BACK TO BALLS IN BILLIARDS

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Abstract. We consider a billiard in the plane with periodic configuration of convex scatterers. This system is recurrent, in the sense that almost every orbit comes back arbitrarily close to the initial point. In this paper we study the time needed to get back in an ε-ball about the initial point, in the phase space and also for the position, in the limit when $\varepsilon \to 0$. We establish the existence of an almost sure convergence rate, and prove a convergence in distribution for the rescaled return times.

1. Introduction

1.1. **Periodic Lorentz gas.** We consider a planar billiard with periodic configuration of scatterers. Such a model is also called a Lorentz process. The motion of a free point particle bouncing on the scatterers according to Descartes' reflection law defines a flow. The flow conserves the initial speed, so that without loss of generality we will assume that the particle moves with unit speed. This is a Hamiltonian flow which preserves a Liouville measure. Observe that the phase space is spatially extended and thus the measure is infinite. We will suppose that the horizon is finite, i.e. the time between two consecutive reflections is uniformly bounded.

We are interested in the quantitative aspect of Poincaré's recurrence for the billiard flow. It is known that this system is recurrent, in particular almost every orbit comes back arbitrarily close to the initial point . In this paper, our goal is to study the return time in balls, in the limit when the radius goes to zero. Our main result is that

(i) the time Z_ε to get back ε-close to the initial point in the phase space is of order $\exp(\frac{1}{\varepsilon^2})$ for Lebesgue almost all initial conditions

(i') the time $\mathcal{Z}_\varepsilon$ to get back ε-close to the initial position is of order $\exp(\frac{1}{\varepsilon})$ for Lebesgue almost all initial conditions

(ii) we determine the fluctuations of $\varepsilon^2 \log Z_\varepsilon$ and of $\varepsilon \log \mathcal{Z}_\varepsilon$ by proving a convergence in distribution to a simple law.

This subject has been well studied recently in the setting of finite measure preserving transformations and typical behavior has been prove in a variety of chaotic systems: exponential statistics of return time, Poisson

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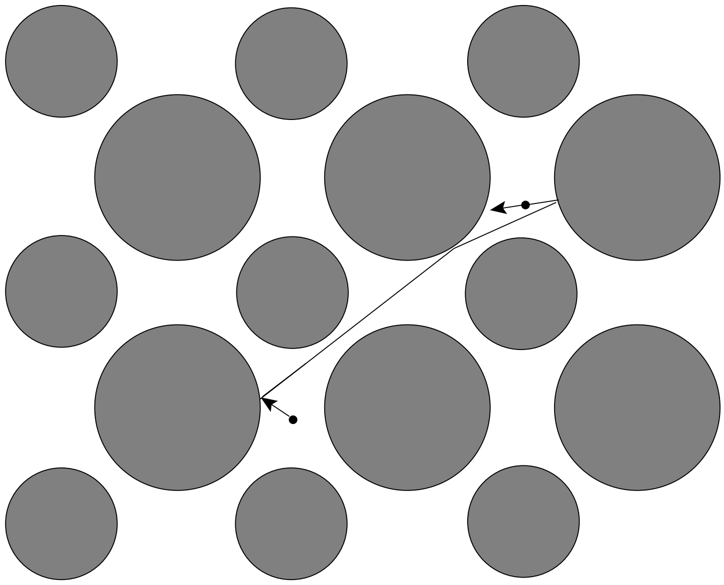

Figure 1. Motion of a point particule in the Lorentz process

law, relation between recurrence rate and dimensions (see e.g. [1] for a state of the art in a probabilistic setting; also [2, 8]). The present work differs by two points from the existing literature. First, the system in question has continuous time; second, the main novelty is that its natural invariant measure is σ-finite. Very few works have appeared on the topic in this situation [3, 12, 23].

A first reduction of the dynamics at the time of collisions with the scatterers (Poincaré section) and a second reduction by periodicity defines the praised billiard map. This map belongs to the class of hyperbolic systems with singularities. Since the work of Sinaï [27] establishing the ergodicity of the billiard map, it has been studied by many authors (let us mention [13], [4, 5], [6] [7]) giving : Bernoulli property, central limit theorem. In the past ten years, the new approach of L.-S. Young [29] has been exploited to get new significant results for the billiard map. Among them, let us mention the exponential decay of correlations [29], a new proof of the central limit theorem [29] and the local limit theorem proved by Szász and Varjú [28].

