5 Stability.

We next come to the question of stability for the solutions obtained in the previous sections. Only Case 1 will be discussed. It turns out that the well known *fluid stability* of the leading order term is the prime mover behind the *kinetic stability*, which in a certain way is uniform down to the fluid level. More precisely we shall devote this section to prove the following new result.

Theorem 5.1 *The steady Couette problem for the Boltzmann equation in the two rolls problem is stable. The stability is uniform in the following sense for small enough mean free path ϵ. When the gap between the cylinders is small and the angular, axial and energy moments are perturbed of order ϵ or ϵ^2, then uniformly in ϵ the perturbation vanishes asymptoticaly in time. Also an initial perturbation of order ϵ^3, with small but otherwise arbitrary fluid dynamic as well as non fluid dynamic part, vanishes asymptotically in time.*

This type of results is expected to carry over to the cases 2-3, where also the fluid stability is well understood.

Among the few earlier rigorous non-linear kinetic stability results outside the situation with global Maxwellian limits, are the studies in [UYY] dealing with stability of half-space Milne problems, and [UYZ] dealing with the Boltzmann equation in full space with an external force.

With $\tilde{\Phi}_s = 1 + \Phi_s$ the rescaled stationary solution, the stability problem consists in proving that the distribution function $\tilde{\Phi}$ tends to $\tilde{\Phi}_s$ when $t \to \infty$, where $\tilde{\Phi}$ solves the evolutionary problem

$$\begin{aligned}
\frac{\partial \tilde{\Phi}}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x \tilde{\Phi} &= \frac{1}{\epsilon^2}(L\tilde{\Phi} + J(\tilde{\Phi}, \tilde{\Phi})), \\
\tilde{\Phi}(0, r, v) = \tilde{\Phi}_s(r, v) + P(r, v), &\quad r \in (r_A, r_B),\ v \in \mathbb{R}^3, \\
\tilde{\Phi}(t, r_A, v) &= \tilde{\Phi}_s(r_A, v),\ t > 0,\ v_r > 0, \\
\tilde{\Phi}(t, r_B, v) &= \tilde{\Phi}_s(r_B, v),\ t > 0,\ v_r < 0,
\end{aligned}$$

and P is a small perturbation of $\tilde{\Phi}_s$.
Denote by $\tilde{\psi} = \tilde{\Phi} - \tilde{\Phi}_s$. It should then be a solution to

$$\frac{\partial \tilde{\psi}}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x \tilde{\psi} = \frac{1}{\epsilon^2}\Big(L\tilde{\psi} + J(\tilde{\psi}, \tilde{\psi}) + 2J(\tilde{\psi}, \tilde{\Phi}_s)\Big), \tag{5.1}$$

$$\tilde{\psi}(0, r, v) = P(r, v), \quad r \in (r_A, r_B),\ v \in \mathbb{R}^3, \tag{5.2}$$

$$\tilde{\psi}(t, r_A, v) = 0,\ t > 0,\ v_r > 0, \quad \tilde{\psi}(t, r_B, v) = 0,\ t > 0,\ v_r < 0, \tag{5.3}$$

and tend to zero when $t \to \infty$.
Here the following perturbations P are considered,

$$P(r,v) = \epsilon(\alpha_1(v^2-5)+\beta_1 v_\theta+\gamma_1 v_z)+\epsilon^2(\alpha_2(v^2-5)+\beta_2 v_\theta+\gamma_2 v_z) \\ +\epsilon^3 p_3(x,v,\epsilon),$$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ 1\leq i\leq 2$ are L^∞-functions of the space variable, and the function $p_3(x,v,\epsilon)$ is measurable with $\parallel p_3 \parallel_{\infty,2}< c$ uniformly in ϵ, where

$$\parallel p_3 \parallel_{\infty,2}= (\int_{\mathbb{R}^3} \sup_{x\in\Omega} p_3^2(x,v,\epsilon)M(v)dv)^{\frac{1}{2}}.$$

As in Section 4, the stationary solution $\tilde{\Phi}_s$ is here determined by an approximate asymptotic expansion Φ_s of terms of up to third order in ϵ with boundary values being those of the same order of $e^{\frac{1}{2}(v_\theta^2-(v_\theta-\epsilon u_{\theta A})^2)}$ at $\{(r_A,v),\ v_r>0\}$ (resp. 0 at $\{(r_B,v),\ v_r<0\}$, plus a rest term ϵS,

$$\tilde{\Phi}_s(r,v) = 1+\Phi_s(r,v)+\epsilon S(r,v),$$

where $\parallel S \parallel_{\infty,2}\leq c|r_B-r_A|\epsilon$, $\parallel S \parallel_{2,2}\leq c|r_B-r_A|\epsilon^2$, and

$$\Phi_s(r,v) = \epsilon\Phi_{H1}(r,v)+\epsilon^2\Phi_2+\epsilon^3\Phi_3,$$
$$\Phi_i = \Phi_{Hi}(r,v)+\Phi_{KiA}(\frac{r-1}{\epsilon},v)+\Phi_{KiB}(\frac{r-r_B}{\epsilon},v), \quad 2\leq i\leq 3.$$

The Hilbert terms Φ_{Hi}, $1\leq i\leq 3$ satisfy

$$L\Phi_{H1} = L\Phi_{H2}+J(\Phi_{H1},\Phi_{H1})-v\cdot\nabla_x\Phi_{H1} \\ = L\Phi_{H3}+2J(\Phi_{H1},\Phi_{H2})-v\cdot\nabla_x\Phi_{H2}=0.$$

They are given by

$$\Phi_{H1}(r,v) = b_1(r)v_\theta,$$
$$\Phi_{H2}(r,v) = a_2+d_2v^2+b_2v_\theta+c_2v_r+\frac{1}{2}b_1^2v_\theta^2+(b_1'-\frac{1}{r}b_1)v_rv_\theta\bar{B},$$
$$\Phi_{H3}(r,v) = a_3+d_3v^2+b_3v_\theta+c_3v_r+d_2'v_r\bar{A}+b_1d_2v_\theta v^2+b_1b_2v_\theta^2 \\ +b_1c_2v_rv_\theta+\frac{1}{6}b_1^3v_\theta^3-b_1(b_1'-\frac{1}{r}b_1)L^{-1}(J(v_\theta,v_rv_\theta\bar{B})) \\ +(b_1b_1'-\frac{1}{r}b_1^2)L^{-1}(v_r(v_\theta^2-1))+\frac{1}{r}c_2L^{-1}((v_\theta^2-v_r^2)) \\ +\frac{1}{r}(b_1'-\frac{1}{r}b_1)L^{-1}((v_\theta^3-3v_r^2v_\theta)\bar{B})+b_2'v_rv_\theta\bar{B}.$$

We take $r_A=1$ (implying $r_B>1$). For compatibility reasons

$$b_1(r) = \frac{u_{\theta A}}{r_B^2-1}(\frac{r_B^2}{r}-r), \tag{5.4}$$

$a_i + 5d_i$ (resp. c_i), $2 \leq i \leq 3$, satisfy first-order differential equations, whereas b_i (resp. d_i), $2 \leq i \leq 3$, satisfy second-order differential equations. Knudsen terms Φ_{KAi} (resp. Φ_{KBi}), $2 \leq i \leq 3$ are added in order to satisfy the given zero ingoing boundary conditions up to third order.
The solution $\tilde{\psi}$ to the evolutionary problem (5.1-3) is determined as the sum of an asymptotic expansion ψ and a rest term ϵR,

$$\tilde{\psi} = \psi + \epsilon R,$$

where

$$\psi(t,r,v) = \epsilon\psi_{H1}(t,r,v) + \epsilon^2\psi_2 + \epsilon^3\psi_3,$$
$$\psi_i = \psi_{Hi}(t,r,v) + \psi_{KiA}(t, \frac{r-1}{\epsilon}, v) + \psi_{KiB}(t, \frac{r-r_B}{\epsilon}, v), \quad 2 \leq i \leq 3.$$

The initial values of ψ_3 is taken as zero, those of ψ_{H1}, ψ_2 are the corresponding orders of P and finally $R_0 := \epsilon^2 p_3$ is taken as initial value for R. For (5.1) to be satisfied up to zeroth order in ϵ included, it is required that

$$\begin{aligned}
0 = L\psi_{H1} &= L\psi_{H2} + J(\psi_{H1}, \psi_{H1} + 2\Phi_{H1}) - v \cdot \nabla_x \psi_{H1} \\
&= L\psi_{H3} + 2J(\psi_{H1}, \psi_{H2} + \Phi_{H2}) + 2J(\psi_{H2}, \Phi_{H1}) - \frac{\partial \psi_{H1}}{\partial t} - v \cdot \nabla_x \psi_{H2} \\
&= L\psi_{K2A} - v_r \frac{\partial \psi_{K2A}}{\partial \eta} = L\psi_{K2B} - v_r \frac{\partial \psi_{K2B}}{\partial \mu} \\
&= L\psi_{K3A} + 2J(\psi_{H1}(r_A), \psi_{K2A} + \Phi_{K2A}) + 2J(\psi_{K2A}, \Phi_{H1}(r_A)) \\
&\quad - \frac{1}{r} N\psi_{K2A} - v_r \frac{\partial \psi_{K3A}}{\partial \eta} \\
&= L\psi_{K3B} + 2J(\psi_{H1}(r_B), \psi_{K2B} + \Phi_{K2B}) + 2J(\psi_{K2B}, \Phi_{H1}(r_B)) \\
&\quad - \frac{1}{r} N\psi_{K2B} - v_r \frac{\partial \psi_{K3B}}{\partial \mu}.
\end{aligned}$$

