KEPLERIAN SHEAR WITH RAJCHMAN PROPERTY

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Abstract. The Keplerian shear was introduced within the context of measure preserving dynamical systems by Damien Thomine [16], as a version of mixing for non erogdic systems. In this study we provide a characterization of the Keplerian shear using Rajchman measure, for some flows on tori bundles. Our work applies to dynamical systems with singularities or with non-absolutely continuous measures. We relate the speed of decay of conditional correlations with the Rajchman order of the measures. Some of these results are extended to the case of compact Lie group bundles.

Contents

1. Introduction

In a dynamical system, the mixing property reveals that trajectories are intermingled and asymptotically distributed somewhat homogeneously. A paradigmatic example is provided by Arnold's cat map on the 2-torus $\mathbb{T}^2$, endowed with the Lebesgue measure,

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

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From a probabilistic perspective, mixing signifies the asymptotic independence of events; long-term evolution forgets the initial conditions. This property, along with its quantitative counterparts (e.g., decay of correlations), forms the foundation for various probabilistic limit theorems within the context of deterministic dynamics, such as the Central Limit Theorem and Borel-Cantelli lemmas (see [6]).

However, there are many systems in which certain quantities are conserved during evolution. This occurs in integrable systems of Hamiltonian dynamics, particularly in some geodesic flows and rational billiards (see [16] and references therein). This phenomenon can also manifest in biological systems, where the preservation of a character results in different evolutionary paths. The presence of these invariants prevents the system from being ergodic and, consequently, from exhibiting mixing. The transvection

$$T := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

acting on the torus $\mathbb{T}^2$, is a paradigmatic example. This map acts on the second coordinate as a rotation on the circle. The evolution in each ergodic component (each individual system) is notably simple and predictable. Nevertheless, Kesten demonstrated that, with some randomness introduced into the rotation angle, trajectories distribute quite homogeneously in the long term. Specifically, the discrepancies, suitably normalized, converge to a Cauchy distribution [12] (see also [8] for multidimensional generalization).

In celestial mechanics, planetary rings (the motion of each dust particle) may be modeled by the flow

$$\begin{aligned} g_t : \quad & [a,b] \times \mathbb{T} && \to [a,b] \times \mathbb{T} \\ & (r,\theta) && \mapsto \left(r, \theta + t r^{-\frac{3}{2}}\right). \end{aligned}$$

While the distance to the center is preserved, and each trajectory is essentially a rotation, the fact that angular velocities vary allows for the aggregation of materials, potentially explaining the formation of larger bodies. Recently Damien Thomine [16] introduced the notion of Keplerian shear, formalizing the fact that in non ergodic systems, trajectories may distribute homogeneously and independently of their past, provided we ignore the invariants.

The aim of our work is to pursue the study of non-ergodic dynamical systems which have the property of Keplerian shear.

The probabilistic dynamical system (X, T, μ) has Keplerian shear if for all $f \in L^2(\mu)$ we have the weak convergence

$$f \circ T^n \to E_\mu(f|\mathscr{I}), \tag{1}$$

$\mathscr{I}$ being the σ-algebra of invariant by the transformation T. The lack of ergodicity is indeed adding a hazard to the dynamics (the choice of the ergodic component), and even if the dynamics on the fiber is not mixing the system can globally appear mixing conditionally to the fibers.

We now present the organisation of the article. In the section 2, we define keplerian shear, relate it to the mixing property and the Rajchman property of a measure, the main notion in this article. We also make a description of the dynamical systems that we consider in this work, which includes locally action-angle dynamics.

In the section 3, we show the main result in the discrete case and we investigate the speed of shearing, using anisotropic Sobolev spaces.

Section 4 addresses in the continuous case the same questions of section 3.

Section 5 generalizes the work to the case where the phase space is a connected-compact Hausdorff Lie group.

And to finish, section 6 provides applications of keplerian shear to Borel-Cantelli lemmas and diophantine approximation.

2. Settings

2.1. Mixing property and keplerian shear.

The mixing property is well known and is satisfied in many situations. However it makes sense only for ergodic systems. From this point of view, keplerian shear may be a right compromise.

We will note in the article $(\Omega, \mathscr{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (resp. $(\Omega, \mathscr{T}, \mu, T)$) a continuous dynamical (resp. discrete). A mixing system is an asymptotically independent system:

Definition 2.1 (Mixing system)**.** *We say that $(\Omega, \mathscr{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (resp. $(\Omega, \mathscr{T}, \mu, T)$) is mixing if*

$$\forall (A, B) \in \mathscr{T}^2, \mu(A \cap g_t^{-1}(B)) - \mu(A)\mu(B) \xrightarrow[t\to+\infty]{} 0$$

$$\left(\textit{resp. } \mu(A \cap (T^n)^{-1}(B)) - \mu(A)\mu(B) \xrightarrow[t\to+\infty]{} 0 \right)$$

We recall that mixing systems are ergodic. Consequently the following systems are not mixing, but we will show that they exhibit keplerian shear.

Example 2.1.1 (Non mixing system because lack of ergodicity)**.** *We endow these systems with the Lebesgue measure on $\mathbb{T}^2$.*

(1) $\begin{array}{lcl} T : \mathbb{T}^2 & \to & \mathbb{T}^2 \\ (x, y) & \mapsto & (x, y + x) \end{array}$

(2) $\begin{array}{lcl} g_t : \mathbb{T}^2 & \to & \mathbb{T}^2 \\ (x, y) & \mapsto & \left(x, y + t\cos\left(2\pi\left(x - \frac{1}{2}\right)\right)\right) \end{array}$

Definition 2.2 (Invariant $\sigma-$algebra)**.** *Invariant $\sigma-$algebra $\mathscr{I}$ is the $\sigma-$algebra of measurable sets invariant by continuous flow (resp. discrete):*
For $(\Omega, \mathscr{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (resp. $(\Omega, \mathscr{T}, \mu, T)$) a continuous dynamical system (resp.discrete)
$\mathscr{I} := \left\{A \in \mathscr{T} : \forall t \in \mathbb{R}, \mu(A\Delta g_t^{-1}(A)) = 0\right\}$ *(resp.* $\left\{A \in \mathscr{T} : \mu(A\Delta T^{-1}(A)) = 0\right\}$*)*

Keplerian shear is a notion of asymptotic independence conditionally to the fibers of the invariant algebra.

Definition 2.3 (Keplerian shear)**.** *Keplerian shear is defined by:*

$$\forall f \in \mathbb{L}^2_\mu(\Omega), f \circ g_t \underset{t\to+\infty}{\rightharpoonup} \mathbb{E}_\mu(f|\mathscr{I}) \left(\textit{resp.} f \circ T^n \underset{n\to+\infty}{\rightharpoonup} \mathbb{E}_\mu(f|\mathscr{I})\right)$$

Definition 2.4 (Conditional correlation)**.** *We define the conditional correlation by:*

$$\forall (f_1, f_2) \in \left(\mathbb{L}^2_\mu(\Omega)\right)^2, Cov_t(f_1, f_2|\mathscr{I}) := \mathbb{E}_\mu(\overline{f_1} \cdot (f_2 \circ g_t)|\mathscr{I}) - \overline{\mathbb{E}_\mu(f_1|\mathscr{I})} \cdot \mathbb{E}_\mu(f_2|\mathscr{I})$$

$$\left(\textit{resp. } Cov_n(f_1, f_2|\mathscr{I}) := \mathbb{E}_\mu(\overline{f_1} \cdot (f_2 \circ T^n)|\mathscr{I}) - \overline{\mathbb{E}_\mu(f_1|\mathscr{I})} \cdot \mathbb{E}_\mu(f_2|\mathscr{I})\right)$$

Proposition 2.1 ([16])**.** *The keplerian shear property is equivalent to the convergence to 0 of the expectation of the conditional correlation, in other words:*
(2)
$$\forall (f_1, f_2) \in \left(\mathbb{L}^2_\mu(\Omega)\right)^2, \mathbb{E}_\mu\left(Cov_t(f_1, f_2|\mathscr{I})\right) \xrightarrow[t\to+\infty]{} 0 \left(\textit{resp. } \mathbb{E}_\mu\left(Cov_n(f_1, f_2|\mathscr{I})\right) \xrightarrow[n\to+\infty]{} 0\right)$$

In ergodic systems, keplerian shear is equivalent to mixing. The mixing property is thus equivalent to the ergodicity and keplerian shear.

2.2. **Rajchman measure.** Fourier theory will be instrumental in our study of correlation decay. In particular measures with the Riemann-Lebesgue property will play an important role in this work.

Definition 2.5 (Rajchman measure)**.** *We are in the space* $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \nu)$ *for the continuous case (resp.* $(\mathbb{T}, \mathscr{B}(\mathbb{T}), \nu)$ *for the discrete case).*
ν is Rajchman if

$$\widehat{\nu}(t) \xrightarrow[t\to\pm\infty]{} 0 \left(resp.\ \widehat{\nu}(n) \xrightarrow[n\to\pm\infty]{} 0 \right)$$

with $\hat{\nu}(t) = \int_{\mathbb{R}} e^{2i\pi tx} d\nu(x)$ $\left(resp. \hat{\nu}(n) = \int_{\mathbb{T}} e^{2i\pi nx} d\nu(x)\right)$

Note that we will only consider one dimensional Rajchman measures.

Definition 2.6 (Rajchman measure in higher dimension)**.** *We are in the space* $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \nu)$ *for the continuous case (resp.* $(\mathbb{T}^d, \mathscr{B}(\mathbb{T}^d), \nu)$ *for the discrete case).*
ν is Rajchman if

$$\widehat{\nu}(w) \xrightarrow[\|w\|\to+\infty]{} 0 \left(resp.\ \widehat{\nu}(n) \xrightarrow[\|n\|\to+\infty]{} 0 \right)$$

with $\hat{\nu}(w) = \int_{\mathbb{R}} e^{2i\pi <w|x>} d\nu(x)$ $\left(resp. \hat{\nu}(n) = \int_{\mathbb{T}} e^{2i\pi <n|x>} d\nu(x)\right)$

2.2.1. *Functional equivalences.* We recall an equivalent interpretation of the Rajchman property in terms of convergence in distribution or equidistribution.

Proposition 2.2. *In the continuous case (resp.discrete), the measure ν is Rajchman if and only if for x distibuted by ν :*

$$\forall a > 0, \frac{nx}{a}[1] \xrightarrow[n\to+\infty]{\mathcal{L}} \mathscr{U}(\mathbb{T})$$

$$\left(resp.\, nx[1] \xrightarrow[n\to+\infty]{\mathcal{L}} \mathscr{U}(\mathbb{T}) \right)$$

Proof. Discrete case. Let $\varphi \in \mathscr{D}(\mathbb{T})$. Its Fourier series decomposition gives

$$\forall x \in \mathbb{T}, \forall n \in \mathbb{Z}, \varphi(nx) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k) e^{2i\pi knx}.$$

By Lebesgue convergence theorem we get

$$\int_{\mathbb{T}} \varphi(nx) d\nu(x) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k) \int_{\mathbb{T}} e^{2i\pi knx} d\nu(x) = \widehat{\varphi}(0) + \sum_{k\in\mathbb{Z}^*} \widehat{\varphi}(k)\widehat{\nu}(kn) \xrightarrow[n\to+\infty]{} \int_{\mathbb{T}} \varphi(x) d\lambda(x).$$

Let's prove the reciprocal implication.
We take $\varphi : x \mapsto e^{2i\pi x}$ and note that $\widehat{\nu}(n) = \int_{\mathbb{T}} \varphi(nx) d\nu(x) \xrightarrow[n\to+\infty]{} 0.$

Continuous case. Let's start with the direct implication.
Let $a > 0$. Let μ be the push forward of ν by $x \mapsto \frac{x}{a}[1]$. We have for any $n \in \mathbb{Z}$

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{2i\pi nx} d\mu(x) = \int_{\mathbb{R}} e^{2i\pi \frac{n}{a} x} d\nu(x) = \widehat{\nu}\left(\frac{n}{a}\right) \xrightarrow[n\to+\infty]{} 0$$

Therefore, μ is Rajchman on $\mathbb{T}$ and we apply the discrete case.

Let's prove the reciprocal implication. Let $\varepsilon > 0$. The uniform continuity of the characteristic function gives

$$\exists \delta > 0, \forall (x, y) \in \mathbb{R}^2, \left(|x - y| < \delta \Longrightarrow |\widehat{\nu}(x) - \widehat{\nu}(y)| < \frac{\varepsilon}{2} \right)$$

In addition, by the Rajchman hypothesis

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, \left(n \geq n_0 \Longrightarrow \widehat{\nu}\left(n\frac{\delta}{2}\right) < \frac{\varepsilon}{2}\right)$$

.

Let $t \geq n_0 \frac{\delta}{2}$. Since $\frac{\delta}{2}\mathbb{N} \cap [t, t+\delta[\neq \emptyset$ there exists $n \in \mathbb{N}, |n\frac{\delta}{2} - t| < \delta$. Thus

$$|\widehat{\nu}(t)| \leq \left|\widehat{\nu}(t) - \widehat{\nu}\left(n\frac{\delta}{2}\right)\right| + \left|\widehat{\nu}\left(n\frac{\delta}{2}\right)\right| < \varepsilon.$$

□

Lemma 2.1 (Weak$-*$ convergence and Rajchman property)**.** *Let μ be a Borel-probabilty measure on $\mathbb{R}$.*

The measure μ is Rajchman if and only if $\left(x \mapsto e^{2i\pi tx}\right) \xrightarrow[t\to+\infty]{} 0$ in the weak-$*$ topology on $\left(\mathbb{L}^1_\mu(\mathbb{R})\right)^* = \mathbb{L}^\infty_\mu(\mathbb{R})$.

Proof. Reciprocal implication is obtained immediately with $(x \in \mathbb{R} \mapsto 1) \in \mathbb{L}^1_\mu(\mathbb{R})$.

Let's prove the direct implication. Let $\varphi \in \mathscr{S}(\mathbb{R})$. Writting φ with its reverse Fourier transform and using Fubini theorem we get

$$\begin{aligned}\int_{\mathbb{R}} e^{2i\pi tx}\varphi(x)d\mu(x) &= \int_{\mathbb{R}} e^{2i\pi tx}\left(\int_{\mathbb{R}} e^{2i\pi xs}\widehat{\varphi}(s)d\lambda(s)\right)d\mu(x) \\ &= \int_{\mathbb{R}} \widehat{\varphi}(s)\widehat{\mu}(t+s)d\lambda(s) \xrightarrow[t\to+\infty]{} 0\end{aligned}$$

by Rajchman property and Lebesgue convergence theorem. We conclude by density of $\mathscr{S}(\mathbb{R})$ in $\mathbb{L}^1_\mu(\mathbb{R})$. □

2.2.2. *Fourier-Rajchman decay.* The notion of Rajchman measure is only qualitative. To estimate the speed of convergence to 0 of the expectation of conditional correlations (2), a quantitative version of the Fourier decay will be needed.

Definition 2.7 (Fourier-Rajchman decay)**.** *A Rajchman speed of a Rajchman measure on $\mathbb{T}^d$ (resp. $\mathbb{R}^d$) is a value $r \geq 0$ such that*

$$\exists C > 0, \exists n_0 \in \mathbb{N}^*, \forall n \in \mathbb{Z}^d, \left(\|n\| \geq n_0 \Longrightarrow |\widehat{\nu}(n)| \leq \frac{C}{\|n\|^r}\right)$$

$$\left(resp. \exists C > 0, \exists t_0 > 0, \forall t \in \mathbb{R}^d, \left(\|t\| \geq t_0 \Longrightarrow |\widehat{\nu}(t)| \leq \frac{C}{\|t\|^r}\right)\right)$$

Definition 2.8 (Rajchman order)**.** *Rajchman order of a Rajchman measure is the supremum of its Rajchman speeds. We denote it $r(\mu)$.*

Remark 2.2.1. *Rajchman order is linked to the Fourier dimension $dimF(\mu)$ defined in Boris Solomyak documents* [14]*, we have $dimF(\mu) = 2r(\mu)$.*

Additionnaly, as mentioned in the work [14]*, when the Rajchman order r verify $r > \frac{1}{2}$, $\mu \ll \lambda$. Consequently a Rajchman measure can be singular continuous only if its Rajchman order r satisfies $r \leq \frac{1}{2}$.*

Definition 2.9 (Diophantine exponent)**.** *We call the Diophantine exponent of $x \in \mathbb{R}$, $Dio(x) = \inf A = \sup B$ where*

$$A = \left\{s > 0 : \exists C > 0, \forall (p,q) \in \mathbb{Z} \times \mathbb{N}^*, |qx - p| \geq \frac{C}{q^s}\right\}$$

and

$$B = \left\{ t > 0 : \exists \textit{ infinitely many} (p,q) \in \mathbb{Z} \times \mathbb{N}^*, |qx - p| < \frac{1}{q^t} \right\}$$

The Rajchman order of a measure control the diophantine exponent a.s. (See Proposition 6.1)

Proposition 2.3. *If $r(\mu) \leq \frac{1}{2}$, then $\mu - a.a\, \alpha \in [0,1[, Dio(\alpha) \leq \frac{1}{r(\mu)} - 1$. In the absolutely continuous case, $Dio(\alpha) = 1$ a.e.*

2.2.3. *Radon-Nikodyme-Lebesgue decomposition.* We discuss the simple relations that can be seen between the Rajchman property and the Radon-Nikodyme-Lebesgue decomposition

$$\nu = \nu_{ac} + \nu_{sc} + \nu_d.$$

(1) Absolutely continue part. Absolutely continuous measure are Rajchman. This is garanteed by Riemann-Lebesgue. When we know the regularity of the density, we can estimate the Rajchman order. The work of Damien Thomine already covers the absolutely continuous case.
(2) Singular continuous part. Many singular continuous measures are Rajchman and other not. We extend Damien Thomine results for very singular systems.
(3) Discrete part. Discrete measure are never Rajchman because the non-convergence of complexe exponential to 0.

Consequently, ν is Rajchman if and only if ν_{sc} is Rajchman and $\nu_d = 0$.

