Existence and uniqueness of the electric potential profile in the edge of tokamak plasmas when constrained by the plasma-wall boundary physics.

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Abstract

The electric potential plays a key role in the confinement properties of tokamak plasmas, with the subsequent impact on the performances of fusion reactors. Understanding its structure in the peripheral plasma – interacting with solid materials – is of crucial importance, since it governs the boundary conditions for the burning core plasma. This paper aims at highlighting the dedicated impact of the plasma-wall boundary layer on this peripheral region. Especially, the physics of plasma-wall interactions leads to non-linear constraints along the magnetic field. In this framework, the existence and uniqueness of the electric potential profile are mathematically investigated. The working model is two-dimensional in space and time evolving.

Keywords : Plasma, Controlled fusion, Fluid model, nonlinear boundary conditions, Existence, Uniqueness, Energy estimate.

1 Introduction

The growing need for new sources of energy is one of the drives of the ITER project [13]. The latter is an experiment dedicated to operate a plasma in conditions close to ignition, namely a state where the energy injected to maintain the fusion reactions is negligible compared to the fusion energy production. To our present knowledge, tokamaks are the best candidates permitting the nuclear reactions to take place, in particular allowing to confine high temperature plasmas. These toroidal devices are characterised by two periodic directions, the toroidal angle – around the main symmetry axis – and the poloidal angle – around the interior cross-section. The confinement properties in the third – radial – direction are achieved thanks to strong magnetic fields. Good confinement is mandatory since thermonuclear reactions impose temperatures in the range of 100 million Kelvins in the core. Strong gradients build-up between the plasma and the plasma facing components. Although the strong magnetic field used in the experiments is sufficient to balance the losses associated to particle Coulomb collisions, it is found that the level of transport is much larger and, as a consequence, confinement performance

reduced. There is now a consensus to attribute this loss of confinement to turbulent activity and in particular to that driven by electric field fluctuations, the so-called electrostatic turbulence.
The understanding of turbulent transport is a matter of active research [6]. In particular the peripheral plasma is widely studied, due to its crucial role to control the interaction of the plasma with the surrounding wall material, but also to define the boundary conditions of the core plasma.

The present paper is dedicated to the mathematical analysis of the slowly varying electric potential that leads to poloidal flows in the edge plasma. These radially sheared flows play an essential role in the saturation of turbulence [16]. Besides, they are suspected to the essential ingredient in the triggering of the so-called H-mode, where the plasma spontaneously develops an edge transport barrier characterised by reduced turbulence and associated transport [4]. These poloidal flows are basically driven by equilibrium constraints, including boundary conditions in the SOL, and by the turbulence itself via non linear interactions [5]. We restrict our analysis to the equilibrium part of this flow. In particular, one investigates the existence and uniqueness of a slowly evolving solution to the key equation of electrostatic turbulence, namely the equation that governs the evolution of the electric potential for such a turbulence. Since the large scale radial structure of the electric potential leads to the large scale poloidal flows associated to the electric drift effects, we focus on the analysis of the mean potential depending only on the radial direction and the parallel one, which introduces nonlinear boundary conditions. Furthermore, assuming that the magnitude of the fluctuations are small compared to the mean value, one can average the evolution equation of the electric potential on all the small scale effects. For small amplitudes of the fluctuations, one can also neglect the mean electric potential generation via the Reynolds stress. The linear part of the equation is then readily averaged. The average of the boundary conditions on the electric current shifts the sheath steady state potential, while conserving the structure of such a term.

The paper is organized as follows. In Section 2 are presented the physics and the equation that governs the evolution of the electric potential. One examines the specific conditions introduced by the contact to the wall material and their impact on the system of equations. The reduction of the $3D$ electrostatic potential equation to a $2D$ equation is analyzed as well as the linearization procedure that allows one to cast the problem in a variational framework. Section 3 summarizes the mathematical nonlinear evolution problem for the electric potential ϕ. Section 4 is concerned with the proof of the existence/uniqueness of a solution of the linearized problem (Theorem 4.5). Finally, Section 5 is dedicated to the mathematical proof of the existence and uniqueness of a solution to the original nonlinear evolution equation. The essential result is stated in Theorem 5.1.

2 Generation of the fluctuating electric field

Two descriptions of the plasma evolution are considered in order to investigate plasma turbulence. The most demanding description is the kinetic description where one computes the probability, or so-called distribution function, to find a particle at a given point of the phase space, namely the 6D space of the position and velocity of the particles [3]. A more convenient representation is given by the moments of this kinetic equation, yielding the so-called fluid equations [1]: the particle balance equation for the zeroth moment, the momentum balance equation for the first moment, the energy balance equation for the second moment, etc. A closure of this infinite series of fluid equations is required to reduce the system to the first fluid moments. The standard closure is that leading to the Navier-Stokes set of equations where the fluid is described in terms of the density, momentum and temperature together with a Fourier law that relates the heat conduction to the temperature gradient. However, the main drive of the turbulence development appears to be a plasma pressure gradient that can be sustained by the density at constant temperature. Therefore, a convenient closure, that still captures the turbulent plasma behavior, is the isothermal closure where the plasma temperature, for all plasma species, is assumed to be constant. The fluid equations are then restricted to the two first balance equations, namely the particle and momentum balance for each plasma species. For simplicity we shall restrict the plasma species to the electrons and to a single ion species. In this case, the two balance equations are

$$\partial_t n_\alpha + \vec{\nabla} \cdot (n_\alpha \vec{v}_\alpha) = S_\alpha \,, \tag{2.1}$$

$$\partial_t \left(m_\alpha n_\alpha \vec{v}_\alpha\right) + \vec{\nabla} \cdot \left(m_\alpha n_\alpha \vec{v}_\alpha \otimes \vec{v}_\alpha + p_\alpha \mathbb{I} + \overline{\overline{\Pi}}_\alpha\right) = q_\alpha n_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right) \,. \tag{2.2}$$

The subscript α stands for the various species of the plasma, in practice e for electrons and i for the hydrogen ions. In the momentum balance, one finds the driving Laplace force on the right hand side, where $\overrightarrow{B}$ is the given magnetic field, and the pressure contribution on the left hand side with the standard scalar contribution p_α, the stress tensor $\overline{\overline{\Pi}}_\alpha$ and the pressure tensor $n_\alpha \overrightarrow{v}_\alpha \otimes \overrightarrow{v}_\alpha$ where $\otimes$ is the tensorial product. One should then introduce the Maxwell-Gauss equation, or Poisson equation in the electrostatic limit, to relate the electric field to the charge density

$$\lambda_D^2 \, \Delta \left(\frac{e\Phi}{T_e}\right) = \frac{n_e - n_i}{n_0} \quad ; \quad \vec{E} = -\vec{\nabla}\Phi \,. \tag{2.3}$$

In this equation, the dimensionless quantity $e\Phi/T_e$ (e the proton charge, T_e the electron temperature) measures the ratio of the electrostatic energy normalized by the electron kinetic energy T_e. As far as electrostatic turbulence is concerned, this ratio is much less than unity in core tokamak plasmas, and can reach several tens of percents at the edge. The Debye length scale reads $\lambda_D^2 = (\epsilon_0 T_e)/(n_0 e^2)$. As a consequence, when one addresses the behaviour of the electrostatic potential on a scale $L \gg \lambda_D$, one is led to consider a quasi-neutral plasma $n_e \approx n_i$ that can be polarized and where the electric potential will be determined by a constraint that stems from the quasineutrality condition.
Without loss of generality the two particle balance equations (2.1) can be replaced by a charge balance equation and a particle balance equation for either species

$$\partial_t \rho_c + \overrightarrow{\nabla} \cdot \overrightarrow{j} = 0 \,, \tag{2.4}$$

$$\partial_t n_e + \vec{\nabla} \cdot (n_e \vec{v}_e) = S_e\,. \tag{2.5}$$

The charge density ρ_c is defined by $\rho_c = e(n_i - n_e)$ and the electric current density by $\vec{j} = e(n_i \vec{v}_i - n_e \vec{v}_e)$. In the quasineutral limit, the charge balance equation is reduced to $\vec{\nabla} \cdot \vec{j} = 0$ since $\rho_c = 0$. The plasma current can be split into a parallel and a transverse component with respect to the magnetic field lines, both depending on the electrostatic potential. From the physical point of view, the electrostatic potential will evolve in order to maintain a divergence free electric current since any departure from this condition will govern a short scale and rapid restoring electric field. The aim now is to determine the electrostatic potential Φ satisfying $\vec{\nabla} \cdot \vec{j}(\Phi) = 0$, as well as the electron density n_e satisfying the balance equation (2.5).
The dependence of the current (resp. fluid velocity $\vec{v}_\alpha$) on the electrostatic potential is obtained by modifying equations (2.2) in order to obtain "the" Navier-Stokes equation

$$(\frac{d}{dt} - \nu\Delta_\perp + \vec{\omega}_\alpha \times)\vec{v}_\alpha = -\frac{1}{m_\alpha n_\alpha}\vec{\nabla}(p_\alpha \mathbb{I} + \overline{\overline{\Pi}}_\alpha) + \frac{q_\alpha}{m_\alpha}\vec{E} - S_\alpha \frac{\vec{v}_\alpha}{n_\alpha}\,. \tag{2.6}$$

In the latter equation $\vec{\omega}_\alpha$ is the Larmor rotation vector proportional to the Larmor frequency and aligned along the magnetic field $\vec{\omega}_\alpha = \frac{q_\alpha}{m_\alpha}\vec{B}$. In this expression, a viscous term has been introduced. Unlike in standard fluids, in the case of plasmas the effect of collisions is negligible. For plasmas, the viscous diffusion term is induced by small scale turbulent fluctuations in the plane transverse to the magnetic field and can be computed in the quasilinear framework. Such a term plays an important role in simulations to damp the small scale fluctuations. The Lagrangian time derivative is defined by

$$\frac{d}{dt} = \partial_t + \vec{v}_\alpha \cdot \vec{\nabla}\,. \tag{2.7}$$

The transverse velocity is obtained now by expanding the projection of (2.6) (perpendicular to the magnetic field lines) with respect to $1/\omega_\alpha$, where the Larmor frequency $\omega_\alpha = \frac{q_\alpha B}{m_\alpha}$ is large compared to all other frequencies that are considered in the present analysis. Such a so-called adiabatic limit is well fulfilled in tokamak plasma turbulence, where turbulence evolves on frequencies typically below $10^5 s^{-1}$, while $\omega_i \approx 5.10^8 s^{-1}$. This gives rise at the first order to the so called drift velocities, the electric drift $\vec{v}_E$ and the diamagnetic drift $\vec{v}_{*\alpha}$

