# RECURRENCE RATES AND HITTING-TIME DISTRIBUTIONS FOR RANDOM WALKS ON THE LINE

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Abstract. We consider random walks on the line given by a sequence of independent identically distributed jumps belonging to the strict domain of attraction of a stable distribution, and first determine the almost sure exponential divergence rate, as $\varepsilon \to 0$, of the return time to $(-\varepsilon, \varepsilon)$. We then refine this result by establishing a limit theorem for the hitting-time distributions of $(x-\varepsilon, x+\varepsilon)$ with arbitrary $x \in \mathbb{R}$.

## 1. Introduction and Results

We consider a recurrent random walk on $\mathbb{R}$, $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$, $n \geq 1$, where the $X_i$ are i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\frac{S_n}{A_n}$ converges, for positive real numbers $A_n$, in distribution to a stable random variable $X$ with index $\alpha$. Necessarily (due to recurrence), $\alpha \in [1, 2]$, and the sequence $(A_n)_{n\geq 1}$ is regularly varying of index $\frac{1}{\alpha}$, satisfying $\sum_{n\geq 1} \frac{1}{A_n} = \infty$.

To capture the speed at which recurrence appears, it is possible to specify, for such a walk, some deterministic sequences $(\varepsilon_n)$ such that $S_n \in (-\varepsilon_n, \varepsilon_n)$ infinitely often, or $S_n \notin (-\varepsilon_n, \varepsilon_n)$ eventually, almost surely. This classical question was addressed, for example, in [7] and [5], the results of which have recently been extended in [6].

Here, we are going to study the number of steps it takes to return to some small neighborhood of the origin (or to hit a different small interval for the first time). For related work on random walks in the plane, intimately related to the $\alpha = 1$ case of the present paper, we refer to [12].

As an additional standing assumption on our walk, we will always require the distribution of the jumps $X_i$ to satisfy the Cramer condition

$$\limsup_{|t|\to\infty} |\mathbb{E}[e^{itX_1}]| < 1. \tag{1}$$

This readily implies, in particular, that the event $\Omega^* := \{S_n \neq 0 \ \forall n \geq 1\}$ has positive probability, and $\Omega^*$ has probability one if and only if no individual path returning to the origin has positive probability.

As a warm-up we first determine the a.s. rate at which the variables

$$\mathbf{T}_\varepsilon := \min\{n \geq 1 \ : \ |S_n| < \varepsilon\}, \quad \varepsilon > 0,$$

diverge on $\Omega^*$ as $\varepsilon \to 0$. Let $\beta \in [2, \infty]$ be the exponent conjugate to $\alpha$, that is, $\alpha^{-1} + \beta^{-1} = 1$.

**Theorem 1.** *In the present setup,*

$$\lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{\log \varepsilon} = -\beta \quad \textit{a.s. on } \Omega^*. \tag{2}$$

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Our main objective then is to determine the precise order of magnitude, and to study the asymptotic distributional behaviour, as $\varepsilon \to 0$, of the more general hitting times of $\varepsilon$-neighbourhoods of arbitrary given points $x$ on the line. We shall, in fact, do so for the walk $S'_n := S'_0 + S_n$, $n \geq 0$, with random initial position $S'_0$, independent of $(S_n)_{n\geq 0}$ and having an arbitrary fixed distribution $P$ on $\mathbb{R}$. For any $x \in \mathbb{R}$ we thus let

$$\mathbf{T}_\varepsilon^x := \inf\{m \geq 1 \ : \ |S'_m - x| < \varepsilon\}$$

and $\Omega_x^* := \{S'_n \neq x \ \forall n \geq 1\}$. Outside $\Omega_x^*$ we clearly have $\lim_{\varepsilon\to 0} \mathbf{T}_\varepsilon^x = \min\{m \geq 1 \ : \ S'_m = x\}$.

It is convenient to state the results in terms of, and work with, the strictly increasing continuous function $G : [0, +\infty) \to [0, +\infty)$ with $G(0) = 0$ which affinely interpolates the values $G(n) = \sum_{k=1}^n \frac{1}{A_k}$, $n \geq 1$. We denote by $G^{-1}$ its inverse function. Evidently, $G(n) = o(n)$. Moreover, by the direct half of Karamata's theorem (cf. Propositions 1.5.8 and 1.5.9a of [2]), $G$ is regularly varying with index $\frac{1}{\beta}$, and satisfies

$$\frac{n}{A_n} = o(G(n)) \ \text{ if } \alpha = 1, \qquad \text{while} \qquad \frac{n}{A_n} \sim \frac{G(n)}{\beta} \ \text{ in case } \alpha \in (1, 2]. \tag{3}$$

We establish a result on convergence in distribution for $\varepsilon\, G(\mathbf{T}_\varepsilon^x)$ conditioned on $\Omega_x^*$ (while $\varepsilon\, G(\mathbf{T}_\varepsilon^x) \to 0$ outside this set). In the case $\alpha = 1$, the limit distribution is the same as for square integrable random walk on the plane, cf. [12]. Recall that $X$ has a density $f_X$. For simplicity we set $\gamma := 2 f_X(0)\, \mathbb{P}(\Omega^*)$.

**Theorem 2.** *Assume that $\alpha = 1$, and fix any $x \in \mathbb{R}$. Conditioned on $\Omega_x^*$, the variables $\varepsilon\, G(\mathbf{T}_\varepsilon^x)$ converge in law,*

$$\lim_{\varepsilon\to 0} \ \mathbb{P}\left(\gamma\varepsilon\, G(\mathbf{T}_\varepsilon^x) \leq t \,|\, \Omega_x^*\right) = \frac{t}{1+t} \quad \forall t > 0.$$

For $\alpha \in (1, 2]$, different limits distributions arise, and we obtain convergence in law of $\mathbf{T}_\varepsilon^x$ to the $\frac{1}{\beta}$-stable subordinator at an independent exponential time:

**Theorem 3.** *Assume that $\alpha \in (1, 2]$, and fix any $x \in \mathbb{R}$. Conditioned on $\Omega_x^*$, the variables $\varepsilon\, G(\mathbf{T}_\varepsilon^x)$ converge in law,*

$$\lim_{\varepsilon\to 0} \ \mathbb{P}\left(\Gamma\!\left(\frac{1}{\beta}\right)\frac{\gamma}{\beta}\, \varepsilon\, G(\mathbf{T}_\varepsilon^x) \leq t \,\middle|\, \Omega_x^*\right) = \Pr\left(\mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta} \leq t\right) \quad \forall t > 0,$$

*or, equivalently,*

$$\lim_{\varepsilon\to 0} \ \mathbb{P}\left(\left(\Gamma\!\left(\frac{1}{\beta}\right)\frac{\gamma}{\beta}\right)^{\beta} \frac{\mathbf{T}_\varepsilon^x}{G^{-1}(1/\varepsilon)} \leq t \,\middle|\, \Omega_x^*\right) = \Pr\left(\mathcal{E}^\beta \mathcal{G}_{1/\beta} \leq t\right) \quad \forall t > 0,$$

