3 Fluid dynamic and non-fluid-dynamic estimates.

This section discusses a priori estimates for the two-rolls cases introduced in Section 2.

We recall that the orthonormal basis $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$ for the kernel of L in $L^2_M(I\!R^3)$ was introduced in Section 2 together with an orthogonal splitting of functions $f \in L^2_M([r_A, r_B] \times I\!R^3)$ into $f = f_{||} + f_\perp = P_0 f + (I - P_0) f$, where for the fluid dynamic part

$$f_{||}(r,v) = f_0(r) - \frac{\sqrt{6}}{2} f_4(r)$$

$$+ f_\theta(r) v_\theta + f_r(r) v_r + f_z(r) v_z + \frac{\sqrt{6}}{6} f_4(r) v^2,$$

$$\int M(v)(1, v, v^2) f_\perp(r, z, v) dv = 0,$$

$$\int M\psi_0 f(r,v) dv = f_0(r), \quad \int M\psi_4 f(r,v) dv = f_4(r),$$

$$\int M\psi_\theta f(r,v) dv = f_\theta(r), \quad \int M\psi_r f(r,v) dv = f_r(r),$$

$$\int M\psi_z f(r,v) dv = f_z(r).$$

Set $Df := v_r \frac{\partial f}{\partial r} (+ v_z \frac{\partial f}{\partial z}) + \frac{1}{r} N f$ with N defined in (2.1). In Case 1 due to the symmetries, the position space may be changed from the two-cylinder domain $\Omega \subset I\!R^3$ with measure dx, to $[r_A, r_B] \subset I\!R^+$ with measure rdr. All functions considered are even in v_z giving in particular $f_z = 0$. The relevant ingoing boundary space becomes

$$L^+ := \{f; \mid f \mid_\sim := \Big(\int_{v_r > 0} v_r M(v) \mid f(r_A, v) \mid^2 dv \Big)^{\frac{1}{2}} +$$

$$\Big(\int_{v_r < 0} \mid v_r \mid M(v) \mid f(r_B, v) \mid^2 dv \Big)^{\frac{1}{2}} < +\infty \}.$$

Set

$$\tilde{L}^q := \{f; |f|_q := \Big(\int M(v) (\int |f(x,v)|^q dx)^{\frac{2}{q}} dv \Big)^{\frac{1}{2}} < +\infty \},$$

$$\mathcal{W}^{q-}([r_A, r_B] \times I\!R^3) = \mathcal{W}^{q-} := \{f; \nu^{\frac{1}{2}} f \in \tilde{L}^q, \nu^{-\frac{1}{2}} Df \in \tilde{L}^q, \gamma^+ f \in L^+\}.$$

Lemma 3.1 *Let $\nu^{-\frac{1}{2}}g \in \tilde{L}^q$, $F_b \in L^+$, $2 \leq q < \infty$, be given. There exists a unique solution $F \in \mathcal{W}^{q-}$ to*

$$DF = \frac{1}{\epsilon}(LF + 2\sum_{j=1}^{j_1} \epsilon^j J(F, \Phi^j) + g), \quad F_{/\partial\Omega^+} = F_b, \tag{3.1}$$

where the terms Φ^j of the axially homogeneous asymptotic expansion were introduced in (2.2), and the boundary data F_b are given on the ingoing boundary $\partial\Omega^+$.

Notice first that the a priori estimates (3.2), (3.4) below imply uniqueness in L^2. Then use the solution formula $F = WF_b + Ug + UKF$ from the proof of Lemma 3.2 below in the case φ of (2.2) equals zero. Here UK is compact in L^2 (e.g by first proving the compactness of UE for $EF := \int MFdv$ and then using the splitting $K = K' + K''$ below), so the L^2 case follows from Fredholm's alternative. The L^∞ case then follows from (3.3), and the intermediate cases hold similarly. Finally the addition of the small perturbation $J(F, \varphi)$ does not change the result.

To obtain uniform control of the final non-linear Boltzmann equation all the way to the fluid dynamic limit, we shall use this section to secure sufficiently strong a priori estimates in $\tilde{L}^q$ for the linear problem (3.1). With regard to the shortest, the most transparent or the most elegant method of proof, various approaches are the best suited depending on the situation. We shall varyingly be using straight forward direct computations, dual estimates, ODE methods or Fourier techniques.

For the non-fluid-dynamic part $F_\perp$ of the solution and for the comparison of the solution in different $\tilde{L}^q$-spaces, in the simplest Case 1 we may use quite explicit computations. Define a specular reflection operator $\mathcal{S}$ at $r = r_A,\ r_B$ as $\mathcal{S}f(r, v) = f(r, -v_r, v_\theta, v_z)$.

Lemma 3.2 *Let $q = 2, \infty$, and let F be a solution in $\mathcal{W}^{q-}$ to (3.1) for $g = g_\perp$. The following estimates hold for small enough $\epsilon > 0$;*

$$\epsilon^{\frac{1}{2}} \mid \mathcal{S}F \mid_\sim + \mid \nu^{\frac{1}{2}} F_\perp \mid_2 \leq c(\mid \nu^{-\frac{1}{2}} g \mid_2 + \epsilon^{\frac{1}{2}} \mid F_b \mid_\sim + \epsilon(\| F_r \|_2 + \| F_\theta \|_2 + \| F_0 \|_2 + \| F_4 \|_2)), \tag{3.2}$$

$$\mid \nu^{\frac{1}{2}} F \mid_\infty \leq c(\mid \nu^{-\frac{1}{2}} g \mid_\infty + \epsilon^{-\frac{2}{q}} \mid \nu^{\frac{1}{2}} F \mid_q + \mid \nu^{\frac{1}{2}} F_b \mid_\sim). \tag{3.3}$$

The estimate (3.3) also holds in this form, when g has a non-vanishing fluid dynamic component $g_\parallel$.

Proof of Lemma 3.2. We first turn to the estimate (3.3). To prove it, we shall need some estimates which are suitably discussed in the original coordinates of (1.2). Consider the exponential form of (3.1) with φ of (2.2) equal zero;

$$\frac{d}{ds}(F(x+sv,v)e^{s\frac{\nu}{\epsilon}}) = \frac{e^{s\frac{\nu}{\epsilon}}(KF+g)}{\epsilon}(x+sv,v),$$

or integrated

$$F(x,v) = e^{-s_0\frac{\nu}{\epsilon}}F_b(x-s_0v,v) + \int_{-s_0}^{0} e^{s\frac{\nu}{\epsilon}}\frac{(KF+g)}{\epsilon}(x+sv,v)ds$$
$$=: WF_b + UKF + Ug.$$

Here s_0 denotes the time to reach the ingoing boundary along the characteristic.

Split the kernel k of K into $k_n = \text{sign}k \min(|k|,n)$ and the remaining part $k-k_n$, and denote the corresponding operators by K' and K''. The operator norm of $K-K' = K''$ tends to zero, and K is compact in L^2_M. It immediately follows that F can be written as

$$F = (UK')^2F + (UK''UK + UK'UK'')F + (UKU+U)g + (UKW+W)F_b$$
$$=: (UK')^2F + Z_1F + Z_2g + Z_3F_b.$$

The K''-factor makes the operator norm of Z_1 in $\tilde{L}^\infty$ tend to zero (uniformly in ϵ) when the cut-off $n \to \infty$. Also by straight forward computations

$$|\nu^{\frac{1}{2}}Z_2g|_\infty \leq c\,|\nu^{-\frac{1}{2}}g|_\infty, \qquad |\nu^{\frac{1}{2}}Z_3F_b|_\infty \leq c\,|\nu^{\frac{1}{2}}F_b|_\sim .$$