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \quad ; \quad \vec{v}_{*\alpha} = \frac{\vec{B} \times \vec{\nabla} p_\alpha}{q_\alpha n_\alpha B^2}\,.$$

At the second order this procedure then leads to the polarization drift [7, 11, 12]

$$\vec{v}_{pol,\alpha} = -\frac{m_\alpha}{q_\alpha B^2}\Big(\partial_t - \nu\Delta_\perp + (\vec{v}_E + \vec{v}_{*\alpha}) \cdot \vec{\nabla}\Big)\vec{\nabla}_\perp \Phi\,. \tag{2.8}$$

Inserting these drift velocities into the charge balance equation, yields

$$\vec{\nabla}_{||} \cdot \vec{j}_{||} + \vec{\nabla}_\perp \cdot \vec{j}_\perp = \vec{\nabla}_{||} \cdot \vec{j}_{||} - \frac{m_i n}{B^2}\Big(\partial_t - \nu\Delta_\perp\Big)\Delta_\perp \Phi - \frac{m_i n}{B^3}[\Phi, \Delta_\perp \Phi] - \mathcal{S}\,, \tag{2.9}$$

where the $\mathcal{S}$-term essentially accounts for the spatial inhomogeneity of the equilibrium magnetic field B and of the density n. It will be discussed later on. Since there is no source of charges injected into the system, the contributions of the particle source terms S_e and S_i cancel out in the charge balance equation. The leading contribution of the polarization current is that of the ions due to the explicit dependence in the mass. Poisson brackets are defined by $[\Phi, f] = B(\vec{v}_E \cdot \vec{\nabla} f) = \frac{\vec{B}}{B} \cdot (\vec{\nabla}\Phi \times \vec{\nabla} f)$.

Finally, the parallel contribution to the divergence of the current is obtained by considering the parallel Ohm's law that takes the standard form $\eta j_\parallel = -\nabla_\parallel \Phi$, where η is the parallel resistivity induced by the collisions. The divergence of the plasma current then takes the form of a differential equation for the electric potential

$$-\frac{m_i n}{B^2}\Big((\partial_t - \nu\Delta_\perp)\Delta_\perp\Phi + \frac{1}{B}[\Phi, \Delta_\perp\Phi]\Big) - \nabla_\parallel \cdot \Big(\frac{1}{\eta}\nabla_\parallel\Phi\Big) - \mathcal{S} = 0\,. \tag{2.10}$$

This equation determines the electric potential, given the source term $\mathcal{S}$ and is coupled through n to the electron balance equation (2.5). Both the mean value and the fluctuating contribution of Φ are governed by this equation.

Regarding the physics of the poloidal flows, we are interested in the large scale contribution of the $\overrightarrow{E} \times \overrightarrow{B}$ flow that is governed by the radial structure of the mean electric potential. More precisely, the present paper aims at investigating the impact of the plasma-wall boundary conditions on the structure and magnitude of the poloidal flows. As discussed in the following, these boundary conditions act in the parallel direction, where the field lines intercept plasma facing components. Therefore, one focuses on the impact of the parallel divergence in the charge balance equation (2.10). In this framework, the sole radial r and parallel z directions need being retained. The radial profile of Φ governs the poloidal $\vec{E} \times \vec{B}$ flow, while the direction parallel to the magnetic field lines enforces the boundary conditions. One is then left with the two-dimensional mean electric potential $\Phi \to \Phi(r, z)$, such that the nonlinear term $[\Phi, \Delta_\perp\Phi]$ vanishes. Especially, we do not account for the part of the poloidal flow self generated by turbulence via the Reynolds stress tensor. This component would especially require to deal with the second transverse direction, namely the poloidal angle.

Consistently with the neglect of the small scale turbulent fluctuations, the source term is assumed to behave smoothly in space. Let's call S this small contribution standing for radial current effects that act as a source for the mean electric potential. As already stated, such a source will mainly be governed by curvature effects combined to large scale density structures. Consequently, this large scale source remains small since governed by polarisation effects only. Indeed, strong electric field can only emerge at sub-Debye scales, for which charge separation – governed by Poisson equation – is effective.

As a result, the mean electric potential is governed by the following reduction of (2.10)

$$\left(\partial_t - \nu\partial_r^2\right)\partial_r^2\phi + \frac{1}{\eta}\partial_z^2\phi = -S\,. \tag{2.11}$$

Here, all variables are dimensionless, with the following normalisations: $\phi = e\Phi/T_e$, $(r, z) \to (r, z)/\rho_s$, $t \to \omega_i t$, $\nu \to \nu/(\rho_s^2\omega_i)$, $\eta = \nu_{coll}/\omega_e$, and finally $S \to S/(en\omega_i)$, with $\rho_s = (m_i T_e)^{1/2}/eB$ the hybrid Larmor radius and ν_{coll} the electron-ion collision frequency. The electron density n will be considered as given in the rest of the paper.

In the here investigated peripheral plasma, plasma-wall interactions constrain the parallel dynamics. The boundary conditions for the electric potential are in fact governed by the parallel current that is adjusted by a boundary layer, the sheath, to minimise the current outflow [15]. This current is the sum of the ion and electron parallel current. At the sheath boundary, it can be shown that the ion flow reaches Mach = 1 so that the ion current writes $j_{\parallel,i} = j_{sat} = enc_s$, where c_s is the sound speed. The electrons are adiabatic and the electron current is governed by the sheath potential so that $j_{\parallel,e} = -j_{sat}\exp(\Lambda_{sheath} - \Phi)$. In this expression, Λ_{sheath} is the floating potential and is given by the ratio of the electron mobility to the ion mobility, $\Lambda_{sheath} \propto \frac{1}{2}\log(m_i/m_e)$. The sheath physics that leads to these expressions ensures that the electron current matches the ion current at equilibrium, so that $\Lambda_{sheath} = \Phi$. Here, Φ stands for the entire electric potential, including small scale fluctuations. Therefore, the large scale component of the here considered electric potential fulfills the modified sheath equilibrium condition $\Lambda = \phi$, where the dimensionless effective potential Λ includes the modification of the sheath potential by small scale turbulent fluctuations. It is taken constant in the following. Experiments have shown that $\Lambda < 0$. In summary, the boundary current leaving the plasma to the wall is given by $\langle j_\parallel \rangle = j_{sat}\,(1 - \exp(\Lambda - \phi))$, the brackets indicating the average over small scales. This condition completes the equation (2.11).

3 The mathematical problem

The objective of this paper is to study from a mathematical point of view the following nonlinear evolution problem, given on the domain Ω (represented in Figure 1)

$$-\partial_t \partial_r^2 \phi - \frac{1}{\eta}\partial_z^2 \phi + \nu \partial_r^4 \phi = S\,, \quad t \geq 0\,, \quad (r,z) \in \Omega\,, \tag{3.1}$$

and completed with the initial condition

$$\partial_r \phi(0,r,z) = \partial_r \phi_0(r,z)\,, \quad (r,z) \in \Omega\,, \tag{3.2}$$

for some given function ϕ_0. The imposed boundary conditions are the no-slip boundary conditions on $\Sigma := \Sigma_0 \cup \Sigma_l \cup \Sigma_L$

$$\partial_r \phi(t,r,z) = \partial_r^3 \phi(t,r,z) = 0\,, \quad t \geq 0\,, \quad (r,z) \in \Sigma\,, \tag{3.3}$$

periodic boundary conditions on $\Gamma_0 \cup \Gamma_1$, that means in the core of the plasma, and the nonlinear sheath boundary conditions on the limiters $\Gamma_a \cup \Gamma_b$

$$\left\{ \begin{array}{ll} \partial_z \phi(t,r,a) = \eta(1 - e^{\Lambda - \phi(t,r,a)})\,, & t \geq 0\,, \quad (r,z) \in \Gamma_a\,, \\ \partial_z \phi(t,r,b) = -\eta(1 - e^{\Lambda - \phi(t,r,b)})\,, & t \geq 0\,, \quad (r,z) \in \Gamma_b\,. \end{array} \right. \tag{3.4}$$

Here Σ_i and Γ_j are parts of the boundary $\partial\Omega$ and stand for

$$\Sigma_i := \{(r = i, z) \in \partial\Omega\} \quad ; \quad \Gamma_j := \{(r, z = j) \in \partial\Omega\}\,.$$

The source term is denoted by S and η, ν and Λ are some given constants. The aim is to prove the existence and uniqueness of a solution ϕ of this problem.

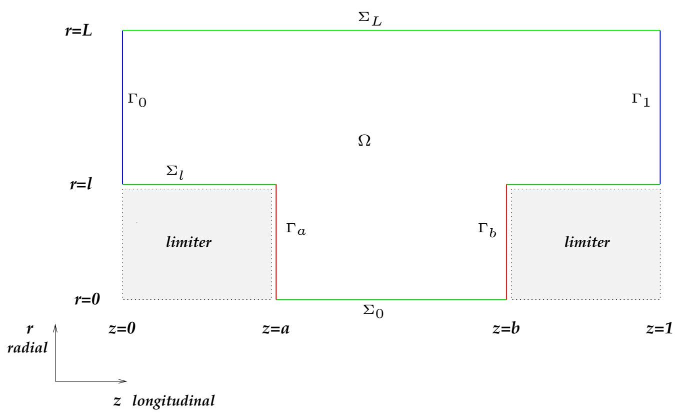

Figure 1: The 2D domain.

4 Existence and uniqueness of the linear problem

We shall start by proving the existence and uniqueness of a weak solution of the linear evolution problem (3.1)-(3.3), completed with the linearized boundary conditions on $\Gamma_a \cup \Gamma_b$

$$\left\{ \begin{array}{ll} \partial_z \phi(t,r,a) = g_a(t,r) + h_a(t,r)\phi(t,r,a)\,, & t \geq 0\,, \quad (r,z) \in \Gamma_a\,, \\ \partial_z \phi(t,r,b) = g_b(t,r) - h_b(t,r)\phi(t,r,b)\,, & t \geq 0\,, \quad (r,z) \in \Gamma_b\,. \end{array} \right. \tag{4.1}$$

This problem is obtained by a linearization of the previous nonlinear evolution problem. Let $Q_t := (0,t) \times \Omega$ be the time-space cylinder. In the rest of this section the following hypothesis is made.