*where $\mathcal{E}$ and $\mathcal{G}_{1/\beta}$ are independent random variables, $\Pr(\mathcal{E} > t) = e^{-t}$, and $\mathcal{G}_{1/\beta}$ having the one-sided stable law of index $\frac{1}{\beta}$ with Laplace transform $\mathbb{E}[e^{-s\mathcal{G}_{1/\beta}}] = e^{-s^{1/\beta}}$, $s > 0$.*

In particular, we have:

**Corollary 1.** *If $(X_n)_{n\geq 1}$ is an i.i.d. sequence of centered random variables with variance 1, satisfying the Cramer condition, and $x \in \mathbb{R}$, then*

$$\lim_{\varepsilon\to 0} \ \mathbb{P}\left(2\mathbb{P}(\Omega_x^*)\, \varepsilon\sqrt{\mathbf{T}_\varepsilon^x} \leq t \,\middle|\, \Omega_x^*\right) = \Pr\left(\frac{\mathcal{E}}{|\mathcal{N}|} \leq t\right) \quad \forall t > 0,$$

*or, equivalently,*

$$\lim_{\varepsilon\to 0} \ \mathbb{P}\left(4\mathbb{P}(\Omega_x^*)^2\, \varepsilon^2 \mathbf{T}_\varepsilon^x \leq t \,\middle|\, \Omega_x^*\right) = \Pr\left(\left(\frac{\mathcal{E}}{|\mathcal{N}|}\right)^2 \leq t\right) \quad \forall t > 0,$$

*where $\mathcal{E}$ and $\mathcal{N}$ are independent variables, $\mathcal{N}$ having a standard Gaussian distribution $\mathcal{N}(0,1)$.*

As Cheliotis does in [6], we will use the following extension of Stone's local limit theorem [13].

**Proposition 1.** *Let $\theta$ be such that $\limsup_{|t|\to\infty} |\mathbb{E}[e^{itX_1}]| < \theta < 1$, and let $c > 1$. Then there exist a real number $h_0 > 0$ and an integer $n_0 \geq 1$ such that, for any $n \geq n_0$, for any interval $I$ contained in $[-h_0, h_0]$, of length larger than $\theta^n$, we have*

$$c^{-1} f_X(0)|I| < \mathbb{P}\left(\frac{S_n}{A_n} \in I\right) < c f_X(0)|I|.$$

## 2. Almost sure convergence : proof of theorem 1

*Proof of Theorem 1.* To begin with, choose $\theta$, $c$, and $h_0$ as in Proposition 1.

To first establish an estimate from below, we fix any $\xi > 1$ and set $\varepsilon_n := G(n)^{-\xi}$. This makes the series $\sum_n \mathbb{P}(|S_n| < \varepsilon_n)$ summable: Indeed, by regular variation and (3), we have $\frac{\varepsilon_n}{A_n} > \theta^n$ for $n$ large, while

$$\frac{\varepsilon_n}{A_n} = O\left(\frac{G(n) - G(n-1)}{G(n-1)^\xi}\right) = O\left(\int_{n-1}^{n} \frac{G'(t)}{G(t)^\xi}\, dt\right),$$

which is summable since $\displaystyle\int_1^\infty \frac{G'(t)}{G(t)^\xi}\, dt = \Big[\frac{G(t)^{1-\xi}}{1-\xi}\Big]_1^\infty < \infty$. In particular, $(\frac{-\varepsilon_n}{A_n}, \frac{\varepsilon_n}{A_n}) \subseteq [-h_0, h_0]$ for large $n$. Proposition 1 therefore applies to these intervals, and shows that $\mathbb{P}(|S_n| < \varepsilon_n) = O(\frac{\varepsilon_n}{A_n})$ is summable as well. Hence, by the Borel-Cantelli lemma, $\mathbb{P}(|S_n| < \varepsilon_n \text{ i.o.}) = 0$. Since $\varepsilon_n \searrow 0$, we can conclude that $\mathbf{T}_{\varepsilon_n} > n$ eventually, almost surely on $\Omega^*$, and we get $\liminf_{n\to\infty} \frac{\log G(\mathbf{T}_{\varepsilon_n})}{-\log \varepsilon_n} \geq \frac{1}{\xi}$ a.s. on $\Omega^*$. Using monotonicity of $\log G(\mathbf{T}_\varepsilon)$ and the fact that $\varepsilon_{n+1} \sim \varepsilon_n$, this extends from the $\varepsilon_n$ to the full limit as $\varepsilon \to 0$, and since $\xi > 1$ was arbitrary, we conclude that

$$\liminf_{\epsilon\to 0} \frac{\log G(\mathbf{T}_\varepsilon)}{-\log \varepsilon} \geq 1 \quad \text{a.s. on } \Omega^*. \tag{4}$$

To control the corresponding lim sup, we now fix any $\xi \in (0,1)$. From Proposition 1, using intervals $(\frac{-\varepsilon_n}{A_n}, \frac{\varepsilon_n}{A_n})$ and regular variation of $(A_n)_{n\geq 1}$, we see that there exists a constant $c' > 0$ such that for every $\varepsilon \in (0,1)$ there is some $m_\varepsilon$ satisfying

$$\mathbb{P}(|S_k| < \varepsilon) \geq \frac{c'\varepsilon}{A_k} \quad \text{for } k \geq m_\varepsilon.$$

More precisely, the dependence of $m_\varepsilon$ on $\varepsilon$ comes from the requirement $2\varepsilon/A_k > \theta^k$ for $k \geq m_\varepsilon$ on the length of intervals, which is met by taking $m_\varepsilon := \kappa(-\log\varepsilon)$ with a suitable constant $\kappa > 0$. Next, choose integers $n_\varepsilon$ in such a way that $G(n_\varepsilon) \leq \varepsilon^{-\frac{1}{\xi}} < G(n_\varepsilon + 1)$. Inspired by a decomposition used by Dvoretski and Erdös [8], we consider the pairwise disjoint events $E_k^\varepsilon := \{|S_k| < \varepsilon \text{ and } \forall j = k+1, \dots, n_\varepsilon : |S_j - S_k| > 2\varepsilon\}$, $1 \leq k \leq n_\varepsilon$. By independence and stationarity we have

$$1 \geq \sum_{k=m_\varepsilon}^{n_\varepsilon} \mathbb{P}(E_k^\varepsilon) \geq \sum_{k=m_\varepsilon}^{n_\varepsilon} \mathbb{P}(|S_k| < \varepsilon)\, \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon - k) \geq c'\varepsilon\, \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon) \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{1}{A_k}.$$

Combining this with $G(m_\varepsilon) = o(G(n_\varepsilon))$ (note that $G(m_\varepsilon)$ is slowly varying), we obtain

$$\mathbb{P}(G(\mathbf{T}_{2\varepsilon}) > \varepsilon^{-\frac{1}{\xi}}) \leq \mathbb{P}(G(\mathbf{T}_{2\varepsilon}) > G(n_\varepsilon)) = \mathbb{P}(\mathbf{T}_{2\varepsilon} > n_\varepsilon) \leq \frac{1}{c'\varepsilon\,(G(n_\varepsilon) - G(m_\varepsilon))} \sim \frac{\varepsilon^{\frac{1}{\xi}-1}}{c'}.$$