The rest term R should then be a solution to

$$\frac{\partial R}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x R = \frac{1}{\epsilon^2} LR + \frac{1}{\epsilon} J(R,R) + \frac{2}{\epsilon} H(R) + \alpha,$$

where

$$H(R) = \frac{1}{\epsilon} J(\psi + \Phi_s, R) + J(S, R),$$

and

$$\begin{aligned}
\alpha = 2\epsilon\Big(&- \frac{\partial \psi_2}{\partial t} - v \cdot \nabla_x \psi_{H3} - \frac{1}{r}(N\psi_{K3A} + N\psi_{K3B}) + J(\psi_2, \psi_2) \\
&+ 2J(\psi_{H1}, \psi_3 + \Phi_3) + 2J(\psi_2, \Phi_2) + 2J(\psi_3, \Phi_{H1})\Big) \\
&+ \epsilon^2 \Big(2J(\psi_2, \psi_3 + \Phi_3) + 2J(\psi_3, \Phi_2) - \frac{\partial \psi_3}{\partial t} \Big) \\
&+ \epsilon^3 \Big(J(\psi_3, \psi_3) + 2J(\psi_3, \Phi_3) \Big) + \frac{2}{\epsilon^2} J(\psi, S).
\end{aligned}$$

The equations involving $L\psi_{Hi}$, $1 \leq i \leq 3$ give the v-dependence of ψ_{Hi},

$$\psi_{H1}(t,r,v) = A_1 + D_1 v^2 + B_1 v_\theta + C_1 v_r + E_1 v_z,$$
$$\psi_{H2}(t,r,v) = A_2 + D_2 v^2 + B_2 v_\theta + C_2 v_r + E_2 v_z + g_2,$$
$$\psi_{H3}(t,r,v) = A_3 + D_3 v^2 + B_3 v_\theta + C_3 v_r + E_3 v_z + g_3.$$

Here A_i, B_i, C_i, D_i and E_i, $1 \leq i \leq 3$ denote functions in the (t,r) variables. By the compatibility conditions

$$\int v \cdot \nabla_x \psi_{H1}(1, v_r) M dv = 0,$$

and the initial and boundary conditions at first order, it holds that

$$A_1 + 5D_1 = C_1 = 0.$$

$\bar{A}(|v|)$ was introduced after Lemma 2.2 from the nonhydrodynamic solution to $L(v_r\bar{A}) = v_r(v^2 - 5)$ together with $\bar{B}(|v|)$ from the corresponding solution to $L(v_r v_\theta \bar{B}) = v_r v_\theta$. Further,

$$g_2 = \frac{1}{2}D_1^2 v^4 + (\frac{1}{2}B_1^2 + B_1 b_1)v_\theta^2 + \frac{1}{2}E_1^2 v_z^2$$
$$+(B_1 + b_1)D_1 v_\theta v^2 + D_1 E_1 v_z v^2 + (B_1 + b_1)E_1 v_\theta v_z$$
$$+\frac{\partial D_1}{\partial r} v_r \bar{A} + (\frac{\partial B_1}{\partial r} - \frac{1}{r}B_1)v_r v_\theta \bar{B} + \frac{\partial E_1}{\partial r} v_r v_z \bar{B},$$

and g_3 is a similar expression depending on ψ_{H1}, ψ_{H2}, Φ_{H1} and Φ_{H2} . By the compatibility conditions

$$\int (\frac{\partial \psi_{H1}}{\partial t} + v \cdot \nabla_x \psi_{H2})(v_\theta, v^2 - 5, v_z) M dv = 0,$$

the functions B_1, D_1 and E_1 are solutions to the parabolic equations

$$\frac{\partial B_1}{\partial t} + w_1(\frac{\partial^2 B_1}{\partial r^2} + \frac{1}{r}\frac{\partial B_1}{\partial r} - \frac{1}{r^2}B_1) = 0,$$
$$B_1(0,r) = \beta_1(r),$$
$$B_1(t, r_A) = B_1(t, r_B) = 0,$$
$$\frac{\partial D_1}{\partial t} + \frac{w_3 - 5w_2}{10}(\frac{\partial^2 D_1}{\partial r^2} + \frac{1}{r}\frac{\partial D_1}{\partial r}) = 0,$$
$$D_1(0,r) = \alpha_1(r),$$
$$D_1(t, r_A) = D_1(t, r_B) = 0,$$
$$\frac{\partial E_1}{\partial t} + w_1(\frac{\partial^2 E_1}{\partial r^2} + \frac{1}{r}\frac{\partial E_1}{\partial r}) = 0,$$
$$E_1(0,r) = \gamma_1(r),$$
$$E_1(t, r_A) = E_1(t, r_B) = 0.$$

Here,

$$w_1 = \int v_r^2 v_\theta^2 \bar{B} M dv, \; w_2 = \int v_r^2 \bar{A} M dv, \; w_3 = \int v_r^2 v^2 \bar{A} M dv.$$

The convergence to zero when $t \to \infty$ of ψ_{H1} is well known from the fluid dynamics context (see e.g. [V]). Here the convergence follows from classical asymptotic properties of the solutions to the above linear parabolic equations [LSU].
The compatibility conditions

$$\int \Big(\frac{\partial \psi_{H1}}{\partial t} + v \cdot \nabla_x \psi_{H2}\Big)(1, v_r) M dv = 0,$$

write

$$\frac{\partial}{\partial r}\Big(r(C_2 + \frac{w_2}{5}\frac{\partial D_1}{\partial r})\Big) = 0,$$
$$\frac{\partial}{\partial r}\Big(A_2 + 5D_2 + \frac{35}{2}D_1^2 + \frac{1}{2}B_1^2 + B_1 b_1 + \frac{1}{2}E_1^2\Big) = \frac{B_1(B_1 + 2b_1)}{r}. \tag{5.5}$$

Let $\lambda(\eta, v)$, $\rho_{A2}(t,\eta,v)$ and $\rho_{B2}(t,\eta,v)$ be the solutions of Theorem 2.4 to the half-space problems

$$v_r \frac{\partial \lambda}{\partial \eta} = L\lambda,$$
$$\lambda(0, v) = 0, \quad v_r > 0,$$
$$\int v_r \lambda(\eta, v) dv = 1,$$

$$v_r \frac{\partial \rho_{A2}}{\partial \eta} = L\rho_{A2},$$
$$\rho_{A2}(t, 0, v) = -\tilde{\psi}_{H2}(t, r_A, v), \quad v_r > 0,$$
$$\int v_r \rho_{A2}(\eta, v) dv = 0,$$

and

$$v_r \frac{\partial \rho_{B2}}{\partial \mu} = L\rho_{B2},$$
$$\rho_{B2}(t, 0, v) = -\tilde{\psi}_{H2}(t, r_B, v), \quad v_r < 0,$$
$$\int v_r \rho_{B2}(\mu, v) dv = 0.$$

As η (resp. μ) tends to $+\infty$ (resp. $-\infty$), λ and ρ_{A2} (resp. ρ_{B2}) tend to some $\alpha_\infty + \delta_\infty v^2 + \beta_\infty v_\theta + v_r + \gamma_\infty v_z$ and $\alpha_{\infty A} + \delta_{\infty A} v^2 + \beta_{\infty A} v_\theta + \gamma_{\infty A} v_z$ (resp. $\alpha_{\infty B} +$

$\delta_{\infty B}v^2 + \beta_{\infty B}v_\theta + \gamma_{\infty B}v_z)$. Give as boundary conditions to A_2, B_2, D_2 and E_2,

$$A_2(t,r_A) = \alpha_\infty C_2(t,r_A) + \alpha_{\infty A}(t), \quad A_2(t,r_B) = \alpha_\infty C_2(t,r_B) + \alpha_{\infty B}(t),$$
$$B_2(t,r_A) = \beta_\infty C_2(t,r_A) + \beta_{\infty A}(t), \quad B_2(t,r_B) = \beta_\infty C_2(t,r_B) + \beta_{\infty B}(t),$$
$$D_2(t,r_A) = \delta_\infty C_2(t,r_A) + \delta_{\infty A}(t), \quad D_2(t,r_B) = \delta_\infty C_2(t,r_B) + \delta_{\infty B}(t),$$
$$E_2(t,r_A) = \gamma_\infty C_2(t,r_A) + \gamma_{\infty A}(t), \quad E_2(t,r_B) = \gamma_\infty C_2(t,r_B) + \gamma_{\infty B}(t),$$

with

$$C_2(t,r_A) = r_B C_2(t,r_B) + \frac{w_2}{5}\Big(r_B \frac{\partial D_1}{\partial r}(t,r_B) - \frac{\partial D_1}{\partial r}(t,r_A)\Big).$$