2.2.4. *Example of singular Rajchman measures.*

Definition 2.10 (Pisot number)**.** *A real θ is a Pisot number if and only if $\theta > 1$ is algebraic and every other roots θ_r of its minimal polynomial satisfy $|\theta_r| < 1$.*

We note that Lebesgue-almost all numbers are not Pisot. Now, to highlight the ambiguity of the Rajchman property for the singular measure, we'll expose examples of continuous singular measures respecting this property.

Example 2.2.1 (Self-similar measure)**.** *Consider for $\theta > 2$, μ_θ the distribution of $\sum_{n\in\mathbb{N}^*} \pm\theta^{-n}$ where signs are chosen with i.i.d probabilities $\frac{1}{2}$. Note that μ_θ has as characteristic function $\widehat{\mu_\theta} : k \in \mathbb{Z} \mapsto \prod_{n\in\mathbb{N}} \cos(2\pi\theta^{-n}k)$.*

μ_θ is Rajchman if and only if θ is not a Pisot number [10].

Additionally, for Lebesgue-almost all real $\theta > 2$, theses measures are singular and the Rajchman order $r(\mu_\theta) > 0$.

Example 2.2.2 (Rajchman measure with Liouville set as support ($Dio = +\infty$))**.** *Christian Bluhm* [3] *has built a Rajchman measure μ_∞ supported on Liouville numbers.*

According to the proposition 2.3, its order is $r(\mu_\infty) = 0$.

2.3. **Tori bundle.**

We study dynamical systems defined in the setting introduced in [16], tori bundles. These systems have local action-angle coordinates.

Definition 2.11 (Tori bundle)**.** *Let $(M, \mathscr{A})$ a $n \in \mathbb{N}^*$ dimensional $\mathscr{C}^1$ Lindelöf*[1] *manifold.*

Let $d \in \mathbb{N}^*$, (Ω, μ) a Borel space and π a continuous projection of Ω in M.

Ω is a (n,d) dimensional tori bundle if

(1) *locally, we have for charts U de $\mathscr{A}$ an homeomorphism : $\psi_U : \pi^{-1}(U) \to U \times \mathbb{T}^d$*
(2) *for all U in $\mathscr{A}, \pi_1 \circ \psi_U = \pi_{|U}$*

[1] Any open cover has a countable subcover.

Bundle of the form $M \times \mathbb{T}^d$ constitute already interesting examples. Note that we don't impose the affine property on tori bundle because not necessary for the sequel.

Definition 2.12 (Compatible flow). *Let $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ be a measure preserving dynamical system.*

The flow $(g_t)_{t\in\mathbb{R}}$ is a compatible flow with (Ω, μ) as a tori bundle if for all charts $U \in \mathscr{A}$ there exists $v_U \in \left(\mathbb{R}^d\right)^U$ measurable such that for $t \in \mathbb{R}, \psi_U \circ g_t \circ \psi_U^{-1}(x, y) = (x, y + tv_U(x))$.

We note $g_t^U : \psi_U \circ g_t \circ \psi_U^{-1}$

Definition 2.13 (Compatibles measure). *μ is a compatible measure if for all charts $U \in \mathscr{A}, \left(\mu_{|\pi^{-1}(U)}\right)_{\psi_U} = (\mu_\pi)_{|U} \otimes \lambda$*

3. Main result in discrete dynamical systems

In this case, we use the Rajchman property with the help of the corresponding Fourier series.

We will highlight the necessary path to show the presence of keplerian shear in this case. We get the main result in the discrete case here.

Theorem 3.1 (The main result). *The discrete dynamical system (Ω, μ, T) with Ω a tori bundle exhibits keplerian shear if and only if for all $\xi \neq 0_{\mathbb{Z}^d}$ and for all charts $U \in \mathscr{A}, m_{\xi,U}^{\mathbb{T}}$ is Rajchman with*

$$m_{\xi,U}^{\mathbb{T}} = \left(\left((\mu_\pi)_{|U}\right)_{\langle\xi|v_U(\cdot)\rangle - \lfloor\langle\xi|v_U(\cdot)\rangle\rfloor}\right)_{|\mathbb{T}\setminus\{0\}}.$$

Proof. Let's start with the direct implication. Let $\xi \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}$ and $U \in \mathscr{A}$. Let take

$$f_1 : x \in \pi^{-1}(U) \mapsto \mathbb{1}_{(\langle\xi|v_U(\cdot)\rangle - \lfloor\langle\xi|v_U(\cdot)\rangle\rfloor)^{-1}(\mathbb{T}\setminus\{0\})}(\pi(x))e^{2i\pi\langle\xi|\pi_2\circ\psi_U(x)\rangle}.$$

Let take $f_2 = f_1$. So

$$\forall n \in \mathbb{N}, \int_{\pi^{-1}(U)} \overline{f_1} \cdot f_2 \circ T^n d\mu(x) = \int_{\mathbb{R}} e^{2i\pi nz} dm_{\xi,U}^{\mathbb{T}}(z)$$

And with Birkhoff-Kakutani and keplerian shear

$$\int_{\mathbb{R}} e^{2i\pi nz} dm_{\xi,U}^{\mathbb{T}}(z) \xrightarrow[n\to+\infty]{} 0.$$

So

$$U \in \mathscr{A}, m_{\xi,U}^{\mathbb{T}} \text{ is Rajchman.}$$

Let threat the reciprocal implication. The weak$-*$ convergence of exponentials of Fourier series in $+\infty$ is equivalent to the Rajchman property of measures $m_{\xi,U}^{\mathbb{T}}$. We show by Fourier decomposition this implication. ☐

3.1. Already known results.

Definition 3.1 (Anisotropic Sobolev spaces). *Let $s > 0$ and $h : (x, y) \in \mathbb{R}^2 \mapsto \begin{cases} \left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} \; if\, y \neq 0 \\ 1\, else \end{cases}$.*

We define anisothropic Sobolev space of order $s > 0$ like this :

$$H^{s,0}(\mathbb{T}^2) := \left\{ f \in \mathbb{L}^2(\mathbb{T}^2) : \sum_{\xi\in\mathbb{Z}^d} \left|\widehat{f}(\xi)\right|^2 h^{2s}(\xi) \in \mathbb{R} \right\}$$

Proposition 3.1. *Considering this dynamical system $(\mathbb{T}^2, \lambda \otimes \lambda, T)$ with $T : (x, y) \in \mathbb{T}^2 \mapsto (x, y + x)$, we have the next result*

$$\forall s > 0, \forall (f_1, f_2) \in (H^{s,0}(\mathbb{T}^2))^2, \forall n \in \mathbb{N}^*, |\mathbb{E}_{\lambda\otimes\lambda}(Cov_n(f_1, f_2|\mathscr{I}))| \leq \frac{4^s}{n^{2s}} \|f_1\|_{H^{s,0}(\mathbb{T}^2)} \|f_2\|_{H^{s,0}(\mathbb{T}^2)}$$

3.2. **Results with another measure than Lebesgue measure.**

Definition 3.2 (Sobolev space on the torus)**.** *We will note for $s>0, d\in\mathbb{N}^*$ the Sobolev space*

$$H^s(\mathbb{T}^d) := \left\{ f\in\mathbb{C}^{\mathbb{T}^d} : \sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 \left(1+\|k\|_2^2\right)^s \in\mathbb{R} \right\}.$$

Definition 3.3. *We will note for $s>0, d\in\mathbb{N}^*$ and $f\in H^s(\mathbb{T}^d), C_f := \sup\left\{ \left|\widehat{f}(\xi)\right| \left(1+\|\xi\|_2^2\right)^{\frac{s}{2}} : \xi\in\mathbb{Z}^d \right\}$*

To work easier and also estimate convergence speeds, we will threat the next proposition.

Proposition 3.2. *Let the dynamical system $(\mathbb{T}^2, \mu\otimes\lambda, T)$ with the transvection function $T: (x,y)\in\mathbb{T}^2 \mapsto (x, y+x)$ yet and μ with $r>0$ as the Rajchman order.*
We get the next proposition :

$$\forall s>2, \exists\gamma\in\left[\min\left\{\frac{s}{2}-1, r\right\}, r\right], \forall(f_1,f_2)\in\left(H^s(\mathbb{T}^2)\right)^2, \exists C>0, \forall n\in\mathbb{N}^*, |\mathbb{E}_{\mu\otimes\lambda}(Cov_n(f_1,f_2|\mathscr{I}))| \leq \frac{C}{n^\gamma}$$

with γ optimal.

Proof. Let $s>2$. Let

$$(f_1,f_2)\in\left(H^s(\mathbb{T}^2)\right)^2.$$

Let $n\in\mathbb{N}^*$. But

$$\int_{\mathbb{T}^2} \overline{f_1}\left(f_2\circ T^n\right) d\left(\mu\otimes\lambda\right) = \sum_{\xi\in\mathbb{Z}^2} \widehat{f_2}(\xi) \int_{\mathbb{T}} e^{2i\pi n\xi_2 x} \left(e^{2i\pi\xi_1 x} \left(\int_{\mathbb{T}} \overline{f_1}(x,y) e^{2i\pi\xi_2 y} d\lambda(y) \right) \right) d\mu(x).$$

And

$$e^{2i\pi\xi_1 x} \left(\int_{\mathbb{T}} \overline{f_1}(x,y) e^{2i\pi\xi_2 y} d\lambda(y) \right) = \sum_{k\in\mathbb{Z}} \widehat{f_1}(k,\xi_2) e^{2i\pi(\xi_1-k)x}.$$

Let pose

$$g_{(\xi_1,\xi_2)} : x\in\mathbb{T} \mapsto e^{2i\pi\xi_1 x} \left(\int_{\mathbb{T}} \overline{f_1}(x,y) e^{2i\pi\xi_2 y} d\lambda(y) \right).$$

But

$$\forall(k,\xi_1,\xi_2)\in\mathbb{Z}^3, \left|\widehat{g_{(\xi_1,\xi_2)}}(k)\right|(1+k^2)^{\frac{s}{2}} = \left|\widehat{f_1}(\xi_1-k,\xi_2)\right|(1+k^2)^{\frac{s}{2}} \leq C_{f_1} \frac{(1+k^2)^{\frac{s}{2}}}{(1+\xi_2^2+(\xi_1-k)^2)^{\frac{s}{2}}}.$$

And

$$\mathbb{E}_{\mu\otimes\lambda}(Cov_n(f_1,f_2|\mathscr{I})) = \sum_{\xi\in\mathbb{Z}\times\mathbb{Z}^*} \widehat{f_2}(\xi) \int_{\mathbb{T}} e^{2i\pi n\xi_2 x} \left(e^{2i\pi\xi_1 x} \left(\int_{\mathbb{T}} \overline{f_1}(x,y) e^{2i\pi\xi_2 y} d\lambda(y) \right) \right) d\mu(x).$$

Let $\delta\in]0,1]$. We assume that

$$|\xi_1-k+n\xi_2| \leq \frac{(n|\xi_2|)^\delta}{4} \, et \, |n\xi_2-k| \leq \frac{(n|\xi_2|)^\delta}{2}.$$

So $|k| \geq n|\xi_2| - \frac{(n|\xi_2|)^\delta}{2} \geq \frac{n|\xi_2|}{2}$. And

$$|\xi_1| \geq n|\xi_2| - |k-n\xi_2| \geq n|\xi_2| - \frac{(n|\xi_2|)^\delta}{4} \geq \frac{3}{4} n|\xi_2|.$$

And $|\widehat{\mu}(\xi_1-k+n\xi_2)| \leq 1$. So we get

$$\sum_{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \, et \, |\xi_1-k+n\xi_2| \leq \frac{(n|\xi_2|)^\delta}{4} \, et \, |n\xi_2-k| \leq \frac{(n|\xi_2|)^\delta}{2}} \left|\widehat{f_2}(\xi_1,\xi_2)\right| \left|\widehat{f_1}(k,\xi_2)\right| \leq \frac{2^{6(s+1)}\zeta(2(s-\delta))C_{f_1}C_{f_2}}{3^s n^{2(s-\delta)}}.$$

Now, we suppose that

$$|\xi_1 - k + n\xi_2| \leq \frac{(n|\xi_2|)^\delta}{4} \, and \, |n\xi_2 - k| > \frac{(n|\xi_2|)^\delta}{2}.$$

So $|\xi_1| \geq |n\xi_2 - k| - \frac{(n|\xi_2|)^\delta}{4} > \frac{(n|\xi_2|)^\delta}{4}$. And $\left|\widehat{f}_1(k,\xi_2)\right| |\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq \frac{C_{f_1}}{(1+k^2+\xi_2^2)^{\frac{s}{2}}}$. So $\left|\widehat{f}_2(\xi_1,\xi_2)\right| \leq \frac{4^s C_{f_2}}{(1+\xi_1^2+\xi_2^2)^{\frac{s}{2}}} \leq \frac{4^s C_{f_2}}{|\xi_2|^{\frac{s}{2}(1+\delta)} n^{\frac{s}{2}\delta}}$.

So

$$\sum_{\substack{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \, et \; |\xi_1 - k + n\xi_2| \leq \frac{(n|\xi_2|)^\delta}{4} \\ et \, |n\xi_2 - k| > \frac{(n|\xi_2|)^\delta}{2}}} \frac{C_{f_1}\left|\widehat{f}_2(\xi_1,\xi_2)\right|}{(1+k^2+\xi_2^2)^{\frac{s}{2}}} \leq \frac{4^s\left(1+\zeta\left(\frac{s}{2}\right)\right)\zeta\left(\frac{s}{2}(1+\delta)-\delta\right)C_{f_1}C_{f_2}}{n^{\left(\frac{s}{2}-1\right)\delta}}.$$

But $s > 2$. So $\frac{s}{2}(1+\delta) - \delta > 1 \, et \, \frac{s}{2} - 1 > 0$. And then, we assume

$$|\xi_1 - k + n\xi_2| > \frac{(n|\xi_2|)^\delta}{4}.$$

So $|\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq \frac{4^r C_\mu}{(n|\xi_2|)^{r\delta}}$. So

$$\sum_{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \, et \, |\xi_1 - k + n\xi_2| > \frac{(n|\xi_2|)^\delta}{4}} \frac{C_{f_1}\left|\widehat{f}_2(\xi_1,\xi_2)\right|}{(1+k^2+\xi_2^2)^{\frac{s}{2}}} \leq \frac{4^r\left(1+\zeta\left(\frac{s}{2}\right)\right)^2 C_{f_1}C_{f_2}}{n^{r\delta}}.$$

Let pose

$$M(r,s) = \sup\left\{4^r\left(1+\zeta\left(\frac{s}{2}\right)\right)^2, 4^s\left(1+\zeta\left(\frac{s}{2}\right)\right)\zeta\left(\frac{s}{2}(1+\delta)-\delta\right), 2^{6(s+1)}\zeta(2(s-\delta))\right\}.$$

With a deep study on extrema, we get

$$|\mathbb{E}_{\mu\otimes\lambda}\left(Cov_n(f_1,f_2|\mathscr{I})\right)| \leq M(r,s)\left(\frac{C_{f_1}C_{f_2}}{n^{\min\left\{\frac{s}{2}-1,r\right\}}}\right).$$

Let note $\gamma > 0$ optimal convergence order optimal of correlations. So

$$\gamma \geq \min\left\{\frac{s}{2}-1, r\right\}.$$

But

$$\left(f : (x,y) \in \mathbb{T}^2 \mapsto e^{2i\pi y}\right) \in H^s(\mathbb{T}^2)$$

because $\mathscr{C}^\infty$. And

$$\left|\int_{\mathbb{T}^2} \overline{f}(x,y) f(T^n(x,y)) d\mu\otimes\lambda(x,y)\right| \leq \frac{C_\mu}{n^r}.$$

And r is the Rajchman order and then optimal. So

$$\gamma \leq r.$$

And then

$$\gamma \in \left[\min\left\{\frac{s}{2}-1, r\right\}, r\right].$$

And so finally

$$\forall s > 2, \exists \gamma \in \left[\min\left\{\frac{s}{2}-1, r\right\}, r\right], \forall (f_1,f_2) \in \left(H^s(\mathbb{T}^2)\right)^2, \exists C > 0, \forall n \in \mathbb{N}^*, |\mathbb{E}_{\mu\otimes\lambda}(Cov_n(f_1,f_2|\mathscr{I}))| \leq \frac{C}{n^\gamma}.$$

And γ is optimal. □

Remark 3.2.1. *The convergence speed is bounded by the order of Rajchman of the measure μ, which makes it unnecessary to take increasingly regular applications to observe convergence speeds greater than r. On the other hand, this is consistent with the result obtained with the Lebesgue measure because its Rajchman order is $+\infty$, which removes the maximum imposed by r for the speed of convergence and allows the observation of ever greater speeds of convergence when the regularity of the functions used increases.*

3.3. **$\sigma-$algebra and invariant functions by discrete flow.** Just like for the continuous flow case, we can identify the $\sigma-$algebra of invariant for the discrete flow using the Fourier series and orthogonality. In the regular case, we always have that the $\sigma-$algebra of invariants is negligible up to $\pi^{-1}(\mathscr{B}(M))$ under the conditions of the theorem 4.1.

Theorem 3.2. *We assume that (Ω, μ, T) exhibits Keplerian shear.*
$f \in \mathbb{L}^2_\mu(\Omega)$ is invariant according to T if and only if for all $U \in \mathscr{A}$,
$\exists (a_k)_{k\in\mathbb{Z}^d} \in \mathbb{L}^2_{(\mu_\pi)_{|U}}(U)^{\mathbb{Z}^d}, (\mu_\pi)_{|U} \otimes \lambda - a.a\,(x,y) \in U \times \mathbb{T}^d$,

$$f_{|\pi^{-1}(U)}\left(\psi_U^{-1}(x,y)\right) = \sum_{k\in\mathbb{Z}^d\cap\{v_U(x)\}^\perp} a_k(x) e^{2i\pi<k|y>}.$$

Proof. Let $f \in \mathbb{L}^2_\mu(\Omega)$ and $U \in \mathscr{A}$.