Hypothesis A : *The given functions $h_a, h_b \in L^\infty((0,T) \times (0,l))$ satisfy*

$$h_a(t,r) \geq c_a > 0 \quad \textit{and} \quad h_b(t,r) \geq c_b > 0\,, \quad \textit{f.a.a. } t \geq 0\,,\ 0 \leq r \leq l\,.$$

Moreover, $g_a, g_b \in L^2((0,T) \times (0,l))$, $S \in L^2(Q_T)$, $\phi_0 \in H^1(\Omega)$, $\eta > 0$, $\nu > 0$, $\Lambda < 0$.

Before considering the existence/uniqueness theorem of the linear case, let us prove the following Poincaré-like inequality, which will be used all along the paper.

Proposition 4.1 *There exists a constant $C_\Omega > 0$, depending only on the domain Ω, such that for every $\phi \in \mathcal{H} := \{\phi \in H^1(\Omega)\ /\ \partial_r^2 \phi \in L^2(\Omega)\,,\ \partial_r \phi = 0 \text{ on } \Sigma\}$ the following inequality holds*

$$||\phi||_{L^2(\Omega)} \leq C_\Omega (||\phi_{|\Gamma_a}||_{L^1(\Gamma_a)} + ||\partial_z \phi||_{L^2(\Omega)} + ||\partial_r^2 \phi||_{L^2(\Omega)})\,. \tag{4.2}$$

Proof: The proof follows by contradiction. Let us suppose that inequality (4.2) is not satisfied. This means, that there exists a sequence $\phi^n \in \mathcal{H}$ with $||\phi^n||_{L^2(\Omega)} = 1$ and satisfying

$$1 = ||\phi^n||_{L^2(\Omega)} \geq n(||\phi^n_{|\Gamma_a}||_{L^1(\Gamma_a)} + ||\partial_z \phi^n||_{L^2(\Omega)} + ||\partial_r^2 \phi^n||_{L^2(\Omega)}) \,. \tag{4.3}$$

Thus, ϕ^n is bounded in $H^1(\Omega)$ as well as $\partial_r^2 \phi^n$ in $L^2(\Omega)$, implying the existence of a weak limit ϕ, such that

$$\phi^n \rightharpoonup \phi \quad \text{weak in } H^1(\Omega)\,, \quad \partial_r^2\phi^n \rightharpoonup \partial_r^2 \phi \quad \text{weak in } L^2(\Omega)\,, \quad \phi^n_{|\Gamma_a} \rightharpoonup \phi_{|\Gamma_a} \quad \text{weak in } L^2(\Gamma_a)\,.$$

Inequality (4.3) implies then $\partial_z \phi \equiv 0$, $\partial_r^2 \phi \equiv 0$ and $\phi_{|\Gamma_a} \equiv 0$, which tells us immediately that $\phi \equiv 0$. Moreover, due to the compact embedding $H^1(\Omega) \subset L^2(\Omega)$, we have

$$\phi^n \to \phi \quad \text{strongly in} \quad L^2(\Omega)\,.$$

which yields that $||\phi||_{L^2(\Omega)} = 1$. This is in contradiction with the fact that $\phi \equiv 0$. ∎

Let us now introduce the mathematical framework of the problem (3.1)-(3.3),(4.1) and clarify what we mean by a weak solution. For this, let V denote the Hilbert space

$$V := \{f \in H^1(\Omega) \,/\, \partial_r^2 f \in L^2(\Omega)\,,\, f \text{ periodic on } \Gamma_0 \cup \Gamma_1\,,\; \partial_r f = 0 \text{ on } \Sigma\}\,,$$

with the scalar product

$$((f,g))_V := \int_\Omega \partial_z f\, \partial_z g\, drdz + \int_\Omega \partial_r^2 f\, \partial_r^2 g\, drdz + \int_0^l f_a\, g_a\, dr + \int_0^l f_b\, g_b\, dr\,, \tag{4.4}$$

where f_a (resp. f_b) denotes the trace function $f(r, z = a)$ (resp. $f(r, z = b)$). Moreover, let the Hilbert space $H := \{f \in L^2(\Omega) \,/\, \partial_r f \in L^2(\Omega)\}$ be associated with the scalar product

$$(f,g)_H := \int_\Omega fg\, drdz + \int_\Omega \partial_r f \partial_r g\, drdz\,. \tag{4.5}$$

It can be shown that V is densely and continuously embedded in H, such that we get by identifying H with H^* the evolution triple

$$V \subset H = H^* \subset V^*\,.$$

In order to write down a weak formulation of the problem (3.1)-(3.3),(4.1) we introduce the following operators and bilinear forms. Let $m : H \times H \to \mathbb{R}$ be the bilinear, continuous, symmetric and non-negative form

$$m(\phi, \vartheta) := \int_\Omega \partial_r \phi\, \partial_r \vartheta\, drdz\,.$$

It is important to remark, that m is not a coercive form on H. The equation (3.1) is degenerate. Furthermore let the continuous, coercive, bilinear form $l(t) : V \times V \to \mathbb{R}$ be given by

$$l(t, \phi, \vartheta) := \frac{1}{\eta} \int_\Omega \partial_z \phi(r,z) \partial_z \vartheta(r,z)\, drdz + \nu \int_\Omega \partial_r^2 \phi(r,z) \partial_r^2 \vartheta(r,z)\, drdz$$
$$+ \frac{1}{\eta} \int_0^l h_b(t,r) \phi(r,b) \vartheta(r,b)\, dr + \frac{1}{\eta} \int_0^l h_a(t,r) \phi(r,a) \vartheta(r,a)\, dr\,,$$

and the corresponding isomorphism $\mathcal{L}(t) : V \to V^*$. Finally, the operator $\mathcal{S}(t) \in V^*$ is defined by

$$\mathcal{S}(t)(\vartheta) := \int_\Omega S(t,\cdot)\,\vartheta(\cdot)\,drdz + \frac{1}{\eta}\int_0^l g_b(t,r)\,\vartheta(r,b)\,dr - \frac{1}{\eta}\int_0^l g_a(t,r)\,\vartheta(r,a)\,dr\,.$$

Definition 4.2 *A function $\phi \in \mathcal{W}_V$, where*

$$\mathcal{W}_V := \{\phi \in L^2(0,T;V) \;/\; \partial_r\phi \in L^\infty(0,T;L^2(\Omega))\}\,,$$

is called a weak solution of the linear problem (3.1)-(3.3),(4.1) if it satisfies in a distributional sens ($\mathcal{D}'(0,T)$) the equation

$$\begin{cases} \dfrac{d}{dt}m(\phi(\cdot),\vartheta) + l(\cdot,\phi(\cdot),\vartheta) = \mathcal{S}(\cdot)(\vartheta)\,, & \textit{for all} \quad \vartheta \in V\,, \\ \partial_r\phi(0) = \partial_r\phi_0\,. \end{cases} \tag{4.6}$$

Remark 4.3 The weak solution ϕ satisfies (4.6) in a distributional sense if and only if for every test function $\xi \in H^1(0,T)$ with $\xi(T) = 0$, the following equation holds for all $\vartheta \in V$

$$-\int_0^T m(\phi(t),\vartheta)\xi'(t)dt + \int_0^T l(t,\phi(t),\vartheta)\xi(t)dt = \int_0^T \mathcal{S}(t)(\vartheta)\xi(t)dt + m(\phi_0,\vartheta)\xi(0)\,. \tag{4.7}$$

Due to the fact that $l(\cdot,\phi(\cdot),\vartheta) \in L^2(0,T)$ and $\mathcal{S}(\cdot)(\vartheta) \in L^2(0,T)$ for all $\vartheta \in V$, equation (4.7) means that $m(\phi(\cdot),\vartheta) \in H^1(0,T)$, thus implying $m(\phi(\cdot),\vartheta) \in C([0,T])$ for all $\vartheta \in V$. ∎

Remark 4.4 The initial condition in (4.6) is well defined in $H^{-1/2}(\Omega)$. Indeed, a function $\phi \in \mathcal{W}_V$ satisfying the equation (4.6) in a distributional sens, satisfies $\partial_r\phi \in L^2(0,T;L^2(\Omega))$, $\partial_t\partial_r\phi \in L^2(0,T;H^{-1}(\Omega))$. Interpolation arguments imply then $\partial_r\phi \in C([0,T];H^{-1/2}(\Omega))$ [10]. ∎

Theorem 4.5 *(***Existence/Uniqueness***)*
Let Hypothesis A be satisfied. Then problem (3.1)-(3.3),(4.1) admits a unique weak solution $\phi \in \mathcal{W}_V$, satisfying the energy estimate

$$\begin{aligned} &||\partial_r\phi||^2_{L^\infty(0,T;L^2(\Omega))} + ||\phi||^2_{L^2(0,T;V)} \le \\ &C(||\partial_r\phi_0||^2_{L^2(\Omega)} + ||S||^2_{L^2(Q_T)} + ||g_a||^2_{L^2((0,T)\times\Gamma_a)} + ||g_b||^2_{L^2((0,T)\times\Gamma_b)})\,, \end{aligned} \tag{4.8}$$

for a constant $C > 0$.

The existence and uniqueness of a weak solution of problem (3.1)-(3.3),(4.1) is based on the standard Galerkin method, applied on the equation

$$\mathcal{M}\phi' + \mathcal{L}\phi = \mathcal{S}\,. \tag{4.9}$$

In order to circumvent the fact that the bilinear form m is not coercive (or equivalently that the operator $\mathcal{M}$ is degenerate), we introduce an additional term in the equation (3.1)

$$\epsilon\partial_t\phi - \partial_t\partial_r^2\phi - \frac{1}{\eta}\partial_z^2\phi + \nu\partial_r^4\phi = S\,,$$

the passage to the limit $\epsilon \to 0$ leading then to the desired result. Another possible way to obtain a regular problem is to factor the equation (4.9) by the kernel of $\mathcal{M}$, as presented in [14].
We shall first of all analyze the modified problem

$$\left\{\begin{array}{l} \dfrac{d}{dt}m_\epsilon(\phi(\cdot),\vartheta) + l(\cdot,\phi(\cdot),\vartheta) = \mathcal{S}(\cdot)(\vartheta)\,, \quad \text{for all} \quad \vartheta \in V\,, \\ \phi(0) = \phi_0 \in H\,, \end{array}\right. \tag{4.10}$$

where

$$m_\epsilon(\phi,\vartheta) := \epsilon\int_\Omega \phi\vartheta drdz + \int_\Omega \partial_r\phi\partial_r\vartheta drdz\,.$$

In the following we shall consider this bilinear form $m_\epsilon(\cdot,\cdot)$ as scalar product on the Hilbert space H, instead of the standard one $(\cdot,\cdot)_H$ given in (4.5), and the corresponding evolution triple $V \subset H = H^* \subset V^*$.