Then there is a solution $A_2 + 5D_2$ to (5.5) if and only if

$$(A_2+5D_2)(t,r_B) - (A_2+5D_2)(t,r_A) = \int_{r_A}^{r_B} B_1(B_1+2b_1)(t,r)\frac{dr}{r},$$

which fixes $C_2(t,r_B)$. Finally, the linear parabolic problems for B_2, D_2, and for E_2 provided by the compatibility conditions

$$\int \Big(\frac{\partial \psi_{H2}}{\partial t} + v\cdot\nabla_x\psi_{H3}\Big)(v_\theta, v^2-5, v_z)M\,dv = 0,$$

have unique solutions;

$$\frac{\partial B_2}{\partial t} + w_1\Big(\frac{\partial^2 B_2}{\partial r^2} + \frac{1}{r}\frac{\partial B_2}{\partial r} - \frac{1}{r^2}B_2\Big) = f_1,$$
$$B_2(0,r) = \beta_2(r),$$
$$B_2(t,r_A) = \beta_\infty C_2(t,r_A) + \beta_{\infty A}(t),$$
$$B_2(t,r_B) = \beta_\infty C_2(t,r_A) + \beta_{\infty B}(t),$$
$$\frac{\partial D_2}{\partial t} + \frac{w_3-5w_2}{10}\Big(\frac{\partial^2 D_2}{\partial r^2} + \frac{1}{r}\frac{\partial D_2}{\partial r}\Big) = \bar{f}_1,$$
$$D_2(0,r) = \alpha_2(r),$$
$$D_2(t,r_A) = \delta_\infty C_2(t,r_A) + \delta_{\infty A}(t),$$
$$D_2(t,r_B) = \delta_\infty C_2(t,r_A) + \delta_{\infty B}(t),$$
$$\frac{\partial E_2}{\partial t} + w_1\Big(\frac{\partial^2 E_2}{\partial r^2} + \frac{1}{r}\frac{\partial E_2}{\partial r}\Big) = \tilde{f}_1,$$
$$E_1(0,r) = \gamma_2(r),$$
$$E_2(t,r_A) = \gamma_\infty C_2(t,r_A) + \gamma_{\infty A}(t),$$
$$E_2(t,r_B) = \gamma_\infty C_2(t,r_A) + \gamma_{\infty B}(t).$$

Here, f_1, $\bar{f}_1$ and $\tilde{f}_1$ are given functions depending on ψ_{H1}, b_1 and c_2.
Let ψ_{K2A} and ψ_{K2B} be defined by

$$\psi_{K2A} = C_2(t,r_A)(\lambda - \alpha_\infty - \delta_\infty v^2 - \beta_\infty v_\theta - v_r - \gamma_\infty v_z)$$
$$+\rho_{2A} - \alpha_{\infty A}(t) - \delta_{\infty A}(t)v^2 - \beta_{\infty A}(t)v_\theta - \gamma_{\infty A}(t)v_z,$$
$$\psi_{K2B}(t,\mu,v) = C_2(t,r_B)(\lambda(-\mu,-v) - \alpha_\infty - \delta_\infty v^2 + \beta_\infty v_\theta + v_r + \gamma_\infty v_z)$$
$$+\rho_{2B}(t,-\mu,-v) - \alpha_{\infty B}(t) - \delta_{\infty B}(t)v^2 + \beta_{\infty B}(t)v_\theta + \gamma_{\infty B}(t)v_z.$$

They satisfy

$$\begin{aligned} v_r \frac{\partial \psi_{K2A}}{\partial \eta} &= L\psi_{K2A}, \\ \psi_{H2}(t, r_A, v) + \psi_{K2A}(t, 0, v) = 0, &\quad t > 0,\ v_r > 0, \\ \lim_{\eta \to +\infty} \psi_{K2A}(t, \eta, v) &= 0, \end{aligned}$$

and

$$\begin{aligned} v_r \frac{\partial \psi_{K2B}}{\partial \mu} &= L\psi_{K2B}, \\ \psi_{H2}(t, r_B, v) + \psi_{K2A}(t, 0, v) = 0, &\quad t > 0,\ v_r < 0, \\ \lim_{\mu \to -\infty} \psi_{K2B}(t, \mu, v) &= 0. \end{aligned}$$

The convergence to zero of $\psi_{H2} + \psi_{K2A} + \psi_{K2B}$ when $t \to \infty$, follows from the convergence of ψ_{H1}, and from the properties of the parabolic problems and of the Knudsen terms.
The Knudsen terms ψ_{K3A} and ψ_{K3B} are defined analogously, so that the boundary conditions at third order be satisfied by ψ. The third order terms are constructed similarly to the second order ones, and analogously converge to zero when $t \to \infty$.

For the a priori estimates of the rest term the following norms will be used,

$$\begin{aligned} \| R \|_{2t,2,2} &= \Big(\int_0^t \int_{\Omega \times I\!R^3} R^2(s, x, v) M(v) ds dx dv \Big)^{\frac{1}{2}}, \\ \| R \|_{\infty,2,2} &= \sup_{t>0} \Big(\int_{\Omega \times I\!R^3} R^2(t, x, v) M(v) dx dv \Big)^{\frac{1}{2}}, \\ \| R \|_{\infty,\infty,2} &= \sup_{t>0} \Big(\int_{I\!R^3} \sup_{x \in \Omega} R^2(t, x, v) M(v) dv \Big)^{\frac{1}{2}}, \\ \| f \|_{2t,2,\sim} &= \Big(\int_0^t \int_{v_r>0} v_r M(v) \mid f(s, r_A, v) \mid^2 dv ds \Big)^{\frac{1}{2}} + \\ &\Big(\int_0^t \int_{v_r<0} \mid v_r \mid M(v) \mid f(s, r_B, v) \mid^2 dv ds \Big)^{\frac{1}{2}} < +\infty. \end{aligned}$$

The rest term R can be split into $R = R_1 + R_2$, where

$$v \cdot \nabla_x R_1 = \frac{1}{\epsilon} L R_1 + 2H(R_1), \tag{5.6}$$

$$R_1(t, r_A, v) = -\frac{1}{\epsilon} \psi(t, r_A, v), \quad t > 0,\ v_r > 0,$$

$$R_1(t, r_B, v) = -\frac{1}{\epsilon} \psi(t, r_B, v), \quad t > 0,\ v_r < 0, \tag{5.7}$$

and

$$\frac{\partial R_2}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x R_2 = \frac{1}{\epsilon^2} L R_2 + \frac{1}{\epsilon} J(R_1 + R_2, R_1 + R_2) + \frac{2}{\epsilon} H(R_2) + \bar{\alpha},$$
$$R_2(0, r, v) = R_0(r, v),$$
$$R_2(t, r_A, v) = 0, \quad t > 0,\ v_r > 0,$$
$$R_2(t, r_B, v) = 0, \quad t > 0,\ v_r < 0,$$

where $\bar{\alpha} = \alpha - \frac{\partial R_1}{\partial t}$. Notice that α can be taken non-hydrodynamic modulo higher order terms in ϵ, which converge uniformly to zero when time tends to infinity. Hence only $\frac{\partial R_1}{\partial t}$ contributes to the hydro-dynamics in $\bar{\alpha}$. A priori bounds on R_1 are first derived, and also hold for $\frac{\partial R_1}{\partial t}$. The ingoing boundary values as given by (5.7) are subexponentially decreasing in ϵ, and tend to zero when time tends to infinity.

Lemma 5.1 *With R_1^{in} (R_1^{out}) the ingoing (outgoing) boundary values of R_1, any solution to (5.6-7) satisfies*

$$\sqrt{\epsilon} \parallel R_1^{out} \parallel_{2t,2,\sim} + \parallel \nu^{\frac{1}{2}}(I - P_0) R_1 \parallel_{2t,2,2} \leq c\sqrt{\epsilon} \parallel R_1^{in} \parallel_{2t,2,\sim},$$
$$\parallel P_0 R_1 \parallel_{2t,2,2} \leq c \parallel R_1^{in} \parallel_{2t,2,\sim},$$
$$\parallel \nu^{\frac{1}{2}} R_1 \parallel_{\infty t,\infty,2} \leq \frac{c}{\epsilon} \parallel R_1^{in} \parallel_{\infty t,2,\sim} .$$

Proof of Lemma 5.1. Denote by

$$| R_1 |_2 = \Big(\int_{\Omega \times I\!R^3} M (R_1)^2 (t, x, v) dx dv \Big)^{\frac{1}{2}}$$

with t acting as a parameter.
By (3.2-4)

$$\sqrt{\epsilon} \mid R_1^{out} \mid_\sim + \mid \nu^{\frac{1}{2}}(I - P_0) R_1 \mid_2 \leq c\Big(\sqrt{\epsilon} \mid R_1^{in} \mid_\sim + \epsilon \mid \nu^{-\frac{1}{2}} H(R_1) \mid_2 \Big),$$
$$\mid P_0 R_1 \mid_2 \leq c\Big(\mid \nu^{-\frac{1}{2}} H(R_1) \mid_2 + \mid R_1^{in} \mid_\sim \Big),$$
$$\mid \nu^{\frac{1}{2}} R_1 \mid_\infty \leq c\Big(\epsilon \mid \nu^{-\frac{1}{2}} H(R_1) \mid_\infty + \frac{1}{\epsilon} \mid \nu^{\frac{1}{2}} H(R_1) \mid_2 + \frac{1}{\epsilon} \mid R_1^{in} \mid_\sim \Big).$$

Then,

$$\mid \nu^{-\frac{1}{2}} H(R_1) \mid_2 \leq c\Big(\mid \nu^{\frac{1}{2}}(\psi_1 + \epsilon\psi_2 + \epsilon^2\psi_3) \mid_\infty$$
$$+ \mid \nu^{\frac{1}{2}}(\Phi_1 + \epsilon\Phi_2 + \epsilon^2\Phi_3) \mid_\infty + \mid \nu^{\frac{1}{2}} S \mid_\infty \Big) \mid \nu^{\frac{1}{2}} R_1 \mid_2$$
$$\leq c\eta \mid \nu^{\frac{1}{2}} R_1 \mid_2,$$

and

$$\mid \nu^{-\frac{1}{2}} H(R_1) \mid_\infty \leq c\eta \mid \nu^{\frac{1}{2}} R_1 \mid_\infty,$$

where $\eta = r_B - r_A$.
Including the estimates in t, this ends the proof of the lemma. □

The a priori bounds on R_2 are obtained by an approach adapted from [M], and involve dual, space-periodic solutions discussed in the following two lemmas.