Therefore, if we let $\varepsilon_p := p^{-\frac{2}{1-\xi}}$, $p \geq 1$ the Borel-Cantelli lemma implies $G(\mathbf{T}_{2\varepsilon_p}) \leq \varepsilon_p^{-\frac{1}{\xi}}$ eventually almost surely, showing that $\limsup_{p\to+\infty} \frac{\log G(\mathbf{T}_{2\varepsilon_p})}{-\log(2\varepsilon_p)} \leq \frac{1}{\xi}$. Using monotonicity as before, we can extend this from the $\varepsilon_p$ to the full limit $\varepsilon \to 0$, and since this is true for any $\xi \in (0,1)$, we obtain

$$\limsup_{\varepsilon\to 0} \frac{\log G(\mathbf{T}_\varepsilon)}{-\log(\varepsilon)} \leq 1 \quad \text{a.s. on } \Omega. \tag{5}$$

To conclude the proof, we note that for any $\alpha \in [1,2]$ we have

$$\lim_{n\to\infty} \frac{\log G(n)}{\log n} = \frac{1}{\beta},$$

which follows readily from regular variation of $G$ (compare Fact 2 in [6]). Together with (4) and (5), this entails

$$\lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{-\log \varepsilon} = \lim_{\varepsilon\to 0} \frac{\log \mathbf{T}_\varepsilon}{\log G(\mathbf{T}_\varepsilon)} \cdot \frac{\log G(\mathbf{T}_\varepsilon)}{-\log \varepsilon} = \beta \quad \text{a.s. on } \Omega^*,$$

as required. □

The first argument can easily be adapted to prove the lower bound (4) also for $\mathbf{T}_\varepsilon^x$ with $x \neq 0$.

## 3. Convergence in distribution for auxiliary processes

We need to introduce auxiliary processes. Let $(M_0^\varepsilon)_{\varepsilon>0}$ be a family of random variables, independent of $(S_n)_{n\geq 0}$, such that $M_0^\varepsilon$ has uniform distribution on the interval $(-\varepsilon, \varepsilon)$. For each $\varepsilon > 0$ we define the walk $(M_n^\varepsilon)_{n\geq 0}$ with random initial position $M_0^\varepsilon$, that is, $M_n^\varepsilon := M_0^\varepsilon + S_n$.

A major step towards Theorems 2 and 3 will be to prove a version which applies to the variables

$$\tau_\varepsilon := \min\{n \geq 1 \colon |M_n^\varepsilon| < \varepsilon\}, \quad \varepsilon > 0.$$

That is, we are interested in the limiting behaviour, as $\varepsilon \to 0$, of the first return time distribution of the walk $(M_n^\varepsilon)_{n\geq 0}$ to the interval $(-\varepsilon, \varepsilon)$. The goal of the present section is to establish

**Theorem 4.** *Assume that $\alpha = 1$. Conditioned on $\Omega^*$, the variables $\varepsilon\, G(\tau_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\gamma\varepsilon\, G(\tau_\varepsilon) \leq t \,|\, \Omega^*\right) = \frac{t}{1+t} \quad \forall t > 0. \tag{6}$$

**Theorem 5.** *Assume that $\alpha \in (1,2]$. Conditioned on $\Omega^*$, the variables $\varepsilon\, G(\tau_\varepsilon)$ converge in law,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\Gamma\Big(\frac{1}{\beta}\Big)\frac{\gamma}{\beta}\, \varepsilon\, G(\tau_\varepsilon) \leq t \,\middle|\, \Omega^*\right) = \Pr\left(\mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta} \leq t\right) \quad \forall t > 0. \tag{7}$$

*Equivalently,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\left(\Gamma\Big(\frac{1}{\beta}\Big)\frac{\gamma}{\beta}\right)^\beta \frac{\tau_\varepsilon}{G^{-1}(1/\varepsilon)} \leq t \,\middle|\, \Omega^*\right) = \Pr\left(\mathcal{E}^\beta \mathcal{G}_{1/\beta} \leq t\right) \quad \forall t > 0.$$

Again we start with considerations valid for any $\alpha \in [1,2]$. To begin with, we define, for $\varepsilon > 0$, $R > 0$, and integers $K > 0$, auxiliary events

$$\Gamma_{\varepsilon,R,K} := \{\forall i = 1, \dots, K : S_i \neq 0 \text{ and } |M_i^\varepsilon| \leq R\},$$

which asymptotically exhaust $\Omega^*$, and on which we can work conveniently. As $\varepsilon \to 0$ we have $\mathbb{P}(\Gamma_{\varepsilon,R,K}) \to \mathbb{P}(\Gamma_{R,K})$ and $\mathbb{P}(\Gamma_{\varepsilon,R,K} \setminus \Omega^*) \to \mathbb{P}(\Gamma_{R,K} \setminus \Omega^*)$, where $\Gamma_{R,K} := \{\forall i = 1, \dots, K : 0 <$

$|S_i| \le R\}$ (except, perhaps, for a countable set of $R$'s which we are going to avoid). Let $n \in \mathbb{N}$. Using again a decomposition similar to that of Dvoretski and Erdös in [8], we find, for $\varepsilon \in (0, \frac{1}{2})$,

$$\mathbb{P}(\Gamma_{\varepsilon,R,K}) = \sum_{k=0}^{n} p_k^- = \sum_{k=0}^{n} p_k^+ \tag{8}$$

with $p_k^\pm = p_{k,n,\varepsilon,R,K}^\pm := \mathbb{P}(\Gamma_{\varepsilon,R,K} \cap \{|M_k^\varepsilon| < \varepsilon \pm 2\varepsilon^2 \text{ and } \forall \ell = k+1, \dots, n : |M_\ell^\varepsilon| \ge \varepsilon \pm 2\varepsilon^2\})$ for $1 \le k \le n$, and $p_0^\pm = p_{0,n,\varepsilon,R,K}^\pm := \mathbb{P}(\Gamma_{\varepsilon,R,K} \cap \{\forall \ell = 1, \dots, n : |M_\ell^\varepsilon| \ge \varepsilon \pm 2\varepsilon^2\})$. The following estimates are the basis of the argument to follow.

**Lemma 1.** *For arbitrary $R$, $K$, and $0 < \gamma' < 2f_X(0) < \gamma''$, there is some $\varepsilon_1$ such that for $0 < \varepsilon < \varepsilon_1$ and $n_\varepsilon > m_\varepsilon \ge (\log \varepsilon)^4$,*

$$\mathbb{P}(\Gamma_{\varepsilon,R,K}) \ge \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma'\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} - \mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma'\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^{K}\{|M_i^\varepsilon| > R\}),$$

*and*

$$\mathbb{P}(\Gamma_{\varepsilon,R,K}) \le \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} + \mathbb{P}(\Gamma_{\varepsilon,R,K} \setminus \Omega^*) + \mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma''\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) + \mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \le m_\varepsilon\}).$$

*Proof.* For the course of this proof, we simplify notations by suppressing the parameters $\varepsilon$, $R$, and $K$ in $m_\varepsilon$, $n_\varepsilon$, $M_i^\varepsilon$, and $\Gamma_{\varepsilon,R,K}$. We will apply (8) with $n = n_\varepsilon$. Also, let $\nu := \varepsilon^2$.