Let's prove the direct assertion. Suppose that $f \circ T = f\ \mu - p.p.$ Let $(x,y) \in U \times \mathbb{T}^d$. So

$$f(T(\psi_U^{-1}(x,y))) = f(\psi_U^{-1}(x,y)).$$

Namely $f(\psi_U^{-1}(x, y + v_U(x))) = f(\psi_U^{-1}(x,y))$. By Fourier series decomposition:

$$\sum_{k\in\mathbb{Z}^d} \widehat{f\circ\psi_U(x,\cdot)}(k) e^{2i\pi<k|y+v_U(x)>} = \sum_{k\in\mathbb{Z}^d} \widehat{f\circ\psi_U(x,\cdot)}(k) e^{2i\pi<k|y>}.$$

By uniqueness of the coefficients of a Fourier series

$$\forall k \in \mathbb{Z}^d, \widehat{f\circ\psi_U(x,\cdot)}(k)\left(e^{2i\pi<k|v_U(x)>} - 1\right) = 0.$$

We assume

$$\widehat{f\circ\psi_U(x,\cdot)}(k) = 0.$$

Conclusion reached! Now, we suppose that

$$\widehat{f\circ\psi_U(x,\cdot)}(k) \neq 0\, et\ (\mu_\pi)_{|U}\left((v_U^{-1}\circ <k|\cdot>)^{-1}(\mathbb{Z}^*)\right) = 0$$

Conclusion reached! And then

$$\widehat{f\circ\psi_U(x,\cdot)}(k) \neq 0\, et\ (\mu_\pi)_{|U}\left((v_U^{-1}((<k|\cdot>)^{-1}(\mathbb{Z}^*))\right) > 0.$$

But $\mathbb{Z}^*$ is countable. So

$$\exists j \in \mathbb{Z}^*, (\mu_\pi)_{|U}\left((v_U^{-1}\circ <k|\cdot>)^{-1}(\{j\})\right) > 0.$$

And then, we suppose that

$$x \in v_U^{-1}((<k|\cdot>)^{-1}(\{j\})).$$

So $e^{2i\pi<k|v_U(x)>} = 1$. So

$$\forall n \in \mathbb{N}, \int_{v_U^{-1}((<k|\cdot>)^{-1}(\{j\}))} e^{2i\pi n<k|v_U(x)>} d\,(\mu_\pi)_{|U}(x) = (\mu_\pi)_{|U}\left((v_U^{-1}((<k|\cdot>)^{-1}(\mathbb{Z}^*))\right) > 0.$$

By Keplerian shear, we have that $m^{\mathbb{T}}_{\xi,U}$ is Rajchman. So

$$\int_{v_U^{-1}((<k|\cdot>)^{-1}(\{j\}))} e^{2i\pi n<k|v_U(x)>} d\,(\mu_\pi)_{|U}(x) \xrightarrow[n\to+\infty]{} 0.$$

And so

$$0 > 0.$$

Impossible!

Let's prove the reciprocal. Suppose that f satisfies for all charts $U \in \mathscr{A}$

$$(\mu_\pi)_{|U} \otimes \lambda - a.a\,(x,y) \in U \times \mathbb{T}^d, f(\psi_U^{-1}(x,y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} a_k(x) e^{2i\pi<k|y>}.$$

Let $z \in \Omega$. So

$$\exists U \in \mathscr{A}, z \in \pi^{-1}(U).$$

And so

$$f(T(z)) = f(T(\psi_U^{-1}(\psi_U(z)))) = f(\psi_U^{-1}(\pi(z), \pi_2 \circ \psi_U(z) + v_U(\pi(z)))).$$

So

$$f(T(z)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi<k|\pi_2 \circ \psi_U(z) + v_U(\pi(z))>}.$$

By orthogonality of the terms

$$f(T(z)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi<k|\pi_2 \circ \psi_U(z)>}.$$

So

$$f(T(z)) = f(\psi_U^{-1}(\psi_U(z))) = f(z).$$

□

Remark 3.3.1. *We note that in the discrete case and in the continuous case, the invariant measurable functions have the same form, ie the same Fourier series decomposition in a dynamical system exhibiting Keplerian shear. Thanks to this consequence, we have the same invariant sets in the discrete case and in the continuous case. We then get the following lemma.*

Definition 3.4 (Orthogonal stability)**.** *For $U \in \mathscr{A}, A \in \mathscr{B}(U \times \mathbb{T}^d)$ is orthogonally stable if*

$$(\mu_\pi)_{|U} - a.a\,x \in U, A_{x\cdot} \overset{\lambda-a.s}{=} A_{x\cdot} + p_{\mathbb{T}^d}\left(\bigcap_{\xi \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \{\xi\}^\perp\right) \tag{3}$$

.

Lemma 3.1. *For $U \in \mathscr{A}$, a measurable $A \in \mathscr{B}(U \times \mathbb{T}^d)$,*

$$(3) \iff \left(\mathbb{1}_A(x,y) = \sum_{k \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \widehat{\mathbb{1}_{A(x,\cdot)}}(k) e^{2i\pi\langle\xi|y\rangle} \left((\mu_\pi)_{|U} \otimes \lambda\right) - p.s.\right)$$

Proof. Let's prove the direct implication. Let $A \in \mathscr{B}(U \times \mathbb{T}^d)$ invariant. So

$$\mathbb{1}_A(x,y) = \sum_{k \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \widehat{\mathbb{1}_{A(x,\cdot)}}(k) e^{2i\pi\langle\xi|y\rangle}\ (\mu_\pi)_{|U} \otimes \lambda - p.s. \tag{4}$$

Obviously we already have

$$\forall x \in U, A_{x\cdot} \subset A_{x\cdot} + p_{\mathbb{T}^d}\left(\bigcap_{\xi \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \{\xi\}^\perp\right).$$

Consider $(x,y)\in U\times\mathbb{T}^d$ which satisfies (4) and

$$y\in A_{x\cdot}+p_{\mathbb{T}^d}\left(\bigcap_{\xi\in\mathbb{Z}^d\cap\{v_u(x)\}^\perp}\{\xi\}^\perp\right).$$

Let take $k\in\mathbb{Z}^d\cap\{v_u(x)\}^\perp$ So

$$\exists(z,r)\in A_{x\cdot}\times p_{\mathbb{T}^d}\left(\{k\}^\perp\right), y=z+r.$$

We find

$$\mathbb{1}_A(x,y)=\sum_{k\in\mathbb{Z}^d\cap\{v_u(x)\}^\perp}\widehat{\mathbb{1}_A(x,\cdot)}(k)e^{2i\pi\langle\xi|z\rangle}=\mathbb{1}_A(x,z).$$

So $y\in A_x$. We conclude that

$$A_{x\cdot}\overset{\lambda-a.s}{=}A_{x\cdot}+p_{\mathbb{T}^d}\left(\bigcap_{\xi\in\mathbb{Z}^d\cap\{v_u(x)\}^\perp}\{\xi\}^\perp\right).$$

For the reciprocal implication, we easily notice that the measurables A satisfying (3) give the Fourier series decomposition got in (7). □

Proposition 3.3 (Identification of invariant sets by orthogonality with fixed x)**.** *Thanks to the lemma 3.1, we deduce that for $U\in\mathscr{A}$, a measurable $A\in\mathscr{B}(U\times\mathbb{T}^d)$ is flow-invariant $(g_t)_{t\in\mathbb{R}}$ (resp. T) if and only if for $(\mu_\pi)_{|U}-a.a$, $x\in U$,*

$$A_{x\cdot}=A_{x\cdot}+p_{\mathbb{T}^d}\left(\bigcap_{k\in\mathbb{Z}^d\cap\{v_U(x)\}^\perp}\{k\}^\perp\right).$$

That is, for $A\in\mathscr{B}(U\times\mathbb{T}^d)$, referring to (7),

$$(3)\iff A\in\mathscr{I}_U$$

4. Main result in continuous dynamical systems

As mentioned in the introduction, many resuts which guarantee keplerian shear were given, mainly in Damien Thomine work [16]. Our next result extends his Theorem 3.3 below.

Theorem 4.1 (Result in the regular case)**.** *Let $(\Omega,\mu,(g_t)_{t\in\mathbb{R}})$ be a compatible flow with a compatible measure on a tori (affine) bundle as in Subsection 2.3.*

Assume that:

1. $\mu\ll\lambda$,
2. *All velocity vectors v_U are $\mathscr{C}^1$,*
3. $\forall U\in\mathscr{A},\mu\left(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0_{\mathbb{Z}^d}\}}\{x\in U: d\langle\xi|v_U(x)\rangle=0\}\right)=0.$

Then the system exhibit keplerian shear. Moreover $\mathscr{I}=\left\{B\in\mathscr{B}(\Omega):\exists A\in\mathscr{B}(M),\mu\left(B\Delta\pi^{-1}(A)\right)=0\right\}$

The proof of this theorem uses Fourier transform, and relies on methods of differential geometry; In particular, the normal form of submersions. We can see that the push forward measures $m_{\xi,U}:=\left((\mu_\pi)_{|U}\right)_{\langle\xi|v_U(\cdot)\rangle}$ (see Figure 1) are absolutely continuous and verify Riemann-Lebesgue lemma. This last property, in other words, Rajchman property will allow lonely to get keplerian shear, it is even an equivalence. We have successfully abstract this property and obtained the keplerian shear under the Rajchman property. Using tools from measure theory (conditional

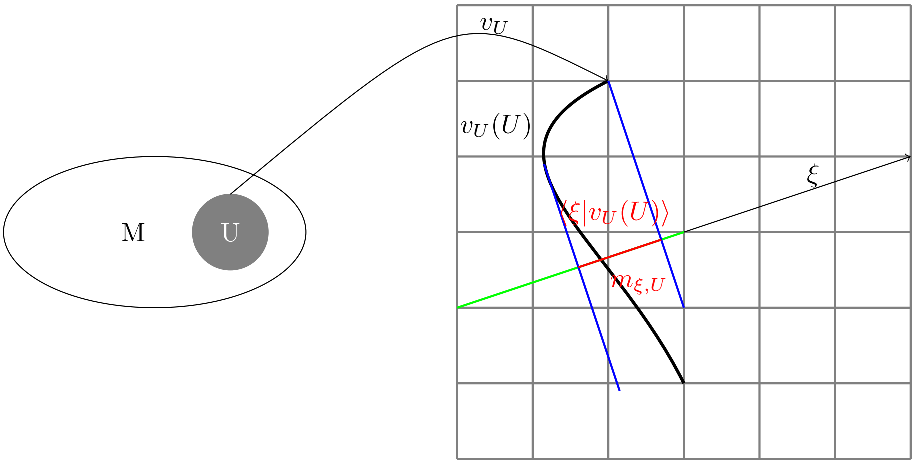

FIGURE 1. Push forward measure

expectation) allow to some extent to get rid of the regularity of the velocity. The next theorem is the main result in continuous case, and refers to the real Rajchman property.

Theorem 4.2 (The main result)**.** *Let* $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ *be a compatible flow with a compatible measure on a tori bundle as in Subsection 2.3.*

The dynamical system exhibit keplerian shear if and only if for all charts $U \in \mathscr{A}$*, for all non zero* $\xi \in \mathbb{Z}^d$*, the push forward measures* $(m_{\xi,U})_{|\mathbb{R}^*}$ *are Rajchman.*

Proof. Let's prove the direct implication. Let $U \in \mathscr{A}$ and non zero $\xi \in \mathbb{Z}^d$. Take

$$f_1 : z \in \pi^{-1}(U) \mapsto \mathbb{1}_{\langle \xi | v_U \rangle \neq 0}(\pi(z)) e^{2i\pi \langle \xi | \pi_2 \circ \psi_U(z) \rangle}.$$

We have thanks to the compatibility of μ

$$\int_\Omega \overline{f_1}(z) \cdot (f_1 \circ g_t(z))\, d\mu(z) = \int_\mathbb{R} \mathbb{1}_{\mathbb{R}^*}(s) e^{2i\pi t s} dm_{\xi,U}(s) = \int_\mathbb{R} e^{2i\pi t s} d\left(m_{\xi,U}\right)_{|\mathbb{R}^*}(s)$$

by push forward theorem. Moreover, with keplerian shear,

$$\int_\Omega \overline{f_1}(z) \cdot (f_1 \circ g_t(z))\, d\mu(z) \xrightarrow[t\to+\infty]{} \int_\Omega \overline{\mathbb{E}_\mu(f_1|\mathscr{I})} \mathbb{E}_\mu(f_1|\mathscr{I}) d\mu = 0$$

because by ergodic Birkhoff theorem

$$\mathbb{E}_\mu(f_1|\mathscr{I})\left(\psi_U^{-1}(x,y)\right) \overset{(\mu_\pi)_{|U}\otimes\lambda - a.s}{=} e^{2i\pi\langle \xi | y \rangle} \mathbb{1}_{\langle \xi | v_U \rangle \neq 0}(x) \lim_{T\to+\infty} \frac{1}{T} \int_0^T e^{2i\pi t \langle \xi | v_U(x) \rangle} dt = 0. \tag{5}$$

So $(m_{\xi,U})_{|\mathbb{R}^*}$ is Rajchman.

Let's prove the reciprocal implication. Let $(U_i)_{i\in I} \in \mathscr{A}^I$ be a countable partition of M modulo μ_π. Let

$$Y := \bigcup_{(j,\xi)\in I\times\mathbb{Z}^d \setminus \{0\}} \left\{(a\circ\pi) \cdot e^{2i\pi\left\langle \xi | \pi_2 \circ \psi_{U_j} \right\rangle} \in L^2_\mu\left(\pi^{-1}(U_j)\right) : a \in \mathbb{L}^\infty_\mu(U_j)\right\}.$$

Let $(i,j) \in I^2$,$(a_1, a_2) \in \mathbb{L}^\infty_\mu(U_i) \times \mathbb{L}^\infty_\mu(U_j)$ and $(\xi_1, \xi_2) \in \left(\mathbb{Z}^d\right)^2$. Take

$$f_l = (a_l \circ \pi) \cdot e^{2i\pi\left\langle \xi | \pi_2 \circ \psi_{U_j} \right\rangle}$$

for $l \in \{1,2\}$. When $i \neq j$, the supports are disjoint, and in this case

$$\mathbb{E}_\mu \left(Cov_t \left(f_1, f_2 | \mathscr{I} \right) \right) = 0.$$

The same happens when $\xi_1 \neq \xi_2$ by periodicity of complex exponentials. Finally, when $\xi_1 = \xi_2 = 0$, $\mathbb{E}_\mu \left(Cov_t \left(f_1, f_2 | \mathscr{I} \right) \right)$ is constant.

Suppose without loss of generality that $i = j$ and $\xi_1 = \xi_2 = \xi \neq 0$. Set $W_i^\xi = U_i \setminus (\langle \xi | v_{U_i} \rangle = 0)$. We have

$$\int_\Omega \overline{f_1} \cdot (f_2 \circ g_t) \cdot \mathbb{1}_{\langle \xi | v_{U_i} \rangle \neq 0} \circ \pi d\mu = \int_\mathbb{R} f(s) e^{2i\pi ts} d(m_{\xi, U_i})_{|\mathbb{R}^*}(s) \xrightarrow[t \to \pm\infty]{} 0$$

because $(m_{\xi,U_i})_{|\mathbb{R}^*}$ is Rajchman thus converges weakly-$*$ according to Lemma 2.1.

Moreover

$$\int_\Omega \overline{f_1} \cdot (f_2 \circ g_t) \cdot \mathbb{1}_{\langle \xi | v_{U_i} \rangle = 0} \circ \pi d\mu = \int_\Omega \overline{f_1} \cdot f_2 \cdot \mathbb{1}_{\langle \xi | v_{U_i} \rangle = 0} \circ \pi d\mu.$$

By proposition 2.4 in [16], we have that

$$f_2 \cdot \mathbb{1}_{\langle \xi | v_{U_i} \rangle = 0} \circ \pi = \mathbb{E}_\mu \left(f_2 | \mathscr{I} \right).$$

We obtains by totality of Y that

$$\forall (f_1, f_2) \in \left(\mathbb{L}^2_\mu(\Omega) \right)^2, \mathbb{E} \left(Cov_t \left(f_1, f_2 | \mathscr{I} \right) \right) \xrightarrow[t \to \pm\infty]{} 0.$$

□

Remark 4.0.1. *We can also note that the result generalises to infinite dimensional tori bundles $\mathbb{T}^\mathbb{N}$ with product topology.*

Remark 4.0.2. *When keplerian shear property holds, the decomposition of Radon-Nikodyme-Lebesgue of the image measures $m_{\xi,U}$ in theorem 4.2 reads as $(m_{\xi,U})_{ac} + (m_{\xi,U})_{sc} + (m_{\xi,U})_d$ with $(m_{\xi,U})_d = \alpha\delta_0$, and $\alpha \geq 0$. Recall that the absolutely continuous part satisfies the Rajchman property. The discrete part is concentrated on 0 because otherwise, a non-trivial periodicity would break the keplerian shear. The behavior of the singular part ν is not obvious, since it may be Rajchman or not, see subsection 2.2.4.*

Remark 4.0.3. *When all the $m_{\xi,U}$'s are of the form $(m_{\xi,U})_{ac} + \alpha\delta_0$, Theorem 4.2 immediately gives Keplerian shear.*

4.1. **$\sigma-$algebra and invariant function by continuous flow.** Under the conditions of Theorem 4.1, in the regular case, the invariant $\sigma-$algebra is just $\pi^{-1}(\mathscr{B}(M))$ modulo zero measure set. On the functional aspect, it amounts to say that a measurable function f is invariant by the flow if and only if it is $\mu-$a.s independent on the second variable on every chart U. However, we will first highlight the invariant functions and then the σ-algebra of invariant sets in order to compare with the regular case mentioned above. In the following theorem, we will use Fourier series to identify a characterization of invariance by an orthogonality property.

Proposition 4.1 (Orthogonality and Fourier characterization of invariant functions)**.** *Under the conditions of Theorem 4.2, a function $f \in \mathbb{L}^2_\mu(\Omega)$ is invariant according to $(g_t)_{t\in\mathbb{R}}$ if and only if for all $U \in \mathscr{A}, \exists (a_k)_{k\in\mathbb{Z}^d} \in \mathbb{L}^2_{(\mu_\pi)_{|U}}(U)^{\mathbb{Z}^d}, (\mu_\pi)_{|U} \otimes \lambda - a.a\, (x,y) \in U \times \mathbb{T}^d$,*

$$f_{|\pi^{-1}(U)}(\psi_U^{-1}(x,y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} a_k(x) e^{2i\pi < k|y>}.$$

In other words, for a chart $U \in \mathscr{A}$ and for $x \in U$ fixed, non zero Fourier coefficients have indices orthogonal with the vector $v_U(x)$.

