boundary conditions plus prescribed incoming data (depending on R_1), zero initial condition and no inhomogeneous term. We recall that R satisfies the vanishing mass condition

$$\int dxdzdvMR(x, z, v, t) = 0, \quad t \in \mathbb{R}^+,$$

so that we have also $\int dxdzdvMR_1(x, z, v, t) = -\int dxdzdvMR_2(x, z, v, t)$. The equations for R_1 and R_2 are

$$\begin{aligned}
&\varepsilon\frac{\partial R_1}{\partial t} + v\cdot\nabla^\mu R_1 - \varepsilon\frac{G}{M}\frac{\partial(MR_1)}{\partial v_z} = \varepsilon\zeta R_1 + \frac{1}{\varepsilon}L_J R_1 + H_1(R_1) + g, \qquad (3.15)\\
&R_1(0, x, z, v) = R_0(x, z, v),\\
&R_1(t, x, \mp\pi, v) = -\frac{1}{\varepsilon}\psi(t, x, \mp\pi, v), \quad t > 0,\ v_z \gtrless 0.
\end{aligned}$$

$$\begin{aligned}
&\varepsilon\frac{\partial R_2}{\partial t} + v\cdot\nabla^\mu R_2 - \varepsilon\frac{G}{M}\frac{\partial(MR_2)}{\partial v_z} = \varepsilon\zeta R_2 + \frac{1}{\varepsilon}L_J R_2 + H_1(R_2), \qquad (3.16)\\
&R_2(0, x, z, v) = 0,\\
&R_2(t, x, \mp\pi, v) = \frac{M_\mp(v)}{M(v)}\int_{w_z \lesseqgtr 0}\Big(R_1(t, x, \mp\pi, w) + R_2(t, x, \mp\pi, w)\\
&\qquad\qquad + \frac{1}{\varepsilon}\bar{\psi}(t, x, \mp\pi, w)\Big)|w_z|Mdw, \quad t > 0,\ v_z \gtrless 0.
\end{aligned}$$

Note that we have multiplied by ε the equations for R_1 and R_2 because, in some arguments we will use the "microscopic time" $\bar{\tau} = \varepsilon^{-1}t$ and such a rescaling corresponds just to replace $\varepsilon\partial_t$ with $\partial_{\bar{\tau}}$ in above equations. In the problems (3.15) and (3.16) the unknowns R_1 and R_2 are sought for as periodic functions in $x \in [-\pi, \pi)$, and g is some given function, periodic on the same interval, such that $\int Mg(\cdot, x, z, v)dxdzdv \equiv 0$. The existence of the solution, is obtained as in [1].

We start by giving a priori estimates obtained by Green's formula, for the nonhydrodynamic part of R_1 and the outgoing flux $\gamma_- R_1$; multiply (3.15) by $2R_1M\kappa$, (where as in [2], $\kappa = \mathrm{e}^{\varepsilon G(z+\pi)}$) integrate with respect to the variables $(\bar{\tau}, x, z, v)$

over $[0,\overline{T}] \times [0,2\pi]^2 \times \mathbb{R}^3$, integrate by parts and use the spectral inequality for L_J and the bounds $1 \le \kappa(z) \le \mathrm{e}^{2\varepsilon G\pi}$ and (3.14), to obtain, for every $\eta_1 > 0$,

$$\begin{aligned} &\| \gamma^- R_1 \|^2_{2\overline{T},2,\sim} + \| R_1(\overline{T}) \|^2_{2,2} + \frac{1}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_1 \|^2_{2\overline{T},2,2} \\ &\le c\Big(\| R_0 \|^2_{2,2} + \varepsilon \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2\overline{T},2,2} \\ &+ \bar{\eta} \| P_J R_1 \|^2_{2\overline{T},2,2} + \frac{1}{2\eta_1} \| P_J g \|^2_{2\overline{T},2,2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{2\overline{T},2,\sim} \Big), \end{aligned} \tag{3.17}$$

where $\bar{\eta} = \frac{\eta_1}{2} + \varepsilon(\zeta + \lambda + \delta + \epsilon + n_0)$.

We consider now the so called *dual problem*, namely we seek for the space-periodic solutions to a linear problem in the rescaled time variable $\bar{\tau} = \varepsilon^{-1} t$. This problem is discussed in the following lemma, where we use the notation introduced in Section 2, but with the function $\mathfrak{q}$ also time dependent. Next lemma follows as in [2], Lemma 4.1, by taking into account the modified spectral inequality (2.9) and the consequent Green inequality.