*(i)* Starting with the $k = 0$ term, we see that

$$p_0^- \ge \mathbb{P}(\Gamma \cap \{\forall \ell = 1, \dots, n : |M_\ell| \ge \varepsilon\}) \ge \mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}).$$

We now consider the case where $m \le k \le n$. Let $\mathcal{A} := (2\nu\mathbb{Z}) \cap (-\varepsilon + 3\nu, \varepsilon - 3\nu)$. Notice that the sets $Q_a := (a - \nu, a + \nu)$ with $a \in \mathcal{A}$ are disjoint and contained in $(-\varepsilon + 2\nu, \varepsilon - 2\nu)$. Therefore the $k$th term in equation (8) satisfies

$$\begin{aligned} p_k^- &\ge \sum_{a\in\mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a \text{ and } \forall \ell = k+1, \dots, n : \ |M_\ell| \ge \varepsilon - 2\nu\}) \\ &\ge \sum_{a\in\mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a \text{ and } \forall \ell = k+1, \dots, n : |S_\ell - S_k + a| \ge \varepsilon - \nu\}) \\ &= \sum_{a\in\mathcal{A}} \mathbb{P}(\Gamma \cap \{M_k \in Q_a\})\ \mathbb{P}(\forall \ell = 1, \dots, n-k : |S_\ell + a| \ge \varepsilon - \nu) \end{aligned} \tag{9}$$

by independence (where we assume that $\varepsilon$ is so small that $(\log \varepsilon)^4 > K$). Note that

$$\mathbb{P}(\Gamma \cap \{M_k \in Q_a\}) = \int_{\{\forall i : x_i \ne x_0, |x_i| \le R\}} \mathbb{P}(S_{k-K} \in Q_a - x_K)\ d\mathbb{P}_{(M_0,\dots,M_K)}(x_0, \dots, x_K),$$

with $d\mathbb{P}_{(M_0,\dots,M_K)}$ denoting the distribution of $(M_0, \dots, M_K)$. Now fix $\theta$ as in Proposition 1, and $c \in (0, 1)$ such that $\gamma' < 2f_X(0)/c$. Elementary considerations (based on our condition on $m = m_\varepsilon$) show that Proposition 1 applies to $I = \frac{1}{A_{k-K}}(Q_a - x_K)$ if $\varepsilon$ is sufficiently small, and in this case gives

$$\mathbb{P}(\Gamma \cap \{M_k \in Q_a\}) \ge \mathbb{P}(\Gamma)\frac{\gamma'\nu}{A_k}. \tag{10}$$

Using this, plus the observation that conditioning on $\{M_0 \in Q_a\}$ amounts to looking at $M_n^* := M_0^* + S_n$, $n \geq 0$, with $M_0^*$ uniformly distributed on $Q_a$, we can continue to estimate, for small $\varepsilon$,

$$\begin{aligned} p_k^- &\geq \mathbb{P}(\Gamma)\frac{\gamma'\nu}{A_k}\sum_{a\in\mathcal{A}}\mathbb{P}(\forall \ell = 1,\dots,n-k : |S_\ell + a| \geq \varepsilon - \nu\}) \\ &\geq \mathbb{P}(\Gamma)\frac{\gamma'\nu}{A_k}\sum_{a\in\mathcal{A}}\mathbb{P}(\{\forall \ell = 1,\dots,n-k : |M_\ell| \geq \varepsilon\} \mid \{M_0 \in Q_a\}) \\ &\geq \mathbb{P}(\Gamma)\frac{\gamma'\varepsilon}{A_k}\sum_{a\in\mathcal{A}}\mathbb{P}(\{\forall \ell = 1,\dots,n-k : |M_\ell| \geq \varepsilon\} \cap \{M_0 \in Q_a\}) \\ &\geq \mathbb{P}(\Gamma)\frac{\gamma'\varepsilon}{A_k}\left(\mathbb{P}(\forall \ell = 1,\dots,n-k : |M_\ell| \geq \varepsilon) - \mathbb{P}(\varepsilon - 4\nu \leq |M_0| \leq \varepsilon)\right) \\ &= \mathbb{P}(\Gamma)\frac{\gamma'\varepsilon}{A_k}\left(\mathbb{P}(\tau_\varepsilon > n-k) - 8\nu\right). \end{aligned} \tag{11}$$

Putting together these estimates via Equation (8) gives

$$\mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}) + \mathbb{P}(\Gamma)\gamma'\varepsilon\sum_{k=m}^{n}\frac{\mathbb{P}(\tau_\varepsilon > n-k)}{A_k} \leq \mathbb{P}(\Gamma) + \mathbb{P}(\Gamma)8\gamma'\varepsilon\nu(G(n) - G(m)).$$

Since $\Gamma^c \cap \{\tau_\varepsilon > n\} \subseteq \bigcup_{i=1}^K \{|M_i| > R\}$ for $\varepsilon$ so small that $n = n_\varepsilon > K$, this proves the first assertion of the lemma.

*(ii)* We only provide a sketch of the proof of the second point since the arguments are very similar to the above. Using Equation (8) gives

$$\mathbb{P}(\Gamma) \leq \mathbb{P}(\Gamma \cap \{\tau_\varepsilon > n\}) + \mathbb{P}(\Gamma \setminus \Omega^*) + \mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m\}) + \sum_{k=m}^{n} p_k^+,$$

since $\sum_{k=1}^m p_k^+ \leq \mathbb{P}(\Gamma \cap \{\tau_{3\varepsilon} \leq m\})$. Next, take $\bar{\mathcal{A}} := (2\nu\mathbb{Z}) \cap (-\varepsilon - 3\nu, \varepsilon + 3\nu)$ and intervals $\bar{Q}_a := [a - \nu, a + \nu]$, $a \in \bar{\mathcal{A}}$, which cover $(-\varepsilon - 2\nu, \varepsilon + 2\nu)$. We can then use arguments parallel to those of part (i) to obtain

$$\begin{aligned} \sum_{k=m}^{n} p_k^+ &\leq \sum_{k=m}^{n}\sum_{a\in\bar{\mathcal{A}}}\mathbb{P}(\Gamma \cap \{M_k \in \bar{Q}_a \text{ and } \forall \ell = k+1, ..., n : \ |M_\ell| > \varepsilon + 2\nu)\}) \\ &\vdots \\ &\leq \mathbb{P}(\Gamma)\gamma''\varepsilon\sum_{k=m}^{n}\frac{\mathbb{P}(\tau_\varepsilon > n-k)}{A_k} + \mathbb{P}(\Gamma)8\gamma''\varepsilon\nu(G(n) - G(m)), \end{aligned}$$

which proves our claim. □

Suitable choice of the $n_\varepsilon$ then enables us to derive an asymptotic bound for the tails of the distributions of the $\varepsilon G(\tau_\varepsilon)$ as $\varepsilon \to 0$.