Proof. Let $f \in \mathbb{L}^2_\mu(\Omega)$. Let's start with the direct implication. Suppose that

$$\forall t \in \mathbb{R}, f \circ g_t = f \, \mu - a.e.$$

Let $U \in \mathscr{A}$, $t \in \mathbb{R}$ and $(x, y) \in U \times \mathbb{T}^d$. Since

$$f(g_t(\psi_U^{-1}(x, y))) = f(\psi_U^{-1}(x, y))$$

we have

$$f(\psi_U^{-1}(x, y + tv_U(x))) = f(\psi_U^{-1}(x, y)).$$

By Fourier serie decomposition, this gives

$$\sum_{k \in \mathbb{Z}^d} \widehat{f \circ \psi_U(x, \cdot)}(k) e^{2i\pi <k|y+tv_U(x)>} = \sum_{k \in \mathbb{Z}^d} \widehat{f \circ \psi_U(x, \cdot)}(k) e^{2i\pi <k|y>}.$$

Hence, by uniqueness of Fourier serie coefficients,

$$\forall k \in \mathbb{Z}^d, \widehat{f \circ \psi_U(x, \cdot)}(k) \left(e^{2i\pi t <k|v_U(x)>} - 1 \right) = 0.$$

When k is such that $\widehat{f \circ \psi_U(x, \cdot)}(k) \neq 0$ we get $\forall t \in \mathbb{R}, t < k|v_U(x) > \in \mathbb{Z}$. So $< k|v_U(x) >= 0$, and then $k \in \{v_U(x)\}^\perp$. And so

$$(\mu_\pi)_{|U} \otimes \lambda - a.a \, (x, y) \in U \times \mathbb{T}^d, f(\psi_U^{-1}(x, y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} \widehat{f \circ \psi_U(x, \cdot)}(k) e^{2i\pi <k|y>}.$$

Let see the reciprocal implication. Let $t \in \mathbb{R}$. Suppose that f satisfies for all charts $U \in \mathscr{A}$

$$(\mu_\pi)_{|U} \otimes \lambda - a.a \, (x, y) \in U \times \mathbb{T}^d, f(\psi_U^{-1}(x, y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} a_k(x) e^{2i\pi <k|y>}.$$

Let $z \in \Omega$ and take $U \in \mathscr{A}$ such that $z \in \pi^{-1}(U)$. We have

$$f(g_t(z)) = f(g_t(\psi_U^{-1}(\psi_U(z)))) = f(\psi_U^{-1}(\pi(z), \pi_2 \circ \psi_U(z) + tv_U(\pi(z)))).$$

So

$$\begin{aligned} f(g_t(z)) &= \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi <k|\pi_2 \circ \psi_U(z) + tv_U(\pi(z))>} \\ &= \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi <k|\pi_2 \circ \psi_U(z)>}. \end{aligned}$$

by orthogonality. So

$$f(g_t(z)) = f(\psi_U^{-1}(\psi_U(z))) = f(z).$$

□

Remark 4.1.1. *We can note that in this proof, we did not use Rajchman property to identify invariant $\sigma-$algebra. This version of the proposition does not depend on keplerian shear, by opposition to the discrete case.*

Remark 4.1.2. *As a byproduct, the proposition gives an explicit form to the conditional expectation, with respect to the invariant σ-algebra, of a function $f \in \mathbb{L}^2_\mu(\Omega)$ locally by the next formula for $U \in \mathscr{A}$*

$$(\mu_\pi)_{|U} \otimes \lambda - a.a \, (x, y) \in U \times \mathbb{T}^d, \left(\mathbb{E}_\mu(f|\mathscr{I}) \circ \psi_U^{-1} \right)(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} (\widehat{f \circ \psi_U^{-1}})(x, \cdot)(k) e^{2i\pi <k|y>}.$$

Lemma 4.1 (Charts invariance). *For any $U \in \mathscr{A}$, $\pi^{-1}(U)$ is $(g_t)_{t \in \mathbb{R}}$-invariant.*

Proof. Let $U \in \mathscr{A}$ and $t \in \mathbb{R}$. We have

$$\mu - a.a\, z \in \pi^{-1}(U), \pi(z) = \pi_1 \circ \psi_U(z).$$

And

$$\mu - a.a\, z \in \pi^{-1}(U), (\psi_U \circ g_t)(z) = (\psi_U \circ g_t \circ \psi_U^{-1})(\pi_1 \circ \psi_U(z), \pi_2 \circ \psi_U(z)).$$

So

$$\mu - a.a\, z \in \pi^{-1}(U), \pi_1((\psi_U \circ g_t)(z)) = \pi(z)$$

because tori bundle property of Ω. Namely

$$\mu - a.a\, z \in \pi^{-1}(U), \pi(g_t(z)) = \pi(z).$$

□

Proposition 4.2. *The invariant σ-algebra is*

$$\mathscr{I} = \left\{ \bigcup_{U \in \mathscr{A}} B_U \in \mathscr{B}(\Omega) : (B_U)_{U \in \mathscr{A}} \in \prod_{U \in \mathscr{A}} \psi_U^{-1}(\mathscr{I}_U) \right\},$$

where for $U \in \mathscr{A}$, the local invariant $\sigma-$algebra on U is

$$\mathscr{I}_U := \left\{ C \in \mathscr{B}(U \times \mathbb{T}^d) : \mu_{\pi_{|U}} - a.a\, x \in U, \widehat{\mathbb{1}_C(x, \cdot)}^{-1}(\mathbb{C}^*) \subset \{v_U(x)\}^\perp \right\}. \tag{6}$$

Proof. We prove the direct inclusion. Let $A \in \mathscr{I}$. Let $U \in \mathscr{A}$ and set $C = \psi_U(A \cap \pi^{-1}(U))$. By proposition 4.1

$$\mathbb{1}_C(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} a_k(x) e^{2i\pi <k|y>} \; (\mu_\pi)_{|U} \otimes \lambda - a.s$$

which shows that $C \in \mathscr{I}_U$. Hence $A \cap \pi^{-1}(U) \in \psi_U^{-1}(\mathscr{I}_U)$, and finally

$$A = \bigcup_{U \in \mathscr{A}} A \cap \pi^{-1}(U).$$

Now, we prove the reciprocal inclusion.
Let $A \in \mathscr{B}(\Omega)$ s.t $\exists (B_U)_{U \in \mathscr{A}} \in \prod_{U \in \mathscr{A}} \psi_U^{-1}(\mathscr{I}_U), A = \bigcup_{U \in \mathscr{A}} B_U$. Then

$$\forall t \in \mathbb{R}, g_t^{-1}(A) = \bigcup_{U \in \mathscr{A}} g_t^{-1}(B_U).$$

Let $U \in \mathscr{A}$. By bijectivity of ψ_U, $\psi_U(B_U) \in \mathscr{I}_U$. Hence

$$(\mu_\pi)_{|U} \otimes \lambda - a.a\,(x, y) \in U \times \mathbb{T}^d, \mathbb{1}_{\psi_U(B_U)}(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} \widehat{\mathbb{1}_{\psi_U(B_U)}(x, \cdot)}(k) e^{2i\pi <k|y>},$$

which proves that $A \in \mathscr{I}$ by Proposition 4.1. □

Remark 4.1.3. *We can see that local invariants $A \in \mathscr{I}_U$ satisfies*

$$\mathbb{1}_A(x, y) = \sum_{k \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \widehat{\mathbb{1}_A(x, \cdot)}(k) e^{2i\pi \langle \xi | y \rangle} \; (\mu_\pi)_{|U} \otimes \lambda - a.s \tag{7}$$

To show the consistency of this result with earlier works, we have the next corollary.

Corollary 4.1. $\pi^{-1}(\mathscr{B}(M)) \subset \mathscr{I}$

Proof. Let $A = \pi^{-1}(B)$ for some $B \in \mathscr{B}(M)$. Let $U \in \mathscr{A}$. We have $A \cap \pi^{-1}(U) = \pi^{-1}(B \cap U)$, hence

$$\psi_U(A \cap \pi^{-1}(U)) = (B \cap U) \times \mathbb{T}^d \in \mathscr{I}_U.$$

Therefore $A \in \mathscr{I}$ by Proposition 4.1. □

Note that this inclusion is generally strict as shown in the following example.

Example 4.1.1. *Consider* $g_t = Id_{\mathbb{T}^2}$, $\Omega = \mathbb{T}^2$, $M = \mathbb{T}$.
Clearly $\mathscr{I} = \mathscr{B}(\mathbb{T}^2)$, *while* $\pi^{-1}(\mathscr{B}(\mathbb{T})) = \{A \times \mathbb{T} \in \mathscr{B}(\mathbb{T}^2) : A \in \mathscr{B}(\mathbb{T})\} \subsetneq \mathscr{I}$.

4.2. **Convergence speed with the real Rajchman property.** We are now interested by the speed of convergence of the conditional correlations. Next result shows that even in C^∞ regularity, the order of convergence is limitated by the Rajchman order.

Proposition 4.3 (Speed roof)**.** *Let* $\xi \neq 0_{\mathbb{Z}^d}$ *and* $U \in \mathscr{A}$. *For all* $\gamma > r\left((m_{\xi,U})_{|\mathbb{R}^*}\right)$, *there exists* $(f_1, f_2) \in (\mathscr{C}^\infty(\pi^{-1}(U)) \cap \mathbb{L}^2_\mu(\Omega))^2$ *such that the decay of conditional correlations is not faster than* $t^{-\gamma}$, *that is,*

$$\forall C > 0, \forall T > 0, \exists t > T, |\mathbb{E}_\mu(Cov_t(f_1 \cdot \mathbb{1}_U \circ \pi, f_2|\mathscr{I}))| > \frac{C}{t^\gamma}.$$

Proof. Let $U \in \mathscr{A}$, $\xi \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}$. Let $\gamma > r\left((m_{\xi,U})_{|\mathbb{R}^*}\right)$. Consider

$$f : z \in \Omega \mapsto e^{2i\pi<\xi|(\pi_2\circ\psi_U^{-1})(z)>}\mathbb{1}_{\pi^{-1}(U)}(z) \in \mathscr{C}^\infty(\pi^{-1}(U)) \cap \mathbb{L}^2_\mu(\Omega).$$

So

$$\int_\Omega \overline{f}(f \circ g_t) d\mu = \int_U e^{2i\pi t<\xi|v_U(x)>} d\left(\mu_\pi\right)(x) = \int_\mathbb{R} e^{2i\pi tz} dm_{\xi,U}(z).$$

By optimality of $r\left((m_{\xi,U})_{|\mathbb{R}^*}\right)$,

$$\forall C > 0, \forall T > 0, \exists t > T, \left|\int_\mathbb{R} e^{2i\pi tz} d\left(m_{\xi,U}\right)_{|\mathbb{R}^*}\right| > \frac{C}{t^\gamma}.$$

The result follows by definition of conditional correlations. □

Remark 4.2.1. *When* $\Omega \simeq M \times \mathbb{T}^d$, *we can take* f_1, f_2 *globally defined of class* $\mathscr{C}^\infty$ *in the proposition 4.3.*

4.3. **Speed of decay of conditional correlations for absolutely continuous measures.** We assume that $\Omega = M \times \mathbb{T}^d$ is the trivial bundle, endowed with an absolutely continuous measures. The speed of decay will depends on the regularity properties of the velocity vector v. We will use stationnary phase method [19] that allow us to evaluate in an optimal way oscillating integrals. The regularity of the velocity vector and the presence of critical points influences the convergence order.

Before this study, we recall that under mild assumption the set of critical points of the velocity vector is discrete.

Lemma 4.2 (Isolation of non-degenerated critical points)**.** *Let* M *be a finite dimensional* $\mathscr{C}^2$ *manifold. Let* $v \in \mathscr{C}^2(M, \mathbb{R})$. *Then every non-degenerated critical point of* v *is isolated.*

Proof. Let $x \in M$ a critical point of v. Let $U \in \mathscr{A}$ s.t $x \in U$. Suppose that x is not an isolated point. So there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of critical points of v convergent to x. Let $n \in \mathbb{N}$. We have $\nabla(v \circ \psi_U^{-1})_{\psi_U(x_n)} = 0$. A first order expansion gives

$$\nabla(v\circ\psi_U^{-1})_{\psi_U(x_n)} = \nabla(v\circ\psi_U^{-1})_{\psi_U(x)} + Hess(v\circ\psi_U^{-1})(\psi_U(x))(\psi_U(x_n), \cdot) + \|\psi_U(x_n) - \psi_U(x)\| h(\psi_U(x_n)),$$

where $h(y) \to 0$ as $y \to \psi_U(x)$. Thus

$$0 = Hess(v \circ \psi_U^{-1})(\psi_U(x))(\psi_U(x_n) - \psi_U(x), \cdot) + \|\psi_U(x_n) - \psi_U(x)\| h(\psi_U(x_n)).$$

So $\forall n \in \mathbb{N}, 0 = Hess(v \circ \psi_U^{-1})(\psi_U(x)) \left(\frac{\psi_U(x_n) - \psi_U(x)}{\|\psi_U(x_n) - \psi_U(x)\|}, \cdot \right) + h(\psi_U(x_n))$. By compactness of the sphere in finite dimension, we can extract a converging subsequence

$$\left(\frac{1}{\left\|\psi_U(x_{\sigma(n)}) - \psi_U(x)\right\|} (\psi_U(x_{\sigma(n)}) - \psi_U(x)) \right)_{n \in \mathbb{N}},$$

we get there exists for this one a limit y in the unit sphere centered in $\psi_U(x)$. By letting n tend towards $+\infty$, we have

$$Hess(v \circ \psi_U^{-1})(\psi_U(x))(y, \cdot) = 0.$$

Since $y \neq 0$ we get

$$det(Hess(v \circ \psi_U^{-1})(\psi_U(x))) = 0.$$

□

Theorem 4.3 (Stationary phase, e.g. [19])**.** *Let $\varphi \in \mathscr{C}^2(\mathbb{R}^n, \mathbb{R})$ with a unique critical point x_c. We suppose that x_c is non degenerated, in other words $det(Hess(x_c)) \neq 0$.*

Then, for any $a \in \mathscr{C}_0^1(\mathbb{R}^n, \mathbb{R})$

$$\forall t > 0, \int_{\mathbb{R}^n} e^{2i\pi t \varphi(x)} a(x) d\lambda(x) = \frac{a(x_c) e^{i\frac{\pi}{4}\left(\sum_{\lambda \in \sigma(Hess(x_c))} sgn(\lambda)\right)}}{t^{\frac{n}{2}} \sqrt{|det(Hess(x_c))|}} + O_{+\infty}\left(\frac{1}{t^n}\right)$$

In the following we say that a is a critical point of order q of a function f if for $1 \leq m \leq q$, $f^{(m)}(a) = 0$ and $f^{(q+1)}(a) \neq 0$.

Lemma 4.3 (Convergence order with singular points in dimension $(1, d)$ with $d \geq 2$)**.** *Suppose that $dim(M) = 1 \leq d$ and M is a compact manifold of class $\mathscr{C}^\infty$.*

Let $\xi \neq 0_{\mathbb{Z}^d}$ and $\ell \geq 2$.

We suppose that v is of class $\mathscr{C}^\ell$, that there exists a unique critical point of order $\ell - 1$ for $< \xi | v(\cdot) >$, and that all the other eventual critical points are of order strictly smaller.

Then

$$r\left(\widehat{\lambda}_{< \xi | v(\cdot) >}\right) = \frac{1}{\ell}.$$

Proof. M is compact Hausdorff, so the number of critical points for functions $< \xi | v >$ is finite. Let $(a_k)_{k \in [\![1, m]\!]}$ the family of critical point of $< \xi | v(\cdot) >$ with respectives orders $l_k - 1 \geq 1$, $1 \leq m < l_k$,

$$< \xi | v^{(m)}(a_k) >= 0 \, and \, < \xi | v^{(l_k)}(a_k) > \neq 0.$$

Let $(U_k)_{k \in [\![1, m]\!]}$ charts such that,

$$\forall k \in [\![1, m]\!], a_k \in U_k$$

We have

$$< \xi | v(\varphi_{U_k}^{-1}(x)) >=< \xi | v(a_k) > + \frac{(x - \varphi_{U_k}(a_k))^{l_k}}{l_k!} < \xi | v^{(l_k)}(a_k) > + (x - \varphi_{U_k}(a_k))^{l_k} h(x - \varphi_{U_k}(a_k)).$$

Let pose for $k \in [\![1, m]\!], w_k : x \in \mathbb{R} \mapsto (x - \varphi_{U_k}(a_k)) \left(\frac{< \xi | v^{(l_k)}(a_k) >}{l_k!} + h(x - \varphi_{U_k}(a_k)) \right)^{\frac{1}{l_k}}$. So $w_k' : x \in \mathbb{R} \mapsto \left(\frac{< \xi | v^{(l_k)}(a_k) >}{l_k!} + h(x - \varphi_{U_k}(a_k)) \right)^{\frac{1}{l_k}} + (x - \varphi_{U_k}(a_k)) \frac{h'(x - \varphi_{U_k}(a_k))}{\left(< \xi | v^{(l)}(a_k) > + h(x - \varphi_{U_k}(a_k))\right)^{\frac{1 - l_k}{l_k}}}$.