It remains the term $UK'UK'$. The first U is (uniformly in ϵ) bounded in $\tilde{L}^\infty$, so it is enough to consider $K'UK'$. Setting $EF(x) = \int F(x,v)M(v)dv$, we can estimate $K'UK'$ by a cut-off dependent multiple of EUE in the operator norm. For fixed ϵ the operator EUE is bounded from L^p into L^q for $p > d$, $q = \infty$, $d \geq 1$, as well as for $1 < p \leq d$, $q < dp(d-p)^{-1}$, $d > 1$. Here d is the dimension of the x-space. For the proof of this estimate of EUE we follow [M Chapter 6]. Let us first consider the case $\epsilon = 1$, $d = 2$, our main concern being the domain Ω equal an open annulus between the radii r_A and r_B.
Let $v' = (v_x, v_y)$ for $v = (v_x, v_y, v_z)$ and let g be a function from $L^p(\Omega)$ where we let $g(x-sv', v)$ for $x \in \Omega$ take the value zero after $x - sv$ has for the first time left Ω. This gives

$$Ug(x,v) = \int_0^\infty g(x-sv',v)e^{(-\nu(v)s)}ds.$$

Set $G(x) = Eg(x,v)$. Then

$$EUG(x) = \int_{\mathbb{R}^3\times(0,\infty)} e^{-\nu(v)s} G(x - sv')M(v)dvds$$
$$\leq \int_{\mathbb{R}^3\times(0,\infty)} e^{-s\nu_0} G(x - sv')M(v)dvds,$$

where $\nu_0 = \inf \nu(v) > 0$. It follows that the v_z-integral can be added after concluding the estimate of the $dv'ds$-integral. We continue the discussion for $v' \in \mathbb{R}^2$ using the notation $v' = v$. A change of variables $(s,v) \to (r,y)$ with $r = |v|, \quad y = x - sv$ gives $EUG \leq G * \varphi$ with

$$\varphi(y) = c_1|y|^{-1} \int_0^\infty k(r,y)dr, \quad c_1 > 0,$$
$$k(r,y) = M(r)e^{-r^{-1}|y|\nu_0}.$$

Since $M(v) \leq c_2 e^{-c_3|v|}$, we get

$$k(r,y) \leq c_4 e^{-\frac{c_3 r}{2}} e^{-\frac{c_3 r}{2} - \frac{\nu_0|y|}{r}}$$
$$\leq c_4 r^{-1} e^{\frac{-c_3 r}{2}} e^{-c_5|y|^{\frac{1}{2}}}.$$

It follows that $\varphi \in L^\rho$ if $\rho < 2$. If $1 < p \leq 2$ the result now follows from Young's inequality (i.e. from $*\varphi : L^p \to L^q$ for $q^{-1} = p^{-1} - \rho'^{-1}$). By Hölder's inequality $EUG \in L^\infty$ if $p > 2$. The proof for $\Omega \in \mathbb{R}^3$ is analogous whereas the case $d = 1$ requires a slightly different estimate of k.
For the desired estimate of the solution in L^∞ by L^2-terms for $d = 2$ we have to apply the estimate of $UK'UK'$ twice (also the solution formula). Including the ϵ-dependence in the above estimate of EUE gives the factor $\epsilon^{-\frac{2}{q}}$.
With this estimate of EUE and choosing the cut-off n large enough, (3.3) follows when $\varphi = 0$. Recalling that φ is of order ϵ, and taking ϵ small enough, it follows that the addition of $J(F,\varphi)$ to g does not change the result in this part of the proof, neither does the addition of a fluid component to g.

Consider next the mapping from $\nu^{-\frac{1}{2}}\tilde{L}^q \times L^+$ into $\mathcal{W}^{q-}$ given by $(g, F_b) \to F$, with F a solution to (3.1) for $\varphi = 0$. Green's formula and the spectral inequality of Lemma 2.2 for the linearized collision operator L, i.e.

$$-\int MfLfdv \geq c \int M\nu f_\perp^2 dv,$$

give

$$\epsilon \mid SF \mid_\sim^2 + \mid \nu^{\frac{1}{2}} F_\perp \mid_2^2 \leq \frac{c}{\delta} \mid \nu^{-\frac{1}{2}} g_\perp \mid_2^2 + \delta \mid \nu^{\frac{1}{2}} F_\perp \mid_2^2 + \epsilon \mid F_b \mid_\sim^2 .$$

This completes the estimate (3.2) when $\varphi = 0$. The inclusion of $J(F,\varphi)$ to g, adds $c\epsilon \mid \nu^{\frac{1}{2}} F_\perp \mid_2^2$, which is incorporated in the left hand side, and a term

$$c\epsilon(\| F_r \|_2 + \| F_\theta \|_2 + \| F_0 \|_2 + \| F_4 \|_2). \qquad \square$$

The control of the fluid part $F_\|$ of the solution, i.e. the kernel of L, is less efficient. In particular Case 2 requires a careful analysis. For this we have chosen a direct computation of each moment in order to obtain sharp estimates. The method is here illustrated in some detail in the following lemmas for the simpler Case 1.

Lemma 3.3 *Let $g = g_\| + g_\perp$ (i.e. with a possible fluid dynamic part $g_\|$ in g, and let F be a solution in $\mathcal{W}^{2-}$ to (3.1). For $\epsilon > 0$ and small enough,*

$$\begin{aligned}\| F_r \|_2 + \| F_\theta \|_2 + \| F_0 \|_2 + \| F_4 \|_2 \leq c(\mid F_\perp \mid_2 \\ + \frac{1}{\epsilon} \mid \nu^{-\frac{1}{2}} g_\perp \mid_2 + \frac{1}{\epsilon^2} \mid g_\| \mid_2 + \mid F_b \mid_\sim).\end{aligned} \tag{3.4}$$

Proof of Lemma 3.3. Define

$$f_{\theta^i r^j}(r) := \int M v_\theta^i v_r^j f_\perp(r,v) dv, \; i+j \geq 2,$$

and $f_{\theta^i r^j 2}(r)$ correspondingly, when there is an extra factor $\mid v \mid^2$ in the integrand. A multiplication of (3.1) with $v_\theta M$ (resp. $v^2 M$) and integration over $\mathbb{R}_v^3$ leads to

$$\begin{aligned} F_{\theta r}(r) &= \frac{F_{\theta r}(1)}{r^2} + \frac{1}{r^2}\int_1^r s^2 \frac{g_\theta}{\epsilon} ds, \\ F_{r2}(r) &= \frac{c_{r2}}{r} + \frac{1}{r\epsilon}\int_1^r s(\sqrt{6} g_4 - 2g_0) ds. \end{aligned}$$

Multiply equation (3.1) with $\bar{A}(|v|) v_r M$ and integrate over $\mathbb{R}_v^3$,

$$\begin{aligned} \Big(\int v_r^2 \bar{A} M F dv\Big)' = \Big(k_4 F_4 + F_{r^2 \bar{A}}\Big)' = \frac{1}{r}\Big(F_{\theta^2 \bar{A}} - F_{r^2 \bar{A}}\Big) \\ + \frac{1}{\epsilon}\Big(\frac{c_{r2}}{r} + \frac{1}{r\epsilon}\int_1^r s(\sqrt{6} g_4 - 2g_0) ds + \int v_r \bar{A} J(F_\perp, \epsilon\Phi^1) M dv\Big) \\ + \sum_{j=2}^{j_1} \epsilon^{j-1} \int v_r \bar{A} J(F, \Phi^j) M dv + \frac{1}{\epsilon}\int g v_r \bar{A} M dv. \end{aligned} \tag{3.5}$$

Using the spectral inequality of Lemma 2.2, we notice that

$$k_4 := \int v_r^2 \psi_4 \bar{A} M dv = \frac{1}{\sqrt{6}} \int v_r v^2 v_r \bar{A} M dv =$$

$$\frac{1}{\sqrt{6}} \int v_r (v^2 - 5) v_r \bar{A} M dv = \frac{1}{\sqrt{6}} \int L(v_r \bar{A}) v_r \bar{A} M dv < -c \int |v_r \bar{A}|^2 M dv < 0.$$

Set $\tilde{F}_4 = k_4 F_4 + F_{r^2 \bar{A}}$ and regroup the terms in (3.5) as

$$\tilde{F}_4' = \frac{c_{r2}}{r\epsilon} + \Big\{ \frac{1}{r} \Big(F_{\theta^2 \bar{A}} - F_{r^2 \bar{A}} \Big)$$

$$+ \frac{1}{\epsilon} \Big(\frac{1}{r\epsilon} \int_1^r s(\sqrt{6} g_4 - 2 g_0) ds + \int v_r \bar{A} J(F_\perp, \epsilon \Phi^1) M dv \Big)$$

$$+ \sum_{j=2}^{j_1} \epsilon^{j-1} \int v_r \bar{A} J(F, \Phi^j) M dv + \frac{1}{\epsilon} \int g v_r \bar{A} M dv \Big\}.$$

Denoting the expression within $\Big\{ ... \Big\}$ by G_4 gives

$$(\tilde{F}_4)' = \frac{c_{r2}}{r\epsilon} + G_4,$$

which integrates to give

$$\tilde{F}_4(r_B) - \tilde{F}_4(r_A) = \frac{c_{r2}}{\epsilon} (\ln r_B - \ln r_A) + \int_{r_A}^{r_B} G_4(s) ds,$$

$$\tilde{F}_4(r) = \tilde{F}_4(r_B) + \frac{c_{r2}}{\epsilon} (\ln r - \ln r_B) + \int_{r_B}^{r} G_4(s) ds.$$