Lemma 5.2 *Let $\pi a > r_B$. Let g be such that*

$$\int_{[0,2\pi a]^2} g(\bar{\tau}, x, v)dx = 0, \quad a.a.\ \bar{\tau} \in [0, \infty),\ v \in I\!R^3. \tag{5.8}$$

Let $\varphi(\bar{\tau}, x, v)$ be periodic of period $(2\pi a)^2$ in the space variable, solution to

$$\frac{\partial \varphi}{\partial \bar{\tau}} + v \cdot \nabla_x \varphi = \frac{1}{\epsilon} L\varphi + g, \qquad \varphi(0, x, v) = 0. \tag{5.9}$$

Then,

$$\| \varphi \|_{\infty,2,2} \leq c\Big(\sqrt{\epsilon} \| \nu^{-\frac{1}{2}}(I - P_0)g \|_{2,2,2} + \frac{1}{\sqrt{\epsilon}} \| P_0 g \|_{2,2,2} \Big),$$

$$\| \nu^{\frac{1}{2}}(I - P_0)\varphi \|_{2,2,2} \leq c\Big(\epsilon \| \nu^{-\frac{1}{2}}(I - P_0)g \|_{2,2,2} + \| P_0 g \|_{2,2,2} \Big),$$

$$\| P_0 \varphi \|_{2,2,2} \leq c\Big(\frac{1}{\sqrt{\epsilon}} \| \nu^{-\frac{1}{2}}(I - P_0)g \|_{2,2,2} + \frac{1}{\epsilon} \| P_0 g \|_{2,2,2} \Big).$$

Proof of Lemma 5.2. First, multiplying (5.9) by φ and integrating the resulting equation on $[0, \bar{T}] \times [0, 2\pi a]^2 \times I\!R^3$ leads to

$$\| \varphi \|^2_{\infty \bar{T},2,2} + \frac{1}{\epsilon} \| \nu^{\frac{1}{2}}(I - P_0)\varphi \|^2_{2\bar{T},2,2}$$
$$\leq c(\epsilon \| \nu^{-\frac{1}{2}}(I - P_0)g \|^2_{2\bar{T},2,2} + \eta_1 \| P_0\varphi \|^2_{2\bar{T},2,2} + \frac{1}{\eta_1} \| P_0 g \|^2_{2\bar{T},2,2}). \tag{5.10}$$

By (5.8) it holds that

$$\frac{\partial}{\partial \bar{\tau}} \int_{[0,2\pi a]^2} P_0\varphi(\bar{\tau}, x, v)dx = 0, \quad \bar{\tau} \geq 0,\ v \in I\!R^3,$$

so that

$$\int_{[0,2\pi a]^2} P_0\varphi(\bar{\tau}, x, v)dx = 0, \quad \bar{\tau} \geq 0,\ v \in I\!R^3. \tag{5.11}$$

Denote by $\bar{\varphi}(\bar{\tau},\xi,v)$, $\xi \in Z^2$ the Fourier series of φ with respect to space, and define $\bar{g}$ analogously. Then for $\xi \neq (0,0)$,

$$\frac{\partial\bar{\varphi}}{\partial\bar{\tau}} = \Big(\frac{1}{\epsilon}L + i\xi\cdot v\Big)\bar{\varphi} + \bar{g}.$$

Let β be a truncation function belonging to $C^1(I\!R)$ with support $[0,\infty]$, and such that $\beta(\bar{\tau}) = 1$ for $\bar{\tau} > \delta$ for some $\delta > 0$. Let $\tilde{\varphi} = \bar{\varphi}\beta$. Then

$$\frac{\partial\tilde{\varphi}}{\partial\bar{\tau}} = (\frac{1}{\epsilon}L + i\xi\cdot v)\tilde{\varphi} + \bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}} + \bar{g}\beta, \quad \xi \in Z^2.$$

Let $\mathcal{F}$ be the Fourier transform in $\bar{\tau}$ with Fourier variable σ. Denote by

$$\Phi = \mathcal{F}\tilde{\varphi},\ \tilde{Z} = \mathcal{F}(\epsilon^{-1}L\tilde{\varphi} + \bar{\varphi}\tfrac{\partial\beta}{\partial\bar{\tau}} + \bar{g}\beta),\ Z = \mathcal{F}(\epsilon^{-1}L\tilde{\varphi} + \bar{g}\beta),\ \hat{U} = (i\sigma + i\xi\cdot v)^{-1}.$$

Let χ be the indicatrix function of the set

$$\{v;\ \mid \sigma + \xi\cdot v \mid < \alpha\},$$

for some positive α to be chosen later. Let $\psi_s(v) = (1 + \mid v \mid)^s$. First,

$$\begin{aligned}\| P_0(\chi\Phi) \|_H \leq\ & c\Big(\mid \int \chi\Phi(\sigma,\xi,v)Mdv \mid \| 1 \|_H + \mid \int \chi\Phi(\sigma,\xi,v)v^2Mdv \mid \| v^2 \|_H \\ & + \mid \int \chi\Phi(\sigma,\xi,v)v_rMdv \mid \| v_r \|_H + \mid \int \chi\Phi(\sigma,\xi,v)v_\theta Mdv \mid \| v_\theta \|_H \Big) \\ & \leq c \| \psi_{-s}\Phi \|_H \Big(\| \chi\psi_s \|_H + \| \chi\psi_{s+2} \|_H \Big) \\ & \leq c\sqrt{\frac{\alpha}{\mid\xi\mid}} \| \psi_{-s}\Phi \|_H .\end{aligned}$$

Now $\Phi = -\hat{U}\tilde{Z}$, and so

$$\begin{aligned}\| P_0(1-\chi)\Phi \|_H^2 \leq\ & c\Big(\| \psi_s(1-\chi)\hat{U} \|_H^2 + \| \psi_{s+2}(1-\chi)\hat{U} \|_H^2 \Big) \| \psi_{-s}Z \|_H^2 \\ & - \sum_0^4 \int \psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^* \\ \leq \frac{c}{\mid\xi\mid\mid\alpha\mid} \| \psi_{-s}Z \|_H^2 & - \sum_0^4 \int \psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.\end{aligned}$$

Choosing $\alpha = \| \psi_{-s}\Phi \|_H^{-1} \| \psi_{-s} Z \|_H$ leads to

$$| \xi | \| P_0\Phi \|_H^2 \leq c \| \psi_{-s}\Phi \|_H \| \psi_{-s} Z \|_H$$
$$- | \xi | \sum_0^4 \int \psi_j (1-\chi) \hat{U} \mathcal{F}(\bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*.$$

Hence,

$$| \xi | \| P_0\Phi \|_H^2 \leq c\Big(\| P_0\Phi \|_H + \| \psi_{-s}(I-P_0)\Phi \|_H \Big) \| \psi_{-s} Z \|_H$$
$$- | \xi | \sum_0^4 \int \psi_j (1-\chi) \hat{U} \mathcal{F}(\bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*.$$

Consequently,

$$| \xi | \| P_0\Phi \|_H^2 \leq c\Big(| \xi |^{-1} \| \psi_{-s} Z \|_H^2 + \| \psi_{-s}(I-P_0)\Phi \|_H \| \psi_{-s} Z \|_H \Big)$$
$$- | \xi | \sum_0^4 \int \psi_j (1-\chi) \hat{U} (\mathcal{F} \bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*.$$

And so,

$$\frac{\xi^2}{1 + | \xi |} \| P_0\Phi \|_H^2 \leq c\Big(\| \psi_{-s} Z \|_H^2 + \| \psi_{-s}(I-P_0)\Phi \|_H^2 \Big)$$
$$- \frac{|\xi|^2}{1+|\xi|} \sum_0^4 \int \psi_j (1-\chi) \hat{U} \mathcal{F}(\bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*.$$

Therefore, for $s \geq \frac{\beta}{2}$,

$$\frac{\xi^2}{1 + | \xi |} \int (P_0\Phi)^2(\sigma, \xi, v) M dv d\sigma$$
$$\leq c\Big(\frac{1}{\epsilon^2} \int \| \psi_{-s}(v) L((I-P_0)\Phi)(\sigma,\xi,\cdot) \|_H^2 \, d\sigma + \int \| \psi_{-s}(v)(I-P_0)\Phi(\sigma,\xi,\cdot) \|_H^2 \, d\sigma$$
$$+ \int \| \psi_{-s} \bar{g} \beta(\bar{\tau},\xi,\cdot) \|_H^2 \, d\bar{\tau} \Big)$$
$$- \frac{|\xi|^2}{1+|\xi|} \sum_0^4 \int d\sigma \int \psi_j (1-\chi) \hat{U} (\mathcal{F} \bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*$$
$$\leq c\Big(\frac{1}{\epsilon^2} \int \| \nu^{\frac{1}{2}}(I-P_0)\Phi(\sigma,\xi,\cdot) \|_H^2 \, d\sigma + \int \| \psi_{-s} \bar{g} \beta(\bar{\tau},\xi,\cdot) \|_H^2 \, d\bar{\tau} \Big)$$
$$- \frac{|\xi|^2}{1+|\xi|} \sum_0^4 \int d\sigma \int \psi_j (1-\chi) \hat{U} (\mathcal{F} \bar{\varphi} \frac{\partial \beta}{\partial \bar{\tau}}) M dv (\int \psi_j (1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U} Z) M dv)^*.$$