Proposition 4.6 *Under Hypothesis A, the modified linear problem (4.10) admits a unique weak solution* $\phi^\epsilon \in W(0,T;V,V^*) := \{\phi \in L^2(0,T;V)\ /\ \partial_t\phi \in L^2(0,T;V^*)\}$*, satisfying the following energy estimate for all* $t \in [0,T]$

$$\begin{aligned} &\epsilon||\phi^\epsilon(t)||^2_{L^2(\Omega)} + ||\partial_r\phi^\epsilon(t)||^2_{L^2(\Omega)} + \frac{1}{\eta}||\partial_z\phi^\epsilon||^2_{L^2(Q_t)} + \nu||\partial_r^2\phi^\epsilon||^2_{L^2(Q_t)} \\ &\qquad\qquad + ||\phi_b^\epsilon||^2_{L^2((0,t)\times\Gamma_b)} + ||\phi_a^\epsilon||^2_{L^2((0,t)\times\Gamma_a)} \\ &\leq c(\epsilon||\phi_0||^2_{L^2(\Omega)} + ||\partial_r\phi_0||^2_{L^2(\Omega)} + ||S||^2_{L^2(Q_T)} + ||g_a||^2_{L^2((0,T)\times\Gamma_a)} + ||g_b||^2_{L^2((0,T)\times\Gamma_b)})\,, \end{aligned} \tag{4.11}$$

with $c > 0$ *a constant independent on* ϵ *and* T*, but depending on* η *and* ν*.*

Remark 4.7 We would like to observe here, that a function ϕ belongs to $W(0,T;V,V^*)$ if and only if $\phi \in L^2(0,T;V)$ and if there exists a function $\psi \in L^2(0,T;V^*)$, denoted by $\partial_t\phi$ and satisfying for every $\xi \in C_0^\infty(0,T)$

$$\int_0^T m_\epsilon(\phi(t),\vartheta)\xi'(t)dt = -\int_0^T <\psi(t),\vartheta>_{V^*,V}\xi(t)dt\,, \quad \text{for all} \quad \vartheta \in V\,.$$

This definition depends on the evolution triple $V \subset H = H^* \subset V^*$ and thus on the ϵ-dependent scalar product on H. ∎

Proposition 4.6 is a standard result of the theory of variational methods (Galerkin method) or semigroup methods, so that we refer to [2, 8, 17] for the proof. The solution satisfies for all test functions $\vartheta \in V$ and $\xi \in H^1(0,T)$ the following variational formulation

$$-\epsilon\int_0^t\int_\Omega \phi^\epsilon\,\vartheta\,\xi'\,drdzd\tau+\epsilon\int_\Omega \phi^\epsilon(t)\,\vartheta\,\xi(t)\,drdz-\int_0^t\int_\Omega \partial_r\phi^\epsilon\,\partial_r\vartheta\,\xi'+\int_\Omega \partial_r\phi^\epsilon(T)\,\partial_r\vartheta\,\xi(T)+$$
$$\frac{1}{\eta}\int_0^t\int_\Omega \partial_z\phi^\epsilon\,\partial_z\vartheta\,\xi+\nu\int_0^t\int_\Omega \partial_r^2\phi^\epsilon\,\partial_r^2\vartheta\,\xi+\frac{1}{\eta}\int_0^t\int_0^l h_b\,\phi_b^\epsilon\,\vartheta_b\,\xi+\frac{1}{\eta}\int_0^t\int_0^l h_a\,\phi_a^\epsilon\,\vartheta_a\,\xi$$
$$=\int_0^t\int_\Omega S\,\vartheta\,\xi+\frac{1}{\eta}\int_0^t\int_0^l g_b\,\vartheta_b\,\xi-\frac{1}{\eta}\int_0^t\int_0^l g_a\,\vartheta_a\,\xi+\epsilon\int_\Omega \phi_0\vartheta\,\xi(0)+\int_\Omega \partial_r\phi_0\,\partial_r\vartheta\,\xi(0)\,, \tag{4.12}$$

or in a simplier way

$$\int_0^t \frac{d}{dt}m_\epsilon(\phi^\epsilon(\tau),\vartheta)\xi(\tau)d\tau+\int_0^t l(\tau,\phi^\epsilon(\tau),\vartheta)\xi(\tau)d\tau=\int_0^t \mathcal{S}(\tau)(\vartheta)\xi(\tau)d\tau\,.$$

Density arguments as well as the Green equality

$$\int_0^t <\partial_t\phi^\epsilon(\tau),\phi^\epsilon(\tau)>_{V^*,V}\,d\tau=\frac{1}{2}m_\epsilon(\phi^\epsilon(t),\phi^\epsilon(t))-\frac{1}{2}m_\epsilon(\phi_0,\phi_0)\,,$$

lead thus to the energy bound (4.11).

Proof of Theorem 4.5 : The energy estimate in the previous proposition implies that ϕ^ϵ, $\partial_z\phi^\epsilon$, $\partial_r^2\phi^\epsilon$ are bounded in $L^2(Q_T)$ independently on ϵ, thus weakly convergent in $L^2(Q_T)$ towards some functions ϕ, $\partial_z\phi$, $\partial_r^2\phi$ respectively. Besides ϕ_a^ϵ (resp. ϕ_b^ϵ) is bounded in $L^2((0,T)\times(0,l))$ and $\partial_r\phi^\epsilon$ bounded in $L^\infty(0,T;L^2(\Omega))$, allowing to pass to the limit in all terms of (4.12). This proves the existence of a weak solution $\phi\in L^2(0,T;V)$ of (4.6) satisfying $\partial_r\phi\in L^\infty(0,T;L^2(\Omega))$. Moreover, the energy estimate (4.8) is deduced immediately from (4.11), due to the fact that $x^\epsilon\rightharpoonup x$ in some Banach space implies $||x||\leq\liminf_{\epsilon>0}||x^\epsilon||$. And finally the uniqueness of the solution is an immediate consequence of the energy estimate as well as the linearity of the problem. ∎

5 Existence and uniqueness of the nonlinear problem

In this section the existence and uniqueness of a solution of the initially introduced nonlinear problem is proven

$$\begin{cases} -\partial_t\partial_r^2\phi-\dfrac{1}{\eta}\partial_z^2\phi+\nu\partial_r^4\phi=S\,, \quad t\geq 0\,, \quad (r,z)\in\Omega\,,\\ \partial_r\phi(0,r,z)=\partial_r\phi_0(r,z)\,, \quad (r,z)\in\Omega\,,\\ \partial_r\phi(t,r,z)=\partial_r^3\phi(t,r,z)=0\,, \quad t\geq 0\,, \quad (r,z)\in\Sigma\,,\\ \partial_z\phi(t,r,a)=\eta(1-e^{\Lambda-\phi(t,r,a)})\,, \quad t\geq 0\,, \quad (r,z)\in\Gamma_a\,,\\ \partial_z\phi(t,r,b)=-\eta(1-e^{\Lambda-\phi(t,r,b)})\,, \quad t\geq 0\,, \quad (r,z)\in\Gamma_b\,. \end{cases} \tag{5.1}$$

Hypothesis B : *Assume that* $S, \partial_r S, \partial_t S, \partial_{tt} S \in L^2(Q_T)$, *satisfy* $||S||_{L^\infty(Q_T)} \le C_S$ *and* $||S(T)||_{L^\infty(\Omega)} \le C_S$, *where* $C_S > 0$ *is a constant to be precised later on and depending on the domain* Ω. *Moreover let* $\phi_0 \in H^2(\Omega)$ *with* $\partial_r^4 \phi_0 \in L^2(\Omega)$.

Let us introduce the Hilbert space U

$$U := \{ f \in H^1(\Omega) \,/\, \partial_r^2 f \in L^2(\Omega) \,,\, f \text{ periodic on} \, \Gamma_0 \cup \Gamma_1 \,,\ f = 0 \text{ on } \Sigma \} \,,$$

differing from the Hilbert space V only in the imposed boundary conditions on Σ and being associated with the same scalar product (4.4) as on V.
The main result of this paper is then the following theorem.

Theorem 5.1 *Let Hypothesis B be satisfied. Then the nonlinear problem (5.1) admits a unique weak solution* $\phi \in \mathcal{X}$, *where*

$$\mathcal{X} := \{ \phi \in \mathcal{W}_V \ / \ \partial_r \phi \in \mathcal{W}_U \,, \partial_t \phi \in \mathcal{W}_V \}.$$

Moreover this solution satisfies the following energy estimate

$$||\partial_r \phi||_{L^\infty(0,T;L^2(\Omega))} + ||\phi||_{L^2(0,T;V)} \le c(1 + ||\partial_r \phi_0||_{L^2(\Omega)} + ||S||_{L^\infty(Q_T)}). \tag{5.2}$$

Definition 5.2 *A function* $\phi \in \mathcal{W}_V$ *is called a weak solution of the nonlinear problem (5.1) if* $e^{\Lambda - \phi(t,r,a)}, e^{\Lambda - \phi(t,r,b)} \in L^1((0,T) \times (0,l))$ *and if* ϕ *satisfies for every test function* $\vartheta \in V \cap H^2(\Omega)$ *and* $\xi \in H^1(0,T)$ *the variational formulation*

$$\begin{aligned}
&- \int_0^T \int_\Omega \partial_r \phi \, \partial_r \vartheta \, \xi' \, drdzdt + \int_\Omega \partial_r \phi(T) \, \partial_r \vartheta \, \xi(T) \, drdz + \frac{1}{\eta} \int_0^T \int_\Omega \partial_z \phi \, \partial_z \vartheta \, \xi \, drdzdt \\
&+ \nu \int_0^T \int_\Omega \partial_r^2 \phi \, \partial_r^2 \vartheta \, \xi \, drdzdt + \int_0^T \int_0^l (1 - e^{\Lambda - \phi(t,r,b)}) \, \vartheta(r,b) \, \xi(t) \, drdt \\
&+ \int_0^T \int_0^l (1 - e^{\Lambda - \phi(t,r,a)}) \, \vartheta(r,a) \, \xi(t) \, drdt = \int_0^T \int_\Omega S \, \vartheta \, \xi \, drdzdt + \int_\Omega \partial_r \phi_0 \, \partial_r \vartheta \, \xi(0) \, drdz \,.
\end{aligned} \tag{5.3}$$