So, we get $w_k'(\varphi_{U_k}(a_k)) = \left(\frac{<\xi|v^{(l_k)}(a)>}{l_k!}\right)^{\frac{1}{l_k}} \neq 0$. We get a local reverse of w_k on a neighbourhood W_k of 0. Let $(V_k)_{k\in[\![1,\, m\,+\,m']\!]}$ a family of charts on M such that,

$$\forall k \in [\![1, m]\!], (V_k = w_k^{-1}(W_k) \subset U_k \, and \forall j \in [\![m+1, m']\!], a_k \notin V_k).$$

Let $(\psi_k)_{k\in[\![1,\, m\,+\,m']\!]}$ a partition of unity subordinated to $(V_k)_{k\in[\![1,\, m\,+\,m']\!]}$. We get immediately that for $k \in [\![1, m]\!]$, $\psi_k(a_k) = 1$. By local reverse, for $k \in [\![1, m]\!]$, we get that on W_k,

$$\forall x \in W_k, < \xi|v(\varphi_{U_k}^{-1}(w_k^{-1}(x))) >=< \xi|v^{(l_k)}(a_k) > +x^{l_k}.$$

So, for $k \in [\![1, m]\!]$,

$$\int_{\varphi_{U_k}^{-1}(w^{-1}(W_k))} e^{2i\pi t<\xi|v(x)>}\psi_k(x)d\widehat{\lambda(x)} = e^{2i\pi t<\xi|v(\varphi_{W_k}^{-1}(w^{-1}(a_k)))>}\int_{W_k} e^{2i\pi t x^{l_k}}\check{\psi}_k(x)d\lambda(x)$$

with

$$\check{\psi}_k : x \in W \mapsto \psi\left(\varphi_{U_k}^{-1}(w_k^{-1}(x))\right) J(\varphi_{U_k}^{-1})(w_k^{-1}(x))J(w_k^{-1}(x)).$$

So

$$\int_{\varphi_{U_k}^{-1}(w_k^{-1}(W_k))} e^{2i\pi t<\xi|v(x)>}\psi(x)d\widehat{\lambda(x)} \quad = e^{2i\pi t<\xi|v(\varphi_{U_k}^{-1}(w_k^{-1}(a_k)))>}\frac{1}{t^{\frac{1}{l_k}}}\int_{\mathbb{R}} e^{2i\pi x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{\frac{1}{l_k}}}\right)d\lambda(x).$$

And

$$\int_{\mathbb{R}} e^{2i\pi x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{\frac{1}{l_k}}}\right)d\lambda(x) = \left(\int_{\mathbb{R}} \frac{(l_k-1)(e^{2i\pi x^l}-1)}{x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{\frac{1}{l}}}\right)d\lambda(x) - \frac{1}{t^{\frac{1}{l_k}}}\int_{\mathbb{R}} \frac{(e^{2i\pi x^{l_k}}-1)}{x^{l_k-1}}\check{\psi}_k{}'\left(\frac{x}{t^{\frac{1}{l_k}}}\right)d\lambda(x)\right).$$

But

$$\forall \alpha > 0, \exists C > 0, \forall x \in \mathbb{R}, \left|\check{\psi}_k{}'(x)\right||1 + x^\alpha| \leq C.$$

And

$$\exists C_2 > 0, \forall x \in \mathbb{R}, \left|\frac{(e^{2i\pi x^{l_k}}-1)}{x^{l_k-1}}\right|\left|1 + x^{l_k-1}\right| \leq C_2.$$

And

$$\exists C_3 > 0, \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^*, \left|\frac{(l_k-1)(e^{2i\pi x^{l_k}}-1)}{x^l}\check{\psi}_k\left(\frac{x}{t^{\frac{1}{l_k}}}\right)\right|\left|1 + x^{l_k-1}\right| \leq C_3\|\psi\|_\infty.$$

By Lebesgue convergence theorem

$$\int_{\mathbb{R}} e^{2i\pi x^l}\check{\psi}_k\left(\frac{x}{t^{\frac{1}{l_k}}}\right)d\lambda(x) \xrightarrow[t\to\pm\infty]{} \check{\psi}_k(0)\int_{\mathbb{R}} \frac{(l_k-1)(e^{2i\pi x^{l_k}}-1)}{x^{l_k}}d\lambda(x).$$

For $k \in [\![m+1, m+m']\!]$, by Greene formula, we get

$$\int_{\varphi_{U_k}^{-1}(V_k)} e^{2i\pi t<\xi|v(x)>}\psi_k(x)d\widehat{\lambda(x)} \in O_{\pm\infty}\left(\frac{1}{t}\right).$$

Thus,

$$\int_M e^{2i\pi<\xi|v(x)>}d\widehat{\lambda(x)} = \sum_{k=1}^m \frac{1}{t^{\frac{1}{l_k}}}\left(J_{\varphi_{U_k}^{-1}}(\varphi_{U_k}(a_k))\left(\frac{l_k!}{<\xi|v^{(l_k)}(a_k)>}\right)^{\frac{1}{l_k}} I_{l_k} + o_{\pm\infty}(1)\right) + O_{\pm\infty}\left(\frac{1}{t}\right)$$

with $I_l := \int_{\mathbb{R}} \frac{e^{2i\pi x^l}-1}{x^l}d\lambda(x)$. A simple analysis shows that I_l does not vanish[2].

[2]For l even $I_l \in (-\infty, 0)$ trivially, while for l odd we have $I_l \in i(0, \infty)$, using a decomposition of the integral on intervals $[2n, 2n+2]$.

By hypothesis, there exists just one critical a point with maximal order ℓ, hence

$$\int_M e^{2i\pi<\xi|v(x)>}d\widehat{\lambda(x)} = \frac{1}{t^{\frac{1}{\ell}}}\left(J_{\varphi_{U_j}^{-1}}\left(\varphi_{U_j}(a)\right)\left(\frac{\ell!}{<\xi|v^{(\ell)}(a)>}\right)^{\frac{1}{\ell}} I_j + o_{\pm\infty}(1)\right).$$

Then,

$$r\left(\widehat{\lambda}_{<\xi|v(\cdot)>}\right) = \frac{1}{\ell}.$$

□

We get then the next proposition.

Proposition 4.4. *Under the hypothesis of lemma 4.3, we get that the order of decay of correlation γ with $f_1, f_2 \in \mathscr{C}^\infty(\Omega)$ satisfies*

$$\gamma \leq \frac{1}{\ell}.$$

Proof. In proof of lemma 4.3, you can take $f_1 = e^{2i\pi<\xi|v(\cdot)>}$ and $f_2 = \psi_j$ with j the index of the chart containing the critical point a with a maximal order ℓ. □

We can now threat the general case.

The following lemma is obtained with the theorem 7.5 p.226 in [1].

Lemma 4.4 (Convergence order around a singular points with analytical functions)**.** *We will use notations and results of* [1]*.*

We place ourselves in $\mathbb{R}^n$*.*

Let $\xi \neq 0_{\mathbb{Z}^d}$*.*

Let v be such that $<\xi|v>$ is analytic with 0 a critical point of multiplicity $m \in \mathbb{N}^*$

Then there exists $\alpha \in \mathbb{Q}_+^*$ and $j \in \mathbb{N}$ in a compact neighbourhood V of 0 such that

$$\forall\varphi \in \mathscr{D}_V(\Omega), \exists C \in \mathbb{R}^*, \frac{\int_{\mathbb{R}^n} e^{2i\pi t<\xi|v(x)>}\varphi(x)d\lambda(x)}{\frac{C(\ln(t))^j}{t^\alpha}} \xrightarrow[t\to+\infty]{} 1.$$

Definition 4.1 (Logarithmic convergence order)**.** *Consider a probability measure ν on* $(\mathbb{R}, \mathscr{B}(\mathbb{R})$

We call logarithmic convergence order of ν the value

$$rl(\nu) := \inf\left\{\beta > 0 : \exists C > 0, \forall t > 0, \left|\widehat{\mu(t)}\right| \leq \frac{C|\ln(t)|^\beta}{t^{r(\nu)}}\right\}.$$

Next result may be obtained by a partition of unity and the previous lemma.

Proposition 4.5 (Convergence order with singular points in dimension (n, d) with $d \geq 2$)**.** *Suppose that M is analytic. Let $\xi \neq 0_{\mathbb{Z}^d}$. Suppose that v such that $<\xi|v>$ is analytic. We have*

$$\exists(j, \alpha) \in \mathbb{N} \times \mathbb{Q}_+^*, \left(r\left(\lambda_{<\xi|v(\cdot)>}\right), rl\left(\lambda_{<\xi|v(\cdot)>}\right)\right) = (\alpha, j).$$

Corollary 4.2. *Let v be a $\mathscr{C}^2$ function. If there exists only one critical points $a \in M$ and it satisfies*

$$\exists\xi \neq 0_{\mathbb{Z}^d}, (\nabla(<\xi|v(\cdot)>)(a) = 0 \, and \, det(Hess <\xi|v(\cdot)> (a)) \neq 0)$$

then $r(\widehat{\lambda}_{<\xi|v(\cdot)>}) = \frac{n}{2}$.

Proof. Let $\xi \neq 0_{\mathbb{Z}^d}$. Applying the stationary phase theorem on respectives charts (U, φ_U) of $\mathscr{A}$ containing at most just one critical point $a \in U$ and test functions ψ on U, we get

$$\int_M e^{2i\pi t<\xi|v(x)>}\psi(x)d\widehat{\lambda}(x) = \int_{\mathbb{R}^n} e^{2i\pi t<\xi|v(\varphi_U^{-1}(x))>}\psi\left(\varphi_U^{-1}(x)\right)J(\varphi_U^{-1})(x)d\lambda(x)$$
$$= \frac{\psi\left(\varphi_U^{-1}(a)\right)e^{i\frac{\pi}{4}\left(\sum_{\lambda\in\sigma(Hess(a))} sgn(\lambda)\right)}}{t^{\frac{n}{2}}\sqrt{|det(Hess(a))|}} + O_{+\infty}\left(\frac{1}{t^n}\right).$$

By partition of unity we obtain

$$\exists C\in\mathbb{R}, \int_M e^{2i\pi<\xi|v(x)>}d\widehat{\lambda}(x) = \frac{C}{t^{\frac{n}{2}}} + O_{+\infty}\left(\frac{1}{t^n}\right).$$

□

Remark 4.3.1. *Note that the result is consistent with the case $n=1$.*

We recall the definition of Damien Thomine [16]

Definition 4.2 (Anisotropic Sobolev space on $\mathbb{R}\times\mathbb{T}$)**.** *Let $s\geq 0$. Let $h : x\in\mathbb{R}\mapsto\sqrt{1+x^2}$. Let*

$$H^{s,0}(\mathbb{R}\times\mathbb{T}) := \left\{ f\in\mathbb{L}^2(\mathbb{R}\times\mathbb{T}) : \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|\widehat{f}(x,k)\right|^2 h^{2s}(x)d\lambda(x)\right\}.$$

Proposition 4.6. *Consider $(\mathbb{T}^2,\mu\otimes\lambda,(g_t)_{t\in\mathbb{R}})$ such that $r=r(\mu)>0$ and for $(x,y)\in\mathbb{T}^2, g_t(x,y)=(x,y+tx)$. Let $s>\frac{1}{2}$. Let for $\varepsilon\in\left]0,\frac{1}{2s}\right[, q_\varepsilon := \min\{s(1-\varepsilon), r(\mu)\}$. Then, we get for all $\varepsilon\in\left]0,\frac{1}{2s}\right[$, there exists $C_\varepsilon>0$ such that $\forall t>0$,*

$$|\mathbb{E}_{\mu\otimes\lambda}(Cov_t(f_1,f_2|\mathscr{I}))| \leq \frac{C_\varepsilon\|f_1\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}\|f_2\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}}{t^{q_\varepsilon}}$$

and if $\gamma>0$ denotes the convergence order on $H^{s,0}(\mathbb{R}\times\mathbb{T})$, we get $\min\left\{s-\frac{1}{2}, r(\mu)\right\}\leq\gamma$. Moreover if $supp(\mu)$ is compact then $\gamma\leq r(\mu)$.

Proof. Let $s>\frac{1}{2}$ and $(f_1,f_2)\in(H^{s,0}(\mathbb{R}\times\mathbb{T}))^2$. Then by Cauchy-Schwarz inequality and Parseval

$$\begin{aligned}|\mathbb{E}_{\mu\otimes\lambda}(Cov_t(f_1,f_2|\mathscr{I}))| &= \left|\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2}\overline{\widehat{f_1}}(x,k)\widehat{f_2}(y,k)\widehat{\mu}(kt-(x+y))d\lambda(x,y)\right| \\ &\leq S(t)\|f_1\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}\|f_2\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}\end{aligned}$$

with

$$S(t) := \sup_{k\in\mathbb{Z}^*} C_\mu\left(\int_{\mathbb{R}^2}\frac{1}{(1+|kt-(x-y)|)^{2r(\mu)}h^{2s}(x)h^{2s}(y)}d\lambda(x,y)\right)^{\frac{1}{2}}.$$

We can consider $k=1$ considering kt instead of t. Let $D_1(t) := \left\{(x,y)\in\mathbb{R}^2 : |t-(x-y)|\leq\frac{t}{2} \textit{ and } |y|\leq\frac{t}{4}\right\}$, $D_2(t) := \left\{(x,y)\mathbb{R}^2 : |t-(x-y)|\leq\frac{t}{2} \textit{ and } |y|>\frac{t}{4}\right\}$ and $D_3(t) := \left\{(x,y)\mathbb{R}^2 : |t-(x-y)|>\frac{t}{2}\right\}$.

Then for all $\varepsilon\in\left]0,\frac{1}{2s}\right[$

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{1}{(1+|t-(x-y)|)^{2r(\mu)}h^{2s}(x)h^{2s}(y)}d\lambda(x,y) &= \int_{D_1(t)}\frac{1}{(1+|t-(x-y)|)^{2r(\mu)}h^{2s}(x)h^{2s}(y)}d\lambda(x,y) \\ &+\int_{D_2(t)}\frac{1}{(1+|t-(x-y)|)^{2r(\mu)}h^{2s}(x)h^{2s}(y)}d\lambda(x,y) \\ &+\int_{D_3(t)}\frac{1}{(1+|t-(x-y)|)^{2r(\mu)}h^{2s}(x)h^{2s}(y)}d\lambda(x,y) \\ &\leq \frac{8}{t^{2s(1-\varepsilon)}}\int_{\mathbb{R}^2}h^{-2s}(x)h^{-2s\varepsilon}(y)d\lambda(x,y) \\ &+\frac{2}{t^{2r(\mu)}}\int_{\mathbb{R}^2}h^{-2s}(x)h^{-2s}(y)d\lambda(x,y)\end{aligned}$$

Then,

$$\forall\varepsilon\in\left]0,\frac{1}{2s}\right[, S(t)\leq\frac{8C_\mu}{t^{q_\varepsilon}}\left(\int_{\mathbb{R}}h^{-2s}(x)d\lambda(x)\int_{\mathbb{R}}h^{-2s\varepsilon}(x)d\lambda(x)\right)^{\frac{1}{2}}$$

with $q_\varepsilon = \min\{s(1-\varepsilon), r(\mu)\}$. Finally,

$$\forall \varepsilon \in \left]0, \frac{1}{2s}\right[, \exists C_\varepsilon > 0, \forall t > 0, |\mathbb{E}_{\mu\otimes\lambda}(Cov_t(f_1, f_2|\mathscr{I}))| \leq \frac{C_\varepsilon}{t^{q_\varepsilon}}.$$

And for $\gamma > 0$ the convergence order on $H^{s,0}(\mathbb{R}\times\mathbb{T})$, we get $\min\left\{s-\frac{1}{2}, r(\mu)\right\} \leq \gamma$ and when $supp(\mu)$ is compact, $\gamma \leq r(\mu)$. □

Remark 4.3.2. *When $(f_1, f_2) \in (H^{s,0}(\mathbb{R}\times\mathbb{T}))^2$ with $s \geq \frac{r(\mu)+1}{2}$, we get that $\gamma = r(\mu)$ as optimal order of convergence when $supp(\mu)$ is compact.*

Remark 4.3.3. *For $s > \frac{1}{2}$, for all integer $k > s + \frac{3}{2}$, $\mathscr{C}_c^k(\mathbb{R}\times\mathbb{T}) \subset H^{s,0}(\mathbb{R}\times\mathbb{T})$, and then, for $k > s+\frac{3}{2}$ the convergence order $\gamma > 0$ for $\mathscr{C}_c^k(\mathbb{R}\times\mathbb{T})$ is such that $\gamma \in \left[\min\left\{s-\frac{1}{2}, r(\mu)\right\}, r(\mu)\right]$ when $supp(\mu)$ is compact.*

Remark 4.3.4. *When $\mu = \lambda_v$ with $v \in \mathscr{C}^\infty$ with a unique critical point of order $l-1$ with $l \geq 3$, then $\gamma = r(\mu) = \frac{1}{l}$.*

5. Flow on compact Lie group bundle

In this part, we extend the notion of Keplerian shear in a more general framework allowing us to cover other cases in which we are not in torus bundles as in the previous sections. The main case we want to cover is the case of Lie group bundles with the non-abelian fibration Lie group. It is precisely this point of non-commutativity which announces the most severe breaking point between the work carried out previously in the article and those which will follow. In this section, however, the aim will be to come back to a case analogous to the classic case seen previously in the article in order to show the fact that the notion exposed in this part is a generalization of that exposed previously. We maintain the properties of compactness, of Lie and of connectedness to maintain properties on the fibration group allowing us to make the analogy with the previous cases which used the torus and also to define a continuous flow/a $\mathbb{R}-$action easily.

5.1. Tools used in connected-compact Hausdorff Lie group bundle.

Definition 5.1. *Let $(M, \mathscr{A})$ be a of Lindelöf manifold of dimension $n \in \mathbb{N}^*$ of class $\mathscr{C}^1$.*

Let (Ω, μ) be a Borelian space, G a connected compact Lie group and π a continuous map from Ω into M.

Ω is a connected compact Lie group bundle if

1. *locally, we have for the charts U of $\mathscr{A}$ a homeomorphism $\psi_U : \pi^{-1}(U) \to U \times G$*
2. *for all U in $\mathscr{A}$, $\pi_1 \circ \psi_U = \pi_{|U}$*

To define the continuous flow, we are going to use the exponential defined on the corresponding Lie algebra of the group. Here the connectedness and the compactness of the group provide good properties to the exponential, allowing us to work in setting similar to the previous one. We will use a local coordinate system, namely coordinates of the second kind, that will make it possible to fiberize the group G as a torus.

Let us consider a basis $(X_j)_{j\in[\![1,\,d]\!]}$ of $Lie(G)$ with $d \in \mathbb{N}^*$ the dimension of $Lie(G)$. We can get a coordinate system in a neighbourhood of the unit 1_G.