We write the analog of (3.17) for the dual problem:

$$\begin{aligned} &\| \gamma^- \varphi \|^2_{2\overline{T},2,\sim} + \| \varphi(\overline{T}) \|^2_{2,2} + \frac{1}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P)\varphi \|^2_{2\overline{T},2,2} \\ &\le c\Big(\varepsilon \| \nu^{-\frac{1}{2}}(I-P)h \|^2_{2\overline{T},2,2} + \bar{\eta} \| P\varphi \|^2_{2\overline{T},2,2} + \frac{1}{2\eta_1} \| Ph \|^2_{2\overline{T},2,2} \Big), \end{aligned}$$

for any solution φ to (3.18) below, with vanishing initial and incoming data. Inequality (3.22) below follows as in [2], page 47.

Lemma 3.1. *Given a x-periodic function h of period 2π, let $\varphi(\bar{\tau}, x, z, v)$ be the x-periodic function solution to*

$$\partial_{\bar{\tau}} \varphi + v \cdot \nabla^{\mu} \varphi - \varepsilon G M^{-1} \frac{\partial(M\varphi)}{\partial v_z} = \frac{1}{\varepsilon} L_J^* \varphi + h, \tag{3.18}$$

with vanishing initial and incoming data. Set $\tilde{\varphi} = \varphi - <\varphi> = \varphi - (2\pi)^{-2} \int \varphi dx dz$.

If the parameters λ, ϵ, δ and n_0 satisfy the assumptions of Theorem 3.1, then there exists η small such that,

$$\begin{aligned} \| \varphi \|_{\infty,2,2} + \| \gamma^- \varphi \|_{2,2,\sim} \le c\Big(\varepsilon^{\frac{1}{2}} \| \nu^{-\frac{1}{2}}(I-P)h \|_{2,2,2} \\ + \varepsilon^{-\frac{1}{2}} \| Ph \|_{2,2,2} + \eta \varepsilon^{\frac{1}{2}} \| <P\varphi> \|_{2,2} \Big), \end{aligned} \tag{3.19}$$

$$\begin{aligned} \| \nu^{\frac{1}{2}}(I-P)\varphi \|_{2,2,2} \le c\Big(\varepsilon \| \nu^{-\frac{1}{2}}(I-P)h \|_{2,2,2} + \| Ph \|_{2,2,2} \\ + \eta\varepsilon \| <P\varphi> \|_{2,2} \Big), \end{aligned} \tag{3.20}$$

$$\begin{aligned} \| \widetilde{P\varphi} \|_{2,2,2} \le c\Big(\| \nu^{-\frac{1}{2}}(I-P)h) \|_{2,2,2} + \varepsilon^{-1} \| Ph \|_{2,2,2} \\ + \eta \| <P\varphi> \|_{2,2} \Big), \end{aligned} \tag{3.21}$$

$$\| <P\varphi> \|_{2,2} \le \frac{c}{\varepsilon} \| h \|_{2,2,2} + \eta \| \nu^{\frac{1}{2}} \varphi \|_{2,2,2} . \tag{3.22}$$

An a priori bound for $P_J R_1$ is obtained in the following lemma based on dual techniques involving the simultaneous considerations of the problems (3.18) and (3.15). Consider first the problem (3.15) without the term $H_1(R_1)$.

Lemma 3.2. *Set* $h := P_J R_1$. *Then*

$$\| h \|^2_{2,2,2} \leq c(\| R_0 \|^2_{2,2} + \| \nu^{-\frac{1}{2}}(I - P_J)g \|^2_{2,2,2} + \frac{1}{\varepsilon^2} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^3} \| \bar{\psi} \|^2_{2,2,\sim}).$$

Proof of Lemma 3.2. In the variables $(\bar{\tau}, x, z, v)$, the function R_1 is 2π-periodic in x and solution to (3.17) with the term $H_1(R)$ missing. Let φ be a 2π-periodic function in x, solution to (3.18) with zero initial values and ingoing boundary values at $z = -\pi, \pi$. We multiply the equation for φ by $\kappa M R_1$ and the one for R_1 by $\kappa M \varphi$, then sum them and integrate on the variables $\bar{\tau} \in [0, \overline{T}]$, $x \in [-\pi, \pi)$, $z \in (-\pi, \pi)$ and $v \in \mathbb{R}^3$. Then we use the periodicity in x to cancel the terms ∂_x and take an integration by parts on the variable z. Using the equilibrium condition

$$v \cdot \nabla^\mu (\kappa M) + \varepsilon G \partial_{v_z}(\kappa M) = 0,$$

we obtain:

$$\begin{aligned}
&\int d\bar{\tau} dx dz dv \big(M \partial_z (v_z \kappa R_1 \varphi) - \varepsilon G \kappa \partial_{v_z}(M R_1 \varphi) \big) + \int d\bar{\tau} dx dz \kappa(z) dv M \partial_{\bar{\tau}}(R_1 \varphi) \\
&= \frac{1}{\varepsilon} \int d\bar{\tau} dx dz dv M \kappa \Big[(L_J((I - P_J)R_1)(I - P)\varphi) + ((I - P_J)R_1 L_J^*(I - P)\varphi) \Big] \\
&+ \int d\bar{\tau} dx dz dv M \kappa \big[g\varphi + h P_J R_1 \big] + \varepsilon\zeta \int d\bar{\tau} dx dz dv M \kappa \varphi R_1.
\end{aligned}$$