Remark 5.3 For $\phi \in \mathcal{X}$, the trace $\phi(t,r,a)$ (resp. $\phi(t,r,b)$) belongs to $H^1((0,T) \times (0,l))$. This does not imply $\phi(t,r,a) \in L^\infty((0,T) \times (0,l))$. But by means of the Trudinger inequality [2], we have

$$\int_0^T \int_0^l e^{|\phi(t,r,a)|^2} drdt < \infty \,,$$

which infers $e^{\Lambda - \phi(t,r,a)} \in L^2((0,T) \times (0,l))$, such that the variational formulation (5.3) is well defined for $\vartheta \in V$ and $\xi \in H^1(0,T)$. ■

Before we start with the existence proof of a weak solution, let us establish an a priori estimate of the unknown ϕ, in order to clarify where the restrictive condition demanded by the theorem, $||S||_{L^\infty(Q_T)} \le C_S$, comes from. Taking in the weak formulation (5.3) ϕ as test function, we get

$$\frac{1}{2}\int_\Omega |\partial_r\phi(T,r,z)|^2 drdz + \frac{1}{\eta}\int_0^T\int_\Omega |\partial_z\phi|^2 drdzdt + \nu\int_0^T\int_\Omega |\partial_r^2\phi|^2 drdzdt +$$
$$+\int_0^T\int_0^l (1-e^{\Lambda-\phi(t,r,b)})\phi(t,r,b)drdt + \int_0^T\int_0^l (1-e^{\Lambda-\phi(t,r,a)})\phi(t,r,a)drdt =$$
$$\frac{1}{2}\int_\Omega |\partial_r\phi_0(r,z)|^2 drdz + \int_0^T\int_\Omega S\phi drdzdt\,.$$

Due to the fact that the function $f(x) = x(1-e^{\Lambda-x})$ behaves at infinity as

$$f(x) \sim |x|e^{\Lambda+|x|} \quad \text{for } x\to-\infty \qquad \text{and} \qquad f(x)\sim x \quad \text{for } x\to\infty\,, \tag{5.4}$$

yields for some constants $c_1, c_2, c_3 > 0$

$$f(x) \ge c_1|x|e^{\Lambda+|x|} \quad \text{for } x < \Lambda-1\,; \quad f(x)\ge c_2|x| \quad \text{for } x>1\,; \quad |f(x)|\le c_3 \quad \text{else}\,,$$

implying with $c_\Lambda = 1-\Lambda > 0$

$$\frac{1}{2}\int_\Omega |\partial_r\phi(T,r,z)|^2 drdz + \frac{1}{\eta}\int_0^T\int_\Omega |\partial_z\phi|^2 drdzdt + \nu\int_0^T\int_\Omega |\partial_r^2\phi|^2 drdzdt +$$
$$+c_1\int_0^T\int_{0\ \phi_b<-c_\Lambda}^l |\phi(t,r,b)|e^{\Lambda+|\phi(t,r,b)|}drdt + c_2\int_0^T\int_{0\ \phi_b\ge -c_\Lambda}^l |\phi(t,r,b)|drdt +$$
$$+c_1\int_0^T\int_{0\ \phi_a<-c_\Lambda}^l |\phi(t,r,a)|e^{\Lambda+|\phi(t,r,a)|}drdt + c_2\int_0^T\int_{0\ \phi_a\ge -c_\Lambda}^l |\phi(t,r,a)|drdt$$
$$\le c + \frac{1}{2}||\partial_r\phi_0||^2_{L^2(\Omega)} + \int_0^T\int_\Omega S\phi drdzdt \le c + \frac{1}{2}||\partial_r\phi_0||^2_{L^2(\Omega)} + ||S||_{L^\infty(Q_T)}||\phi||_{L^1(Q_T)}\,.$$

As one can observe, on the left hand side of the previous inequality only the L^1-norm of ϕ_a (resp. ϕ_b) occurs. This is rather restrictive and will constrain us to assume a bound on the source term S, in order to get an *a priori* estimate for ϕ. Indeed, using the Poincaré-like inequality (4.2), we can deduce with $c_4 := \min\{c_1, c_2\} > 0$

$$\frac{1}{2}\int_\Omega |\partial_r\phi(T,r,z)|^2 drdz + \frac{1}{\eta}\int_0^T\int_\Omega |\partial_z\phi|^2 drdzdt + \nu\int_0^T\int_\Omega |\partial_r^2\phi|^2 drdzdt +$$
$$+c_4\int_0^T\int_0^l |\phi(t,r,b)|drdt + c_4\int_0^T\int_0^l |\phi(t,r,a)|drdt$$
$$\le c + \frac{1}{2}||\partial_r\phi_0||^2_{L^2(\Omega)} + C_\Omega||S||_{L^\infty(Q_T)}\left[||\phi_a||_{L^1((0,T)\times\Gamma_a)} + \sqrt{T}(||\partial_z\phi||_{L^2(Q_T)} + ||\partial_r^2\phi||_{L^2(Q_T)})\right]\,.$$

Using the Young inequality, the term $\sqrt{T}||S||_{L^\infty(Q_T)}(||\partial_z\phi||_{L^2(Q_T)} + ||\partial_r^2\phi||_{L^2(Q_T)})$ can be introduced without any problem on the left hand side, whereas the problematic term is $||S||_{L^\infty(Q_T)}||\phi_a||_{L^1((0,T)\times(0,l))}$, which can only be introduced on the left hand side by assuming that the source term S satisfies the following bound

$$||S||_{L^\infty(Q_T)} \le \frac{c_4}{2C_\Omega} := C_S\,.$$

Assuming this, allows to get the a priori energy estimate (5.2). Indeed

$$\int_\Omega |\partial_r \phi(T,r,z)|^2 drdz + \frac{1}{\eta}\int_0^T \int_\Omega |\partial_z \phi|^2 drdzdt + \nu \int_0^T \int_\Omega |\partial_r^2 \phi|^2 drdzdt +$$
$$+\int_0^T\int_0^l |\phi(t,r,b)|drdt + \int_0^T\int_0^l |\phi(t,r,a)|drdt \leq C(\Omega,\nu,\eta,T)[1 + ||\partial_r \phi_0||^2_{L^2(\Omega)} + ||S||_{L^\infty(Q_T)}].$$

Let us now pass to the existence part of the proof of Theorem 5.1, which involves several steps. The idea is to construct a fixed point setting $\mathcal{T} : X \to X$ by linearizing the problem (5.1). To $\varphi \in X$ we associate $\phi := \mathcal{T}(\varphi) \in X$, solution of problem (5.1) with the modified linearized boundary condition on $\Gamma_a \cup \Gamma_b$

$$\partial_z \phi_{a,b} = \pm\eta(1 - e^{\Lambda - \varphi_{a,b}} - \varphi_{a,b} + \phi_{a,b})\,.$$

This allows to use the linear theory presented in Section 4. Here we denote by $\phi_{a,b}$ the trace functions $\phi(t,r,z=a)$ or $\phi(t,r,z=b)$ and the sign $''+''$ (resp. $''-''$) is associated to the boundary Γ_a (resp. Γ_b). In order to prove the existence of a fixed point of $\mathcal{T}$, the boundary condition is truncated in the following way

$$\partial_z \phi^n_{a,b} = \pm\eta(1 - e^{\Lambda - \tilde{\varphi}_{a,b}} - \tilde{\varphi}_{a,b} + \phi^n_{a,b})\,,$$

where for fixed $n \in \mathbb{N}$

$$\tilde{\varphi}_{a,b}(t,r) := \frac{\varphi_{a,b}(t,r)}{1 + |\varphi_{a,b}(t,r)|/n}\,.$$

To circumvent as in the linear case the degeneracy of the problem, the regularization term $\epsilon\partial_t\phi$ is introduced. Thus we shall first prove the existence and uniqueness of a weak solution $\phi^{\epsilon,n}$ to the modified nonlinear problem

$$\begin{cases} \epsilon\partial_t\phi^{\epsilon,n} - \partial_t\partial_r^2\phi^{\epsilon,n} - \dfrac{1}{\eta}\partial_z^2\phi^{\epsilon,n} + \nu\partial_r^4\phi^{\epsilon,n} = S\,, \quad \text{in } \Omega\,, \\ \partial_z\phi^{\epsilon,n}_{a,b} = \pm\eta(1 - e^{\Lambda - \tilde{\phi}^{\epsilon,n}_{a,b}} - \tilde{\phi}^{\epsilon,n}_{a,b} + \phi^{\epsilon,n}_{a,b})\,, \quad \text{on } \Gamma_a \cup \Gamma_b\,, \end{cases} \tag{5.5}$$

completed with the remaining initial/boundary conditions. This will be done with a fixed point argument. Afterwards we shall pass to the limit $n \to \infty$ and prove the existence of a solution of

$$\begin{cases} \epsilon\partial_t\phi^{\epsilon} - \partial_t\partial_r^2\phi^{\epsilon} - \dfrac{1}{\eta}\partial_z^2\phi^{\epsilon} + \nu\partial_r^4\phi^{\epsilon} = S\,, \quad \text{in } \Omega\,, \\ \partial_z\phi^{\epsilon}_{a,b} = \pm\eta(1 - e^{\Lambda - \phi^{\epsilon}_{a,b}})\,, \quad \text{on } \Gamma_a \cup \Gamma_b\,, \end{cases} \tag{5.6}$$

completed again with the remaining initial/boundary conditions. Finally, the limit $\epsilon \to 0$ will be performed.