Proposition 5.1 (Coordinate systems of the second kind, e.g. [4]). *There exists a neighborhood V of the neutral element 1_G such that all the elements $a \in G$ are written in a unique way as follows*

$$a = \prod_{j=1}^{d} exp\,(x_j X_j) \quad with \;\; (x_j)_{j\in[\![1,\,d]\!]} \in \mathbb{R}^{[\![1,\,d]\!]}$$

(See Figure 2)

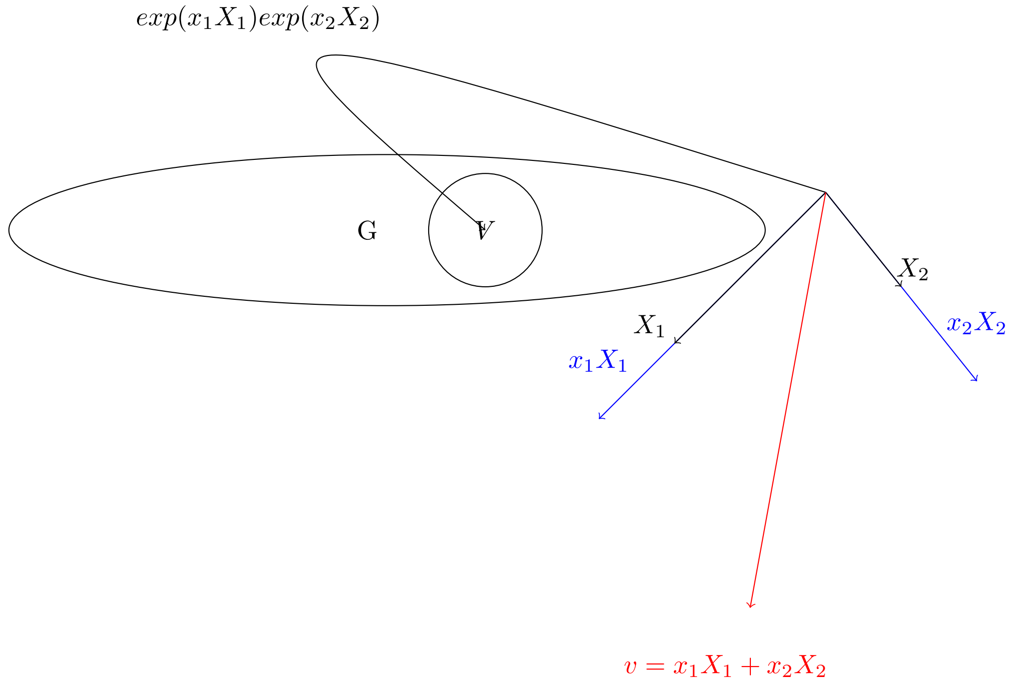

Figure 2. Coordinate system of the Second kind in Lie algebra

Now, we explain why we assume that the group is compact Hausdorff and connected. Compacity ensures the surjectivity of the exponential over a certain domain. More precisely, we use the following result.

Theorem 5.1. *The compactness of G implies the surjectivity of the exponential on the connected component of the neutral element of G.*

By considering the neighborhood of the neutral element on which the points are written in a unique way with the coordinate system of the second kind, we can with all the right translations of this neighborhood cover the whole group G. By compacity, we can extract a finite subcover from it, which allows us to approach more easily the properties already encountered in the case of tori bundles.

The fact that G is connected complement the previous property. The only connected component which will necessarily be that of the neutral element, so that the exponential become surjective over the whole group G.

5.2. **Main properties of the flow on the Lie group.** This step enables us to identify the possible flows on the Lie group. This will give us a more simplified approach to the dynamic system under study.

Theorem 5.2. *All flows $(g_t)_{t\in\mathbb{R}}$ on G satisfying for all $y, z \in G, g_t(yz) = g_t(y)z$ are of the form $(g_t(x, y) \mapsto exp(tv)x)$ with $g_t(x, y) \mapsto \exp(tv)x)$.*

We can then continue by defining the notion of compatible flow.

Definition 5.2 (Compatible flow)**.** *Let $(g_t)_{t\in\mathbb{R}}$ a flow on (Ω, μ)*
$(g_t)_{t\in\mathbb{R}}$ is a compatible flow iff :

(1) *There exists for all* $U \in \mathscr{A}$ *a measurable function* $v_U : U \to G$
(2) $\forall U \in \mathscr{A}, \forall (x, y) \in U \times G, \psi_U \circ g_t \circ \psi_U^{-1}(x, y) = (x, \exp(tv_U(x))y)$

Definition 5.3 (Compatible measure)**.** *Consider the notations previously established with the connected compact Lie group bundle.*
Let $U \in \mathscr{A}$*. Let pose* $\mu^U := \mu_{\psi_U}$*. And let pose* $\mu' := \mu_\pi$*.* μ *on* Ω *is compatible iff*

$$\mu^U_{|\pi^{-1}(U)} = \mu'_{|U} \otimes \mathscr{H} \, with \, \mathscr{H} \, the \, Haar \, measure \, on \, G$$

The next theorem written in paragraph 2 of the article by Antoine Delzant [7] allow us to come back to an analogous case with torus.

Theorem 5.3 (Isomorphism with torus)**.** *Any compact, connected and abelian Lie group is isomorphic to a torus.*

Then, we use the following theorem ensuring that we have the properties of Lie on the subgroups of G to get tori.

Theorem 5.4 (Lie subgroup)**.** *Any closed subgroup of* G *admits a group structure of Lie and* $Lie(H)$ *is a sev of* $Lie(G)$

Definition 5.4 (Flow orbit on the Lie group)**.** *We call for the flow previously defined the orbital group of a direction* $v \in Lie(G)$ *the set*

$$H_v := \overline{\{\exp(tv) \in G : t \in \mathbb{R}\}}$$

H_v *is an abelian Lie group, compact and connected and so isomorphic to a torus.*
Let d_v *be the dimension of this torus.*
Note $\chi_v : H_v \to \mathbb{T}^{d_v}$ *the isomorphism*

With the flow orbit, we can make a semi-fibration on Lie group to use torus property as in the previous theorem in the tori bundle case.

We consider $v \in Lie(G)$. Consider then the orbital group in the direction v. H_v has a group structure of Lie by observing the previous theorem. We consider a basis $(X_j^v)_{j \in [\![1,\, d_v]\!]}$ of $Lie(H_v)$ that we complete to make it a basis $(X_j^v)_{j \in [\![1,\, d]\!]}$ of $Lie(G)$. Consider the neighborhood V of the local coordinate system of the second kind associated with $(X_j^v)_{j \in [\![1,\, d]\!]}$.

We cover G with a finite family of elements $(g_k)_{k \in [\![1,\, l]\!]} \in G^{[\![1,\, l]\!]}$ s.t

$$G = \bigcup_{k=1}^{l} V g_k \tag{8}$$

Let $(W_k)_{k \in [\![1,\, l]\!]}$ s.t $W_1 = V g_1$ and

$$\forall k \in [\![2, l]\!], W_k = V g_k \setminus \left(\bigcup_{j \in [\![1,\, k-1]\!]} V g_j \right). \tag{9}$$

Let $\varphi : G^2 \to G$ defined by $\varphi(x, y) = xy$. By uniqueness of the coordinate system in V of the second kind we can define

$$\begin{array}{rccl} \psi_k^v : & V g_k & \to & Lie(H_v) \times G \\ & x & \mapsto & \left(\sum\limits_{k \in [\![1,\, d_v]\!]} x_j^{(v,k)} X_j^v, \prod\limits_{k \in [\![d_v+1,\, d]\!]} \exp(x_j^{(v,k)} X_j^v) g_k \right). \end{array}$$

Since the orbits are abelian subgroups we have

$$\forall k \in [\![1, l]\!], \forall x \in Vg_k, \forall v \in Lie(G), \prod_{k \in [\![1,\, d_v]\!]} \exp(x_j^{(v,k)} X_j^v) = \exp\left(\sum_{k \in [\![1,\, d_v]\!]} x_j^{(v,k)} X_j^v\right)$$

Let $\phi_t^v : Lie(G) \times G \to Lie(G) \times G$ defined by $\phi_t^v(\theta, y) = (\theta + tv, y)$ and $\eta : Lie(G) \times G \to G^2$ defined by $\eta(\theta, y) = (\exp(\theta), y)$. We immediately get

$$\forall k \in [\![1, l]\!], \forall x \in Vg_k, (\varphi \circ \eta \circ \phi_t^v \circ \psi_k^v)(x) = \exp(tv)x.$$

We finally set $\check{\varphi} := \varphi \circ \eta$.

5.3. **Keplerian shear on compact Lie group bundle.** We can now study the Keplerian shear problem.

Let pose for $U \in \mathscr{A}$ and $y \in G, w_U = \chi_{(v_U(\cdot),y)} \circ v_U(\cdot)$

Let pose for nonzero $\xi \in \mathbb{Z}^d$ and $U \in \mathscr{A}$, $\nu^{(\xi,U)} := \mu'_{|U}\ {}_{\langle \xi | w_U(\cdot) \rangle}$

Theorem 5.5. *The dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits keplerian shear iff for all nonzero $\xi \in \mathbb{Z}^d$, and for all $U \in \mathscr{A}$, $\nu^{(\xi,U)}_{|\mathbb{R}^*}$ is Rajchman*

The plan of the demonstration will initially be to come back to a situation very similar to that of the torus bundle by relying on the maximal torus. To do this, we are going to use the orbit in the vector $v_U(x)$ in the Lie algebra in order to identify an abelian subgroup which is isomorphic to a torus. The second step will be for a map $U \in \mathscr{A}$ and $x \in U$ fixed to work on the sections of the Lie group G by the translations of the orbit of $v_U(x)$. Once immersed in this configuration provided by Lemma 5.1, we will be able to carry out the same reasoning as in the torus in order to arrive at the property of Rajchman of $\nu^{(\xi,U)}_{|\mathbb{R}^*}$. In order to clarify and highlight the configuration by the orbits, we will show an equality allowing us to realize this link.

Lemma 5.1. *Let $U \in \mathscr{A}$ and $(f_1, f_2) \in \left(\mathbb{L}^2_{\mu'_{|U} \otimes \mathscr{H}}(U \times G)\right)^2$. We define for $t \in \mathbb{R}$*

$$b_t(f_1, f_2) := \int_{U \times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x, y) d\mu'_{|U} \otimes \mathscr{H}(x, y)$$

Then

$$b_t(f_1, f_2) = \sum_{k \in [\![1,\, l]\!]} \int_U \left(\int_{\mathbb{T}^{d_{v_U(x)}} \times W_k} \overline{\check{f}_1}(x, z + tw_U(x), r)) \check{f}_2(x, z, r)) dm'_{(x,k)}(z, r)\right) d\mu'_{|U}(x) \tag{10}$$

with

$$\check{f}_1 : (x, z, r) \mapsto f_1(x, \varphi(\chi^{-1}_{v_U(x)}(z), r)) \text{ and } \check{f}_2 : (x, z, r) \mapsto f_2(x, \varphi(\chi^{-1}_{v_U(x)}(z), r)).$$

and $m'_{(x,k)} = \left(\chi_{v_U(x)} \circ \exp\right)_* m_{(x,k)}$ is the push forward measure obtained by transfer theorem, where $m_{(x,k)} = \left(\psi_k^{v_U(x)}\right)_* \mathscr{H}$ is the push forward measure associated.

Proof. Let non-zero $\xi \in \mathbb{Z}^d$. Let $U \in \mathscr{A}$ and $(f_1, f_2) \in \left(\mathbb{L}^2_{\mu'_{|U} \otimes \mathscr{H}}(U \times G)\right)^2$. We have the following expansion

$$\begin{aligned}
b_t(f_1, f_2) &:= \int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x, y) d\mu'_{|U} \otimes \mathscr{H}(x, y) \\
&= \int_U \int_G \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x, y) d\mathscr{H}(y) d\mu'_{|U}(x) \\
&= \sum_{k=1}^{l} \int_U \left(\int_{W_k} \overline{f_1}(x, \check{\varphi} \circ \phi_t^{v_U(x)} \circ \psi_k^{v_U(x)}(y)) f_2(x, \check{\varphi} \circ \psi_k^{v_U(x)}(y)) d\mathscr{H}(y) \right) d\mu'_{|U}(x) \\
&= \sum_{k=1}^{l} \int_U \left(\int_{Lie(H_{v_U(x)})\times W_k} \overline{f_1}(x, \check{\varphi} \circ \phi_t^{v_U(x)}(\theta, r)) f_2(x, \check{\varphi}(\theta, r)) dm_{(x,k)}(\theta, r) \right) d\mu'_{|U}(x),
\end{aligned}$$

In this part, we have established the first step by placing ourselves in the Lie algebra of the orbit of $v_U(x)$ which is the denoted group $H_{v_U(x)}$ and which is abelian, so

$$\check{\varphi} \circ \phi_t^{v_U(x)}(\theta, r) = \check{\varphi}(\theta + tv_U(x), r) = \varphi(\exp(\theta + tv_U(x)), r) = \varphi(\exp(tv_U(x)) \exp(\theta), r).$$

We are now at the second step placing ourselves in the sections of the Lie group G by $H_{v_U(x)}$. So

$$\begin{aligned}
&b_t(f_1, f_2) \\
&= \sum_{k=1}^{l} \int_U \left(\int_{Lie(H_{v_U(x)})\times W_k} \overline{f_1}(x, \varphi(\exp(tv_U(x)) \chi_{v_U(x)}^{-1}\left(\chi_{v_U(x)}\left(e^\theta\right)\right), r)) f_2(x, \varphi(e^\theta), r)) dm_{(x,k)}(\theta, r) \right) d\mu'_{|U}(x) \\
&= \sum_{k=1}^{l} \int_U \left(\int_{\mathbb{T}^{d_{v_U(x)}}\times W_k} \overline{f_1}(x, \varphi(\exp(tv_U(x)) \chi_{v_U(x)}^{-1}(z), r)) f_2(x, \varphi(\chi_{v_U(x)}^{-1}(z), r)) dm'_{(x,k)}(z, r) \right) d\mu'_{|U}(x)
\end{aligned}$$

We pass here in the torus isomorphic to $H_{v_U(x)}$. Since

$$z + \chi_{v_U(x)}(\exp(tv_U(x))) = z + t\chi_{v_U(x)}(\exp(v_U(x))) = z + tw_U(x)$$

we get

$$\begin{aligned}
&b_t(f_1, f_2) \\
&= \sum_{k=1}^{l} \int_U \left(\int_{\mathbb{T}^{d_{v_U(x)}}\times W_k} \overline{f_1}(x, \varphi(\chi_{v_U(x)}^{-1}(z + tw_U(x)), r)) f_2(x, \varphi(\chi_{v_U(x)}^{-1}(z), r)) dm'_{(x,k)}(\theta, r) \right) d\mu'_{|U}(x)
\end{aligned}$$

This proves (10). □

We are then in the case of a torus foliation by the variable x which will then allow us to apply the same reasoning as in the case of the torus.

Proof of Theorem 5.5. Let begin with the direct implication. Suppose that the dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits keplerian shear. Let $U \in \mathscr{A}$ and let

$$\xi \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}.$$

Let $(f_1, f_2) \in \left(\mathbb{L}^2_{\mu'_{|U} \otimes \mathscr{H}}(U \times G)\right)^2$ such that :

$$\check{f}_1 : (x, z, r) \mapsto e^{2i\pi\langle \xi | z\rangle}$$

and

$$\check{f}_2 : (x, z, r) \mapsto \mathbb{1}_{\langle \xi | _U(\cdot)\rangle \neq 0}(x) e^{2i\pi\langle \xi | z\rangle}$$

taking notations of Lemma 5.1. Then

$$\int_{\mathbb{R}} e^{-2i\pi tz} d\nu^{\xi,U}(z) = \int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} g(x) d\mu'(x)$$

with

$$g : x \in U \mapsto \mathbb{1}_{\langle\xi|w_U(\cdot)\rangle\neq 0}(x).$$

Thus, by Lemma 5.1

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} g(x) d\mu'_{|U}(x) = \int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y).$$

By the Keplerian shear property the right hand term converges and as in (5), the limit is zero. So

$$\int_{\mathbb{R}} e^{-2i\pi tz} d\nu^{\xi,U}(z) \xrightarrow[t\to\pm\infty]{} 0.$$

Let us continue with the reciprocal implication. Let d be the dimension of the maximal torus of G. Consider the level set of torus dimensions (see Definition 5.4)

$$Y_m := \left\{x \in U : d_{v_U(x)} = m\right\}, m \in [\![1,d]\!].$$

Consider $(m,k) \in [\![1,d]\!] \times [\![1,l]\!]$ and define the measure ω_m^k on $U \times \mathbb{T}^m \times W_k$ by

$$\forall A \in \mathscr{B}(U\times\mathbb{T}^m\times W_k), \omega_m^k(A) = \int_U \int_{\mathbb{T}^m\times W_k} \mathbb{1}_A(x,y,z) dm'_{(x,k)}(y,z) d\mu'_{|Y_m}(x).$$

Let $U \in \mathscr{A}$ and $(\xi_1,\xi_2) \in \mathbb{Z}^m \times \mathbb{Z}^{m'}$. Let $(f_1,f_2) \in \left(\mathbb{L}^2_{\mu'_{|U}\otimes\mathscr{H}}(U\times G)\right)^2$ such that,

$$\check{f}_1 : (x,z,r) \mapsto a_1(x,r) e^{2i\pi\langle\xi_1|z\rangle} \mathbb{1}_{Y_m}(x)$$

and

$$\check{f}_2 : (x,z,r) \mapsto a_2(x,r) e^{2i\pi\langle\xi_2|z\rangle} \mathbb{1}_{Y_{m'}}(x)$$

with a_1 and a_2 square summable functions in the appropriate Lebesgue space. If $m \neq m'$, the scalar product between f_1, f_2 is zero. Therefore we assume that $m = m'$. So, by Lemma 5.1

$$\int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y) = \int_U e^{-2i\pi t\langle\xi_1|w_U(x)\rangle} g(x) d\mu'_{|U}(x) \tag{11}$$

with

$$g : x \in U \mapsto \mathbb{1}_{Y_m}(x) \sum_{k=1}^{l} \left(\int_{W_k} \overline{a_1}(x,r) a_2(x,r) e^{2i\pi\langle\xi_2-\xi_1|z\rangle} dm'_{(x,k)}(z,r)\right).$$

By hypothesis, when $\xi_1 \neq 0$,

$$\int_{\mathbb{R}} e^{-2i\pi tz} d\nu^{\xi_1,U}(z) \xrightarrow[t\to\pm\infty]{} 0.$$

And then, by Lemma 2.1, we have that $\exp(2i\pi t\cdot)$ converge weakly-$*$ in $\mathbb{L}^\infty_{\nu^{\xi_1,U}}(\mathbb{R})$ to 0. Thus,

$$\int_U e^{-2i\pi t\langle\xi_1|w_U(x)\rangle} g(x) d\mu'_{|U}(x) \xrightarrow[t\to\pm\infty]{} \int_{\{<\xi_1|w_U(\cdot)>=0\}\times G} \overline{f_1}(x,y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y)$$

Then, remembering (11), we get

$$\int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y) \xrightarrow[t\to\pm\infty]{} \int_{\{<\xi_1|w_U(\cdot)>=0\}\times G} \overline{f_1}(x,y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y).$$

By totality obtained by the Fourier series expansion, this gives that for all functions $f_1, f_2 \in \mathbb{L}^2_{\mu'_{|U}\otimes\mathscr{H}}(Y_m \times G)$

(12)

$$\int_{Y_m\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y) \xrightarrow[t\to\pm\infty]{} \int_{Y_m\times G} \overline{E_{\mu'_{|U}\otimes\mathscr{H}}(f_1|\mathscr{I}_U)} E_{\mu'_{|U}\otimes\mathscr{H}}(f_2|\mathscr{I}_U) d\mu'_{|U} \otimes \mathscr{H}.$$

Let $(f_1, f_2) \in \left(\mathbb{L}^2_{\mu'_{|U}\otimes\mathscr{H}}(U \times G)\right)^2$. Finally we apply (12) on each term of the following sum

$$\begin{aligned}&\int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y)\\ &= \sum_{m=1}^{d} \int_{Y_m\times G} \overline{f_1}(x, \exp(tv_U(x))\, y) f_2(x,y) d\mu'_{|U} \otimes \mathscr{H}(x,y).\end{aligned}$$

□

Now, the aim here is to give an analogue of the fundamental theorem 3.3 in [16], which guarantees Keplerian shear for flows with regular velocities and the negligible critical points.