We use the above equation to get an estimate for the term before the last in the r.h.s: $hP_J R_1 = h^2$. All the terms are estimated as in [2] but we give the explicit computation here for sake of completeness. We need to track the ζ-term and take care of the terms due to the modified spectral inequality. The last term is bounded as

$$\varepsilon\zeta \Big| \int d\bar{\tau} dx dz dv M \kappa \varphi R_1 \Big| \leq \zeta C \Big(\| R_1 \|^2_{2\overline{T},2,2} + \varepsilon^2 \| \varphi \|^2_{2\overline{T},2,2} \Big).$$

Therefore, for any arbitrary choice of K_i, $i = 0, 1, \ldots, 4$ we get, for ζ small,

$$\begin{aligned}
\| h \|^2_{2\overline{T},2,2} \quad &\leq \frac{K_1}{2} \| R_1(\overline{T}, \cdot, \cdot) \|^2_{2,2} + \frac{1}{2K_1} \| \varphi(\overline{T}, \cdot, \cdot) \|^2_{2,2} \\
&+ \frac{K_0}{2} \| \gamma^- R_1 \|^2_{2\overline{T},2,\sim} + \frac{1}{2K_0} \| \gamma^- \varphi \|^2_{2\overline{T},2,\sim} \\
&+ \Big(\frac{K_3}{2\varepsilon} + \zeta C \Big) \| \nu^{\frac{1}{2}}(I - P_J)R_1 \|^2_{2\overline{T},2,2} + \Big(\frac{1}{2K_3\varepsilon} \Big) \| \nu^{\frac{1}{2}}(I - P)\varphi \|^2_{2\overline{T},2,2} \\
&+ \frac{K_4}{2} \| \nu^{-\frac{1}{2}}(I - P_J)g \|^2_{2\overline{T},2,2} + \Big(\frac{1}{2K_4} + \varepsilon^2 \zeta C \Big) \| \nu^{\frac{1}{2}}(I - P)\varphi \|^2_{2\overline{T},2,2} \\
&+ \frac{K_2}{2} \| P_J g \|^2_{2\overline{T},2,2} + \Big(\frac{1}{2K_2} + \varepsilon^2 \zeta C \Big) \| P\varphi \|^2_{2\overline{T},2,2}.
\end{aligned}$$

All the φ-terms computed at time $\overline{T}$ on the l.h.s can be estimated using (3.19)–(3.22) in Lemma 3.1. Using the Green inequality (3.17) to bound the R_1-terms, we obtain, for $\bar{T} \to \infty$,

$$
\begin{aligned}
&\| h \|^2_{2,2,2} \\
&\leq c\Bigg[(K_0+K_1+K_3+\varepsilon\zeta) \| R_0 \|^2_{2,2} + \Big(\frac{K_1+K_0+K_3}{\varepsilon^2} + \zeta\varepsilon^{-1}\Big) \| \bar\psi \|^2_{2,2,\sim} \\
&\quad + \Big(\varepsilon K_1 + \varepsilon K_0 + \varepsilon K_3 + K_4 + \zeta\Big) \| \nu^{-\frac12}(I-P_J)g \|^2_{2,2,2} \\
&\quad + \Big(\frac{1}{\varepsilon K_0} + \frac{1}{\varepsilon K_1} + \frac{1}{\varepsilon^2 K_2} + \frac{1}{\varepsilon K_3} + \frac{1}{K_4} + \varepsilon^2\zeta\Big) \| h \|^2_{2,2,2} \\
&\quad + \Big(\frac{K_1+K_0}{\eta_1} + \frac{K_3}{\eta_1} + K_2 + \varepsilon\zeta\Big) \| P_J g \|^2_{2,2,2} \\
&\quad + \Big(\bar\eta K_0 + \bar\eta K_1 + \bar\eta K_3 + \zeta\varepsilon\bar\eta\Big) \| P_J R_1 \|^2_{2,2,2} \\
&\quad + \eta\Big(\frac{\varepsilon}{K_1} + \frac{\varepsilon}{K_3} + \frac{\varepsilon^2}{K_4} + \frac{1}{K_2} + \varepsilon^2\zeta\Big) \|< P\varphi >\|^2_{2,2} \Bigg].
\end{aligned}
$$