**Lemma 2.** *For all $\alpha \in [1, 2]$ and any $t > 0$ we have*

$$\limsup_{\varepsilon\to 0}\mathbb{P}(\gamma\varepsilon G(\tau_\varepsilon) > t) \leq \frac{\mathbb{P}(\Omega^*)}{1+t}.$$

*Proof.* Fix $t$, $R$, $K$, and $0 < \gamma' < 2f_X(0)$. For $\varepsilon > 0$ we take $m_\varepsilon := (\log\varepsilon)^4$ and choose $n_\varepsilon$ so that $G(n_\varepsilon) \leq \frac{t}{\gamma\varepsilon} \leq G(n_\varepsilon + 1)$, whence $\mathbb{P}(\varepsilon\gamma G(\tau_\varepsilon) > t) \sim \mathbb{P}(\tau_\varepsilon > n_\varepsilon)$. As in the proof of Theorem

1 we see that $G(m_\varepsilon) = o(G(n_\varepsilon))$. Therefore

$$\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \geq \varepsilon(G(n_\varepsilon) - G(m_\varepsilon))\mathbb{P}(\tau_\varepsilon > n_\varepsilon) \sim \frac{t}{\gamma}\,\mathbb{P}(\tau_\varepsilon > n_\varepsilon). \tag{12}$$

Together with the first part of Lemma 1, this yields

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\varepsilon\gamma G(\tau_\varepsilon) > t) \leq \frac{\mathbb{P}(\Gamma_{R,K}) + \mathbb{P}(\exists 1 \leq i \leq K : |S_i| > R-1)}{1 + t\frac{\gamma'}{\gamma}\mathbb{P}(\Gamma_{R,K})},$$

since $8\gamma'\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) \to 0$.

Taking successively $R \to \infty$, then $K \to \infty$, and finally $\gamma' \to 2f_X(0)$, we obtain the lemma. □

When $\alpha = 1$, this upper bound actually is the limit:

**Lemma 3.** *If $\alpha = 1$, then for any $t > 0$ we have*

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\gamma\varepsilon G(\tau_\varepsilon) > t) \geq \frac{\mathbb{P}(\Omega^*)}{1+t}.$$

*Proof.* Fix $t$, $R$, $K$, and $\gamma'' > 2f_X(0)$, and choose $m_\varepsilon$ and $n_\varepsilon$ as in the previous proof. Similar to that situation we have $\mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma''\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) \to 0$, and, as a consequence of Theorem 1, also $\mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m_\varepsilon\}) \to 0$.

Since $\alpha = 1$ means that $G$ is slowly varying, we have $G(2n_\varepsilon) - G(n_\varepsilon) = o(G(n_\varepsilon))$. Hence

$$\begin{aligned}
&\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{2n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon - k)}{A_k} \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon \left( \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon)}{A_k} + \sum_{k=n_\varepsilon}^{2n_\varepsilon} \frac{1}{A_k} \right) \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma''\varepsilon G(n_\varepsilon)[\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1)] \\
&\leq \mathbb{P}(\tau_\varepsilon > n_\varepsilon) + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{\varepsilon,R,K})\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1).
\end{aligned} \tag{13}$$

Combining these observations with the second estimate of Lemma 1 (replacing $n_\varepsilon$ by $2n_\varepsilon$) entails

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\tau_\varepsilon > n_\varepsilon) \geq \frac{\mathbb{P}(\Gamma_{R,K}) - \mathbb{P}(\Gamma_{R,K} \setminus \Omega^*)}{1 + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{R,K})}.$$

We conclude by successively taking $R \to \infty$, $K \to \infty$, and $\gamma'' \to 2f_X(0)$. □

*Proof of Theorem 4.* Immediate from Lemmas 2 and 3, as $\varepsilon G(\tau_\varepsilon) \to 0$ outside $\Omega^*$. □

When $\alpha \in (1, 2]$, Lemma 1 does not yet give the limit distribution. Still, it immediately implies the tightness of the family of distributions with the normalisation given there:

**Lemma 4.** *The family of distributions of the random variables $\varepsilon G(\tau_\varepsilon)$, $\varepsilon \in (0, 1)$, is tight.*

Hence it will be enough to prove that the advertised limit law is the only possible accumulation point of our distributions. We henceforth abbreviate

$$Z_\varepsilon := \frac{\gamma}{\beta}\,\varepsilon G(\tau_\varepsilon), \quad \varepsilon > 0.$$

**Lemma 5.** *Suppose that $\alpha \in (1,2]$. Let $(\varepsilon_p)_{p\geq 1}$ be a positive sequence with $\lim_{p\to\infty} \varepsilon_p = 0$, and such that the conditional distributions of the $Z_{\varepsilon_p}$ on $\Omega^*$ converge to the law of some random variable $Y$. Then its tail satisfies the integral equation*

$$1 = \Pr(Y > t) + t \int_0^1 \frac{\Pr(Y > t(1-u)^{\frac{1}{\beta}})}{u^{\frac{1}{\alpha}}}\, du \quad \forall t > 0.$$

*Proof. (i)* We write $f(t) := \Pr(Y > t)$, and first prove that

$$\forall t > 0, \quad 1 \geq f(t) + t \int_0^1 u^{-\frac{1}{\alpha}} f(t(1-u)^{\frac{1}{\beta}})\, du.$$

Let us only consider $\varepsilon$ belonging to $\{\varepsilon_p,\ p \geq 1\}$. Note that by monotonicity and right continuity of $f$ it suffices to prove the inequality for all $t \in (0,\infty)$ such that, for all $N \geq 1$ and all $r = 0, ..., N-1$, the function $f$ is continuous at $t\left(1 - \frac{r}{N}\right)^{\frac{1}{\beta}}$. Henceforth such a $t$ will be fixed.

Now take some $\delta > 0$, and choose $N_\delta > 1$ such that for all $N \geq N_\delta$,

$$\left| \int_0^1 \frac{f(t(1-u)^{\frac{1}{\beta}})}{u^{\frac{1}{\alpha}}}\, du - \frac{1}{N} \sum_{r=1}^{N-1} \frac{f(t(1-(r/N))^{\frac{1}{\beta}})}{((r+1)/N)^{\frac{1}{\alpha}}} \right| \leq \delta.$$

Now fix integers $N \geq N_\delta$, $K \geq 1$, and some $0 < \gamma' < 2f_X(0)$. For $\varepsilon > 0$ small enough take $n_\varepsilon$ such that $G(n_\varepsilon) \leq \frac{\beta t}{\gamma\varepsilon} < G(n_\varepsilon + 1)$ (and hence $G(n_\varepsilon) \sim \frac{\beta t}{\gamma\varepsilon}$). Finally, let $m_\varepsilon := n_\varepsilon/N$.