Making δ tend to zero implies that

$$\int_0^\infty \int (P_0\bar{\varphi})^2(\bar{\tau},\xi,v)Mdvd\bar{\tau}$$
$$\leq c\Big(\frac{1}{\epsilon^2}\int_0^\infty \int \nu((I-P_0)\bar{\varphi})^2(\bar{\tau},\xi,v)Mdvd\bar{\tau}$$
$$+\int_0^\infty \psi_{-s}\bar{g}^2(\bar{\tau},\xi,v)Mdvd\bar{\tau}\Big).$$

Summing the former inequalities over all $\xi \in Z^2$ with $\xi \neq (0,0)$ and taking (5.11) into account, implies by Parseval that

$$\int_0^\infty \int (P_0\varphi)^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}$$
$$\leq c\Big(\frac{1}{\epsilon^2}\int_0^\infty \int \nu((I-P_0)\varphi)^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}$$
$$+\int_0^\infty \int \nu^{-1}g^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}\Big).$$

Together with (5.10) this ends the proof of the lemma. □

Lemma 5.3 *Let $\pi a > r_B$. Let g be such that*

$$\int_{[0,2\pi a]^2} g(\bar{\tau},x,v)dx = 0, \quad a.a.\ \bar{\tau} \in [0,\infty),\ v \in I\!R^3. \tag{5.12}$$

Let $\varphi(\bar{\tau},x,v)$ be periodic of period $(2\pi a)^2$ in the space variable x and solution to

$$\frac{\partial\varphi}{\partial\bar{\tau}} + v\cdot\nabla_x\varphi = \frac{1}{\epsilon}L\varphi + g, \tag{5.13}$$
$$\varphi(0,x,v) = 0.$$

Then,

$$\int_0^\infty \int_{|x|=r_B}\int_{v_r>0} v_r\varphi^2(\bar{\tau},x,v)Mdvd\sigma(x)d\bar{\tau} + \int_0^\infty \int_{|x|=r_A}\int_{v_r<0} \mid v_r \mid \varphi^2(\bar{\tau},x,v)Mdvd\sigma(x)d\bar{\tau}$$
$$\leq \frac{c}{\epsilon^2}\int_0^\infty \int g^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}).$$

(Here $d\sigma(x)$ is the surface measure of the circles.)

Proof of Lemma 5.3. Let $C_{(0,1)}$ be the set in the (x,y)-plane consisting of the half with $y \geq 0$ of the circle with radius r_B and center at the origin together with the rectangle given by $|x| \leq r_B, -\eta < y < 0$, where $\eta > 0$ taken small enough that any rotation of the set $C_{(0,1)}$ around the origin stays within the square $\{|x|, |y| < \pi a\}$. Let $C_{(v_x,v_y)}$ be the set $C_{(0,1)}$ rotated from the $(0,1)$-direction to the (v_x, v_y)-direction. Let $\chi_{(0,1)}$ be defined and continuous in $C_{(0,1)}$, monotone and continuously differentiable in the y-direction, equal zero at $y = -\frac{\eta}{2}$ and equal one at $y \geq 0$. Define $\chi_{(v_x,v_y)}(x,y)$ correspondingly by rotation. Then

$$\frac{\partial}{\partial \bar{T}}(\chi^2_{(v_x,v_y)}\varphi^2) + v \cdot \nabla_x(\chi^2_{(v_x,v_y)}\varphi^2) =$$

$$\frac{2}{\epsilon}\chi^2_{(v_x,v_y)}\varphi L\varphi + 2\chi^2_{(v_x,v_y)}\varphi g + 2(v \cdot \nabla_x \chi_{(v_x,v_y)})\chi_{(v_x,v_y)}\varphi^2.$$

Hence,

$$\int_0^{\bar{T}} \int_{|x|=r_B} v_r \chi^2_{(v_x,v_y)}\varphi^2(\bar{\tau}, x, v) M d\sigma(x) d\bar{\tau} \leq A_v + B_v + C_v,$$

where by Lemma 5.2

$$\int_{v_r>0} A_v dv := \frac{2}{\epsilon}\int_0^{\bar{T}} \int \chi^2_{(v_x,v_y)}\varphi L\varphi M dv dx d\bar{\tau}$$

$$= \frac{2}{\epsilon}\int_0^{\bar{T}} \int \chi^2_{(v_x,v_y)}\varphi L((I - P_0)\varphi) M dv dx d\bar{\tau}$$

$$\leq \frac{c}{\epsilon^2} \| \nu^{-\frac{1}{2}} g \|^2_{2,2,2},$$

$$\int B_v dv := 2\int_0^{\bar{T}} \int \chi^2_{(v_x,v_y)}\varphi g M dv dx d\bar{\tau} \leq \frac{c}{\epsilon} \| \nu^{-\frac{1}{2}} g \|^2_{2,2,2},$$

$$\int C_v dv := 2\int_0^{\bar{T}} \int (v \cdot \nabla_x \chi_{(v_x,v_y)})\chi_{(v_x,v_y)}\varphi^2 M dv dx d\bar{\tau}$$

$$\leq c \| \nu^{\frac{1}{2}} \varphi \|^2_{2\bar{T},2,2} \leq \frac{c}{\epsilon^2} \| \nu^{-\frac{1}{2}} g \|^2_{2,2,2}.$$

Here the C_v-estimate was carried out for hard spheres, but holds also for hard forces for the particular g appearing in the applications below. The r_A-part is treated similarly. □

For the iteration procedure to obtain R_2 we shall be using systems of the type

$$\frac{\partial R_2}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x R_2 = \frac{1}{\epsilon^2} L R_2 + \frac{1}{\epsilon} G, \tag{5.14}$$

$$R_2(0, r, v) = R_0(r, v), \tag{5.15}$$

$$R_2(t, r_A, v) = 0, \quad t > 0, \ v_r > 0,$$

$$R_2(t, r_B, v) = 0, \quad t > 0, \ v_r < 0. \tag{5.16}$$

Multiply (5.14) with $R_2 M$, integrate over $[0,t]\times\Omega\times I\!R^3$ and use the spectral inequality, so that

$$\| R_2(t) \|_{2,2}^2 + \frac{1}{\epsilon} \| R_2^{out} \|_{2t,2,2}^2 + \frac{1}{\epsilon^2} \| \nu^{\frac{1}{2}}(I-P_0)R_2 \|_{2t,2,2}^2$$
$$\leq c\Big(\| R_0 \|_{2,2}^2 + \| \nu^{-\frac{1}{2}}(I-P_0)G \|_{2t,2,2}^2 + \frac{\eta_1}{2\epsilon} \| P_0R_2 \|_{2t,2,2}^2 + \frac{1}{2\eta_1\epsilon} \| P_0G \|_{2t,2,2}^2 \Big),$$

for every $\eta_1 > 0$.
The a priori bounds on P_0R_2 are discussed in the following two lemmas. They are based on dual techniques using the space periodic solutions introduced above. Denote by

$$h(t,x,v) := P_0R_2 - < P_0R_2 >, \quad < f(t,v) > := \int_\Omega f(t,x,v)dx.$$

Lemma 5.4 *For any $0<\eta<1$ there is ϵ_η such that, for $0<\epsilon<\epsilon_\eta$,*

$$\| h \|_{2,2,2}^2 \leq \frac{c}{\epsilon}\Big(\| R_0 \|_{2,2}^2 + \| \nu^{-\frac{1}{2}}(I-P_0)G \|_{2,2,2}^2 + \frac{1}{\epsilon^3} \| P_0G \|_{2,2,2}^2 \Big) + \eta \, \|< P_0R_2 >\|_{2,2,2}^2 .$$

Proof of Lemma 5.4. In the variables $(\bar\tau,x,v) := (\frac{t}{\epsilon},x,v)$, the function R_2 is solution to

$$\frac{\partial R_2}{\partial\bar\tau} + v\cdot\nabla_x R_2 = \frac{1}{\epsilon}LR_2 + G, \quad R_2(0,r,v) = R_0(r,v),$$
$$R_2(\bar\tau,r_A,v) = 0, \quad \bar\tau>0,\ v_r>0,$$
$$R_2(\bar\tau,r_B,v) = 0, \quad \bar\tau>0,\ v_r<0.$$

Let φ be the $(2\pi a)^2$-periodic φ function solution to

$$\frac{\partial\varphi}{\partial\bar\tau} + v\cdot\nabla_x\varphi = \frac{1}{\epsilon}L\varphi + h, \quad \varphi(0,x,v) = 0,$$

where h is taken as zero outside the gap between the cylinders and periodically continued. Denote by

$$(f,g)_H = \int f(v)g(v)M(v)dv.$$

Then,

$$\frac{\partial}{\partial\bar\tau}(R_2,\varphi)_H + \int div_x(vR_2\varphi)Mdv = \frac{2}{\epsilon}(LR_2,(I-P_0)\varphi)_H$$
$$+(G,\varphi)_H + (h,P_0R_2)_H.$$