Lemma 5.4 *Let n be an integer bigger than 5 and $\epsilon > 0$ be fixed. For $\phi_0 \in H^1(\Omega)$ and $S \in L^2(Q_T)$ with $||S||_{L^\infty(Q_T)} \leq C_S$, the modified nonlinear problem (5.5) admits a weak* *solution $\phi^{\epsilon,n} \in W(0,T;V,V^*)$, which satisfies the following estimate for all $t \in [0,T]$*

$$
\begin{aligned}
&\epsilon||\phi^{\epsilon,n}(t)||^2_{L^2(\Omega)} + ||\partial_r\phi^{\epsilon,n}(t)||^2_{L^2(\Omega)} + \frac{1}{\eta}||\partial_z\phi^{\epsilon,n}||^2_{L^2(Q_t)} + \nu||\partial_r^2\phi^{\epsilon,n}||^2_{L^2(Q_t)} \\
&+\int_0^t\int_{0\ \ \phi_b^{\epsilon,n}<-c_\Lambda}^l |\phi_b^{\epsilon,n}|e^{\Lambda+\frac{|\phi_b^{\epsilon,n}|}{1+|\phi_b^{\epsilon,n}|/n}}drd\tau + \int_0^t\int_{0\ \ \phi_b^{\epsilon,n}\geq -c_\Lambda}^l |\phi^{\epsilon,n}(\tau,r,b)|drd\tau + \\
&+\int_0^t\int_{0\ \ \phi_a^{\epsilon,n}<-c_\Lambda}^l |\phi_a^{\epsilon,n}|e^{\Lambda+\frac{|\phi_a^{\epsilon,n}|}{1+|\phi_a^{\epsilon,n}|/n}}drd\tau + \int_0^t\int_{0\ \ \phi_a^{\epsilon,n}\geq -c_\Lambda}^l |\phi^{\epsilon,n}(\tau,r,a)|drd\tau \leq c\,,
\end{aligned}
\tag{5.7}
$$

for a constant $c > 0$ independent on n, ϵ and $c_\Lambda > 0$ depending only on Λ. In particular,

$$
||\phi^{\epsilon,n}||^2_{L^2(0,T;V)} + ||\partial_t\phi^{\epsilon,n}||^2_{L^2(0,T;V^*)} \leq C(1+\epsilon||\phi_0||^2_{L^2(\Omega)} + ||\partial_r\phi_0||^2_{L^2(\Omega)} + ||S||^2_{L^\infty(Q_T)})\,, \tag{5.8}
$$

for a constant $C(\eta,\nu,T,\Lambda) > 0$ independent on n and ϵ.

Proof: Let $n \geq 5$ and $\epsilon > 0$ be fixed. For notational simplicity, we omit the indices n and ϵ in the rest of this proof. To prove the existence of a solution, we shall use a fixed point argument. Let us define the application $\mathcal{T} : X \to X$ with

$$
X := \{\phi \in W(0,T;V,V^*)\,/\,||\phi||_{L^2(0,T;V)} + ||\partial_t\phi||_{L^2(0,T;V^*)} \leq M\}\,,
$$

and $M > 0$ a constant to be defined later. For $\varphi \in X$ the image $\phi := \mathcal{T}(\varphi)$ is defined as the solution of the problem (5.5) with the linearized boundary conditions

$$
\partial_z\phi_{a,b} = \pm\eta(1 - e^{\Lambda-\tilde{\varphi}_{a,b}} - \tilde{\varphi}_{a,b} + \phi_{a,b})\,.
$$

The existence of such a solution is assured by the linear case. Moreover, the image ϕ belongs to X, if M is well chosen. Indeed, the energy estimate from Theorem 4.5 yields for all $t \in [0,T]$

$$
\begin{aligned}
&\epsilon||\phi(t)||^2_{L^2(\Omega)} + ||\partial_r\phi(t)||^2_{L^2(\Omega)} + ||\phi||^2_{L^2(0,T;V)} \leq C\Big(\epsilon||\phi_0||^2_{L^2(\Omega)} + ||\partial_r\phi_0||^2_{L^2(\Omega)} \\
&\qquad +||S||^2_{L^2(Q_T)} + ||1 - e^{\Lambda-\tilde{\varphi}_b} - \tilde{\varphi}_b||^2_{L^2((0,T)\times\Gamma_b)} + ||1 - e^{\Lambda-\tilde{\varphi}_a} - \tilde{\varphi}_a||^2_{L^2((0,T)\times\Gamma_a)}\Big) \\
&\leq C\Big(\epsilon||\phi_0||^2_{L^2(\Omega)} + ||\partial_r\phi_0||^2_{L^2(\Omega)} + ||S||^2_{L^2(Q_T)} + 2(1+n+e^{\Lambda+n})^2 lT\Big) := M\,.
\end{aligned}
$$

Moreover $\mathcal{T} : X \to X$ is a continuous map (with respect to the L^2-norm) on the bounded, closed, convex space X, compactly embedded in $L^2(Q_T)$ (Aubin theorem). The continuity of the application $\mathcal{T}$ is a straightforward consequence of the fact, that $W(0,T;V,V^*)$ is compactly embedded in $L^2(0,T;L^2(\Gamma_{a,b}))$. Indeed, $W(0,T;V,V^*) \subset L^2(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and interpolation arguments imply that $W(0,T;V,V^*)$ is compactly embedded in $L^2(0,T;H^{1-\delta}(\Omega))$ for $0 < \delta < 1$. Further, this latter space is continuously embedded in $L^2(0,T;L^2(\Gamma_{a,b}))$. Thus we can pass to the limit in the exponential term of the variational formulation

$$
\begin{aligned}
&-\epsilon\int_0^T\int_\Omega \phi\,\vartheta\,\xi'\,drdzdt + \epsilon\int_\Omega \phi(T)\,\vartheta\,\xi(T)\,drdz - \int_0^T\int_\Omega \partial_r\phi\,\partial_r\vartheta\,\xi' + \int_\Omega \partial_r\phi(T)\,\partial_r\vartheta\,\xi(T) \\
&+\frac{1}{\eta}\int_0^T\int_\Omega \partial_z\phi\,\partial_z\vartheta\,\xi + \nu\int_0^T\int_\Omega \partial_r^2\phi\,\partial_r^2\vartheta\,\xi + \int_0^T\int_0^l (1 - e^{\Lambda-\tilde{\varphi}_b} - \tilde{\varphi}_b + \phi_b)\,\vartheta(r,b)\,\xi(t) \\
&+\int_0^T\int_0^l (1 - e^{\Lambda-\tilde{\varphi}_a} - \tilde{\varphi}_a + \phi_a)\,\vartheta(r,a)\,\xi(t) = \int_0^T\int_\Omega S\,\vartheta\,\xi + \epsilon\int_\Omega \phi_0\,\vartheta\,\xi_0 + \int_\Omega \partial_r\phi_0\,\partial_r\vartheta\,\xi_0\,,
\end{aligned}
$$

establishing the continuity of the map $\mathcal{T}$. The Schauder fixed point theorem then implies the existence of a fixed point $\mathcal{T}(\phi) = \phi$, solution of the modified problem (5.5). Finally, the proof of the estimate (5.7) is straightforward by taking as test function $\phi^{\epsilon,n}$ in the variational formulation. We observe only that

$$\int_0^t \int_0^l (\phi^{\epsilon,n}_{a,b} - \tilde{\phi}^{\epsilon,n}_{a,b})\phi^{\epsilon,n}_{a,b} dr d\tau \geq 0 \,,$$

and

$$1 - e^{\Lambda - x} \geq 1 - e^{\Lambda} \,, \ \text{ for } \ x \geq 0 \,; \quad \left(1 - e^{\Lambda + \frac{|x|}{1+|x|/n}}\right) x \geq \frac{1}{2}|x| e^{\Lambda + \frac{|x|}{1+|x|/n}} \,, \ \text{ for } \ x \leq -c_\Lambda \,,$$

for a constant $c_\Lambda > 0$. ∎

The next step concerns the passage to the limit $n \to \infty$, for fixed $\epsilon > 0$.

Lemma 5.5 *Let $\epsilon > 0$ be fixed and let $\phi_0 \in H^1(\Omega)$, $S \in L^2(Q_T)$ with $||S||_{L^\infty(Q_T)} \leq C_S$. Then problem (5.6) admits a unique weak solution $\phi^\epsilon \in W(0,T;V;V^*)$, satisfying the bound*

$$\epsilon||\phi^\epsilon(t)||^2_{L^2(\Omega)} + ||\partial_r \phi^\epsilon(t)||^2_{L^2(\Omega)} + ||\phi^\epsilon||^2_{L^2(0,T;V)} \leq C(1 + \epsilon||\phi_0||^2_{L^2(\Omega)} + ||\partial_r \phi_0||^2_{L^2(\Omega)} + ||S||^2_{L^\infty(Q_T)}) \,, \tag{5.9}$$

for a constant $C > 0$ independent on ϵ, but depending on T.

Proof: Let $\epsilon > 0$ be fixed. The existence of a solution ϕ^ϵ is proven by passing to the limit $n \to \infty$ in the variational formulation of the problem (5.5). Denote by $\{\phi^{\epsilon,n}\}_{n\in\mathbb{N}}$ a sequence of solutions of (5.5) determined in Lemma 5.4. Since this sequence is bounded in $W(0,T;V,V^*)$, there exists a subsequence $\{\phi^{\epsilon,n_k}\}_{k\in\mathbb{N}}$, such that

$$\phi^{\epsilon,n_k} \to \phi^\epsilon \,, \quad \partial_z \phi^{\epsilon,n_k} \rightharpoonup \partial_z \phi^\epsilon \,, \quad \partial_r \phi^{\epsilon,n_k} \rightharpoonup \partial_r \phi^\epsilon \,, \quad \partial_r^2 \phi^{\epsilon,n_k} \rightharpoonup \partial_r^2 \phi^\epsilon \quad \text{in } L^2(Q_T) \,,$$

as well as

$$\partial_r \phi^{\epsilon,n_k}(t,\cdot,\cdot) \rightharpoonup \partial_r \phi^\epsilon(t,\cdot,\cdot) \quad \text{in } L^2(\Omega) \,, \forall t \in [0,T] \,.$$