According to the first point of Lemma 1, we have

$$\begin{aligned}\mathbb{P}(\Gamma_{\varepsilon,R,K}) \geq \mathbb{P}(Z_\varepsilon > t) + \mathbb{P}(\Gamma_{\varepsilon,R,K})\gamma'\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \\ - \mathbb{P}(\Gamma_{\varepsilon,R,K}) 8\gamma'\varepsilon^3 (G(n_\varepsilon) - G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^{K} \{|M_i^\varepsilon| > R\}),\end{aligned}$$

Due to our assumption on the $Z_{\varepsilon_p}$ and $t$, we see that $\mathbb{P}(Z_\varepsilon > t) \to \mathbb{P}(\Omega^*) f(t)$ as $\varepsilon_p \to 0$. Next, by monotonicity,

$$\begin{aligned}\sum_{k=n_\varepsilon/N}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} &\geq \sum_{r=1}^{N-1} \sum_{k=0}^{n_\varepsilon/N-1} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k - (rn_\varepsilon/N))}{A_{k+(rn_\varepsilon/N)}} \\ &\geq \sum_{r=1}^{N-1} \left( G\left(\frac{r+1}{N}\, n_\varepsilon\right) - G\left(\frac{r}{N}\, n_\varepsilon\right)\right) \mathbb{P}\left(\tau_\varepsilon > \left(1 - \frac{r}{N}\right) n_\varepsilon\right).\end{aligned}$$

By regular variation, the first term of the product is asymptotically equivalent to

$$G(n_\varepsilon) \left[ \left(\frac{r+1}{N}\right)^{\frac{1}{\beta}} - \left(\frac{r}{N}\right)^{\frac{1}{\beta}} \right] \geq \frac{G(n_\varepsilon)}{\beta N \left(\frac{r+1}{N}\right)^{\frac{1}{\alpha}}}$$

as $\varepsilon_p \to 0$. On the other hand, the second term is equal to

$$\mathbb{P}\left( Z_\varepsilon > \varepsilon \frac{\gamma}{\beta} G\left(\left(1 - \frac{r}{N}\right) n_\varepsilon\right)\right) \to \mathbb{P}(\Omega^*)\ f\left(t\left(1 - \frac{r}{N}\right)^{\frac{1}{\beta}}\right),$$

since $G\left(\left(1-\frac{r}{N}\right)n_\varepsilon\right) \sim \left(1-\frac{r}{N}\right)^{\frac{1}{\beta}} G(n_\varepsilon)$. As a consequence, we see that

$$\begin{aligned}\liminf_{p\to\infty} \varepsilon_p \sum_{k=n_{\varepsilon_p}/N}^{n_{\varepsilon_p}} \frac{\mathbb{P}(\tau_{\varepsilon_p} > n_{\varepsilon_p}-k)}{A_k} &\geq \mathbb{P}(\Omega^*)\frac{t}{\gamma}\frac{1}{N}\sum_{r=1}^{N-1} \frac{f(t\left(1-\frac{r}{N}\right)^{1-\frac{1}{\alpha}})}{\left(\frac{r+1}{N}\right)^{\frac{1}{\alpha}}} \\ &\geq \mathbb{P}(\Omega^*)\frac{t}{\gamma}\left(\int_0^1 \frac{f(t(1-u)^{1-\frac{1}{\alpha}})}{u^{\frac{1}{\alpha}}}\,du - \delta\right).\end{aligned} \tag{14}$$

Furthermore, we again have $\mathbb{P}(\Gamma_{\varepsilon,R,K})8\gamma'\varepsilon^3(G(n_\varepsilon)-G(m_\varepsilon)) \to 0$. Moreover, $\mathbb{P}(\bigcup_{i=1}^K\{|M_i^\varepsilon| > R\}) \to \mathbb{P}(\bigcup_{i=1}^K\{|S_i| > R\})$. Combining all these asymptotic estimates and taking the limit $\varepsilon_p \to 0$, we end then up with

$$\mathbb{P}(\Gamma_{R,K}) \geq \mathbb{P}(\Omega^*)\left[f(t) + \frac{\mathbb{P}(\Gamma_{R,K})\gamma' t}{\gamma}\left(\int_0^1 \frac{f(t(1-u)^{1-\frac{1}{\alpha}})}{u^{\frac{1}{\alpha}}}\,du - \delta\right)\right] - \mathbb{P}(\bigcup_{i=1}^K\{|S_i| > R\}).$$

Successively letting $R\to\infty$, $K\to\infty$, $\gamma'\to 2f_X(0)$ and $\delta\to 0$ we obtain the desired inequality.

*(ii)* The converse inequality is proved analogously, using the other half of Lemma 1 and the fact that $\mathbb{P}(\Omega^* \cap \{\tau_{3\varepsilon} \leq m_\varepsilon\}) = o(1)$. □

Now let us identify the limit distribution satisfying the equality given by Lemma 5. To this end we consider the variables

$$Z'_\varepsilon := \left(\frac{\gamma}{\beta}\right)^\beta \frac{\tau_\varepsilon}{G^{-1}(1/\varepsilon)}, \quad \varepsilon > 0.$$

**Lemma 6.** *The conditional distributions of the $Z_{\varepsilon_p}$ converge to a random variable $Y$ iff the conditional distributions of the $Z'_{\varepsilon_p}$ converge to $Y^\beta$. The latter then satisfies*

$$1 = \Pr(Y^\beta > t) + \int_0^t \frac{\Pr(Y^\beta > t-v)}{v^{\frac{1}{\alpha}}}\,dv \quad \forall t > 0.$$

*Proof.* The equivalence of the two conditional distributional convergence statements follows from regular variation of $G^{-1}$, see e.g. Lemma 1 of [4]. Suppose that they hold. Then, according to Lemma 5, for any $t > 0$, we have

$$1 = \Pr(Y^\beta > t) + t^{\frac{1}{\beta}}\int_0^1 \frac{\Pr(Y^\beta > t(1-u))}{u^{\frac{1}{\alpha}}}\,du,$$

and the conclusion follows by a change of variables, $v = tu$. □

**Lemma 7.** *Let $W$ be a random variable with values in $[0,\infty)$ satisfying*

$$\Pr(W \leq t) = \int_0^t \frac{\Pr(W > t-v)}{v^{\frac{1}{\alpha}}}\,dv \quad \forall t > 0. \tag{15}$$

*Then*

$$\mathbb{E}\left[e^{-sW}\right] = \frac{1}{1+c_\beta s^{\frac{1}{\beta}}} \quad \forall s > 0,$$

*with $c_\beta := \Gamma(\frac{1}{\beta})^{-1}$. In particular, the distribution of $W$ coincides with that of $c_\beta^\beta \mathcal{E}^\beta \mathcal{G}_{\frac{1}{\beta}}$, where the independent variables $\mathcal{E}$ and $\mathcal{G}_{\frac{1}{\beta}}$ are as in the statement of Theorem 3.*

*Proof.* Let $s > 0$. We have

$$\mathbb{E}[e^{-sW}] = \int_0^{+\infty} \Pr(e^{-sW} \geq u)\, du = \int_0^{+\infty} \Pr\left(W \leq -\frac{\log(u)}{s}\right) du = \int_0^{+\infty} \Pr\left(W \leq v\right) s e^{-sv}\, dv.$$