Integrating with respect to $\bar{\tau}$ and x gives

$$\begin{aligned}\| h \|^2_{2\bar{\tau},2,2} \leq \frac{K_1}{2} \| R_2(\bar{\tau},\cdot,\cdot) \|^2_{2,2} + \frac{1}{2K_1} \| \varphi(\bar{\tau},\cdot,\cdot) \|^2_{2,2} \\ + \frac{K_2}{2} \| R_2^{out} \|^2_{2\bar{\tau},2,\sim} + \frac{1}{2K_2} \| \varphi^{out} \|^2_{2\bar{\tau},2,\sim} \\ + \frac{K_3}{2\epsilon} \| \nu^{\frac{1}{2}}(I-P_0)R_2 \|^2_{2\bar{\tau},2,2} + \frac{1}{2K_3\epsilon} \| \nu^{\frac{1}{2}}(I-P_0)\varphi \|^2_{2\bar{\tau},2,2} \\ + \frac{K_4}{2} \| \nu^{-\frac{1}{2}}(I-P_0)G \|^2_{2\bar{\tau},2,2} + \frac{1}{2K_4} \| \nu^{\frac{1}{2}}(I-P_0)\varphi \|^2_{2\bar{\tau},2,2} \\ + \frac{K_5}{2} \| P_0 G \|^2_{2\bar{\tau},2,2} + \frac{1}{2K_5} \| P_0\varphi \|^2_{2\bar{\tau},2,2},\end{aligned}$$

for any positive constants K_j, $j = 1, ..., 5$.
It then follows from the preceeding estimates that

$$\begin{aligned}\| h \|^2_{2,2,2} \leq c\Big(\frac{1}{\epsilon^2} \| R_0 \|^2_{2,2} + \frac{1}{\epsilon} \| \nu^{-\frac{1}{2}}(I-P_0)G \|^2_{2,2,2} + \frac{1}{\epsilon^4} \| P_0 G \|^2_{2,2,2} \Big) \\ + \eta \| < P_0 R_2 > \|^2_{2,2,2} .\end{aligned}$$

This ends the proof of Lemma 5.4 when coming back to the t-variable. □

Lemma 5.5

$$\| < P_0 R_2 > \|^2_{2,2,2} \leq c(\frac{1}{\epsilon} \| R_0 \|^2_{2,2} + \frac{1}{\epsilon} \| \nu^{-\frac{1}{2}}(I-P_0)G \|^2_{2,2,2}) + \frac{1}{\epsilon^4} \| P_0 G \|^2_{2,2,2}).$$

Proof of Lemma 5.5. For $t > 0$ let $\varphi(t,x,v)$ be the solution to the (stationary) problem

$$\begin{aligned}v \cdot \nabla_x \varphi &= \frac{1}{\epsilon} L\varphi - \epsilon < P_0 R_2 >, \\ \varphi(t, r_A, v) &= 0, \quad t > 0,\ v_r > 0, \\ \varphi(t, r_B, v) &= 0, \quad t > 0,\ v_r < 0.\end{aligned}$$

By (3.2-4),

$$\begin{aligned}\| \nu^{\frac{1}{2}}(I-P_0)\varphi \|_{2,2} &\leq \epsilon \| < P_0 R_2 > \|_{2,2}, \\ \| P_0 \varphi \|_{2,2} &\leq \| < P_0 R_2 > \|_{2,2}, \\ \| \varphi^{out} \|_{\sim} &\leq \sqrt{\epsilon} \| < P_0 R_2 > \|_{2,2} .\end{aligned} \tag{5.17}$$

Then

$$\begin{aligned}\epsilon \frac{\partial}{\partial t}(R_2, \varphi) - \epsilon(R_2, \frac{\partial \varphi}{\partial t}) + \int div_x(v R_2 \varphi) M dv \\ = \frac{2}{\epsilon}(LR_2, (I-P_0)\varphi)_H + (G,\varphi)_H - \epsilon(< P_0 R_2 >, R_2)_H .\end{aligned}$$

Hence for η_1 of order ϵ

$$\|< P_0R_2 >\|^2_{2t,2,2}\leq c\Big(\| R_0 \|^2_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_0)G \|^2_{2t,2,2} + \frac{1}{\eta_1\epsilon} \| P_0G \|^2_{2t,2,2}$$
$$+\frac{\eta_1}{\epsilon} \| P_0R_2 \|^2_{2t,2,2} + \int_0^t \int R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds\Big).$$

And so, by Lemma 5.4, and for η_1 of order ϵ and small enough,

$$\|< P_0R_2 >\|^2_{2t,2,2}\leq \frac{c}{\epsilon}\Big(\| R_0 \|^2_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_0)G \|^2_{2,2,2} + \frac{1}{\epsilon^3} \| P_0G \|^2_{2,2,2} \Big)$$
$$+ \int_0^t \int R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds.$$

It remains to bound the term $\int_0^t R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds$ from above. Differentiate the equation satisfied by φ with respect to t. Similarly to (5.17),

$$\| P_0\frac{\partial\varphi}{\partial t} \|_{2t,2,2}\leq\|< P_0\frac{\partial R_2}{\partial t} >\|_{2t,2,2} .$$

Taking the hydrodynamic part of the equation (5.14) leads to

$$P_0\frac{\partial R_2}{\partial t} + \frac{1}{\epsilon}P_0(v\cdot\nabla_xR_2) = \frac{1}{\epsilon}P_0G.$$

Moreover,

$$< P_0(v\cdot\nabla_xR_2 >= c\Big(r_BP_0(v_rR_2(t,r_B,v)) - r_AP_0(v_rR_2(t,r_A,v))\Big).$$

Hence,

$$\| P_0\frac{\partial\varphi}{\partial t} \|^2_{2t,2,2}\leq c \|< P_0\frac{\partial R_2}{\partial t} >\|^2_{2t,2,2}\leq c\Big(\frac{1}{\epsilon^2} \| R_2^{out} \|^2_{2t,2} + \frac{1}{\epsilon^2} \| P_0G \|^2_{2t,2,2} \Big).$$

And so, Lemma 5.5 follows. □

Lemma 5.6 *Any solution R_2 to the system*

$$\frac{\partial R_2}{\partial t} + \frac{1}{\epsilon}v\cdot\nabla_xR_2 = \frac{1}{\epsilon^2}LR_2 + \frac{2}{\epsilon}H(R_2) + \frac{1}{\epsilon}G,$$
$$R_2(0,r,v) = R_0(r,v),$$
$$R_2(t,r_A,v) = 0, \quad t>0,\ v_r>0,$$
$$R_2(t,r_B,v) = 0, \quad t>0,\ v_r<0,$$

satisfies

$$\| R_2 \|_{2,2,2}\leq c\Big(\frac{1}{\sqrt{\epsilon}} \| R_0 \|_{2,2} + \frac{1}{\sqrt{\epsilon}} \| \nu^{-\frac{1}{2}}(I-P_0)G \|_{2,2,2} + \frac{1}{\epsilon^2} \| P_0G \|_{2,2,2} \Big),$$
$$\| R_2 \|_{\infty,2,2}\leq c\Big(\| R_0 \|_{2,2} + \| \nu^{-\frac{1}{2}}(I-P_0)G \|_{2,2,2} + \frac{1}{\epsilon\sqrt{\epsilon}} \| P_0G \|_{2,2,2} \Big),$$
$$\| R_2 \|_{\infty,\infty,2}\leq c\Big(\frac{1}{\epsilon} \| R_0 \|_{2,2} + \| R_0 \|_{\infty,2} + \frac{1}{\epsilon} \| \nu^{-\frac{1}{2}}(I-P_0)G \|_{2,2,2}$$
$$+\frac{1}{\epsilon^2\sqrt{\epsilon}} \| P_0G \|_{2,2,2} + \epsilon \| G \|_{\infty,\infty,2} \Big).$$

Proof of Lemma 5.6. Consider first for $H = 0$ the solution R_2 to

$$\frac{\partial R_2}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x R_2 = \frac{1}{\epsilon^2} L R_2 + \frac{1}{\epsilon} G,$$
$$R_2(0, r, v) = R_0(r, v),$$
$$R_2(t, r_A, v) = 0, \quad t > 0,\ v_r > 0,$$
$$R_2(t, r_B, v) = 0, \quad t > 0,\ v_r < 0.$$

It satisfies

$$\sup_{t \geq 0} \| R_2(t) \|_{2,2} + \frac{1}{\epsilon} \| \nu^{-\frac{1}{2}} (I - P_0) R_2 \|_{2,2,2} \leq c\Big(\| R_0 \|_{2,2}$$
$$+ \| (I - P_0) G \|_{2,2,2} + \frac{\eta}{\sqrt{\epsilon}} \| P_0 R_2 \|_{2,2,2} + \frac{1}{\eta \sqrt{\epsilon}} \| P_0 G \|_{2,2,2} \Big),$$

for any $\eta > 0$. Moreover, it follows from Lemmas 5.4-5 that

$$\| P_0 R_2 \|_{2,2,2} \leq c\Big(\frac{1}{\sqrt{\epsilon}} \| R_0 \|_{2,2} + \frac{1}{\sqrt{\epsilon}} \| \nu^{-\frac{1}{2}} (I - P_0) G \|_{2,2,2} + \frac{1}{\epsilon^2} \| P_0 G \|_{2,2,2} \Big).$$