The term $< P\varphi >$ is bounded by using (3.22) in Lemma 3.1 as

$$\|< P\varphi >\|^2_{2,2} \leq c\frac{1}{\varepsilon^2} \| h \|^2_{2,2,2} .$$

We recall that $\bar\eta = \eta_1 + \varepsilon(\zeta+\lambda+\delta+\epsilon+n_0)$. So choosing ε small, then K_1, K_0 and K_3 (resp. K_2) of order ε^{-1} (resp. ε^{-2}) times a big constant, K_4 big and η_1, η_2 of order ε times a small constant and $\zeta+\lambda+\delta+\epsilon+n_0$ sufficiently small, leads to

$$
\begin{aligned}
\| h \|^2_{2,2,2} \quad &\leq c\Big(\frac{1}{\varepsilon} \| R_0 \|^2_{2,2} + \| \nu^{-\frac12}(I-P_J)g \|^2_{2,2,2} + \frac{1}{\varepsilon^2} \| P_J g \|^2_{2,2,2} \\
&\quad + \frac{1}{\varepsilon^3} \| \bar\psi \|^2_{2,2,\sim} + \eta \|< P_J R_1 >\|^2_{2,2,2} \Big).
\end{aligned}
$$

□

The final estimates for R_1 are summarized in

Lemma 3.3. *The solution R_1 to (3.15) satisfies*

$$
\begin{aligned}
\| \nu^{\frac12} R_1 \|_{2,2,2} \quad &\leq c\Big(\| R_0 \|_{2,2} + \| \nu^{-\frac12}(I-P_J)g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon} \| P_J g \|_{2,2,2} + \varepsilon^{-\frac32} \| \bar\psi \|_{2,2,\sim} \Big), \\
\| R_1 \|_{\infty,2,2} \quad &\leq c\Big(\| R_0 \|_{2,2} + \| \nu^{-\frac12}(I-P_J)g \|_{2,2,2} + \frac{1}{\varepsilon} \| P_J g \|_{2,2,2} \\
&\quad + \varepsilon^{-\frac32} \| \bar\psi \|_{2,2,\sim} \Big), \\
\| \nu^{\frac12} R_1 \|_{\infty,\infty,2} \quad &\leq c\Big(\varepsilon^{-1} \| R_0 \|_{2,2} + \| R_0 \|_{\infty,2} + \frac{1}{\varepsilon} \| \nu^{-\frac12}(I-P_J)g \|_{2,2,2} \\
&\quad + \frac{1}{\varepsilon^2} \| P_J g \|_{2,2,2} + \varepsilon \| \nu^{-\frac12} g \|_{\infty,\infty,2} + \varepsilon^{-\frac52} \| \bar\psi \|_{2,2,\sim} \\
&\quad + \frac{1}{\varepsilon} \| \bar\psi \|_{\infty,2,\sim} \Big).
\end{aligned}
$$

Proof of Lemma 3.3. The solution R_1 of (3.15) without H_1-term satisfies

$$\begin{aligned}&\frac{1}{\sqrt{\varepsilon}}\parallel \gamma^- R_1 \parallel_{2,2\sim} + \sup_{t\geq 0}\parallel R_1(t)\parallel_{2,2} + \frac{1}{\varepsilon}\parallel \nu^{\frac{1}{2}}(I-P_J)R_1\parallel_{2,2,2}\\ &\leq c\Big(\parallel R_0\parallel_{2,2} + \varepsilon^{-\frac{3}{2}}\parallel \bar{\psi}\parallel_{2,2,\sim} + \parallel \nu^{-\frac{1}{2}}(I-P_J)g\parallel_{2,2,2}\\ &+\frac{\bar{\eta}}{\sqrt{\varepsilon}}\parallel P_J R_1\parallel_{2,2,2} + \frac{1}{\eta\sqrt{\varepsilon}}\parallel P_J g\parallel_{2,2,2}\Big),\end{aligned}$$

with $\bar{\eta} = \eta + \varepsilon(\zeta+\lambda+\delta+\epsilon+n_0)$, for any $\eta > 0$. Moreover, it follows from Lemma 3.2 that

$$\begin{aligned}\parallel P_J R_1\parallel_{2,2,2} \quad &\leq c\Big(\parallel R_0\parallel_{2,2} + \parallel \nu^{-\frac{1}{2}}(I-P_J)g\parallel_{2,2,2} + \frac{1}{\varepsilon}\parallel P_J g\parallel_{2,2,2}\\ &+\frac{1}{\varepsilon\sqrt{\varepsilon}}\parallel \bar{\psi}\parallel_{2,2,\sim}\Big).\end{aligned}$$

Choosing $\eta = \sqrt{\varepsilon}$ leads to the first two inequalities of Lemma 3.3. The last inequality of Lemma 3.3 is obtained as in [1], by studying the solution along the characteristics. Adding the term $\varepsilon^{-1}H_1(R_1)$ does not change these results. □

The remaining part R_2 of R satisfies the problem (3.16). Its analysis is more involved and will use a careful study of the Fourier transform of R_2. The existence for the problem (3.16) can be adapted from the corresponding study in [12], if one includes into that approach the spectral estimate for L_J, and the characteristics due to the force term.