Hence, for any $s > 0$, we find

$$\begin{aligned}
\mathbb{E}[e^{-sW}] &= \int_0^{+\infty} \left[\int_0^v \frac{\Pr(W \geq v - w)}{w^{\frac{1}{\alpha}}}\, dw\right] s e^{-sv}\, dv \\
&= \int_0^{+\infty} \frac{1}{w^{\frac{1}{\alpha}}} \left[\int_w^{+\infty} \Pr(W \geq v - w) s e^{-sv}\, dv\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}} \left[\int_0^{+\infty} \Pr(W \geq z) s e^{-sz}\, dz\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}} \left[1 - \int_0^{+\infty} \Pr(W \leq z) s e^{-sz}\, dz\right] dw \\
&= \int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}}\, dw \cdot \left[1 - \mathbb{E}[e^{-sW}]\right],
\end{aligned}$$

and our claim about the Laplace transform of $W$ follows since

$$\int_0^{+\infty} \frac{e^{-sw}}{w^{\frac{1}{\alpha}}}\, dw = \frac{\beta}{s^{\frac{1}{\beta}}} \int_0^{+\infty} e^{-z^\beta}\, dz = \frac{1}{c_\beta s^{\frac{1}{\beta}}} \quad \text{with} \quad c_\beta := \frac{1}{\Gamma(\frac{1}{\beta})}.$$

Given this, a routine calculation (cf. XIII.11.10 of [9]) shows that $W$ indeed has the same Laplace transform as $c_\beta^\beta \mathcal{E}^\beta \mathcal{G}_{\frac{1}{\beta}}$. □

*Proof of Theorem 5.* According to Lemma 4 the family of distributions of the $Z_\varepsilon$, $\varepsilon \in (0,1)$, is tight. By Lemma 5, Lemma 6 and Lemma 7, the law of $c_\beta\, \mathcal{E}\mathcal{G}_{1/\beta}^{1/\beta}$ is the only possible accumulation point of these distributions. □

## 4. Convergence in distribution for $\mathbf{T}_\varepsilon^x$

To complete the proof of Theorems 2 and 3 we now utilize Theorems 4 and 5. Note first that it suffices to prove Theorems 2 and 3 under the additional assumption that $S_0' = 0$, in which case

$$\mathbf{T}_\varepsilon^x = \hat{\mathbf{T}}_\varepsilon^x := \inf\{n \geq 1 : |S_n - x| < \varepsilon\} \text{ and } \Omega_x^* = \hat{\Omega}_x^* := \{S_n \neq x\ \forall n\}.$$

Indeed, in the situation of Theorem 2, with arbitrary distribution $P$ of $S_0'$, we then have

$$\mathbb{P}\left(\gamma\varepsilon\, G(\mathbf{T}_\varepsilon^x) \leq t\right) = \int_{\mathbb{R}} \mathbb{P}\left(\gamma\varepsilon\, G(\hat{\mathbf{T}}_\varepsilon^{x-y}) \leq t\right) dP(y) \to \int_{\mathbb{R}} \mathbb{P}\left(\hat{\Omega}_{x-y}^*\right) dP(y) \cdot \frac{t}{1+t}$$

by the $P = \delta_0$ case of Theorem 2 and dominated convergence. Analogously for Theorem 3.

Therefore, for the remainder of this section we assume that $S_0' = 0$.

Next, we observe that our key lemma (Lemma 1) can be adapted as follows. Let $\Gamma_{R,K}^x$ be the event defined by

$$\Gamma_{R,K}^x := \{\forall i = 1, \ldots, K : S_i \neq x \text{ and } |S_i| \leq R\}.$$

**Lemma 8.** *For arbitrary $R$, $K$, and $0 < \gamma' < 2f_X(0) < \gamma''$, there is some $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ and $n_\varepsilon > m_\varepsilon \geq (\log \varepsilon)^4$,*

$$\mathbb{P}(\Gamma^x_{R,K}) \geq \mathbb{P}(\mathbf{T}^x_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma^x_{R,K})\gamma'\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k}$$
$$- \mathbb{P}(\Gamma^x_{R,K})8\gamma'\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) - \mathbb{P}(\bigcup_{i=1}^{K}\{|S_i| > R\}),$$

*and*

$$\mathbb{P}(\Gamma^x_{R,K}) \leq \mathbb{P}(\mathbf{T}^x_\varepsilon > n_\varepsilon) + \mathbb{P}(\Gamma^x_{R,K})\gamma''\varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} + \mathbb{P}(\Gamma^x_{R,K} \setminus \Omega^*_x)$$
$$+ \mathbb{P}(\Gamma^x_{R,K})8\gamma''\varepsilon^3(G(n_\varepsilon) - G(m_\varepsilon)) + \mathbb{P}(\Omega^*_x \cap \{\mathbf{T}_{3\varepsilon} \leq m_\varepsilon\}).$$

*Proof of Lemma 8.* We have the following analogue of formula (8),

$$\mathbb{P}(\Gamma^x_{R,K}) = \sum_{k=0}^{n_\varepsilon} p_k^{x,-} = \sum_{k=0}^{n_\varepsilon} p_k^{x,+}, \tag{16}$$

with

$$p_0^{x,\pm} := \mathbb{P}(\Gamma^x_{R,K} \cap \{\forall \ell = 1, \ldots, n_\varepsilon : |S_\ell - x| \geq \varepsilon \pm 2\varepsilon^2\})$$

and

$$p_k^{x,\pm} := \mathbb{P}(\Gamma^x_{R,K} \cap \{|S_k - x| < \varepsilon \pm 2\varepsilon^2 \text{ and } \forall \ell = k+1, \ldots, n_\varepsilon : |S_\ell - x| \geq \varepsilon \pm 2\varepsilon^2\}).$$

We follow the proof of Lemma 1.

*(i)* Observe first that

$$p_0^{x,-} \geq \mathbb{P}(\Gamma^x_{R,K} \cap \{\mathbf{T}^x_\varepsilon > n_\varepsilon\}).$$

Now consider indices with $m_\varepsilon \leq k \leq n_\varepsilon$. With the same set $\mathcal{A}$ as in the proof of Lemma 1, we find, arguing as in (9), that

$$p_k^{x,-} \geq \sum_{a\in\mathcal{A}} \mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a \text{ and } \forall \ell = k+1, \ldots, n_\varepsilon : \ |S_\ell - x| \geq \varepsilon - 2\nu\})$$
$$\geq \sum_{a\in\mathcal{A}} \mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a\})\, \mathbb{P}(\forall \ell = 1, \ldots, n_\varepsilon - k : |S_\ell + a| \geq \varepsilon - \nu).$$

A proof parallel to that of (10) shows that

$$\mathbb{P}(\Gamma^x_{R,K} \cap \{S_k - x \in Q_a\}) \geq \mathbb{P}(\Gamma^x_{R,K})\frac{\gamma'\nu}{A_k}$$

if $\varepsilon$ is sufficiently small. Therefore,

$$p_k^{x,-} \geq \mathbb{P}(\Gamma^x_{R,K})\frac{\gamma'\nu}{A_k} \sum_{a\in\mathcal{A}} \mathbb{P}(\forall \ell = 1, \ldots, n_\varepsilon - k : |S_\ell + a| \geq \varepsilon - \nu\})$$
$$\geq \mathbb{P}(\Gamma^x_{R,K})\frac{\gamma'\varepsilon}{A_k}\left(\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k) - 8\nu\right),$$

where the second step uses an estimate contained in (11). Continuing as in the proof of Lemma 1, we obtain the first assertion of our lemma.