Eliminating c_{r2}, it follows that

$$\tilde{F}_4(r) = \tilde{F}_4(r_B) + \frac{\ln r - \ln r_B}{\ln r_B - \ln r_A} \Big(\tilde{F}_4(r_B) - \tilde{F}_4(r_A) + \int_{r_A}^{r_B} G_4(s) ds \Big)$$

$$- \int_r^{r_B} G_4(s) ds. \qquad (3.6)$$

With $w_1 = (v_r^2 v_\theta^2 \bar{B}, 1)$, an analogous solution formula for $\frac{\tilde{F}_\theta}{r} := \frac{w_1 F_\theta}{r} + \frac{F_{\theta r^2 \bar{B}}}{r}$ can be obtained in the same way. Namely, multiply the equation (3.1) with $M v_r v_\theta \bar{B}(|v|)$ and integrate over $I\!R_v^3$. It follows that

$$(\frac{\tilde{F}_\theta}{r})' = \frac{F_{\theta^3\bar{B}} - 3F_{\theta r^2\bar{B}}}{r^2} +$$

$$\frac{1}{r\epsilon}\Big(\frac{c_{\theta r}}{r^2} + \frac{1}{r^2}\int_1^r s^2\frac{g_\theta}{\epsilon} + 2\int v_r v_\theta \bar{B} J(v_r F_r + F_\perp, \epsilon\Phi^1) M dv\Big)$$

$$+2\sum_{j=2}^{j_1}\epsilon^{j-1}\int v_r v_\theta \bar{B} J(F, \Phi^j) M dv$$

$$+\frac{1}{\epsilon}\int v_r v_\theta \bar{B} M g dv = \frac{c_{\theta r}}{r^3\epsilon} + G_\theta.$$

And so

$$\frac{\tilde{F}_\theta(r_B)}{r_B} - \frac{\tilde{F}_\theta(r_A)}{r_A} = \frac{2c_{\theta r}}{\epsilon}(\frac{1}{r_A^2} - \frac{1}{r_B^2}) + \int_{r_A}^{r_B} G_\theta(s)ds,$$

$$\frac{\tilde{F}_\theta(r)}{r} - \frac{\tilde{F}_\theta(r_B)}{r_B} = \frac{2c_{\theta r}}{\epsilon}(\frac{1}{r_B^2} - \frac{1}{r^2}) + \int_{r_B}^{r} G_\theta(s)ds.$$

Eliminating $c_{\theta r}$ gives

$$\frac{\tilde{F}_\theta(r)}{r} = \frac{\tilde{F}_\theta(r_B)}{r_B} + \frac{(r^2 - r_B^2)r_A^2}{(r_B^2 - r_A^2)r^2}\Big(\frac{\tilde{F}_\theta(r_B)}{r_B} - \frac{\tilde{F}_\theta(r_A)}{r_A} - \int_{r_A}^{r_B} G_\theta(s)ds\Big) + \int_{r_B}^{r} G_\theta(s)ds. \quad (3.7)$$

Multiplying the equation (3.1) with M and integrating over $I\!R_v^3$, leads to $(rF_r)' = r\frac{g_0}{\epsilon}$, i.e.

$$F_r(r) = \frac{F_r(1)}{r} + \frac{1}{r}\int_1^r s\frac{g_0}{\epsilon}ds. \quad (3.8)$$

By definition of $F_r(1)$,

$$| F_r(1) | = | \int v_r F(1,v) M dv |$$

$$\leq c\Big(\int | v_r | F^2(1,v) M dv\Big)^{\frac{1}{2}} \leq c(| SF |_\sim + | F_b |_\sim).$$

And so by (3.8)

$$\| F_r \|_2 \leq c(\frac{1}{\epsilon} | g_{\|} |_2 + | SF |_2 + | F_b |_\sim). \quad (3.9)$$

Multiply the equation (3.1) with $v_r M$ and integrate with respect to v. It follows that

$$(\int v_r^2 F(r,v)Mdv)' = \Big(F_0 + \sqrt{\frac{2}{3}}F_4 + F_{r^2}\Big)'$$
$$= \frac{F_{\theta^2} - F_{r^2}}{r} + \frac{g_r}{\epsilon}.$$

Multiply this with $2\Big(F_0 + \sqrt{\frac{2}{3}}F_4 + F_{r^2}\Big)$ and integrate with respect to r on (r, r_B), then on (r_A, r_B), to obtain

$$\| F_0 + \sqrt{\frac{2}{3}}F_4 \|_2 \leq c\Big(\mid F_\perp \mid_2 + \frac{1}{\epsilon} \| g_r \|_2 + \mid \int v_r^2 F(r_B, v)Mdv \mid \Big).$$

But

$$\mid \int v_r^2 F(r_B,v)Mdv \mid \leq c(\int M \mid v_r \mid F^2(r_B,v)dv)^{\frac{1}{2}}$$
$$\leq c(\mid SF \mid_\sim + \mid F_b \mid_\sim).$$

Hence

$$\| F_0 + \sqrt{\frac{2}{3}}F_4 \|_2 \leq c\Big(\mid F_\perp \mid_2 + \frac{1}{\epsilon} \| g_r \|_2 + \mid SF \mid_\sim + \mid F_b \mid_\sim \Big). \tag{3.10}$$

It follows from (3.6) and (3.7) that

$$\| F_4 \|_2 + \| F_\theta \|_2 \leq c\Big(\mid F_\perp \mid_2 + \frac{1}{\epsilon^2} \mid g_{||} \mid_2 + \frac{1}{\epsilon} \mid g_\perp \mid_2$$
$$+ \mid SF \mid_\sim + \mid F_b \mid_\sim + \epsilon \mid F_{||} \mid_2 \Big).$$

This together with (3.9-10) gives (3.4). □

Analogous estimates hold in the axially homogeneous **Case 2**. Care is here needed to remove terms of low ϵ-order in the proof of the fluid dynamic estimates. This complication has its origin in the fact that the boundary scalings (of order ϵ) here are larger than the Knudsen number (ϵ^4). For upcoming negative order terms in $F_\perp$ the example $\alpha = \int MdvJ(F_\perp, v_\theta)v_r\bar{A}$ will suffice to clarify the technique. That moment can obviously be written as $\int MdvF_\perp\chi$ for some non-fluid-dynamic function χ. Projecting the whole equation along $L^{-1}\chi$, increases the epsilon order of the term α by one. This can be repeated until all appearing moments of $F_\perp$ are of non-negative order. A corresponding raising of order for the fluid dynamic estimates is more involved (see [AN2]). The resulting a priori estimates are

Lemma 3.4 *If $0 < \delta'$ is small enough and $g = g_\perp$, then for small enough $\epsilon > 0$ the following estimates hold for a solution of (3.1) in $\mathcal{W}^2$,*

$$| \tilde{\nu}^{\frac{1}{2}} F_\perp |_2 \leq c\Big(\epsilon^{-3} \mid \tilde{\nu}^{-\frac{1}{2}} g_\perp |_2 + \epsilon^2 \mid F_b |_\sim \Big), \qquad (3.11)$$

$$\| F_0 \|_2 + \| F_r \|_2 + \| F_4 \|_2 + \| F_\theta \|_2 \leq c\Big(\epsilon^{-5} \mid \tilde{\nu}^{-\frac{1}{2}} g_\perp |_2 + \mid F_b |_\sim \Big). \qquad (3.12)$$

If F is a solution of (3.1) in $\mathcal{W}^{\infty-}$, then the following estimate holds for small enough $\epsilon > 0$;

$$| \tilde{\nu}^{\frac{1}{2}} F |_\infty \leq c(| \tilde{\nu}^{-\frac{1}{2}} g |_\infty + \epsilon^{-\frac{8}{q}} | \tilde{\nu}^{\frac{1}{2}} F |_q + | \tilde{\nu}^{\frac{1}{2}} F_b |_\sim), \quad q \leq \infty. \qquad (3.13)$$

A fluid dynamic component in g does not change the results in (3.14-15).