In (3.16) the given indata part is

$$\begin{aligned}f^-(t,x,\mp\pi,v) \quad &= \frac{M_\mp}{M}\int_{w_z\lessgtr 0}\Big(R_1(t,x,\mp\pi,w) + \frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,w)\Big)|w_z|M dw,\\ &v_z \gtrless 0.\end{aligned}$$

By Green's formula for (3.16), and noting that $H_1(R_2)$ only depends on $(I-P)R_2$, we get

$$\begin{aligned}&\varepsilon\|R_2(t)\|^2_{2,2} + \parallel \gamma^- R_2\parallel^2_{2t,2,\sim} + \frac{c}{\varepsilon}\parallel \nu^{\frac{1}{2}}(I-P_J)R_2\parallel^2_{2t,2,2}\\ &\leq \parallel \gamma^+ R_2\parallel^2_{2t,2,\sim} + \varepsilon\bar{\delta}\|P_J R_2\|^2_{2t,2,2}\ ,\end{aligned} \tag{3.23}$$

with $\bar{\delta} = \zeta+\lambda+\delta+\varepsilon+n_0$. By arguing along the same lines of [2], pag. 142-143, we can estimate the outgoing flux part of R_2 appearing in the r.h.s. of (3.23) and thus obtain

$$\varepsilon\|R_2\|^2_{2,2}(t) + \frac{c}{\varepsilon}\parallel \nu^{\frac{1}{2}}(I-P_J)R_2\parallel^2_{2t,2,2} \leq \frac{1}{\varepsilon\eta}\parallel f^-\parallel^2_{2t,2,\sim} + C\varepsilon(\bar{\delta}+\eta)\parallel P_J R_2\parallel^2_{2t,2,2},$$

$$\parallel \gamma^- R_2\parallel^2_{2t,2,\sim} \leq \frac{1}{\varepsilon^2}\parallel f^-\parallel^2_{2t,2,\sim} + C\parallel P_J R_2\parallel^2_{2t,2,2}\ . \tag{3.24}$$

The hydrodynamic estimates for R_2 are obtained in two steps: first we consider a 1-d (x-independent) case, with an inhomogeneous term g_1 which will take into account the x-dependence in later proofs,

$$\varepsilon\frac{\partial R_2}{\partial t} + v_z\frac{\partial R_2}{\partial z} - \varepsilon G M^{-1}\frac{\partial(MR_2)}{\partial v_z} = \frac{1}{\varepsilon}L_J R_2 + H_1(R_2) + \tilde{g}_1, \tag{3.25}$$

where $\tilde{g}_1 = g_1 + \varepsilon\zeta R_2$.

The 1 -dimensional Lemma 4.4 in [2] holds without changes and we just quote it without proof:

Lemma 3.4.

$$\| P_J R_2 \|_{2,2,2}^2 \leq \frac{c_1}{\varepsilon^2} \| f^- \|_{2,2\sim}^2 + c_2(\| P_J R_1 \|_{2,2,2}^2 + \| \nu^{-\frac{1}{2}} \tilde{g}_1 \|_{2,2,2}^2).$$

By the relation between $\tilde{g}_1$ and g_1 and the Green inequality, we also have

$$\| P_J R_2 \|_{2,2,2}^2 \leq c\Big(\frac{1}{\varepsilon^2} \| f^- \|_{2,2\sim}^2 + \| P_J R_1 \|_{2,2,2}^2 + \| \nu^{-\frac{1}{2}} g_1 \|_{2,2,2}^2 \Big).$$

Now we have to examine the 2-dimensional case. This is treated in [2] in Lemma 4.5. The inclusion of the term $\varepsilon\zeta R_2$ can be handled as before. We sketch the approach and give the details for the R_{20}-moment.

Lemma 3.5. *Let R_2 be solution to (3.16). Then there is $c > 0$ such that*

$$\| P_J R_2 \|_{2,2,2}^2 \leq c(\frac{1}{\varepsilon^2} \| f^- \|_{2,2\sim}^2 + \| P_J R_1 \|_{2,2,2}^2).$$

Proof. The equation for $\hat{R}_2 = \mathcal{F}_x \mathcal{F}_z R_2$, the Fourier transform in x, z of R_2, is

$$\varepsilon \frac{\partial}{\partial t} \hat{R}_2 + i\mu v_x \xi_x \hat{R}_2 + i v_z \xi_z \hat{R}_2 - \varepsilon G M^{-1} \frac{\partial}{\partial v_z} (M \hat{R}_2) \\ = \varepsilon^{-1} \widehat{L_J R_2} + \widehat{H_1(R_2)} - v_z r (-1)^{\xi_z} + \varepsilon\zeta \hat{R}_2, \quad (3.26)$$

r denoting the difference between the ingoing and outgoing boundary values,

$$r(t,x,v) = R_2(t,x,\pi,v) - R_2(t,x,-\pi,v). \quad (3.27)$$

For any function $\phi(v)$ we denote $R_{2\phi} = \int dv M R_2 \phi(v)$. In particular, we denote $R_{20} = \int dv M R_2$ and $R_{24} = \int dv M R_2 v^2$. All the functions below depend on t but we omit such a dependence.