*(ii)* Similar adaptations give the second assertion of the lemma. □

We can now complete the proofs of our main distributional limit theorems:

*Proof of Theorem 2.* We go back to Lemmas 2 and 3, observing that we already have (6) at our disposal. Take $t \in (0,\infty)$, $R, K \geq 1$, and $\gamma' < 2f_X(0) < \gamma''$. For $\varepsilon > 0$ let $m_\varepsilon := (\log \varepsilon)^4$ and choose $n_\varepsilon$, such that $G(n_\varepsilon) \leq \frac{t}{\gamma\varepsilon} \leq G(n_\varepsilon + 1)$, meaning that $\mathbb{P}(\varepsilon\gamma G(\mathbf{T}_\varepsilon^x) > t) \sim \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon)$.

In view of (6), the estimate (12) of Lemma 2 becomes

$$\liminf_{\varepsilon\to 0} \varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} \geq \frac{\mathbb{P}(\Omega^*)}{\gamma} \frac{t}{1+t}.$$

Combining this with the first part of Lemma 8 leads to

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Gamma_{R,K}^*) \left(1 - \frac{\gamma'}{2f_X(0)} \frac{t}{1+t}\right) + \mathbb{P}(\exists 1 \leq i \leq K : |S_i| > R-1).$$

Successively letting $R \to \infty$, then $K \to \infty$, and finally $\gamma' \to 2f_X(0)$, we obtain

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \frac{\mathbb{P}(\Omega_x^*)}{1+t}.$$

To get the corresponding lower bound, recall that $\mathbb{P}(\Omega^* \cap \{\mathbf{T}_{3\varepsilon}^x \leq m_\varepsilon\}) \to 0$ by Theorem 1. Parallel to (13) we have

$$\mathbb{P}(\mathbf{T}_\varepsilon^x > 2n_\varepsilon) + \mathbb{P}(\Gamma_{R,K}^x)\gamma''\varepsilon \sum_{k=m_\varepsilon}^{2n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > 2n_\varepsilon - k)}{A_k} \leq \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) + t\frac{\gamma''}{\gamma}\mathbb{P}(\Gamma_{R,K}^x)\mathbb{P}(\tau_\varepsilon > n_\varepsilon) + o(1).$$

Together with the second part of Lemma 8 (with $n_\varepsilon$ replaced by $2n_\varepsilon$) and (6), this implies

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \geq \frac{\mathbb{P}(\Omega_x^*)}{1+t},$$

completing the proof. □

*Proof of Theorem 3.* We fix $t \in (0,\infty)$, and choose $n_\varepsilon$ such that $G(n_\varepsilon) \leq \frac{\beta t}{\gamma\varepsilon} < G(n_\varepsilon + 1)$.

According to the proof of Theorem 5 (see, in particular, (14) in Lemma 5), we know that for $m_\varepsilon$ with $m_\varepsilon = o(n_\varepsilon)$,

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{k=m_\varepsilon}^{n_\varepsilon} \frac{\mathbb{P}(\tau_\varepsilon > n_\varepsilon - k)}{A_k} = \frac{\mathbb{P}(\Omega^*)}{\gamma} \Pr(Y \geq t) =: \psi,$$

where $Y = \Gamma(\frac{1}{\beta})^{-1}\mathcal{EG}_{1/\beta}^{1/\beta}$ is the limiting random variable of the $\gamma\beta^{-1}\varepsilon G(\tau_\varepsilon)$. Therefore, if we take $m_\varepsilon := (\log\varepsilon)^4$, then Lemma 8 implies that for $R, K \geq 1$ and $\gamma' < 2f_X(0) < \gamma''$,

$$\limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Gamma_{R,K}^x)\left(1 - \gamma'\psi\right) + \mathbb{P}\left(\bigcup_{i=1}^K \{|S_i| > R\}\right)$$

and

$$\liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \geq \mathbb{P}(\Gamma_{R,K}^x)\left(1 - \gamma''\psi\right) - \mathbb{P}\left(\Gamma_{R,K} \setminus \Omega_x^*\right).$$

Since $\lim_{K\to+\infty}\lim_{R\to+\infty}\mathbb{P}(\Gamma_{R,K}^x) = \mathbb{P}(\Omega_x^*)$ and $\lim_{K\to+\infty}\lim_{R\to+\infty}\mathbb{P}\left(\bigcup_{i=1}^K\{|S_i| > R\}\right) = 0$, we get

$$\mathbb{P}(\Omega_x^*)\left(1 - \gamma''\psi\right) \leq \liminf_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \limsup_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) \leq \mathbb{P}(\Omega_x^*)\left(1 - \gamma'\psi\right),$$

and hence

$$\lim_{\varepsilon\to 0} \mathbb{P}(\mathbf{T}_\varepsilon^x > n_\varepsilon) = \mathbb{P}(\Omega_x^*)\left(1 - 2f_X(0)\psi\right) = \mathbb{P}(\Omega_x^*)\Pr(Y > t),$$

as required. □

*Proof of Corollary 1.* This is an $\alpha = 2$ case with $A_n = \sqrt{n}$ and $f_X(0) = \frac{1}{\sqrt{2\pi}}$. Recalling that $\mathcal{G}_{1/2} = \frac{1}{2\mathcal{N}^2}$ in distribution (cf. XIII.3.b of [9]) proves our claim. □

## References


[1] J. Aaronson, M. Denker, *Characteristic functions of random variables attracted to 1-stable laws*, Ann. Probab. 26 (1998), no. 1, 399–415.

[2] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge University Press, 1987.

[3] L. Breiman, *Probability*, Addison-Wesley, Reading Mass., 1968

[4] X. Bressaud, R. Zweimüller, Non exponential law of entrance times in asymptotically rare events for intermittent maps with infinite invariant measure, *Ann. Henri Poincaré* **2** (2001) 501–512.

[5] J. Bretagnolle and D. Dacunha-Castelle, *Théorèmes limites à distance finie pour les marches aléatoires*, Ann. Instit. H. Poincaré B **4**-1 (1968) 25–73.

[6] D. Cheliotis, A note on recurrent random walks, *Statistics and Probability Letters* **76** (2006) 1025–1031

[7] K.L. Chung and P. Erdös, On the lower limit of sums of independent random variables, *Ann. Math.* **48**, no. 4 (1947) 1003-1013.

[8] A. Dvoretzky & P. Erdös, Some problems on random walk in space, *Proc. Berkeley Sympos. math. Statist. Probab.* (1951) 353–367

[9] W. Feller, An introduction to probability theory and its applications, vol. 2, John Wiley and Sons, Inc., New York-London-Sydney, second edn., 1971

[10] I.A. Ibragimov, Yu. Linnik, Independent and stationary sequences of random variables. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov. Translation from the Russian edited by J.F. C. Kingman, Wolters-Noordho Publishing, Groningen, 1971

[11] J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux*, Bulletin de la Société Mathématique de France, 61 (1933), p. 55-62

[12] F. Pène, B. Saussol, Quantitative recurrence in two-dimensional extended processes, *Ann. Instit. H. Poincaré - Proba. stat.* **45**-4 (2009) 1065-1084.

[13] C. Stone, A local limit theorem for non-lattice multidimensional distribution functions, *Ann. Math. Statist* **36** (1965) 546–551



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