For the sake of simplicity, the index ϵ is omitted in the rest of the proof and the subsequence $\{\phi^{n_k}\}_{k\in\mathbb{N}}$ is denoted again by $\{\phi^n\}_{n\in\mathbb{N}}$. The passage to the limit $n \to \infty$ in the variational formulation is straightforward, except for the limit

$$\int_0^T \int_0^l (1 - e^{\Lambda - \tilde{\phi}^n_{a,b}} - \tilde{\phi}^n_{a,b} + \phi^n_{a,b})\, \vartheta_{a,b}\, \xi(t)\, dr dt \to \int_0^T \int_0^l (1 - e^{\Lambda - \phi_{a,b}})\, \vartheta_{a,b}\, \xi(t)\, dr dt \,. \tag{5.10}$$

Due to the compact embedding $W(0,T;V,V^*) \subset L^2(0,T;L^2(\Gamma_{a,b}))$, we know that

$$\phi^n_{a,b} \to \phi_{a,b} \quad \text{strongly in } L^2((0,T) \times (0,l)) \,,$$

implying immediately

$$\int_0^T \int_0^l (-\tilde{\phi}^n_{a,b} + \phi^n_{a,b}) \vartheta_{a,b} \xi(t) dr dt \to 0 \,.$$

To prove the convergence of the exponential part, we shall use

$$\int_0^T \int_{0\ \phi_b^n<-c_\Lambda}^l |\phi_b^n| e^{\Lambda+\frac{|\phi_b^n|}{1+|\phi_b^n|/n}} drdt + \int_0^T \int_{0\ \phi_a^n<-c_\Lambda}^l |\phi_a^n| e^{\Lambda+\frac{|\phi_a^n|}{1+|\phi_a^n|/n}} drdt \leq C\,. \tag{5.11}$$

First we observe that $e^{\Lambda-\tilde{\phi}_{a,b}^n}$ is bounded in $L^1((0,T)\times(0,l))$, so that Fatou's lemma implies $e^{\Lambda-\phi_{a,b}} \in L^1((0,T)\times(0,l))$. Choosing now regular enough test functions ϑ and ξ (for example $\vartheta \in H^2(\Omega)\cap V$ and $\xi \in H^1(0,T)$) and fixing $\delta > 0$, we have for $K > 0$

$$\begin{aligned}\left|\int_0^T \int_0^l \left(e^{\Lambda-\tilde{\phi}_b^n} - e^{\Lambda-\phi_b}\right) \vartheta_b\, \xi\, drdt\right| \leq & \left|\int_0^T \int_{0\ \phi_b^\epsilon<-K}^l (e^{\Lambda-\tilde{\phi}_b^n} - e^{\Lambda-\phi_b})\, \vartheta_b\, \xi\, drdt\right| \\ & +\left|\int_0^T \int_{0\ \phi_b^\epsilon\geq-K}^l (e^{\Lambda-\tilde{\phi}_b^n} - e^{\Lambda-\phi_b})\, \vartheta_b\, \xi\, drdt\right|.\end{aligned} \tag{5.12}$$

The first term on the right hand side is estimated as follows

$$\left|\int_0^T \int_{0\ \phi_b^n<-K}^l (e^{\Lambda-\tilde{\phi}_b^n} - e^{\Lambda-\phi_b})\vartheta_b \xi\right| \leq \frac{1}{K}\int_0^T \int_{0\ \phi_b^n<-K}^l e^{\Lambda-\tilde{\phi}_b^n} |\phi_b^n|\,|\vartheta_b \xi| + \int_0^T \int_{0\ \phi_b^n<-K}^l e^{\Lambda-\phi_b} |\vartheta_b \xi|\,.$$

Due to (5.11) we can choose K in such a manner that this term is smaller than $\frac{\delta}{2}$. For this fixed K, the second term on the right hand side of (5.12) will be smaller than $\frac{\delta}{2}$ for sufficiently large n, due to the Lebesgue dominated convergence theorem. Thus the convergence of the exponential term in (5.10) holds. The uniqueness of the just found weak solution $\phi^\epsilon \in W(0,T;V,V^*)$ is a straightforward consequence of the fact that the function $-e^{-x}$ is monoton. Moreover, the energy bound (5.9) follows immediately by passing to the limit in (5.8). This ends the proof of Lemma 5.5. ■

And finally, let ϵ tend to zero.

Proof of theorem 5.1 :
The reasoning of the proof of Lemma 5.5 cannot be followed for passing to the limit $\epsilon \to 0$ in the variational formulation of problem (5.6), because the ϵ-independent bound of ϕ^ϵ in $W(0,T;V,V^*)$ has no more sense. Indeed, the definition of $\partial_t \phi^\epsilon$ is based on the evolution triple $V \subset H = H^* \subset V^*$ and thus ϵ-dependent. However, the ϵ-independent bound of the sequence $\{\phi^\epsilon\}_{\epsilon>0}$ in $L^2(0,T;V)$ can be used, such that we will only have $\phi_{a,b}^\epsilon \rightharpoonup \phi_{a,b}$ weak in $L^2(Q_T)$ instead of the strong convergence. The difficult point is to prove that

$$\int_0^T \int_0^l (1-e^{\Lambda-\phi_{a,b}^\epsilon})\vartheta_{a,b}\xi(t)drdt \to \int_0^T \int_0^l (1-e^{\Lambda-\phi_{a,b}})\vartheta_{a,b}\xi(t)drdt\,, \tag{5.13}$$

when $\epsilon \to 0$, the convergence of the other terms of the variational formulation being straigtforward. To prove (5.13) we shall show, by strengthening the assumptions on S and ϕ_0, that the functions $g_{a,b}^\epsilon(t,r) := e^{\Lambda-\phi_{a,b}^\epsilon}$ are bounded in $W^{1,1}((0,T)\times(0,l))$ and thus up to a subsequence strongly convergent in $L^1((0,T)\times(0,l))$ towards some function $g_{a,b}$. Due to the fact, that $\phi_{a,b}^\epsilon \rightharpoonup \phi_{a,b}$ in $L^2((0,T)\times(0,l))$ we can identify

$g_{a,b}(t,r) = e^{\Lambda-\phi_{a,b}}$, which ends the proof. Let thus prove the boundedness of $g^\epsilon_{a,b}$ in $W^{1,1}((0,T)\times(0,l))$, independently on ϵ. First of all, taking ϕ^ϵ as test function in the variational formulation of (5.6) and using the fact that ϕ^ϵ is bounded in $L^2(Q_T)$, gives rise to the estimate

$$\int_0^T\int_0^l (1-e^{\Lambda-\phi^\epsilon_a})\,\phi^\epsilon_a\,drdt + \int_0^T\int_0^l (1-e^{\Lambda-\phi^\epsilon_b})\,\phi^\epsilon_b\,drdt \le C\,,$$

independently on ϵ. These integrals are well-defined due to Fatou's Lemma. Hence

$$\int_0^T\int_{0\ \phi^\epsilon_{a,b}\le -c_\Lambda}^l |\phi^\epsilon_{a,b}|e^{\Lambda-\phi^\epsilon_{a,b}}drdt + \int_0^T\int_{0\ \phi^\epsilon_{a,b}>-c_\Lambda}^l |\phi^\epsilon_{a,b}|drdt \le C\,,$$

which implies the ϵ-independent bound of $g^\epsilon_{a,b}$ in $L^1((0,T)\times(0,l))$.
Denote now by $\psi^\epsilon := \partial_r\phi^\epsilon$. Deriving "formally" the equation (5.6) with respect to r yields the system

$$\begin{cases} \epsilon\partial_t\psi^\epsilon - \partial_t\partial_r^2\psi^\epsilon - \dfrac{1}{\eta}\partial_z^2\psi^\epsilon + \nu\partial_r^4\psi^\epsilon = \partial_r S\,, \quad \text{in } Q_T\,, \\ \partial_z\psi^\epsilon_{a,b} = \pm\eta\psi^\epsilon_{a,b}e^{\Lambda-\int_R^r\psi^\epsilon_{a,b}dr'-\phi^\epsilon_{a,b}(t,R)}\,, \quad \text{on } \Gamma_a\cup\Gamma_b\,, \\ \psi^\epsilon = \partial_r^2\psi^\epsilon = 0\,, \quad \text{on } \Sigma\,, \end{cases} \tag{5.14}$$

completed with the residual initial-boundary conditions. Here $R\in(0,l)$ is fixed. A similar fixed point argument as in the proof of Lemma 5.4 implies for every $\epsilon>0$ the existence of a weak solution $\psi^\epsilon\in W(0,T;U,U^*)$ of (5.14). Moreover it can be shown rigorously that $\psi^\epsilon=\partial_r\phi^\epsilon$, where ϕ^ϵ is solution of (5.6), yielding thus the regularity of ϕ^ϵ. Furthermore, the solution ψ^ϵ of (5.14) satisfies

$$\begin{aligned} &\frac{\epsilon}{2}||\psi^\epsilon(T)||^2_{L^2(\Omega)} + \frac{1}{2}||\partial_r\psi^\epsilon(T)||^2_{L^2(\Omega)} + \frac{1}{\eta}||\partial_z\psi^\epsilon||^2_{L^2(Q_T)} + \nu||\partial_r^2\psi^\epsilon||^2_{L^2(Q_T)} \\ &+ \int_0^T\int_0^l e^{\Lambda-\phi^\epsilon_a}|\psi^\epsilon_a|^2drdt + \int_0^T\int_0^l e^{\Lambda-\phi^\epsilon_b}|\psi^\epsilon_b|^2drdt \\ &= \frac{\epsilon}{2}||\psi_0||^2_{L^2(\Omega)} + \frac{1}{2}||\partial_r\psi_0||^2_{L^2(\Omega)} + \int_0^T\int_\Omega \partial_r S\psi^\epsilon drdzdt\,. \end{aligned} \tag{5.15}$$