First, we consider the case $\xi_x = 0$. We apply Lemma 3.4 to $\hat{R}_2(0, \xi_z, v) = \int dx R_2(x,z,v)$. By integrating (3.16) over x and taking into account the periodic conditions in the direction x, we get the 1-dimensional equation (3.25), where the term g_1 comes from the x-dependent terms in the expansion appearing in L_J. Since the limiting solution is close to the laminar 1-dimensional solution up to order δ, g_1 is of order δ and is linear in R_2. Thus, by Lemma 3.4 we get a bound for the Fourier components $P_J \hat{R}_2(0, \xi_z)$, for δ small.

Then we need to estimate $P_J \hat{R}_2(\xi_x, \xi_z)$ for $\xi_x \neq 0$. Arguments similar to those used in the proof of Lemma 4.1 in [2] (Lemma 3.1 here) imply that large values of ξ can be dealt with by taking advantage of the factor $|\xi|^{-2}$ and the estimates for r due to the inequality (3.24). Therefore we need only to consider finitely many (ξ_x, ξ_z) with $\xi_x \neq 0$.

The strategy used in [2] and repeated here is to get estimates of all the hydrodynamic moments R_{2v_x}, R_{2v_y}, R_{2v_z}, R_{2v^2} in terms of R_{20} which is estimated at the end.

The first moment considered is the v_x-moment for $\xi_z = 0$. Multiplying (3.26) by M and integrating over the velocity we get an equation for $\hat{R}_{20}$. Multiplying the conjugate of (3.26) by $v_x M$ and integrating over the velocity we get an equation for $\hat{R}^*_{2v_x}(\xi_x, 0)$. Then, we multiply the first by $\hat{R}^*_{2v_x}(\xi_x, 0)$ and the second by $\hat{R}_{20}$.

Summing the two it results,

$$\varepsilon\frac{\partial}{\partial t}\big(\hat{R}_{20}(\xi_x,0)\hat{R}^*_{2v_x}(\xi_x,0)\big) = i\mu\xi_x\hat{R}_{2v_x}(\xi_x,0)\hat{R}^*_{2v_x}(\xi_x,0)$$
$$-i\mu\xi_x\hat{R}^*_{2v_x^2}(\xi_x,0)\hat{R}_{20}(\xi_x,0) + \hat{R}^*_{2v_x}(\xi_x,0)\int dvMv_zr$$
$$+\hat{R}_{20}(\xi_x,0)\int dvMv_xv_zr^* + 2\varepsilon\zeta\hat{R}_{20}(\xi_x,0)\hat{R}^*_{2v_x}(\xi_x,0).$$

We want to get an estimate of the time integral of the first term on the r.h.s. and hence of $\|\ \hat{R}_{2v_x}(\xi_x,0)\ \|^2_{2,2,2}$. To this end, we integrate over the time variable on the interval $[0,t]$. The integration of the time derivative produces a term at time $t=0$ which vanishes because R_2 has 0 initial conditions and a term computed at time t. Such a term is estimated by using the Green inequality. The first boundary term is estimated by noticing that $\int dvMv_zr$ depends only on f_- and the second boundary term is estimated by using (3.24). The result is

$$\int |\hat{R}_{2v_x}|^2(\xi_x,0)dt \le C\int dt\Big(|\hat{R}_{20}|^2(\xi_x,0) + \eta\ \|\ PR_2\ \|^2_{2,2}$$
$$+\ \|\ \nu^{\frac12}(I-P)R_2\ \|^2_{2,2} + \frac{1}{\varepsilon^2}\ \|\ \nu^{\frac12}(I-P_J)R_2\ \|^2_{2,2} + \frac{1}{\varepsilon^2}\ \|\ f^-\ \|^2_{2\sim}\Big), \tag{3.28}$$

where η is some constant that can be made small by assuming the parameters λ, ζ, δ, ε and n_0 sufficiently small.