Choosing $\eta = \sqrt{\epsilon}$ leads to the first inequality of Lemma 5.6, and choosing $\eta = \epsilon$ leads to the second one. Then, by some additional computations similar to what we have done in previous sections,

$$\| R_2 \|_{\infty,\infty,2} \leq c\Big(\frac{1}{\epsilon} \| R_2 \|_{\infty,2,2} + \| R_0 \|_{\infty,2} + \epsilon \| G \|_{\infty,\infty,2} \Big),$$

which leads to the last inequality of Lemma 5.6. Adding the small perturbation $\frac{1}{\epsilon} H(R_2)$ does not change the results. □

Proof of Theorem 5.1. The convergence to zero when $t \to \infty$ of the asymptotic expansion ψ for the difference $\tilde{\Phi} - \tilde{\Phi}_s$ was discussed at the beginning of this section. The corresponding rest term ϵR was split into $\epsilon R_1 + \epsilon R_2$, where by Lemma 5.1 and by the boundary conditions being satisfied by ψ up to third order in ϵ,

$$\| \nu^{\frac{1}{2}} R_1 \|_{2,2,2} \leq c \mid R_1^{in} \mid_{2,\sim}, \quad \| \nu^{\frac{1}{2}} R_1 \|_{\infty,\infty,2} \leq \frac{c}{\epsilon} | R_1^{in} |_{\infty,\sim},$$

i.e. subexponential decrease in ϵ and convergence to zero when time tends to infinity.

So it only remains to show the existence of R_2 and its convergence to zero when $t \to +\infty$. We shall prove that R_2 can be obtained as the limit of an approximating sequence and that

$$\int_0^{+\infty} \int_\Omega \int_{I\!R^3} (R_2)^2 (t, x, v) M(v) dt dx dv < c\epsilon^2. \tag{5.18}$$

This in turn implies the L^2-convergence to zero of R_2 when time tends to infinity, i.e. $\lim_{t\to\infty}\int R_2(t,x,v)^2 M dxdv = 0$.
Let the approximating sequence (R_2^n) be defined by $R_2^0 = 0$, and

$$\begin{aligned}\frac{\partial R_2^{n+1}}{\partial t} + \frac{1}{\epsilon} v\cdot\nabla_x R_2^{n+1} = \frac{1}{\epsilon^2} L R_2^{n+1} + \frac{2}{\epsilon} H(R_2^{n+1})\\ +\frac{1}{\epsilon} J(R_1 + R_2^n, R_1 + R_2^n) + \bar{\alpha},\\ R_2^{n+1}(0,r,v) = R_0(r,v),\\ R_2^{n+1}(t,r_A,v) = 0, \quad t>0,\ v_r>0,\\ R_2^{n+1}(t,r_B,v) = 0, \quad t>0,\ v_r<0,\end{aligned}$$

where R_0 is of ϵ-order two and

$$\bar{\alpha} = \alpha - \frac{\partial R_1}{\partial t}.$$

The function R_2^1 is solution to

$$\begin{aligned}\frac{\partial R_2^1}{\partial t} + \frac{1}{\epsilon} v\cdot\nabla_x R_2^1 = \frac{1}{\epsilon^2} L R_2^1 + \frac{2}{\epsilon} H(R_2^1) + \frac{1}{\epsilon} J(R_1, R_1) + \bar{\alpha},\\ R_2^1(0,r,v) = R_0(r,v),\\ R_2^1(t,r_A,v) = 0, \quad t>0,\ v_r>0,\\ R_2^1(t,r_B,v) = 0, \quad t>0,\ v_r<0,\end{aligned}$$

so that by Lemma 5.6 and the subexponential decrease of R^{in} together with the orders 2 of R_0 and 1 of $\alpha_\perp$ and 2 of $\alpha_{||}$,

$$\| R_2^1 \|_{\infty,\infty,2} \le c_1 \epsilon^{\frac{1}{2}}, \quad \| R_2^1 \|_{2,2,2} \le c_1 \epsilon,$$

for some constant c_1. A closer inspection shows that $c_1 = O(r_B - r_A)$ when the coefficients in the perturbation P are $O(r_B - r_A)$.
By induction, for $r_B - r_A$ small enough

$$\begin{aligned}\| R_2^j \|_{\infty,\infty,2} \le 2c_2 |r_B - r_A| \epsilon^{\frac{1}{2}}, \quad j \le n,\\ \| R_2^{n+1} - R_2^n \|_{2,2,2} \le c_3\sqrt{r_B - r_A}\, \| R_2^n - R_2^{n-1} \|_{2,2,2}, \quad n \ge 1,\end{aligned}$$

for some constants c_2, c_3. Namely, if this holds up to n^{th} order, then

$$\begin{aligned}\frac{\partial}{\partial t}(R_2^{n+2} - R_2^{n+1}) + \frac{1}{\epsilon} v\cdot\nabla_x (R_2^{n+2} - R_2^{n+1})\\ = \frac{1}{\epsilon^2} L(R_2^{n+2} - R_2^{n+1}) + \frac{2}{\epsilon} H(R_2^{n+2} - R_2^{n+1}) + \frac{1}{\epsilon} G^{n+1},\\ (R_2^{n+2} - R_2^{n+1})(0,r,v) = 0,\\ (R_2^{n+2} - R_2^{n+1})(t,r_A,v) = 0, \quad t>0,\ v_r>0,\\ (R_2^{n+2} - R_2^{n+1})(t,r_B,v) = 0, \quad t>0,\ v_r<0,\end{aligned}$$

with

$$G^{n+1} = (I - P_0)G^{n+1} = 2J(R_1, R_2^{n+1} - R_2^n) + J(R_2^{n+1} + R_2^n, R_2^{n+1} - R_2^n),$$

and where by Lemma 5.6

$$\| R_2^{n+2} - R_2^{n+1} \|_{2,2,2} \leq \frac{c}{\sqrt{\epsilon}} \| G^{n+1} \|_{2,2,2}$$

$$\leq \frac{c}{\sqrt{\epsilon}} \Big(\| R_1 \|_{\infty,\infty,2} + \| R_2^{n+1} \|_{\infty,\infty,2} + \| R_2^n \|_{\infty,\infty,2} \Big) \| R_2^{n+1} - R_2^n \|_{2,2,2}$$

$$\leq c_2 \sqrt{r_B - r_A} \| R_2^{n+1} - R_2^n \|_{2,2,2} .$$

This ends the first induction step, and also implies that

$$\| R_2^{n+2} \|_{2,2,2} \leq \| R_2^{n+2} - R_2^{n+1} \|_{2,2,2} + \ldots + \| R_2^2 - R_2^1 \|_{2,2,2} + \| R_2^1 \|_{2,2,2} \leq 2c_1 \epsilon,$$

for $r_B - r_A$ small enough. Similarly $\| R_2^{n+2} \|_{\infty,\infty,2} \leq 2c_2 |r_B - r_A| \epsilon^{\frac{1}{2}}$. In particular (R_2^n) is a Cauchy sequence in $L^2([0, +\infty[\times \Omega \times I\!R^3_M)$. The existence of R_2 follows, and the estimate (5.18) holds. This completes the study of the R_2-term and Theorem 5.1 follows. □

The dependence on a small enough $r_B - r_A$ was introduced to be able to use a short ϵ-expansion. With an ϵ-expansion of higher order the same proof shows that the existence of R_2 and the stability result of Theorem 5.1 hold for an arbitrary fixed $r_B - r_A$, when ϵ is small enough.

6 Positivity.

We shall in this final section discuss the positivity of the earlier solutions.

In the time-dependent small data case, positivity of all sufficiently nice solutions can be proved by Gronwall based ideas, see [LZ]. But in the stationary small data case the question whether all nice solutions are positive remains an interesting open problem. For general time-dependent problems, positivity is usually introduced at the beginning of the approximation procedure and then kept throughout, so the solutions *constructed* are positive, but not necessarily other solutions. When there is uniqueness around, time-dependent positivity may alternatively be obtained by comparison with some other equation already known to have only positive solutions (see [A]). That turns out to be a possible approach also here for our stationary solutions using a new type of comparison equation. The proof starts by considering a variant of the stationary Boltzmann equation with a particular extra term depending only on the negative part of the solution. This new equation is then proved only to have positive solutions, the extra term disappears and the solutions solve the BE. The proof goes on to construct a solution to the new equation of the type we already discussed for the original problem, and to show that this new solution coincides with the original solution. There is the following technical problem. In one step of the proof, growth estimates are needed for terms like $v_r v_\theta \bar{B} = L^{-1} v_r v_\theta$. For Maxwellian molecules such estimates are proved in [C], and that can be used to complete our positivity proof in the Maxwellian case. But for strictly hard forces, suitable growth estimates still seem to be an open problem - also of interest in other contexts.

Write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Suppose f satisfies the related problem (6.1-2) below. Then $f^- = 0$ by Theorem 6.1 below, and $f = f^+$ is a non-negative solution also to (2.1), (2.3). If the contraction mapping approach used above can be extended to the construction of suitable solutions for the problem (6.1-2), then as a consequence, any solution from the previous sections would coincide with such a non-negative solution.