Due to the uniform bound of $\psi^\epsilon=\partial_r\phi^\epsilon$ in $L^2(Q_T)$, we get the estimate

$$\int_0^T\int_0^l |\partial_r\phi^\epsilon_a|^2e^{\Lambda-\phi^\epsilon_a}drdt + \int_0^T\int_0^l |\partial_r\phi^\epsilon_b|^2e^{\Lambda-\phi^\epsilon_b}drdt \le C\,,$$

independent on ϵ, giving rise to the bound of $\partial_r g^\epsilon_{a,b} = \partial_r\phi^\epsilon_{a,b}e^{\Lambda-\phi^\epsilon_{a,b}}$ in $L^1((0,T)\times(0,l))$. To do the same for the variable t, we have to be much more precise. Denote $\mu^\epsilon := \partial_t\phi^\epsilon$. Derivating the problem (5.6) with respect to t yields a similar system as (5.14), completed with a slightly changed initial condition

$$\begin{cases} \epsilon\partial_t\mu^\epsilon - \partial_t\partial_r^2\mu^\epsilon - \dfrac{1}{\eta}\partial_z^2\mu^\epsilon + \nu\partial_r^4\mu^\epsilon = \partial_t S\,, \quad \text{in } Q_T\,, \\ \partial_z\mu^\epsilon_{a,b} = \pm\eta\mu^\epsilon_{a,b}e^{\Lambda-\int_0^t\mu^\epsilon_{a,b}d\tau-\phi_{a,b}(t=0,r)}\,, \quad \text{on } \Gamma_a\cup\Gamma_b\,, \\ \epsilon\mu^\epsilon_{t=0} - \partial_r^2\mu^\epsilon_{t=0} = \dfrac{1}{\eta}\partial_z^2\phi_0 - \nu\partial_r^4\phi_0 + S_{t=0}\,, \quad \text{in } \Omega\,. \end{cases} \tag{5.16}$$

The initial condition for μ^ϵ is deduced from equation (5.6) at $t=0$. Similar arguments as for the regularity in r yield the existence of a solution $\mu^\epsilon = \partial_t\phi^\epsilon \in W(0,T;V,V^*)$, satisfying the estimate

$$\frac{\epsilon}{2}||\mu^\epsilon(T)||^2_{L^2(\Omega)} + \frac{1}{2}||\partial_r\mu^\epsilon(T)||^2_{L^2(\Omega)} + \frac{1}{\eta}||\partial_z\mu^\epsilon||^2_{L^2(Q_T)} + \nu||\partial_r^2\mu^\epsilon||^2_{L^2(Q_T)} + \int_0^T\int_0^l e^{\Lambda-\phi_a^\epsilon}|\mu_a^\epsilon|^2$$
$$+ \int_0^T\int_0^l e^{\Lambda-\phi_b^\epsilon}|\mu_b^\epsilon|^2 drdt = \int_0^T\int_\Omega \partial_t S\mu^\epsilon drdzdt + \frac{\epsilon}{2}||\mu^\epsilon_{t=0}||^2_{L^2(\Omega)} + \frac{1}{2}||\partial_r\mu^\epsilon_{t=0}||^2_{L^2(\Omega)}\,. \tag{5.17}$$

The right hand side is controlled in the following manner. First we have

$$\int_0^T\int_\Omega \partial_t S\partial_t\phi^\epsilon drdzdt = -\int_0^T\int_\Omega \partial_t^2 S\phi^\epsilon drdzdt + \int_\Omega \partial_t S_{t=T}\phi^\epsilon_{t=T}drdz - \int_\Omega \partial_t S_0\phi_0 drdz\,.$$

Taking $\partial_t\phi^\epsilon \in L^2(0,T;V)$ as test function in the variational formulation of (5.6), enables to estimate $||\phi^\epsilon_{t=T}||_{L^2(\Omega)}$. Indeed,

$$\begin{aligned}
&\epsilon||\partial_t\phi^\epsilon||^2_{L^2(Q_T)} + ||\partial_t\partial_r\phi^\epsilon||^2_{L^2(Q_T)} + \frac{1}{2\eta}||\partial_z\phi^\epsilon_{t=T}||^2_{L^2(\Omega)} + \frac{\nu}{2}||\partial_r^2\phi^\epsilon_{t=T}||^2_{L^2(\Omega)} + \\
&\qquad\int_0^T\int_0^l \partial_t\phi_a^\epsilon(1-e^{\Lambda-\phi_a^\epsilon})\,drdt + \int_0^T\int_0^l \partial_t\phi_b^\epsilon(1-e^{\Lambda-\phi_b^\epsilon})\,drdt \\
&= \int_0^T\int_\Omega S\partial_t\phi^\epsilon\,drdzdt + \frac{1}{2\eta}||\partial_z\phi_0||^2_{L^2(\Omega)} + \frac{\nu}{2}||\partial_r^2\phi_0||^2_{L^2(\Omega)} \\
&= -\int_0^T\int_\Omega \partial_t S\phi^\epsilon\,drdzdt + \int_\Omega S_{t=T}\phi^\epsilon_{t=T}\,drdz - \int_\Omega S(0)\phi_0\,drdz + \frac{1}{2\eta}||\partial_z\phi_0||^2_{L^2(\Omega)} + \frac{\nu}{2}||\partial_r^2\phi_0||^2_{L^2(\Omega)}\,.
\end{aligned} \tag{5.18}$$

Then

$$\int_0^T\int_0^l \partial_t\phi^\epsilon_{a,b}(1-e^{\Lambda-\phi^\epsilon_{a,b}})drdt = \int_0^l\left[\phi^\epsilon_{a,b}(T) + e^{\Lambda-\phi^\epsilon_{a,b}(T)}\right]dr - \int_0^l\left[\phi_{a,b}(0) + e^{\Lambda-\phi_{a,b}(0)}\right]dr\,.$$

Assuming like in the case for the energy estimate (5.2) that the term $S_{t=T}$ is sufficiently small, i.e. $||S_{t=T}||_{L^\infty(\Omega)} \le C_S$ and since $x + e^{\Lambda-x} \ge |x|$ for all $|x| \ge c_\Lambda$, it holds that

$$\epsilon||\partial_t\phi^\epsilon||^2_{L^2(Q_T)} + ||\partial_t\partial_r\phi^\epsilon||^2_{L^2(Q_T)} + \frac{1}{\eta}||\partial_z\phi^\epsilon_{t=T}||^2_{L^2(\Omega)} + \nu||\partial_r^2\phi^\epsilon_{t=T}||^2_{L^2(\Omega)} + \int_0^l |\phi^\epsilon_{a,b}(T)|dr \le C\,,$$

with a constant $C>0$ independent on ϵ. Thus $||\phi^\epsilon_{t=T}||_V \le C$, independently on ϵ.

It remains to show that the term $\epsilon||\partial_t\phi^\epsilon_{t=0}||^2_{L^2(\Omega)} + ||\partial_t\partial_r\phi^\epsilon_{t=0}||^2_{L^2(\Omega)}$ on the right hand side of (5.17) is bounded independently on ϵ. First it follows from (5.6), that $\zeta^\epsilon(r,z) := \partial_t\phi^\epsilon_{t=0}(r,z)$ satisfies the following system

$$\begin{cases} \epsilon\zeta^\epsilon(r,z) - \partial_r^2\zeta^\epsilon(r,z) = f(r,z)\,, \quad (r,z)\in\Omega \\ \partial_r\zeta^\epsilon = 0\,, \quad \text{on} \quad \Sigma\,, \end{cases} \tag{5.19}$$

where the right hand side $f(r,z) = S_0 + \frac{1}{\eta}\partial_z^2\phi_0 - \nu\partial_r^4\phi_0$ satisfies

$$\int_{0/l}^{L} f(r,z)dr = 0\,, \quad \text{f.a.a.} \quad z \in [0,1]\,.$$

Indeed, the initial condition ϕ_0 satisfies equation (5.1) at $t = 0$. The integration limits of $\int_{0/l}^{L}$ depend on z, taking for $z \in [a,b]$ the integration interval $r \in [0,L]$ and for $z \in [0,1]\backslash[a,b]$ the interval $r \in [l,L]$. For every $z \in [0,1]$, the system (5.19) has a unique weak solution $\zeta^\epsilon(\cdot,z) \in H_r^2$ satisfying the energy estimate

$$\epsilon\int_{0/l}^{L} |\zeta^\epsilon(r,z)|^2 dr + \int_{0/l}^{L} |\partial_r\zeta^\epsilon(r,z)|^2 dr = \int_{0/l}^{L} f(r,z)\zeta^\epsilon(r,z)dr \le ||f(\cdot,z)||_{L_r^2}||\zeta^\epsilon(\cdot,z)||_{L_r^2}\,. \tag{5.20}$$

Integrating (5.19) with respect to r gives rise to $\int_{0/l}^{L}\zeta^\epsilon(r,z)dr = 0$. Consequently, the Poincaré-Wirtinger inequality yields from (5.20) the estimate

$$\epsilon||\zeta^\epsilon(\cdot,z)||_{L_r^2}^2 + c||\partial_r\zeta^\epsilon(\cdot,z)||_{L_r^2}^2 \le c||f(\cdot,z)||_{L_r^2}^2\,.$$

Integrating this inequality with respect to z finally yields $||\partial_r\zeta^\epsilon||_{L^2(\Omega)}^2 \le c$ and again by the Poincaré-Wirtinger inequality $||\zeta^\epsilon||_{L^2(\Omega)}^2 \le c$, independently on ϵ. This is nothing else but

$$||\partial_t\phi_{t=0}^\epsilon||_{L^2(\Omega)}^2 \le c\,, \qquad ||\partial_t\partial_r\phi_{t=0}^\epsilon||_{L^2(\Omega)}^2 \le c\,.$$

Altogether we get from (5.17)

$$\int_0^T\int_0^l |\partial_t\phi_{a,b}^\epsilon|^2 e^{\Lambda-\phi_{a,b}^\epsilon} \le C\,,$$

with a constant $C > 0$ independent on ϵ. Hence $\partial_t g_{a,b}^\epsilon = \partial_t\phi^\epsilon e^{\Lambda-\phi_{a,b}^\epsilon}$ is uniformly bounded in $L^1((0,T)\times(0,l))$. We have thus shown that the functions $g_{a,b}^\epsilon$ are bounded in $W^{1,1}((0,T)\times(0,l))$ and conclude the existence proof. The uniqueness of this solution $\phi \in \mathcal{X}$ is a straightforward consequence of the monotonicity of the function $-e^{-x}$. ∎

6 Conclusion

Existence and uniqueness of a solution for the nonlinear equation describing the evolution of the electric potential of a turbulent plasma flow, was investigated. The results are obtained under an appropriate smallness condition on the source term S as well as by requiring more regularity of the source term and the initial condition ϕ_0. An interesting question is whether it is possible to avoid these hypothesis or at least to get them weaker. Another open question concerns the numerical implementation of the here analyzed problem. Due to the fact that the parallel resistivity η is rather small, the problem is strongly anisotropic, inducing thus numerical problems (it becomes degenerate in the limit $\eta \to 0$). All these open problems will be considered in a forthcoming work.

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