This is the simplest case, but the other moments R_{2v_x}, for $\xi_z \neq 0$, and R_{2v_y}, R_{24} are obtained by a similar approach, see [2], and the contribution from $\varepsilon\zeta R_2$ produces a term of the form $\varepsilon\zeta\ \|\ P_JR_2\ \|^2_{2,2,2}$ which is absorbed under the smallness assumption for the parameters. We conclude:

$$\int_0^\infty dt\Big(|\hat{R}_{2v_z}|^2 + |\hat{R}_{2v_x}|^2 + |\hat{R}_{2v_y}|^2 + |R_{24}|^2\Big) \tag{3.29}$$
$$\le C\int dt\Big(\ \|\ R_{20}\ \|^2_{2,2} + \eta\ \|\ PR_2\ \|^2_{2,2}$$
$$+\ \|\ \nu^{\frac12}(I-P)R_2\ \|^2_{2,2} + \frac{1}{\varepsilon^2}\ \|\ \nu^{\frac12}(I-P_J)R_2\ \|^2_{2,2} + \frac{1}{\varepsilon^2}\ \|\ f^-\ \|^2_{2\sim}\ \Big).$$

The moment $\hat{R}_{20}(\xi_x,\xi_z)$ for $\xi_x \neq 0$ requires a different analysis. Below, for any function $h(t,x,z,v)$ we denote

$$\hat{h}(\sigma,\xi_x,\xi_z,v) = \mathcal{F}_t\mathcal{F}_x\mathcal{F}_zh(\sigma,\xi_x,\xi_z,v), \quad \text{and} \quad \hat{h}^z(\sigma,\xi_x,z,v) = \mathcal{F}_t\mathcal{F}_xh(\sigma,\xi_x,z,v).$$

Let us start with $\xi_z = 0$. We introduce the cutoff function $\beta(\bar\tau)$ supported in $(0,+\infty)$ with value 1 for $\bar\tau > \tau_0$ for some positive τ_0 and consider the partial Fourier transform of βR_2, $(\widehat{\beta R_2})^z$ and the full Fourier transform $\widehat{\beta R_2}$. They satisfy the equations

$$(i\varepsilon\sigma + i\mu\xi_xv_x)(\widehat{\beta R_2})^z + v_z\partial_z(\widehat{\beta R_2})^z =$$
$$\varepsilon GM^{-1}\partial_{v_z}(M(\widehat{\beta R_2})^z) + \frac{1}{\varepsilon}L_J\widehat{(\beta R_2)}^z + H_1\widehat{(\beta R_2)}^z + \varepsilon\zeta(\widehat{\beta R_2})^z + i\varepsilon(\widehat{\beta' R_2})^z.$$
$$(\varepsilon i\sigma + \mu v_xi\xi_x + iv_z\xi_z)\widehat{\beta R_2} + v_z\widehat{\beta r}(-1)^{\xi_z} = \tag{3.30}$$
$$\varepsilon M^{-1}G\partial_{v_z}\widehat{\beta R_2} + \varepsilon^{-1}\widehat{L_J\beta R_2} + \beta\widehat{H_1(R_2)} + \varepsilon\zeta\widehat{\beta R_2} + \varepsilon\widehat{\beta' R_2}.$$

When $\tau_0 \to 0$, β tends to the Heaviside function and its derivative to the δ-function in $\bar\tau = 0$. Thus the last term in the first of (3.30) vanishes because R_2 is initially 0. Therefore, we get

$$(i\varepsilon\sigma + i\mu\xi_x v_x)\hat{R}_2^z + v_z\partial_z \hat{R}_2^z = \varepsilon G M^{-1}\partial_{v_z}(M\hat{R}_2^z) + \frac{1}{\varepsilon}L_J\hat{R}_2^z + H_1\hat{R}_2^z + \varepsilon\zeta\hat{R}_2^z.$$

Now we multiply by M and integrate on z and v. Thus we obtain:

$$(i\varepsilon\sigma - \varepsilon\zeta)\hat{R}_{20}^z + i\mu\xi_x\hat{R}_{2v_x}^z + \mathcal{F}_{\bar{\tau}}\mathcal{F}_x f_{v_z}^- = 0. \tag{3.31}$$

Recall that, by the argument in the beginning of the proof, we only need to consider bounded $|\xi_x|$ and let $\bar{\xi}_x$ be the maximum value for $|\xi_x|$ that we have to deal with. Now we consider σ's such that $|\sigma| > \sigma_1$ where σ_1 is chosen so that

$$\left|\frac{\mu\bar{\xi}_x}{\varepsilon\sigma_1}\right|^2 < \frac{1}{4C},$$

C being the constant appearing in inequality (3.29). Therefore, denoting by $\chi_{\sigma_1}(\sigma)$ the characteristic function of $[-\sigma_1, \sigma_1]$, we have

$$\begin{aligned}\int d\sigma(1-\chi_{\sigma_1})|\hat{R}_{20}^z(\sigma,\xi_x,0)|^2 \le \frac{1}{2}\int |\hat{R}_{20}^z(\sigma,\xi_x,0)|^2 + C\Big(\eta \parallel PR_2 \parallel_{2,2,2}^2 \\ + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2}\parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2}\parallel f^- \parallel_{2\sim}^2\Big).\end{aligned} \tag{3.32}$$