Theorem 6.1 *Let Ω be a bounded set in $I\!R^n$ with smooth boundary, and f_b a nonnegative function defined on $\partial\Omega^+$. If $M^{-1}f \in \tilde{L}^\infty(\Omega \times I\!R^3)$ and f solves the boundary value problem*

$$v \cdot \nabla_x f = Q(f^+, f^+) - ML(M^{-1} f^-), \quad (x, v) \in \Omega \times I\!R^3, \qquad (6.1)$$

$$f = f_b, \quad \partial\Omega^+, \qquad (6.2)$$

then $f^- = 0$, and $f = f^+$ solves the corresponding boundary value problem for the Boltzmann equation,

$$\begin{aligned} &v \cdot \nabla_x f = Q(f,f), \quad \Omega \times I\!R^3, \\ &f = f_b, \quad \partial\Omega^+. \end{aligned}$$

<u>Proof of Theorem 6.1</u> The function $F = M^{-1} f$ satisfies

$$v \cdot \nabla_x F = J(F^+, F^+) - L(F^-), \quad F = M^{-1} f_b, \quad \partial\Omega^+.$$

Define J^+ and J^- by $J(\varphi,\varphi) = J^+(\varphi,\varphi) - J^-(\varphi,\varphi)$, where

$$\begin{aligned} J^+(\varphi,\varphi)(v) &:= \int \mid v - v_* \mid^\beta b(\theta) M_* \varphi' \varphi'_* dv_* d\omega, \\ J^-(\varphi,\varphi)(v) &:= \varphi(v) \int \mid v - v_* \mid^\beta b(\theta) M_* \varphi_* dv_* d\omega. \end{aligned}$$

Also, F^- satisfies

$$\begin{aligned} &-v \cdot \nabla_x F^- = \chi_{F^- \neq 0}(J^+(F^+, F^+) - L(F^-)), \\ &F^- = 0, \quad \partial\Omega^+. \end{aligned} \tag{6.3}$$

Multiplying (6.3) with $-MF^-$, integrating on $\Omega \times I\!R^3$ and using that

$$\begin{aligned} -\int MF^- \chi_{F^- \neq 0} L(F^-) dv &= -\int MF^- L(F^-) dv \\ &\geq c \int M\nu \mid (I - P_0) F^- \mid^2 dv, \end{aligned}$$

implies that

$$\begin{aligned} &\frac{1}{2} \int_{\partial\Omega^-} \mid v \cdot n \mid M(F^-)^2 + c \int_{\Omega \times I\!R^3} M\nu \mid (I - P_0) F^- \mid^2 \\ &\leq -\int MF^- \chi_{F^- \neq 0} J^+(F^+, F^+) \leq 0. \end{aligned}$$

It follows that

$$F^- = 0 \text{ on } \partial\Omega^-, \quad L(F^-) = 0.$$

And so, F^- satisfies

$$F^- = 0, \ \partial\Omega^- \cup \partial\Omega^+, \quad v \cdot \nabla_x F^- \leq 0.$$

This implies that F^- is identically zero. □

Corollary 6.2 *If there is a solution f to (6.1-2) in a ball of contraction from the proofs of Theorem 4.2-4, then $f^- = 0$ and $f = f^+$ is the unique and strictly positive solution in that ball of the corresponding boundary value problem (2.1), (2.3).*

Theorem 6.3 *The solutions obtained in Theorem 4.2-4 are strictly positive in the case of Maxwellian molecules.*

Proof of Theorem 6.3. For the case of Maxwellian molecules there is indeed in all three cases a solution to (6.1-2), i.e. the hypothesis of the corollary holds. We start with the axially homogeneous situation of Case 1. Set $\bar{\chi} = \chi_{|v|<\epsilon^{-\frac{1}{n}}}$ and denote again by φ the previous asymptotic expansion of order two,

$$\varphi(r,v) = \sum_{i=1}^{2} \epsilon^i \Phi^i.$$

If the terms in Φ^i, $1 \leq i \leq 2$ are polynomially bounded in the v-variable, with bounded coefficients in the r-variable, then for ϵ and $\frac{1}{n}$ small enough and positive, it would hold that

$$1 + \bar{\chi}\varphi = 1 + \bar{\chi}\Big(\sum_{i=1}^{2} \epsilon^i \Phi^i\Big) \geq 0. \tag{6.4}$$

The required bounds follow from the previous discussion of the terms in φ except the $\bar{B}$-term in Φ^2 (and also some $\bar{A}$-terms in Case 2-3). But it is well known that also such $\bar{A}$ and $\bar{B}$ terms are polynomially bounded in the Maxwellian case (cf [C]). Notice that the $\tilde{L}^q$-norm of $(1-\bar{\chi})\Phi$ for any q is of arbitrarily high order in ϵ because of the factor M in the v-integrand.
Using the approach of Section 4, the positivity under the cut-off $\bar{\chi}$ in (6.4), and the corresponding splitting

$$f = M(1 + \bar{\chi}\varphi + \epsilon R),$$

lead to a nonnegative solution of (6.1-2) with $M^{-1}f \in \tilde{L}^\infty$ as follows. Namely, the rest term R should be a solution to

$$DR = \frac{1}{\epsilon}\Big(LR + 2J(\bar{R}, \bar{\chi}\varphi) + \epsilon J(\bar{R}, \bar{R}) + \bar{l}\Big), \tag{6.5}$$

where

$$\bar{l} = \frac{1}{\epsilon}\Big(L(\bar{\chi}\varphi) + J(\bar{\chi}\varphi, \bar{\chi}\varphi) - \epsilon D(\bar{\chi}\varphi)\Big),$$

and

$$\bar{R}(r,v) = R(r,v) \ \text{ when } \ \epsilon R(r,v) \geq -\Big(1 + \bar{\chi}\sum_{i=1}^{2} \epsilon^i \Phi^i(r,v)\Big),$$
$$\bar{R}(r,v) = -\frac{1}{\epsilon}\Big(1 + \bar{\chi}\sum_{i=1}^{2} \epsilon^i \Phi^i(r,v)\Big) \ \text{ otherwise.}$$

Here $\bar{l}$ can be decomposed as $\bar{l}_\perp$ as in Section 4, and $\bar{l}_{||}$ which in $\tilde{L}^q$ is of arbitrarily high order in ϵ. The approximating sequences $(R^n)_{n\in I\!N}$ and $(\bar{R}^n)_{n\in I\!N}$ are defined by $R^0 = \bar{R}^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon}\Big(LR^{n+1} + 2\sum_{j=1}^{2} \epsilon^j J(\bar{R}^{n+1}, \bar{\chi}\Phi^j) + g^n\Big), \tag{6.6}$$
$$R^{n+1}(1,v) = R_A(v), \ v_r > 0, \quad R^{n+1}(r_B,v) = R_B(v), \ v_r < 0, \tag{6.7}$$

with

$$g^n := \epsilon J(\bar{R}^n, \bar{R}^n) + \bar{l},$$
$$\epsilon R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u_{\theta A1}^2 v_\theta^2} - 1 - \bar{\chi}\Phi(r_A, v), \quad v_r > 0,$$
$$\epsilon R_B(v) := -\bar{\chi}\Phi(r_B, v), \quad v_r < 0,$$

and

$$\bar{R}^n(r,v) = R^n(r,v) \ \text{ when } \ \epsilon R^n(r,v) \geq -\Big(1 + \bar{\chi}\sum_{i=1}^{2} \epsilon^i \Phi^i(r,v)\Big),$$
$$\bar{R}^n(r,v) = -\frac{1}{\epsilon}\Big(1 + \bar{\chi}\sum_{i=1}^{2} \epsilon^i \Phi^i(r,v)\Big) \ \text{ otherwise.}$$

From here the only difference with respect to the contraction mapping analysis of Section 4, is related to the appearance of factors $\bar{R}^n$ instead of the previous R^n in J. The existence result in Lemma 3.1 is not changed by the replacements $\bar{R}$. Arguing similarly to the previous cases, the contribution to the a priori non fluid dynamic estimate (3.2) due to $g_{||}$ gives rise to an extra term $| g_{||} |_2 \epsilon^{-1}$, hence

$$\epsilon^{\frac{1}{2}} | \mathcal{S}F |_\sim + | \tilde{\nu}^{\frac{1}{2}} F_\perp |_2 \leq c(| \tilde{\nu}^{-\frac{1}{2}} g_\perp |_2 + \epsilon^{-1} | \tilde{\nu}^{-\frac{1}{2}} g_{||} |_2 + \epsilon | F_{||} |_2 + \epsilon^{\frac{1}{2}} | F_b |_\sim).$$

The proof of the fluid dynamic Lemma 3.3 is essentially unchanged in the present situation (with the $\bar{R}$-terms included in $g_{\perp}$), and its estimate (3.4) follows.

We turn to the existence proof for (6.5), (6.7). In the new situation the contraction mapping arguments from the proof of Theorem 4.2 still hold. That leads to an isolated solution for (6.5), (6.7) which defines the positive solution of Corollary 6.2. The solution lies in the same ball of contraction as the solution constructed in Section 4, so they coincide and the solution of Section 4 is positive. That completes the proof of Theorem 6.3 in the axially homogeneous case.
The other cases for Maxwellian molecules are similarly proved. □

Remark. The only obstacle for extending the above approach to hard forces, is a lack of growth estimates at zero and infinity for certain terms in the asymptotic expansion φ, like the terms $v_r\bar{A}$ and $v_\theta v_r\bar{B}$.

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