In **Case 3** the partial differential nature of the problem requires more work than the ordinary differential equations appearing in Cases 1 and 2. But the two-roll domain is bounded and has a simple geometry that allows the use of a direct approach involving orthogonal (Fourier) expansions. For more complicated geometries in other bounded domains one may first by similar Fourier based methods study the dual problem in say a box containing the domain in question, and then via dual estimates and trace theorems obtain corresponding results for more arbitrary bounded domains (cf [M]).
With the change of variables from $(r, z) \in (1, r_B) \times (-\frac{r_B-1}{2}, \frac{r_B-1}{2})$ to $(s, Z) \in (-\pi, \pi)^2$ and with $\eta = \frac{r_B-1}{2\pi}$, we will be interested in the case when the new unknown $\tilde{F}(s, Z, v) := F(\eta s + \frac{r_B+1}{2}, \eta Z, v)$ solves

$$v_r \frac{\partial \tilde{F}}{\partial s} + v_z \frac{\partial \tilde{F}}{\partial Z} + \eta\mu(s) N\tilde{F} = \frac{\eta}{\epsilon}(L\tilde{F} + \tilde{g}), \qquad (3.14)$$

where $\mu(s) = \frac{2}{2\eta s + r_B + 1}$. The control of the fluid dynamic moments will be obtained by Fourier series expansions. Write (in the new variables) the Fourier expanded density function $\tilde{F}$ as

$$\tilde{F}(s, Z, v) = \sum_{(n,j)\in Z^2} \alpha^{nj}(v) e^{i(ns+jZ)}.$$

The fluid dynamic moments $\tilde{F}_0$, $\tilde{F}_4$, $\tilde{F}_r$, $\tilde{F}_\theta$, and $\tilde{F}_z$ become

$$\tilde{F}_0(s, Z) = \sum_{(n,j)} m_0^{nj} e^{i(ns+jZ)}, \quad \tilde{F}_4(s, Z) = \sum_{(n,j)} m_4^{nj} e^{i(ns+jZ)},$$

$$\tilde{F}_r(s, Z) = \sum_{(n,j)} u_r^{nj} e^{i(ns+jZ)}, \ \tilde{F}_\theta(s, Z) = \sum_{(n,j)} u_\theta^{nj} e^{i(ns+jZ)}, \ \tilde{F}_z(s, Z) = \sum_{(n,j)} u_z^{nj} e^{i(ns+jZ)},$$

where

$$m_0^{nj} := (\alpha^{nj}, 1), \quad m_4^{nj} := (\alpha^{nj}, \psi_4),$$
$$u_r^{nj} := (\alpha^{nj}, \psi_r), \quad u_\theta^{nj} := (\alpha^{nj}, \psi_\theta), \quad u_z^{nj} := (\alpha^{nj}, \psi_z).$$

Recall that (α, β) denotes the scalar product $\int \alpha(v)\beta(v)M(v)dv$, and notice that $u_z^{n0} = 0$ due to the symmetry $\tilde{F}(s, Z, v_r, v_\theta, v_z) = \tilde{F}(s, -Z, v_r, v_\theta, -v_z)$. Notice that the Fourier coefficients of the first r-derivative contain a multiple of the boundary value difference,

$$\alpha^{nj}(\frac{\partial F}{\partial r}) = in\alpha^{nj}(F) + \frac{(-1)^n}{2\pi}d^j, \quad (n,j) \in Z^2,$$

whereas for the first z-derivative no such term is present. Set $d = (\tilde{F}(\pi - 0) - \tilde{F}(-\pi + 0))\frac{1}{2\pi}$ with d^j its j'th Fourier coefficient in the Z-direction.
Denote by $\lambda := (v_r^2\bar{A}, \psi_4)$, $w_1 = (v_r^2 v_\theta^2 \bar{B}, 1)$, and by $Q = I - P_0$, and write

$$e(Z, v) := \frac{1}{2\pi}\Big((\mu\tilde{F})(\pi - 0, Z, v) - (\mu\tilde{F})(-\pi + 0, Z, v)\Big) = \sum_{j \in Z} e^j(v)e^{ijZ}.$$

Set

$$\begin{aligned}
\Lambda_r^{nj} := &-\frac{3i}{\epsilon}(g^{nj}, v_r) - 3j(g^{nj}, v_r v_z \bar{B}) + n(g^{nj}, (2v_r^2 - v_\theta^2 - v_z^2)\bar{B}) \\
&-in\epsilon(-1)^n d^j_{v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}} + 3i(-1)^n d^j_{r^2} - 3ij\epsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} \\
&-i\epsilon n^2(Qv_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Q\alpha^{nj}) - i\epsilon nj(Qv_z(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Q\alpha^{nj}) \\
&-3i\epsilon nj(Qv_r^2 v_z \bar{B}, Q\alpha^{nj}) - i\epsilon j^2(Qv_r v_z^2 \bar{B}, Q\alpha^{nj}) \\
&-\epsilon\eta n(\mu\tilde{F})^{nj}_{v_r(2v_r^2 - 7v_\theta^2 - v_z^2)\bar{B}} - 3\epsilon\eta j(\mu\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} + 3i\eta(\mu\tilde{F})^{nj}_{v_r^2 - v_\theta^2},
\end{aligned}$$

$$\begin{aligned}
\Lambda_z^{nj} := &-\frac{3i}{\epsilon}(g^{nj}, v_z) + j(g^{nj}, (2v_z^2 - v_r^2 - v_\theta^2)\bar{B}) + 3n(g^{nj}, v_r v_z \bar{B}) \\
&-3in\epsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} - ij\epsilon(-1)^n d^j_{v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}} + 3i(-1)^n d^j_{rz} \\
&-3i\epsilon n^2(Qv_r^2 v_z \bar{B}, Q\alpha^{nj}) - 3i\epsilon nj(Qv_r v_z^2 \bar{B}, Q\alpha^{nj}) \\
&-i\epsilon nj(Qv_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Q\alpha^{nj}) - i\epsilon j^2(Qv_z(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Q\alpha^{nj}) \\
&-3\epsilon\eta n(\mu\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} - \epsilon\eta j(\mu\tilde{F})^{nj}_{v_r(-v_r^2 - v_\theta^2 + 2v_z^2)\bar{B}} + i\eta(\mu\tilde{F})^{nj}_{v_r v_z}.
\end{aligned}$$

Lemma 3.5 *Let $\tilde{F}$ be a solution to (3.14). Denote by $\epsilon_1 = \frac{\epsilon}{\eta}$. For $(n, j) \neq (0, 0)$,*

$$m_0^{nj} = -\frac{4}{3}w_1(g^{nj}, 1) + \frac{4}{3}w_1(-1)^n d_r^j + \frac{n\Lambda_r + j\Lambda_z}{3(n^2 + j^2)} + \frac{4}{3}w_1(\mu\tilde{F})^{nj}_{v_r}$$
$$+\sqrt{\frac{2}{3}}\frac{1}{\lambda(n^2 + j^2)}\Big(\frac{1}{\epsilon_1^2}(g^{nj}, v^2 - 5) + i\frac{n}{\epsilon_1}g^{nj}_{v_r\bar{A}} + \frac{ij}{\epsilon_1}(g^{nj}, v_z\bar{A}) - \frac{\eta}{\epsilon_1}(\mu g)^{nj}_{v_r\bar{A}}$$