Theorem 5.6. *If for all $U \in \mathscr{A}$, w_U is of class $\mathscr{C}^1$, $\mu'_{|U} \ll \lambda$ and*

$$\mu\left(\bigcup_{\xi\in\mathbb{Z}^d\setminus\left\{0_{\mathbb{Z}^d}\right\}} \{x \in U : d\langle\xi|w_U(x)\rangle = 0\}\right) = 0,$$

then the dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ has Keplerian shear.

Proof. Let $U \in \mathscr{A}$. Let us show that the measures $\nu^{(\xi,U)}$ have the Rajchman property. We assume v_U of class $\mathscr{C}^1$. So $w_U : x \in U \mapsto \chi_{v_U(x)}(v_U(x))$ is $\mathscr{C}^1$. Let

$$\xi \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}.$$

By Radon-Nikodyme

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} d\mu'_{|U}(x) = \int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x) d\lambda(x).$$

But

$$\mu\left(\bigcup_{\xi\in\mathbb{Z}^d\setminus d\left\{0_{\mathbb{Z}^d}\right\}} \{x \in U : d\langle\xi|w_U(x)\rangle = 0\}\right) = 0.$$

So

$$\forall a \in \mathbb{R}, \mu\left(\bigcup_{\xi\in\mathbb{Z}^d\setminus\left\{0_{\mathbb{Z}^d}\right\}} \{x \in U : \langle\xi|w_U(x)\rangle = a\}\right) = 0.$$

And so

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x) d\lambda(x) = \int_{\mathbb{R}^d} e^{-2i\pi t<\xi|z>} \sum_{j\in L} \frac{\mathbb{1}_{w_U(U)\cap V_j}(z)\left(\left(w_U\right)_{|w_U^{-1}(V_k)}\right)^{-1}(z)}{\left|det\left(J_{w_U}\left(\left(\left(w_U\right)_{|w_U^{-1}(V_k)}\right)^{-1}(z)\right)\right)\right|} d\lambda(x).$$

And so by Riemann-Lebesgue,

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x) d\lambda(x) \xrightarrow[t\to+\infty]{} 0.$$

And then,

$$\int_U e^{-2i\pi t\langle \xi | w_U(x)\rangle} d\mu'_{|U}(x) \xrightarrow[t\to+\infty]{} 0.$$

□

Remark 5.3.1. *We can note that we could use same arguments as Damien Thomine* [16] *in the proof of his theorem analogous to this one, i.e., we could use the normal forms of submergences by relying on the regularity properties of w_U. Thanks to the property of Rajchman, we were able to more expeditiously prove the above theorem.*

5.4. **Main examples.**

Example 5.4.1 (Torus)**.** *The tori also form an abelian example, it is still a connected compact Lie group. The most notorious torus is the torus of dimension 2 denoted $\mathbb{T}^2$ which we can easily represent in $\mathbb{R}^3$. The first example in the abelian framework would be the one used in the article by Damien Thomine* [16] *with $\mathbb{T}^2$, a measurable velocity vector $v(x)$, and for a single fiber to torus, the flow:*

$$g_t : (x, y) \in \mathbb{T}^2 \mapsto (x, y + tv(x))$$

To know if this dynamical system exhibits Keplerian shear, it is necessary and it suffices that μ_v Let of Rajchman. We can then study the speeds of convergence.

Example 5.4.2 (Spinorial groups)**.** *The spin group for $n \geq 2$, $Spin(n)$ is a compact and connected Lie group, which allows us to use it as an example.*

Example 5.4.3 (Orthogonal groups)**.** *The orthogonal groups for $n \geq 2$ $SO_n(\mathbb{R})$ are connected compact Lie groups. The most usual example is that of the orthogonal group $SO_3(\mathbb{R})$ which is compact, connected with a structure of Lie but not abelian. We consider as flow $g_t : M \in SO_3(\mathbb{R}) \mapsto exp(tA)M$ with A an antisymmetric matrix, fundamental property for the stability of the exponential in the orthogonal group $SO_3(\mathbb{R})$. As before, we can leaf it in torus as with the general cases previously treated in bundles in compact and connected Lie groups. The other classic examples are in the abelian framework which, according to the work of Antoine Delzant* [7]*, amounts to torus bundles as before. As before, we consider as flow $g_t : M \in Spin(3) \mapsto exp(tA)M$ with A an antisymmetric matrix.*

Example 5.4.4 (Special unitary group)**.** *A special unitary group for $n \geq 2$, $SU_n(\mathbb{R})$ is also a connected compact Lie group and so we can use the previous theorems. Moreover, these Lie groups are even simply connected. They are always non-abelian groups, which means that they are not a torus. The most common example is $SU_2(\mathbb{R})$ which is isomorphic to the hypersphere $\mathbb{S}^3$ of $\mathbb{R}^4$. The flow for $Ain\mathscr{M}_2(\mathbb{R})$ such that ${}^tA = -A^*$ on it will be defined by*

$$g_t : M \in SU_2(\mathbb{R}) \mapsto \exp(tA)M.$$

6. Keplerian shear, Rajchman property and Diophantine approximation

In this section we present an application of Keplerian shear (with speed estimates) to dynamical Borel Cantelli lemmas. The latter is linked to Diophantine approximation and we discuss the relations between the Rajchman property and diophantine properties.

Let the probabilistic space (Ω, μ). The main application consists in readjusting the dynamic Borel-Cantelli theorem in the non-ergodic case. We find an adaptation of Sprindzuk's theorem inspired by the thesis of Victoria Xing [18]. The keplerian shear will ensure that for almost any x, there exists an infinity of integers n satisfying $T^n(x) \in A_n$.

Theorem 6.1 (Variable Sprindzuk)**.** *Let $(f_k)_{k\in\mathbb{N}^*}$ and $(g_k)_{k\in\mathbb{N}^*}$ measurable and positive application sequencies. Let $(\varphi_k)_{k\in\mathbb{N}^*}$ a real sequence such that : $\forall k \in \mathbb{N}^*, 0 \leq g_k \leq \varphi_k \leq 1\, \mu - p.s$ Let $\delta > 1$ and $C > 0$. Suppose that for all $(m,n) \in (\mathbb{N}^*)^2$ satisfying $n \geq m$,*

$$\int_\Omega \left(\sum_{k=m}^{n} f_k(x) - g_k(x)\right)^2 d\mu(x) \leq C \left(\sum_{k=m}^{n} \varphi_k\right)^\delta .$$

Then

$$\forall n \in \mathbb{N}, \forall \varepsilon > 0, \sum_{k=1}^{n} f_k = \sum_{k=1}^{n} g_k + O\left(\phi(n)^{\frac{\delta}{2}} \left(\log\left(\phi(n)\right)\right)^{1+\varepsilon}\right)\ \mu - a.e.$$

with $\phi(n) = \sum_{k=1}^{n} \varphi_k$.

For the proof, we will draw on the proof of the original theorem in the book by Sprindzuk [15] on page 45 formula (68)

Proof. Let $\delta > 1$ and let us denote for $I \subset \mathbb{N}^*, \phi(I) = \sum_{k\in I} \varphi_k$. By the fact that $\forall k \in \mathbb{N}^*, 0 \leq \varphi_k \leq 1$, we have that

$$\exists (n_v)_{v\in\mathbb{N}^*} \in {\mathbb{N}^*}^{\mathbb{N}^*}, \forall v \in \mathbb{N}^*, \phi(n_v) < v \leq \phi(n_v + 1).$$

We have also

$$\forall v \in \mathbb{N}^*, n_{v+1} \geq n_v + 1.$$

And

$$\forall v \in \mathbb{N}^*, \phi(n_v) + 1 < v + 1 \leq \phi(n_{v+1} + 1).$$

So

$$\forall (u,v) \in (\mathbb{N}^*)^2, (u < v \Longrightarrow [\![n_u + 1, n_v]\!] \neq \emptyset).$$

Let

$$\begin{array}{rll} \sigma : & \mathscr{P}(\mathbb{N}^*) & \to \mathscr{P}(\mathbb{N}^*) \\ & I & \mapsto \{n_w \in \mathbb{N} : w \in I\} \end{array} .$$

Let for $r \in \mathbb{N}^*, s \in [\![0, r]\!]$ sets of parts

$$J_{r,s} := \left\{[\![i2^s + 1, (i+1)2^s]\!] \in \mathscr{P}(\mathbb{N}^*) : i \in [\![0, 2^{r-s} - 1]\!]\right\}.$$

Let $r \in \mathbb{N}^*$ and $s \in [\![0, r]\!]$. We notice that

$$\bigcup_{I\in J_{r,s}} \sigma(I) = [\![1, n_{2^r}]\!].$$

Let $i \in [\![0, 2^{r-s} - 1]\!]$. But

$$\phi\left(n_{i2^s}\right) < i2^s \leq \phi\left(n_{i2^s} + 1\right) \leq \phi\left(n_{i2^s}\right) + 1.$$

And then

$$i2^s - 1 \leq \phi\left(n_{i2^s}\right) < i2^s$$

And

$$\phi\left(n_{(i+1)2^s}\right) < (i+1)2^s \leq \phi\left(n_{(i+1)2^s} + 1\right) \leq \phi\left(n_{(i+1)2^s}\right) + 1.$$

And

$$(i+1)2^s - 1 \leq \phi\left(n_{(i+1)2^s}\right) < (i+1)2^s.$$

And then,

$$\phi\left(\sigma\left([\![n_{i2^s} + 1, n_{(i+1)2^s}]\!]\right)\right) = \phi\left(n_{(i+1)2^s}\right) - \phi\left(n_{i2^s}\right) \leq 2^s + 1 \leq 2^{s+1}.$$

So

$$\sum_{I\in J_{r,s}} \left(\phi(\sigma(I))\right)^\delta \leq 2^\delta 2^{r+s(\delta-1)}.$$

Denote

$$J_r := \bigcup_{s\in[\![0,\,r]\!]} J_{r,s}.$$

So

$$\sum_{I\in J_r} \left(\phi(\sigma(I))\right)^{\delta} = \sum_{s=0}^{r}\sum_{I\in J_{r,s}} \left(\phi(\sigma(I))\right)^{\delta} \leq \frac{2^{\delta}2^{r\delta}}{2^{\delta-1}-1}.$$

Let

$$\begin{array}{llll} h: & \mathbb{N}^*\times\Omega & \to \mathbb{R} & \\ & (l,x) & \mapsto \sum_{I\in J_l}\left(\sum_{k\in I} f_k(x)-g_k(x)\right)^2 & . \end{array}$$

So

$$\int_{\Omega} h(r,x)d\mu(x) \leq C\left(\sum_{I\in J_r}\phi(\sigma(I))\right) \leq \frac{2^{\delta}C2^{r\delta}}{2^{\delta-1}-1}.$$

Let $\varepsilon>0$. By the Markov inequality,

$$\mu\left(h(r,X)\geq \frac{2^{\delta}Cr^{1+\varepsilon}2^{r\delta}}{2^{\delta-1}-1}\right)\leq r^{-(1+\varepsilon)}.$$

By Borel-Cantelli,

$$\mu-a.a\, x\in\Omega, \exists r_x\in\mathbb{N}^*, \forall r\geq r_x, h(r,x)\leq \frac{2^{\delta}Cr^{1+\varepsilon}2^{r\delta}}{2^{\delta-1}-1}.$$

For $v\in\mathbb{N}^*$, we have that $[\![1,v]\!]$ can be split into a finite number r_v of interval J_r such that

$$r_v\leq \lfloor\log_2(v)\rfloor+1.$$

Let $J(v)$ denote the set of these intervals. We then get

$$\sum_{k=1}^{n_v} f_k-g_k = \sum_{I\in J(v)}\left(\sum_{k\in I} f_k-g_k\right).$$

By the Cauchy-Schwarz inequality,

$$\mu-a.a\, x\in\Omega, \left(\sum_{k=1}^{n_v} f_k-g_k\right)^2 \leq r_v\sum_{I\in J(v)}\left(\sum_{k\in I} f_k-g_k\right)^2 = r_v h\left(r_v,x\right).$$

So

$$\mu-a.a\, x\in\Omega, \exists r_x\in\mathbb{N}^*, \forall r\geq r_x, \left(\sum_{k=1}^{n_v} f_k(x)-g_k(x)\right)^2 \leq \frac{2^{\delta}Cr^{2+\varepsilon}2^{r\delta}}{2^{\delta-1}-1}\leq \frac{2^{\delta}C\log_2^{2+\varepsilon}(v)v^{\delta}}{2^{\delta-1}-1}.$$

And so

$$\mu-a.a\, x\in\Omega, \exists r_x\in\mathbb{N}^*, \forall r\geq r_x, \left|\sum_{k=1}^{n_v} f_k(x)-g_k(x)\right| \leq \frac{2^{\frac{\delta}{2}}C^{\frac{1}{2}}\log_2^{1+\frac{\varepsilon}{2}}(v)v^{\frac{\delta}{2}}}{(2^{\delta-1}-1)^{\frac{1}{2}}}.$$

Let $n\in\mathbb{N}^*$. So

$$\exists v\in\mathbb{N}^*, n_v\leq n\leq n_{v+1}.$$

So

$$\mu-a.a\, x\in\Omega, \sum_{k=1}^{n_v} f_k(x)\leq \sum_{k=1}^{n} f_k(x)\leq \sum_{k=1}^{n_{v+1}} f_k(x).$$

We also have

$$\mu-a.a\, x\in\Omega, 0\leq \sum_{k=n_v}^{n_{v+1}} g_k(x)\leq \phi\left(n_{v+1}\right)-\phi\left(n_v\right).$$

In addition,

$$v-1 \leq \phi\left(n_{v}\right) < v \leq \phi\left(n_{v}+1\right) \leq \phi\left(n_{v+1}\right) < v+1.$$

And so

$$0 < \phi\left(n_{v+1}\right)-\phi\left(n_{v}\right) < 2.$$

And then,

$$\mu - a.a\, x \in \Omega, 0 \leq \sum_{k=n_v+1}^{n_{v+1}} g_k(x) < 2.$$

So

$$\mu - a.a\, x \in \Omega, \sum_{k=1}^{n_v} g_k(x) + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_v)\right)\left(\phi(n_v)\right)^{\frac{\delta}{2}}\right) \leq \sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{n_{v+1}} g_k(x) + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_{v+1})\right)\left(\phi(n_{v+1})\right)^{\frac{\delta}{2}}\right)$$

We then find

$$\mu - a.a\, x \in \Omega, -2 + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_v)\right)\left(\phi(n_v)\right)^{\frac{\delta}{2}}\right) \leq \sum_{k=1}^{n}(f_k(x)-g_k(x)) \leq 2 + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_{v+1})\right)\left(\phi(n_{v+1})\right)^{\frac{\delta}{2}}\right).$$

We easily find

$$-2 + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_v)\right)\left(\phi(n_v)\right)^{\frac{\delta}{2}}\right) = O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n)\right)\left(\phi(n)\right)^{\frac{\delta}{2}}\right).$$

And on the other hand

$$2 + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n_{v+1})\right)\left(\phi(n_{v+1})\right)^{\frac{\delta}{2}}\right) = O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n)\right)\left(\phi(n)\right)^{\frac{\delta}{2}}\right).$$

Finally

$$\mu - a.a\, x \in \Omega, \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} g_k(x) + O\left(\log_2^{1+\frac{\varepsilon}{2}}\left(\phi(n)\right)\left(\phi(n)\right)^{\frac{\delta}{2}}\right).$$

□

Theorem 6.2 (Dynamical Borel-Cantelli by keplerian shear)**.** *Suppose that (Ω, T, μ) is a discrete measure preserving dynamical system with keplerian shear. Let (A_n) be a sequence of measurable sets and note $S_{M,N} = \sum_{k=M}^{N} \mathbb{1}_{A_k} \circ T^k$.*

We suppose that 1a or 1b holds and 2

(1) (a) *There exists $\gamma > 0, C > 0$ such that for all $(j,k) \in \mathbb{N}^2$ satisfying $k \neq j$, $\left|\mathbb{E}\left(Cov_{k-j}\left(\mathbb{1}_{A_j}, \mathbb{1}_{A_k} | \mathscr{I}\right)\right)\right| \leq \frac{C}{|k-j|^{\gamma}}$*

(b) *There exists $\gamma > 0, D > 0$ and $\mathscr{B} \subset \mathbb{L}^2_{\mu}(\Omega)$, a Banach space such that : $\exists C > 0, \forall (f,g) \in \mathscr{B}^2, \forall n \in \mathbb{N}, |\mathbb{E}(Cov_n(f,g|\mathscr{I}))| \leq \frac{C\|f\|_{\mathscr{B}}\|g\|_{\mathscr{B}}}{n^{\gamma}}$ and $\forall j \in \mathbb{N}, \left\|\mathbb{1}_{A_j}\right\|_{\mathscr{B}} \leq D$*

(2) *There exists $\beta > \max(\frac{1}{2}, 1-\frac{\gamma}{2})$ such that $\liminf_{N\to\infty}\left(\frac{\mathbb{E}(S_N|\mathscr{I})}{N^{\beta}}\right) > 0$ $\mu - a.s.$*

We have

$$\frac{S_N}{\mathbb{E}(S_N|\mathscr{I})} \xrightarrow[N\to+\infty]{\mu-a.s} 1.$$

Proof. Without loss of generality we assume that $\gamma \in (0,1)$. Note that 1b implies 1a so 1a holds in both cases.