Hence

$$\begin{aligned}\frac{1}{2}\int d\sigma(1-\chi_{\sigma_1})|\hat{R}_{20}^z(\sigma,\xi_x,0)|^2 \le \frac{1}{2}\int \chi_{\sigma_1}|\mathcal{F}_x R_{20}(\sigma,\xi_x,0)|^2 + C\Big(\eta \parallel PR_2 \parallel_{2,2,2}^2 \\ + \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2}\parallel \nu^{\frac{1}{2}}(I-P_J)R_2 \parallel_{2,2}^2 + \frac{1}{\varepsilon^2}\parallel f^- \parallel_{2\sim}^2\Big).\end{aligned} \tag{3.33}$$

Finally, we have to deal with the $|\sigma| < \sigma_1$. The right-hand sides of (3.30) contain only terms that can be estimated by contributions either involving the non hydrodynamic part or the hydrodynamic one multiplied by a small factor. Therefore, we can use the arguments of [2] to conclude that, for $\varepsilon\sigma \le \sigma_1$,

$$\begin{aligned}&\parallel \chi_{\sigma_1}\hat{R}_{20}(\cdot,\xi_x,0)\parallel_2^2 \\ \le & c\Big(\frac{1}{\varepsilon^2}\parallel (I-P_J)R_2\parallel_{2,2,2}^2 + \parallel (I-P)R_2\parallel_{2,2,2}^2 + \eta\parallel R_2\parallel_{2,2,2}^2\Big).\end{aligned} \tag{3.34}$$

By summing the last two bounds, we get the estimate of $\hat{R}_{20}$ for $\xi_z = 0$. Then one can repeat the same argument for $\xi_z \ne 0$. The boundary term can be removed by subtracting the equation for $\xi_z = 0$ from the second of (3.30). Using the estimate of R_{20} in (3.29) the proof of the lemma is concluded. □

We summarize the results on R_2 in the following

Lemma 3.6. *If the parameters satisfy the conditions of Theorem 3.1, any solution R_2 to the problem (3.16) satisfies the a priori estimates*

$$\begin{aligned}\parallel \nu^{\frac{1}{2}}(I-P_J)R_2\parallel_{2,2,2}^2 &\le c\Big(\varepsilon\parallel R_0\parallel_{2,2}^2 + \varepsilon\parallel \nu^{-\frac{1}{2}}(I-P_J)g\parallel_{2,2,2}^2 \\ &\quad + \frac{1}{\varepsilon}\parallel P_J g\parallel_{2,2,2}^2 + \frac{1}{\varepsilon^2}\parallel \bar{\psi}\parallel_{2,2,\sim}^2\Big), \\ \parallel P_J R_2\parallel_{2,2,2}^2 &\le c\Big(\frac{1}{\varepsilon}(\parallel R_0\parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}}(I-P_J)g\parallel_{2,2,2}^2) \\ &\quad + \frac{1}{\varepsilon^3}\parallel P_J g\parallel_{2,2,2}^2 + \frac{1}{\varepsilon^4}\parallel \bar{\psi}\parallel_{2,2,\sim}^2\Big),\end{aligned}$$

$$\begin{aligned}\| \nu^{\frac{1}{2}} R_2 \|^2_{\infty,\infty,2} &\le c(\frac{1}{\varepsilon^2} \| R_2 \|^2_{\infty,2,2} + \| \gamma^- R_1 \|^2_{\infty,2,\sim} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty,2,\sim}) \\ &\le c\Big(\frac{1}{\varepsilon^3} \| R_0 \|^2_{2,2} + \frac{1}{\varepsilon^3} \| \nu^{-\frac{1}{2}} (I - P_J) g \|^2_{2,2,2} \\ &+ \frac{1}{\varepsilon^5} \| P_J g \|^2_{2,2,2} + \frac{1}{\varepsilon^6} \| \bar{\psi} \|^2_{2,2,\sim} + \| R_0 \|^2_{\infty,2} \\ &+ \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|^2_{\infty,\infty,2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty,2,\sim} \Big).\end{aligned}$$

Using the previous lemmas it is standard to prove the following

Theorem 3.2. *There exists a solution Y to the rest term problem (3.6) such that*

$$\int_0^{+\infty} \int_{[-\pi,\pi]} \int_{[-\pi,\pi]} \int_{\mathbb{R}^3} |e^{\frac{1}{2}\zeta t} Y(t,x,z,v)|^2 M(v) dt dx dz dv < c\varepsilon^7. \tag{3.35}$$

Thus the proof of Theorem 3.1 is complete.

The positivity of the solution to the problem (1.1) is obtained by a suitable modification of the argument in [3] to which we refer for details.

Acknowledgments. We wish to warmly thank Yan Guo who pointed out to us the error in the spectral inequality in [2].

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Received May 2012; revised July 2012.

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