$$
\begin{aligned}
&-\frac{(-1)^n}{\epsilon_1}d^j_{v_r(v^2-5)} - i(-1)^n n d^j_{v_r^2\bar{A}} - i(-1)^n j d^j_{v_r v_z \bar{A}} + \eta(-1)^n e^j_{v_r^2\bar{A}} \\
&\quad +n^2(Qv_r^2\bar{A}, Q\alpha^{nj}) + j^2(Qv_z^2\bar{A}, Q\alpha^{nj}) + 2nj(v_r v_z\bar{A}, Q\alpha^{nj}) \\
&\quad -i\eta n(\mu\tilde{F})^{nj}_{v_\theta^2-v_r^2} - i\eta j(\mu\tilde{F})^{nj}_{v_r v_z} + i\eta j(\mu\tilde{F})^{nj}_{v_r v_z\bar{A}} \\
&\quad -\eta^2(\mu^2\tilde{F})^{nj}_{v_r^2-v_\theta^2} + i\eta n(\mu\tilde{F})^{nj}_{v_r^2\bar{A}} - \eta(\mu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big), \quad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
m_4^{nj} = &\frac{1}{\lambda(n^2+j^2)}\Big(-\frac{1}{\epsilon_1^2}(g^{nj}, v^2-5) - i\frac{n}{\epsilon_1}g^{nj}_{v_r\bar{A}} - \frac{ij}{\epsilon_1}(g^{nj}, v_z\bar{A}) + \frac{\eta}{\epsilon_1}(\mu g)^{nj}_{v_r\bar{A}} \\
&+\frac{(-1)^n}{\epsilon_1}d^j_{v_r(v^2-5)} + i(-1)^n n d^j_{v_r^2\bar{A}} + ij(-1)^n d^j_{v_r v_z\bar{A}} - \eta(-1)^n e^j_{v_r^2\bar{A}} \\
&-n^2(Qv_r^2\bar{A}, Q\alpha^{nj}) - j^2(Qv_z^2\bar{A}, Q\alpha^{nj}) - 2nj(v_r v_z\bar{A}, Q\alpha^{nj}) \\
&+i\eta n(\mu\tilde{F})^{nj}_{v_\theta^2-v_r^2} + i\eta j(\mu\tilde{F})^{nj}_{v_r v_z} - i\eta j(\mu\tilde{F})^{nj}_{v_r v_z\bar{A}} + \eta^2(\mu\tilde{F})^{nj}_{v_r^2-v_\theta^2} \\
&-i\eta n(\mu\tilde{F})^{nj}_{v_r^2\bar{A}} + \eta(\mu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big) \quad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
u_\theta^{nj} = &\frac{1}{w_1(n^2+j^2)}\Big(-\frac{1}{\epsilon_1^2}(g^{nj}, v_\theta) - \frac{in}{\epsilon_1}(g^{nj}, v_r v_\theta\bar{B}) - \frac{ij}{\epsilon_1}(g^{nj}, v_\theta v_z\bar{B}) - 2\frac{\eta}{\epsilon_1}(\mu g)^{nj}_{v_r v_\theta\bar{B}} \\
&+\frac{(-1)^n}{\epsilon_1}d^j_{r\theta} + in(-1)^n d^j_{v_r^2 v_\theta\bar{B}} + ij(-1)^n d^j_{v_r v_\theta v_z\bar{B}} + 2\eta(-1)^n e^j_{v_r^2 v_\theta\bar{B}} \\
&-n^2(Qv_r^2 v_\theta\bar{B}, Q\alpha^{nj}) - j^2(Qv_\theta v_z^2\bar{B}, Q\alpha^{nj}) - 2nj(v_r v_\theta v_z\bar{B}, Q\alpha^{nj}) \\
&+i\eta n(\mu\tilde{F})^{nj}_{(v_\theta^3-2v_r^2v_\theta)\bar{B}} + 4i\eta j(\mu\tilde{F})^{nj}_{v_r v_\theta v_z\bar{B}} + 2i\eta n(\mu\tilde{F})^{nj}_{v_r^2 v_\theta\bar{B}} \\
&-2\eta(\mu'\tilde{F})^{nj}_{v_r^2 v_\theta\bar{B}} + 2\eta^2(\mu^2\tilde{F})^{nj}_{v_r v_\theta\bar{B}}\Big), \quad (3.17)
\end{aligned}
$$

$$
u_r^{nj} = \frac{i}{n^2+j^2}\Big(-\frac{n}{\epsilon_1}(g^{nj}, 1) + \frac{-j^2\Lambda_r^{nj} + nj\Lambda_z^{nj}}{3\epsilon_1 w_1(n^2+j^2)} + n(-1)^n d_r^j + \eta n(\mu\tilde{F})^{nj}_{v_r}\Big), \quad (3.18)
$$

$$
u_z^{nj} = \frac{i}{n^2+j^2}\Big(-\frac{j}{\epsilon_1}g^{nj}, 1) + \frac{nj\Lambda_r^{nj} - n^2\Lambda_z^{nj}}{3\epsilon_1 w_1(n^2+j^2)} + j(-1)^n d_r^j + \eta j(\mu\tilde{F})^{nj}_{v_r}\Big). \quad (3.19)
$$

Proof of Lemma 3.5. This is proved by moment projections and direct computations from the Fourier expanded (3.14), see [AN3]. □

Lemma 3.6 *Let $\tilde{F}$ be a solution to (3.14). Then for η small enough,*

$$
\begin{aligned}
&| m_0^{00} | + | m_4^{00} | + | u_\theta^{00} | + | u_r^{00} | + | u_z^{00} | \\
\leq c\Big(&\frac{| g_{\|} |_2}{\epsilon_1^2} + \frac{| \nu^{-\frac{1}{2}} g_\perp |_2}{\epsilon_1} + | S\tilde{F} |_\sim + \frac{| \tilde{F}_b |_\sim}{\sqrt{\epsilon_1}} + \eta \| \tilde{F} \|_2\Big).
\end{aligned}
$$

Proof of Lemma 3.6 For $(n,j)=(0,0)$, it holds that

$$\alpha^{00}=\frac{1}{2\pi}\int_{-\pi}^{\pi}dZ[\frac{1}{2}(\tilde{F}(\pi-0)+\tilde{F}(-\pi+0))-\sum_{n\neq 0}\alpha^{n0}e^{in\pi}]$$
$$=\Delta-\sum_{n\neq 0}\alpha^{n0}e^{in\pi}, \qquad (3.20)$$

where $\Delta=\frac{1}{4\pi}\int_{-\pi}^{\pi}dZ(\tilde{F}(\pi-0)+\tilde{F}(-\pi+0))$. First,

$$\alpha^{00}_{v_r^2\bar{A}}=\frac{1}{\sqrt{6}}\alpha_4^{00}\int v_r^2v^2\bar{A}Mdv+\alpha^{00}_{\perp v_r^2\bar{A}}.$$

A multiplication of (3.20) with $Mv_r^2\bar{A}$ and v-integration gives

$$\alpha^{00}_{v_r^2\bar{A}}=\Delta_{v_r^2\bar{A}}-\sum_{n\neq 0}\alpha^{n0}_{v_r^2\bar{A}}(-1)^n.$$

To proceed, take the scalar product of (3.14) with $v_r\bar{A}$ and identify the Fourier coefficients,

$$(-1)^n d^j_{v_r^2\bar{A}}+in(v_r^2\bar{A},\alpha^{nj})+ij(v_rv_z\bar{A},\alpha^{nj})+\eta(\mu\tilde{F}_{v_\theta^2-v_r^2})^{nj}=$$
$$\frac{1}{\epsilon_1}\Big((v_r(v^2-5),\alpha^{nj})+(g^{nj},v_r\bar{A})\Big). \qquad (3.21)$$

Also take the scalar product of (3.16) with v^2-5, and identify the Fourier coefficients,

$$-i(-1)^n d^j_{v_r(v^2-5)}+n(v_r(v^2-5),\alpha^{nj})+j(v_z(v^2-5),\alpha^{nj})$$
$$=-\frac{i}{\epsilon_1}\Big((g^{nj},v^2-5)+\epsilon_1\eta(\mu\tilde{F})^{nj}_{v_r(v^2-5)}\Big). \qquad (3.22)$$

Moreover, (3.14) writes

$$v_r\frac{\partial}{\partial s}(\mu\tilde{F})+v_z\frac{\partial}{\partial Z}(\mu\tilde{F})-v_r\mu'\tilde{F}+\eta\mu^2N\tilde{F}=\frac{1}{\epsilon_1}(L(\mu\tilde{F})+\mu g),$$

so that

$$i(nv_r+jv_z)(\mu\tilde{F})^{nj}+(-1)^n v_r e^j(v)-v_r(\mu'\tilde{F})^{nj}$$
$$+\eta(\mu^2N\tilde{F})^{nj}=\frac{1}{\epsilon_1}(L(\mu\tilde{F})^{nj}+(\mu g)^{nj}),$$

where

$$e(Z,v) := \frac{1}{2\pi}\Big((\mu\tilde{F})(\pi-0,Z,v) - (\mu\tilde{F})(-\pi+0,Z,v)\Big) = \sum_{j\in Z} e^j(v)e^{ijZ}.$$

Taking the scalar product with $v_r\bar{A}$ leads to

$$(-1)^n e^j_{v_r^2\bar{A}} + in(v_r^2\bar{A},(\mu\tilde{F})^{nj}) + ij(v_r v_z\bar{A},(\mu\tilde{F})^{nj} - (v_r^2\bar{A},(\mu'\tilde{F})^{nj})$$
$$+\eta(\mu^2\tilde{F}_{v_r^2-v_\theta^2})^{nj} = \frac{1}{\epsilon_1}\Big((v_r(v^2-5),(\mu\tilde{F})^{nj}) + (v_r\bar{A},(\mu g)^{nj})\Big). \qquad (3.23)$$