By expansion of the square as a double sum and the series-integral comparison criterion, we get $\forall (M,N) \in \mathbb{N}^2$ s.t. $M < N$,

$$\begin{aligned}\|S_{M,N} - \mathbb{E}\left(S_{M,N}|\mathscr{I}\right)\|_2^2 &\leq \mathbb{E}(S_{M,N}) + 2C\sum_{l=1}^{N-M+1}\frac{(N-M+1)}{l^{\gamma}} \\ &\leq \mathbb{E}(S_{M,N}) + 2C(N-M+1)^{2-\gamma}.\end{aligned}$$

By setting $\delta = 2-\gamma$, and noticing that $E(S_{M,N}) \leq N-M$ we get that there exists $D>0$ satisfying

$$\forall N \in \mathbb{N}, \|S_{M,N} - \mathbb{E}\left(S_{M,N}|\mathscr{I}\right)\|_2^2 \leq D(N-M+1)^{\delta}.$$

By variable Sprindzuk theorem 6.1 with $\varphi_k = 1$ for all k, we get

$$S_N - \mathbb{E}(S_N|\mathscr{I}) \overset{a.s}{=} O(N^{\frac{\delta}{2}} \log^{1+\varepsilon}(N)).$$

Then

$$\frac{S_N}{\mathbb{E}(S_N|\mathscr{I})} = 1 + O\left(\frac{N^{\frac{\delta}{2}}\log^{1+\varepsilon}(N)}{N^{\beta}}\right) \xrightarrow[N\to+\infty]{a.s} 1.$$

□

Example 6.0.1. *Let us place ourselves in the case of* $(\mathbb{T}^2, \lambda\otimes\lambda, T)$ *with*

$$Mat(T) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consider a sequence $(b_n) \in \mathbb{T}^{\mathbb{N}}$ *and let*

$$A_n = \mathbb{T} \times \left[b_n - \frac{1}{n^p}, b_n + \frac{1}{n^p}\right]$$

with $0 < p < \frac{1}{2}$. *We get that* $\forall N \in \mathbb{N}^*$,

$$\mathbb{E}(S_N|\mathscr{I}) = E(S_N) = 2\sum_{k=1}^{N} \frac{1}{k^p} \sim \frac{2}{1-p} N^{1-p}.$$

Let $s := \frac{1}{2}$ *and* $n \in \mathbb{N}^*$. *By direct computation of the Fourier coefficients of* 1_{A_n}, *we see that*

$$\mathbb{1}_{A_n} \in H^{s,0}(\mathbb{T}^2)$$

since

$$\|\mathbb{1}_{A_n}\|^2_{H^{s,0}(\mathbb{T}^2)} \leq \frac{1}{\pi^2}\zeta(2) + \frac{2}{n^p} \leq 3.$$

By the decorrelation estimates in [16], $\forall (k,j) \in \mathbb{N}^2$,

$$\mathbb{E}\left(Cov_{k-j}\left(\mathbb{1}_{A_k}, \mathbb{1}_{A_j}|\mathscr{I}\right)\right) \leq \frac{4^s 3^2}{|k-j|^{2s}}.$$

The hypotheses are then brought together, applying theorem 6.2 with $\gamma = 1$ *and* $\beta \in (\frac{1}{2}, 1-p)$, *we get*

$$S_N \sim \frac{2}{1-p} N^{1-p} \quad \mu - a.s.$$

Definition 6.1 ($s-$Diophantine number)**.** *Let* $s > 0$.

A number $x \in \mathbb{R}$ *is* $s-$*diophantine if*

$$\exists C > 0, \forall (p,q) \in \mathbb{Z}\times\mathbb{N}^*, |qx - p| \geq \frac{C}{q^s}. \tag{13}$$

Let $\mathscr{D}_i(s)$ *be the set of diophantine numbers.*

Definition 6.2 (Liouville Number)**.** *A number* $x \in \mathbb{R}$ *is Liouville if and only if for all* $s > 0$, x *is not diophantine, that is, it does not satisfy for any* $s > 0$ *the assertion* (13). *In other words*

$$x \in \mathbb{L} := \mathbb{R} \setminus \left(\bigcup_{s>0} \mathscr{D}_i(s)\right).$$

Or even, $Dio(x) = +\infty$.

In the case of any measure of Rajchman, we have as an application a proposition based on theorem 4.2 page 551 of Athreya [2] and also the theorem of Kurzweil ([13] and [17], theorem 1.3 page 3) concerning diophantine numbers. To introduce this theorem, we will also have to recall the Shrinking Target Property (STP) and its monotone version (MSTP) of a dynamical system.

Definition 6.3 (Shrinking Target Properties)**.** *A discrete dynamical system (Ω,μ,T) is called STP if for all sequence of balls $(B_n)_{n\in\mathbb{N}}$ of radius $r_n>0$ tending to 0 and satisfying*

$$\sum_{n\in\mathbb{N}}\mu(B_n)=+\infty \text{ we have } \mu\left(\overline{\lim_{n\to+\infty}}T^{-n}(B_n)\right)=1.$$

The MSTP property is defined in the same way, assuming in addition that the sequence r_n is monotone.

Let us now recall Kurzweil's theorem in dimension 1.

Theorem 6.3 (Kurzweil-Tseng)**.** *Consider the dynamical system $(\mathbb{T},\lambda,T_\alpha)$ with $\alpha\in\mathbb{T}$ and $T_\alpha: x\mapsto x+\alpha$. Then the dynamical system is $s-$MSTP if and only if α is s-diophantine (i.e., $\alpha\in\mathscr{D}_i(s)$).*

We also deduce with the documents of Chaika [5] the following theorem.

Theorem 6.4 (Kurzweil)**.** *For $\lambda-a.a\,\alpha\in[0,1[,(\mathbb{T},\lambda,T_\alpha)\,is\,MSTP$.*

In the case of singular continuous Rajchman measures, we will have a weakening of the theorem 6.4 of Kurzweil. First, we present a dynamical Borel Cantelli for Rajchman measures. As in Theorem 6.2, it is based on covariance estimates. Here they do not rely on the keplerian shear but are obtained directly from the Rajchman property.

Theorem 6.5 (Dynamical Borel-Cantelli for Rajchman measures)**.** *Consider a Rajchman measure μ on $\mathbb{T}$ of order $0\leq r(\mu)\leq\frac{1}{2}$. Let T be the transvection on $\mathbb{T}^2$ defined by $T(x,y)=(x,x+y)$. Let $C>0$ and $s>\frac{1}{r(\mu)}-1$. We have*

$$\frac{S_N}{\mathbb{E}(S_N)}\xrightarrow[N\to+\infty]{\mu\otimes\lambda-p.s}1$$

where

$$S_N=\sum_{k=1}^{N}\mathbb{1}_{A_k}\circ T^k,\quad \text{and}\quad A_n=\mathbb{T}\times\left]z_n-\frac{C}{n^{\frac{1}{s}}},z_n+\frac{C}{n^{\frac{1}{s}}}\right[.$$

Proof. Let $p=\frac{1}{s}$. Let for $(m,n)\in(\mathbb{N}^*)^2$ such that $m>n$ the sum

$$S_{(m,n)}=\sum_{k=m}^{n}\mathbb{1}_{A_k}\circ T^k.$$

We get

$$\begin{aligned}\int_{\mathbb{T}^2}\left(S_{(m,n)}-\mathbb{E}\left(S_{(m,n)}\right)\right)^2d\mu\otimes\lambda &=\textstyle\sum_{k=m}^{n}\sum_{j=m}^{n}\int_\Omega\mathbb{1}_{A_k}\circ T^{k-j}\mathbb{1}_{A_j}-\mathbb{E}_\mu\left(\mathbb{1}_{A_k}|\mathscr{I}\right)\mathbb{E}_\mu\left(\mathbb{1}_{A_j}|\mathscr{I}\right)d\mu\\ &\leq\mathbb{E}_\mu(S_{(m,n)})+2\sum_{j=m}^{n}\sum_{l=1}^{n-m}\int_\Omega\mathbb{1}_{A_{j+l}}\circ T^l\mathbb{1}_{A_j}-\mathbb{E}_\mu\left(\mathbb{1}_{A_{j+l}}|\mathscr{I}\right)\mathbb{E}_\mu\left(\mathbb{1}_{A_j}|\mathscr{I}\right)d\mu.\end{aligned}$$

And then, we have

$$\begin{aligned}\left|\int_\Omega\mathbb{1}_{A_{j+l}}\circ T^l\mathbb{1}_{A_j}-\mathbb{E}_\mu\left(\mathbb{1}_{A_{j+l}}|\mathscr{I}\right)\mathbb{E}_\mu\left(\mathbb{1}_{A_j}|\mathscr{I}\right)d\mu\right| &=\left|\textstyle\sum_{k\in\mathbb{Z}^*}\frac{\sin\left(\frac{2k\pi C}{j^p}\right)\sin\left(\frac{2k\pi C}{(j+l)^p}\right)}{k^2\pi^2}\widehat{\mu}(kl)\right|\\ &\leq 4C_\mu C\textstyle\sum_{k\in\mathbb{N}^*}\frac{1}{l^{r(\mu)}}\frac{\left|\sin\left(\frac{2\pi kC}{j^p}\right)\right|}{k^{1+r}(j+l)^p\pi}\\ &\simeq\frac{2^{2+r}C_\mu C^{1+r}\pi^{1-r}}{j^{pr(\mu)}(j+l)^pl^{r(\mu)}}\int_{\mathbb{R}_+^*}\frac{|\sin(x)|}{x^{1+r}}d\lambda(x)\end{aligned}$$

And we get

$$\begin{aligned}\int_{\mathbb{T}^2} \left(S_{(m,n)} - \mathbb{E}\left(S_{(m,n)}\right)\right)^2 d\mu \otimes \lambda &\ll \mathbb{E}_\mu\left(S_{(m,n)}\right) + \int_{[0,n-m]} \int_{[0,n-m]} \frac{1}{x^{r(\mu)}} \frac{1}{(x+y)^p} \frac{1}{y^{pr(\mu)}} d\lambda(y) d\lambda(x) \\ &\ll \int_{[0,\frac{\pi}{2}]} \frac{1}{(\cos(\theta))^{r(\mu)}(\cos(\theta)+\sin(\theta))^p(\sin(\theta))^{pr(\mu)}} d\lambda(\theta)(n-m)^{2-r(\mu)-p(1+r(\mu))}\end{aligned}.$$

Let's consider $\delta = 2 - r(\mu) - p(1+r(\mu))$. But $p < \frac{r(\mu)}{1-r(\mu)}$. So $\frac{\delta}{2} < 1-p$. By Sprindzuk's theorem,

$$S_N \overset{\mu\otimes\lambda - a.s}{=} \mathbb{E}(S_N) + O\left(N^{\frac{\delta}{2}} \log_2^{1+\varepsilon}(N)\right).$$

So

$$\frac{S_N}{\mathbb{E}(S_N)} \xrightarrow[N\to+\infty]{\mu\otimes\lambda-a.s} 1.$$

□

Remark 6.0.1. *We observe that this "loss of power" in the Borel-Cantelli lemma above increases as the Rajchman order decreases.*

Next result relates the order of a Rajchman measure to diophantine properties of its support and thus the SMTP property. This result will therefore also show the proposition 2.3.

Proposition 6.1 (General Kurzweil for Rajchman measures)**.** *Consider a Rajchman measure μ on $\mathbb{T}$ of order $0 \le r(\mu) \le \frac{1}{2}$. Let $s > \frac{1}{r(\mu)} - 1$. We have $\mu(\mathscr{D}_i(s)) = 1$, or equivalently*

$$\mu - a.a\, \alpha \in [0,1[, (\mathbb{T}, \lambda, T_\alpha)\ s - MSTP.$$

Proof. Let $C > 0$, $j \in \mathbb{N}^*$ and $p \in [\![0, j-1]\!]$. Consider

$$\sum_{p=0}^{q-1} \mu\left(\left]\frac{p}{j} - \frac{C}{j^{s+1}}, \frac{p}{j} + \frac{C}{j^{s+1}}\right[\right) = \int_{\mathbb{T}} \mathbb{1}_{\left]\frac{p}{j} - \frac{C}{j^{s+1}}, \frac{p}{j} + \frac{C}{j^{s+1}}\right[} d\mu = 2\frac{C}{j^s} + \sum_{k\in\mathbb{Z}^*} \frac{\sin\left(2\pi k \frac{C}{j^s}\right)}{k\pi} \widehat{\mu}(kj).$$

So

$$\sum_{p=0}^{q-1} \mu\left(\left]\frac{p}{j} - \frac{C}{j^{s+1}}, \frac{p}{j} + \frac{C}{j^{s+1}}\right[\right) \le 2\frac{C}{j^{s+1}} + \frac{2C_\mu C^\varepsilon}{\pi j^{r(\mu)+s\varepsilon}} \zeta(1 + r(\mu) - \varepsilon)$$

with $\varepsilon \in]0, r(\mu)[$. And

$$\mu\left(\bigcup_{p\in[\![0,\, j-1]\!]} \left]\frac{p}{j} - \frac{C}{j^{s+1}}, \frac{p}{j} + \frac{C}{j^{s+1}}\right[\right) \le 2\left(\frac{C}{j^s} + \frac{C_\mu C^\varepsilon}{\pi j^{r(\mu)+s\varepsilon}} \zeta(1 + r(\mu) - \varepsilon)\right)$$

And we have by hypothesis that

$$r(\mu) > \frac{1}{s+1}.$$

By strict inequality, we get that there exists $\varepsilon \in]0, r(\mu)[$ such that

$$r(\mu) > \frac{1}{s+1} \text{ and } r(\mu) > 1 - s\varepsilon.$$

We also have the assumption that $r \le \frac{1}{2}$. By Borel-Cantelli

$$\mu\left(\overline{\lim_{n\to+\infty}}\left(\bigcup_{p\in[\![0,\, n-1]\!]} \left]\frac{p}{n} - \frac{C}{n^{s+1}}, \frac{p}{n} + \frac{C}{n^{s+1}}\right[\right)\right) = 0.$$

So for

$$\mu - a.a\, \alpha \in [0,1[, \exists n \in \mathbb{N}^*, \forall k \ge n, d(k\alpha, \mathbb{Z}) \ge \frac{C}{k^s}.$$

Let then take $\alpha \in [0,1[$ such that

$$\exists n \in \mathbb{N}^*, \forall k \geq n, d(k\alpha, \mathbb{Z}) \geq \frac{C}{k^s}.$$

This proves that $\alpha \in \mathscr{D}_i(s)$, hence $\mu(\mathscr{D}_i(s)) = 1$. By Kurzweil-Tseng theorem 6.3,

$$(\mathbb{T}, \lambda, T_\alpha)\ s - MSTP.$$

□

Recall that when we have $r(\mu) > \frac{1}{2}$, the measure μ is absolutely continuous and therefore by Kurzweil theorem cited above $\mu - a.a\, \alpha \in [0,1[$, the system $(\mathbb{T}, \lambda, T_\alpha)$ is $1 - MSTP$.

Remark 6.0.2. *We can note that the result obtained in proposition 6.1 remains consistent with that obtained with the Keplerian shear in the theorem 6.5 because*

$$\sum_{k=1}^{n} \lambda \left(\left] z_k - \frac{C}{k^{\frac{1}{s}}}, z_k + \frac{C}{k^{\frac{1}{s}}} \right[\right)^s = \sum_{k=1}^{n} \frac{2^s C^s}{k} \xrightarrow[n \to +\infty]{} +\infty.$$

Remark 6.0.3. *For a Rajchman measure with positive order $r(\mu) > 0$, $\mu - a.a\, \alpha \in [0,1[$ is Diophantine, that is, $Dio(\alpha) < +\infty$ $\mu-$almost surely.*

The following result shows that Proposition 6.1 is optimal.

Proposition 6.2 (Optimality of Rajchman order [11])**.** *Let $0 < r < \frac{1}{2}$. Then for all $1 < s < \frac{1}{r} - 1$, there exists μ a Rajchman measure such that $r(\mu) = r$ and $\mu(\mathscr{D}_i(s)) = 0$.*

Proof. Let $\varepsilon = \frac{1}{r} - (s+1) > 0$. We can consider μ_α with $\alpha = s - 1 + \varepsilon$ constructed in the document of Kaufman [11] with its support in $E(\alpha) \subset [0,1[\setminus \mathscr{D}_i(s)$. □

Remark 6.0.4. *With convex combination, for any $s > 1$ we can construct a Rajchman measure μ for which $\mu(\mathscr{D}_i(s))$ is equal to any value in $[0,1]$.*

Example 6.0.2 (Diophantine property with self-similar measures)**.** *The self-similar measure μ_θ with $\theta > 1$ not being Pisot, according to the work of Pablo Shmerkin and Jean-Pierre Kahane* [10]*, has an order of Rajchman $r(\mu_\theta) > 0$. In particular $\mu_\theta - a.a\, \alpha \in [0,1[$ is diophantine.*

Example 6.0.3 (Liouville property with Rajchman measures on S_∞)**.** *The measure of Rajchman μ_∞ mentioned in 2.2.2 is Rajchman but since its support S_∞ almost surely contains Liouville numbers, then we deduce by contrapositive of the proposition 6.1 that μ_∞ does not have a strictly positive Rajchman order, i.e. that $r(\mu_\infty) = 0$.*

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