By (3.21-23) for $n\neq 0$,

$$\alpha^{n0}_{v_r^2\bar{A}} = -\frac{1}{\epsilon_1^2 n^2}g^{n0}_{v^2-5} - \frac{i}{\epsilon_1}g^{n0}_{v_r\bar{A}} + \frac{(-1)^n}{\epsilon_1 n^2}d^0_{v_r(v^2-5)}$$
$$+i\frac{(-1)^n}{n}d^0_{v_r^2\bar{A}} - \frac{\eta}{\epsilon_1 n^2}(\mu\tilde{F})^{n0}_{v_r(v^2-5)} + i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_\theta^2-v_r^2}$$
$$= -\frac{1}{\epsilon_1^2 n^2}g^{n0}_{v^2-5} - \frac{i}{\epsilon_1}g^{n0}_{v_r\bar{A}} + \frac{\eta}{n^2}(\mu g)^{n0}_{v_r\bar{A}}$$
$$+\frac{(-1)^n}{\epsilon_1 n^2}d^0_{v_r(v^2-5)} + i\frac{(-1)^n}{n}d^0_{v_r^2\bar{A}} - \eta\frac{(-1)^n}{n^2}e^0_{v_r^2\bar{A}}$$
$$+i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_\theta^2-v_r^2} - i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_r^2\bar{A}}$$
$$+\frac{\eta}{n^2}(\mu'\tilde{F})^{n0}_{v_r^2\bar{A}} - \frac{\eta^2}{n^2}(\mu^2\tilde{F})^{n0}_{v_r^2-v_\theta^2}.$$

From here, using

$$d^0_{v_r(v^2-5)} + \eta(\mu\tilde{F})^{00}_{v_r(v^2-5)} = \frac{1}{\epsilon_1}g^{00}_{(v^2-5)},$$

it follows that

$$| m_4^{00} |_2 \leq c\Big(\frac{| g_0 |_2 + | g_4 |_2}{\epsilon_1^2} + \frac{\eta | g_r |_2 + | \nu^{-\frac{1}{2}}g_\perp |_2}{\epsilon_1} + | \tilde{F}_\perp |_2$$
$$+ | S\tilde{F} |_\sim + | \tilde{F}_b |_\sim + \eta | \tilde{F}_{||} |\Big).$$

Since

$$m_0^{00} = \alpha^{00}_{r^2} - \frac{\sqrt{6}}{3}m_4^{00} - \alpha^{00}_{\perp r^2}, \quad u_\theta^{00} = \frac{1}{w_1}(\alpha^{00}_{v_r^2 v_\theta\bar{B}} - \alpha^{00}_{\perp v_r^2 v\theta\bar{B}}),$$
$$u_r^{00} = \Delta_r - \sum_{n\neq 0}(-1)^n u_r^{n0}, \quad u_z^{00} = \frac{1}{w_1}(\alpha^{00}_{v_r^2 v_z\bar{B}} - \alpha^{00}_{\perp v_r^2 v_z\bar{B}}),$$

similar inequalities can be obtained for m_4^{00}, u_θ^{00}, u_r^{00}, and u_z^{00} and the lemma follows. □

Lemma 3.7 *Let $\nu^{\frac{1}{2}}\beta \in \tilde{L}^\infty$ be given. Then there is $\eta_0 > 0$ such that for $\eta < \eta_0$, a solution $\tilde{F}$ in $\mathcal{W}^{2-}$ to*

$$v_r \frac{\partial \tilde{F}}{\partial s} + v_z \frac{\partial \tilde{F}}{\partial Z} + \eta\mu N\tilde{F} = \frac{\eta}{\epsilon}\Big(L\tilde{F} + \epsilon J(\tilde{F},\beta) + g\Big), \qquad (3.24)$$
$$\tilde{F}_{/\partial\Omega^+} = \tilde{F}_b,$$

satisfies

$$| \nu^{\frac{1}{2}}\tilde{F} |_2 \leq c\Big(\frac{\eta^2}{\epsilon^2} | g_\| |_2 + \frac{\eta}{\epsilon} | \nu^{-\frac{1}{2}} g_\perp |_2 + \sqrt{\frac{\eta}{\epsilon}} | \tilde{F}_b |_\sim \Big). \qquad (3.25)$$

Proof of Lemma 3.7 Consider first the case where $\beta = 0$. As in the axially homogeneous situation, Green's formula and Lemma 2.2 imply that

$$\epsilon_1 | S\tilde{F} |_\sim^2 + | \nu^{\frac{1}{2}}\tilde{F}_\perp |_2^2 \leq c(| \nu^{-\frac{1}{2}} g_\perp |_2^2 + \int (g_\|, \tilde{F}_\|) + \epsilon_1 | \tilde{F}_b |_\sim^2). \qquad (3.26)$$

Then Parseval's identity, Lemma 3.6 for $(n,j) = (0,0)$, and an estimate of the Fourier coefficients $(n,j) \neq (0,0)$ as given in Lemma 3.5, imply that

$$| \tilde{F}_\| |_2 \leq c\Big(\frac{| g_\| |_2}{\epsilon_1^2} + \frac{\| \nu^{-\frac{1}{2}} g_\perp \|_2}{\epsilon_1} + \frac{| \tilde{F}_b |_\sim}{\sqrt{\epsilon_1}} + | \nu^{\frac{1}{2}}\tilde{F}_\perp |_2 + \eta | \tilde{F}_\| |_2 \Big).$$

And so (3.25) holds in the $\beta = 0$ case, since $| F_\| |_2 \simeq | \nu^{\frac{1}{2}} F_\| |_2$,. The case $\beta \neq 0$ can be handled as the case $\beta = 0$ with g in the right hand side, by taking instead $g + \epsilon\tilde{J}(\tilde{F},\beta)$ in the right hand side. This gives

$$| \tilde{F}_\| |_2 \leq c\Big(\frac{| g_\| |_2}{\epsilon_1^2} + \frac{\| \nu^{-\frac{1}{2}}(g_\perp + \epsilon\tilde{J}(\tilde{F},\beta)) \|_2}{\epsilon_1} + \frac{1}{\sqrt{\epsilon_1}} | \tilde{F}_b |_\sim \Big)$$
$$\leq c\Big(\frac{| g_\| |_2}{\epsilon_1^2} + \frac{| \nu^{-\frac{1}{2}} g_\perp |_2}{\epsilon_1} + \frac{1}{\sqrt{\epsilon_1}} | \tilde{F}_b |_\sim + \eta | \nu^{\frac{1}{2}}\tilde{F} |_2 | \nu^{\frac{1}{2}}\beta |_\infty \Big).$$

Thus the lemma holds for η small enough. □
Remark. If we had access to the estimates in this section of the non-hydrodynamic part with respect to $\tilde{L}^q$ for (large) $q > 2$, then the actual asymptotic expansions required in the existence proofs of the following Section 4 would be considerably shortened in the Cases 2 and 3.

4 Existence theorems and fluid dynamic limits.

Based on the discussions about asymptotic expansions and a priori estimates in Sections 2-3, this section studies existence results and fluid dynamic limits for our three choices of archetypical two-rolls behaviour.
Given the asymptotic expansion φ of (2.4), the aim for Case 1 is to prove the existence of a rest term R, so that

$$f = M(1 + \varphi + \epsilon R) \tag{4.1}$$

is a solution to (2.1), (2.3) in Case 1 with $M^{-1}f \in \tilde{L}^\infty$. This corresponds to the function R being a solution to

$$DR = \frac{1}{\epsilon}\Big(LR + 2J(R,\varphi) + \epsilon J(R,R) + l\Big),$$

where l was defined in (2.10). Recall that the asymptotic expansion φ is of order two in ϵ with correct boundary values up to order two and that l of (2.10) - the pure φ- part of the equation - is of ϵ-order two and η-order one in $\tilde{L}^q$, where $\eta = r_B - r_A$. Notice that Φ^j, $j = 1, 2$, may be constructed so that 'practically' $D\Phi^j = (I - P_0)D\Phi^j$, hence $l = l_\perp$. This holds modulo a possible higher order fluid dynamic component, neglected in this section, that does not change the line of reasoning or its results.
Let the sequences $(R^n)_{n\in I\!N}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\Big(LR^{n+1} + 2\sum_{j=1}^{2}\epsilon^j J(R^{n+1},\Phi^j) + g^n\Big), \tag{4.2}$$

$$R^{n+1}(1,v) = R_A(v),\ v_r > 0, \quad R^{n+1}(r_B,v) = R_B(v),\ v_r < 0. \tag{4.3}$$

In (4.2-3)

$$g^n := \epsilon J(R^n,R^n) + l,$$

$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1}v_\theta - \frac{\epsilon^2}{2}u_{\theta A1}^2} - 1 - \sum_{j=1}^{2}\epsilon^j\Phi^j(r_A,v), \quad v_r > 0,$$

$$\epsilon R_B(v) := -\sum_{j=1}^{2}\epsilon^j\Phi^j(r_B,v), \quad v_r < 0,$$

with R_A, R_B of ϵ-order two.
For the rest term iteration scheme (4.2-3) the following holds.

Lemma 4.1 *For $0 < \epsilon$, $0 < r_B - r_A$ small enough, there is a unique sequence (R^n) of solutions to (4.2-3) in the set $X := \{R; \mid \tilde{\nu}^{\frac{1}{2}}R \mid_q \leq C\}$ for some constant*

C. The sequence converges in $\tilde{L}^q$ for $2 \le q \le \infty$, to an isolated solution of

$$DR = \frac{1}{\epsilon}\Big(LR + \epsilon J(R,R) + 2J(R,\varphi) + l\Big), \tag{4.4}$$
$$R(1,v) = R_A(v),\ v_r > 0, \quad R(r_B,v) = R_B(v),\ v_r < 0. \tag{4.5}$$

Proof of Lemma 4.1 Denote by $\eta = r_B - r_A$. The existence result of Lemma 3.1 holds for the boundary value problem

$$DF = \frac{1}{\epsilon}\Big(LF + 2J(F,\varphi) + g\Big),$$

$$F(1,v) = R_A(v),\ v_r > 0, \quad F(r_B,v) = R_B(v),\ v_r < 0.$$

Here $g = g_\perp$ and by Lemma 3.2-3

$$|\, \nu^{\frac{1}{2}} F \,|_2 \le c_1\Big(\frac{1}{\epsilon} \,|\, \nu^{-\frac{1}{2}} g_\perp \,|_2 + |\, R_b \,|_\sim\Big),$$
$$|\, \nu^{\frac{1}{2}} F \,|_\infty \le c_1\Big(\,|\, \nu^{-\frac{1}{2}} g \,|_\infty + \frac{1}{\epsilon} \,|\, \nu^{\frac{1}{2}} F \,|_2 + |\, \nu^{\frac{1}{2}} R_b \,|_\sim\Big). \tag{4.6}$$

We note the obvious L^2-norm equivalence $|\, F_{||} \,|_2 \simeq |\, \nu^{\frac{1}{2}} F_{||} \,|_2$, and the Grad type inequality

$$|\, \nu^{-\frac{1}{2}} J(g,h) \,|_q \le C \,|\, \nu^{\frac{1}{2}} g \,|_\infty |\, \nu^{\frac{1}{2}} h \,|_q, \tag{4.7}$$

which follows by an easy, direct computation. This will next be used to show by induction that

$$|\, \nu^{\frac{1}{2}}(R^{n+1} - R^n) \,|_2 \le c\eta \,|\, \nu^{\frac{1}{2}}(R^n - R^{n-1}) \,|_2, \quad |\, \nu^{\frac{1}{2}} R^n \,|_\infty \le c\eta, n \in I\!N, n > 0. \tag{4.8}$$

For $n = 0$, R^1 is the solution to

$$DR^1 = \frac{1}{\epsilon}(LR^1 + 2J(\varphi, R^1) + l),$$
$$R^1(1,v) = R_A(v),\ v_r > 0, \quad R^1(r_B,v) = R_B(v),\ v_r < 0,$$

so that by (4.6-7) $|\, \nu^{\frac{1}{2}} R^1 \,|_2 \le c\eta\epsilon$, $|\, \nu^{\frac{1}{2}} R^1 \,|_\infty \le c\eta$, where $\eta = r_B - r_A$. Also, $R^{n+2} - R^{n+1}$ is a solution to

$$D(R^{n+2} - R^{n+1}) = \frac{1}{\epsilon}(L(R^{n+2} - R^{n+1}) + 2J(\varphi, R^{n+2} - R^{n+1})$$
$$+\epsilon J(R^{n+1} + R^n, R^{n+1} - R^n)),$$
$$R^{n+2} - R^{n+1} = 0, \quad \partial\Omega^+,$$

which by (4.6-7) and the induction hypothesis (4.8) leads to

$$|\, \nu^{\frac{1}{2}}(R^{n+2} - R^{n+1}) \,|_2 \le c \,|\, \nu^{-\frac{1}{2}} J(R^{n+1} + R^n, R^{n+1} - R^n) \,|_2$$
$$\le c(|\, \nu^{\frac{1}{2}} R^{n+1} \,|_\infty + |\, \nu^{\frac{1}{2}} R^n \,|_\infty) \,|\, \nu^{\frac{1}{2}}(R^{n+1} - R^n) \,|_2$$
$$\le 2c^2\eta \,|\, \nu^{\frac{1}{2}}(R^{n+1} - R^n) \,|_2 \,.$$

Moreover,

$$| \nu^{\frac{1}{2}} R^{n+2} |_\infty \leq | \nu^{\frac{1}{2}} (R^{n+2} - R^{n+1}) |_\infty + ...$$
$$+ | \nu^{\frac{1}{2}} (R^2 - R^1) |_\infty + | \nu^{\frac{1}{2}} R^1 |_\infty \leq c\eta,$$

for sufficiently small $\eta > 0$. And so (R^n) converges to some R, solution to (4.4-5) in $\tilde{L}^q$ for $q \leq \infty$. The contraction mapping construction guarantees that the solution is isolated. □

The existence of isolated solutions to (2.1), (2.3) is an immediate consequence of Lemma 4.1. It also follows that, when ϵ tends to zero the fluid dynamic moments converge to the (Hilbert type) corresponding leading (first) order limiting fluid solution given by (2.5). This is obvious in L^2 from the estimate of R^1 in Lemma 4.1, and holds in L^∞ for the following reason. If the asymptotic expansion were carried out to third order, then R^1 would be of order ϵ also in L^∞. Grouping it together with the new third order term from the asymptotic expansion, shows that the R^1 of our present Lemma 4.1 also is of order ϵ. We have thus proved

Theorem 4.2 *For $0 < \epsilon$, $0 < r_B - r_A$ small enough and $j = 1$, there is an isolated axially homogeneous solution of (2.1).(2.3). When ϵ tends to zero, the corresponding fluid dynamic moments of ϕ converge to solutions of the limiting fluid equations at the leading order ϵ.*

In Case 1 the (incompressible) fluid dynamics behaviour is given by the limiting first order (angular) velocity $\frac{u_{\theta A}}{r} \frac{r_B^2 - r^2}{r_B^2 - r_A^2} = \frac{U_{\theta A}}{r} \frac{r_B^2 - r^2}{r_B + r_A}$.

Using similar arguments but more extended asymptotic expansions, the same type of results can be proved in the other cases. In Case 2 our present estimates give (see [AN2])

Theorem 4.3 *Assume that $r_B - r_A$ is small enough and that $(A + 5D) < 0$. There is a negative value Δ_{bif} of the parameter Δ, such that for the quantity $\Delta_{bif} - \Delta$ positive and small enough, there are for ϵ positive and small enough, two isolated, non-negative L^1-solutions f_ϵ^j, $j = 1, 2$ of (2.1), (2.3) coexisting with $M^{-1} f_\epsilon^j \in \tilde{L}^\infty$,*

$$\int M^{-1} supess_{r \in (r_A, r_B)} | f_\epsilon^j(r, v) |^2 \, dv < +\infty.$$

The two solutions have different outward radial bulk velocities of order ϵ^3. For fixed ϵ, they converge to the same solution, when Δ increases to Δ_{bif}. Their fluid dynamic moments converge to solutions of the corresponding limiting fluid equations at leading order, when $\epsilon \to 0$.

Here the leading order (in ϵ) fluid dynamics behaviour is given by the first order angular velocity $\frac{u_{\theta A1}}{r} - \frac{u_{\theta A1}}{r_B} e^{\frac{u_3 Y}{w_1 r_B}}$ and the two possible third order radial velocities $\frac{u_3}{r}$, where u_3 solves (2.40).

Finally in Case 3 one obtains

Theorem 4.4 *For $j = 1$, $0 < \epsilon$, $0 < r_B - r_A$ small enough, there is a smallest bifurcation value $u_{\theta Ab} > 0$, such that the axially homogeneous solution to the problem (2.1), (2.3) bifurcates at $u_{\theta Ab}$ with a steady secondary solution appearing locally for $u_{\theta Ab} < u_{\theta A}$, which is axially symmetric and axially $(r_B - r_A)$−periodic. When ϵ tends to zero, the corresponding fluid dynamic moments converge to solutions of the limiting fluid equations at the leading order ϵ (bifurcated solution of Taylor-Couette type).*

In this case the limiting fluid Taylor-Couette equations of incompressible Navier-Stokes type are

$$
\begin{aligned}
u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} &= -\frac{1}{2}\frac{\partial P_1}{\partial r} + \mu\Big(\Delta u_r - \frac{u_r}{r^2}\Big), \\
\frac{u_r}{r}\frac{\partial (ru_\theta)}{\partial u_\theta} + u_z\frac{\partial u_\theta}{\partial z} &= \mu\Big(\Delta u_\theta - \frac{u_\theta}{r^2}\Big), \\
u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} &= -\frac{1}{2}\frac{\partial P_1}{\partial z} + \mu\Delta u_z, \\
\frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} &= 0,
\end{aligned}
\tag{4.9}
$$

where μ depends on the molecular model, and P_1 is the next order term in ϵ of the perturbed relative pressure.

Proof of Theorem 4.4 Given the asymptotic expansion (4.1) in Case 3 and its bifurcation point, the aim is to prove the existence of a rest term R, so that for the parameters near the bifurcation point, there is an axially periodic solution

$$f = M(1 + \varphi + \epsilon R)$$

to (2.1) with an added $\frac{\partial}{\partial z}$-term and boundary values (2.3) with $M^{-1}f \in \tilde{L}^\infty$. This corresponds to the rest term R being a solution of the same type to

$$DR = \frac{1}{\epsilon}\Big(LR + 2\tilde{J}(R, \varphi) + \epsilon J(R, R) + l\Big).$$

In Section 2 a third order asymptotic expansion in ϵ was constructed in a δ^2-neighbourhood of the bifurcation velocity $u_{\theta Ab}$ with correct boundary values up to ϵ-order three, and so that l - the φ-part of the equation - is smooth in r, z and of order ϵ^3 in $\tilde{L}^q$. Notice that Φ^j can be constructed so that $D\Phi^j = (I - P_0)D\Phi^j$, hence that $l = l_\perp$.

Let the sequences $(R^n)_{n\in I\!N}$ be defined as in the earlier Couette case by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\Big(LR^{n+1} + 2\sum_{j=1}^{3}\epsilon^j J(R^{n+1}, \Phi^j) + g^n\Big), \tag{4.10}$$

$$R^{n+1}(1, v) = R_A(v),\ v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v),\ v_r < 0. \tag{4.11}$$

In (4.10-11)

$$g^n := \epsilon^2 J(R^n, R^n) + l,$$

$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u_{\theta A1}^2} - 1 - \sum_{j=1}^{3} \epsilon^j \Phi^j(r_A, v), \quad v_r > 0,$$

$$\epsilon R_B(v) := 0, \quad v_r < 0,$$

with $R = (R_A, R_B)$ of ϵ-order three.
For the rest term iteration scheme (4.10-11) the following proposition holds and with it the proof of Theorem 4.4 is complete.

Proposition 4.5 *For $\epsilon > 0$ and small enough together with $\eta = r_B - r_A$, there is a unique sequence (R^n) of solutions to (4.10-11) in the set $X := \{R; \mid \nu^{\frac{1}{2}} R \mid_q \leq K\epsilon\}$ for some constant K. The sequence converges in $\tilde{L}^q$ for $2 \leq q \leq \infty$, to an isolated solution of*

$$DR = \frac{1}{\epsilon}\Big(LR + \epsilon J(R, R) + 2J(R, \varphi) + l\Big), \tag{4.12}$$

$$R(1, v) = R_A(v), \ v_r > 0, \quad R(r_B, v) = R_B(v), \ v_r < 0. \tag{4.13}$$

Proof of Proposition 4.5. The existence result of Lemma 3.1 holds for the boundary value problem

$$Df = \frac{1}{\epsilon}\Big(Lf + 2\sum_{j=1}^{3} \epsilon^j J(f, \Phi^j) + g\Big),$$

$$f(1, v) = R_A(v), \ v_r > 0, \quad f(r_B, v) = R_B(v), \ v_r < 0.$$

Rescale in space to $(-\pi, \pi)^2$ and consider the approximation (4.10-11) in the case $n = 0$ with $g^0 = l$. As discussed before (4.10), this $g^0 = g^0_\perp$ is of order ϵ^3 in $\tilde{L}^\infty$, and

$$\mid \nu^{-\frac{1}{2}} l \mid_\infty + \mid R_b \mid_\sim \leq c_1 \epsilon^3,$$

for some constant c_1. By (3.25) and (3.3) it holds that for some constant c_2

$$\mid \nu^{\frac{1}{2}} R^1 \mid_2 \leq c_1 c_2 \eta \epsilon^2, \quad \mid \nu^{\frac{1}{2}} R^1 \mid_\infty \leq 2 c_1 c_2 \eta \epsilon, \tag{4.14}$$

for η and ϵ small enough. Let us prove by induction that

$$\mid \nu^{\frac{1}{2}} R^n \mid_\infty \leq 4 c_1 c_2 \epsilon,$$

$$\mid \nu^{\frac{1}{2}} (R^{n+1} - R^n) \mid_2 \leq 2 c_1 c_2 \epsilon \mid \nu^{\frac{1}{2}} (R^n - R^{n-1}) \mid_2, \quad n \geq 1. \tag{4.15}$$

For $n=1$, R^2-R^1 satisfies

$$D(R^2-R^1)=\frac{\eta}{\epsilon}\Big(L(R^2-R^1)+2\sum_{j=1}^{3}\epsilon^j J(R^2-R^1,\Phi^j)+\epsilon J(R^1,R^1)\Big),$$
$$(R^2-R^1)(r_A,z,v)=0,\ v_r>0,\quad (R^2-R^1)(r_B,z,v)=0,\ v_r<0,$$

so that, by (3.25),

$$\mid \nu^{\frac{1}{2}}(R^2-R^1)\mid_2\leq c_2\eta\mid \nu^{-\frac{1}{2}}J(R^1,R^1)\mid_2 .$$

Recall that for any $g\in\tilde{L}^\infty$ (resp. $h\in\tilde{L}^q$,

$$\mid \nu^{-\frac{1}{2}}J(g,h)\mid_q\leq c_3\mid \nu^{\frac{1}{2}}g\mid_\infty\mid \nu^{\frac{1}{2}}h\mid_q . \tag{4.16}$$

Hence

$$\mid \nu^{\frac{1}{2}}(R^2-R^1)\mid_2\leq c_1\eta^2\epsilon\mid \nu^{\frac{1}{2}}(R^1-R^0)\mid_2,$$

for η small enough. If (4.15) holds until n, then

$$\begin{aligned}\mid \nu^{\frac{1}{2}}R^{n+1}\mid_\infty&\leq\mid \nu^{\frac{1}{2}}(R^{n+1}-R^n)\mid_\infty+...+\mid \nu^{\frac{1}{2}}(R^1-R^0)\mid_\infty\\&\leq\frac{c_4}{\epsilon}(\mid \nu^{\frac{1}{2}}(R^{n+1}-R^n)\mid_2+...+\mid \nu^{\frac{1}{2}}(R^1-R^0)\mid_2)\\&\leq 4c_1c_2\epsilon,\end{aligned}$$

for η small enough. Then $R^{n+2}-R^{n+1}$ satisfies

$$\begin{aligned}D(R^{n+2}-R^{n+1})=\frac{1}{\epsilon}\Big(L(R^{n+2}-R^{n+1})+2\sum_{j=1}^{3}\epsilon^j J(R^{n+2}-R^{n+1},\Phi^j)\\+\epsilon J(R^{n+1}+R^n,R^{n+1}-R^n)\Big)\end{aligned}$$
$$(R^{n+2}-R^{n+1})(r_A,z,v)=0,\ v_r>0,\quad (R^{n+2}-R^{n+1})(r_B,z,v)=0,\ v_r<0,$$

so that by (3.25) and the bound on $\mid \nu^{\frac{1}{2}}R^n\mid_\infty$ and $\mid \nu^{\frac{1}{2}}R^{n+1}\mid_\infty$,

$$\begin{aligned}\mid \nu^{\frac{1}{2}}(R^{n+2}-R^{n+1})\mid_2\leq c_3\eta(\mid \nu^{\frac{1}{2}}R^{n+1}\mid_\infty+\mid \nu^{\frac{1}{2}}R^n\mid_\infty)\mid \nu^{\frac{1}{2}}(R^{n+1}-R^n)\mid_2\\\leq 2c_1c_2\epsilon\mid \nu^{\frac{1}{2}}(R^{n+1}-R^n)\mid_2,\end{aligned}$$

for ϵ and η small enough.
And so (R^n) converges for sufficiently small $\eta>0$ to some R, solution to (4.12-13) in $\tilde{L}^q$ for $q\leq\infty$. The contraction mapping construction guarantees that this solution is isolated.