Conze [9] and Schmidt [25] proved that recurrence of the Lorentz process follows from some central limit theorem for the billiard map. Szász and Varjú [28] used their local limit theorem to give another proof of the recurrence. As proved by Simányi [26] and the first named author [20], once its recurrence proved, it is not difficult to prove the total ergodicity of the Lorentz process. More recently, estimates on the first return time in the initial cell have been established by Dolgopyat, Szász and Varjú in [10] and an analogous estimate

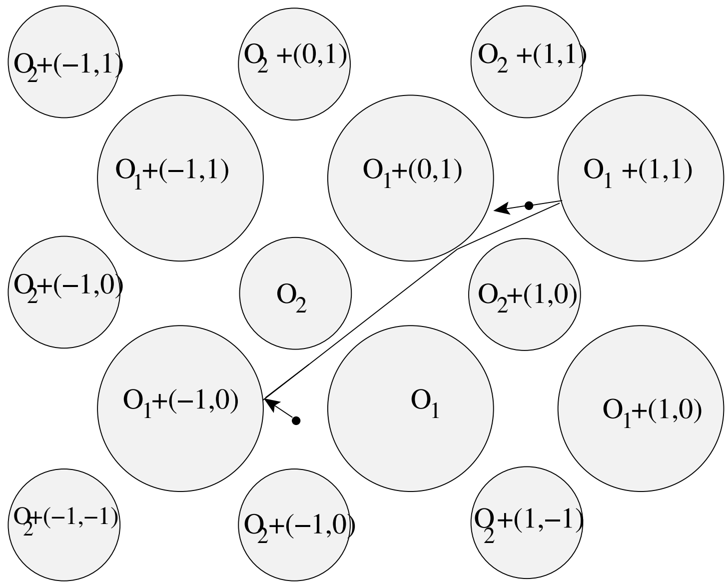

FIGURE 2. Labeling of the obstacles

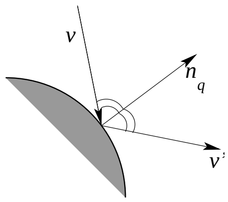

FIGURE 3. Elastic reflection

for the return time in the initial obstacle follows from a paper of the first named author [22].

1.2. **Precise description of the model and statement of the results.** We now precisely define the billiard flow Φ_t. Let $(O_i)_{i\in I}$ be a finite number of **open**, **convex** subsets of $\mathbb{R}^2$ with C^3 boundaries and **non-null curvature**. We let $Q = \mathbb{R}^2 \setminus \bigcup_{i\in I, \ell\in\mathbb{Z}^2} \ell + O_i$ be the billiard domain in the plane. We suppose that the sets $\ell + O_i$ in this union have pairwise disjoint closure. The flow is given by the motion of a point particle with position $q \in Q$ and velocity $v \in S^1$. Namely, the motion is ballistic if there are no collisions with an obstacle in the time interval $[0,t]$: $\Phi_t(q,v) = (q+tv, v)$. At the time of a collision the velocity changes according to reflection law $v \mapsto v'$: If n_q denotes the normal to the boundary of the obstacle at the point of collision $q \in \partial Q$, pointing inside the domain (i.e. outside the obstacle) then the angle $\angle(n_q, v') = \pi - \angle(n_q, v)$; see Figure 3. We assume that the billiard has **finite horizon**, in the sense that the time between two consecutive collisions is uniformly bounded.

We endow the space $X = Q \times S^1$ with the product metric

$$d((q,v),(q',v')) = \max(d(q,q'), d(v,v')),$$

where for simplicity we denote all the distances by d. The flow preserves the Lebesgue measure on $Q \times S^1$; it is σ-finite but nevertheless the system is well known to be recurrent [9, 25, 28].

For $x \in X$ and $\varepsilon > 0$ we define the minimal time to get back ε-close to the initial point by

$$Z_\varepsilon(x) := \inf\{t > \varepsilon \colon d(\Phi_t(x), x) < \varepsilon\}. \tag{1}$$

The quantity $Z_\varepsilon(\cdot)$ is well defined and finite for, at least, Lebesgue a.e. x. We denote by $\Pi_Q : X = Q \times S^1 \to Q$ the canonical projection. We also define the minimal time to get back ε-close to the initial position by

$$\mathcal{Z}_\varepsilon(x) := \inf\{t > \varepsilon \colon d(\Pi_Q(\Phi_t(x)), \Pi_Q(x)) < \varepsilon\}. \tag{2}$$

In the paper we give a precise asymptotic analysis of the return times Z_ε and $\mathcal{Z}_\varepsilon$ expressed by our main theorem. We say that a random variable Y_ε defined on X converges in the strong distribution sense to a random variable Y if for any probability $\mathbb{P} \ll Leb$, $Y_\varepsilon \to Y$ in distribution under $\mathbb{P}$.

Theorem 1.1. *The billiard flow satisfies*

(i) for Lebesgue a.e. $x \in X$ *we have* $\lim_{\varepsilon\to 0} \dfrac{\log\log Z_\varepsilon(x)}{-\log\varepsilon} = 2$*;*

(ii) the random variable $\varepsilon^2 \log Z_\varepsilon$ *converges as* $\varepsilon \to 0$ *in the strong distribution sense to a random variable* Y_0 *with distribution* $P(Y_0 > t) = \frac{1}{1+\beta_0 t}$ *for some constant* $\beta_0 > 0$*;*

(iii) for Lebesgue a.e. $x \in X$ *we have* $\lim_{\varepsilon\to 0} \dfrac{\log\log \mathcal{Z}_\varepsilon(x)}{-\log\varepsilon} = 1$*;*

(iv) the random variable $\varepsilon \log \mathcal{Z}_\varepsilon$ *converges as* $\varepsilon \to 0$ *in the strong distribution sense to a random variable* Y_1 *with distribution* $P(Y_1 > t) = \frac{1}{1+\beta_1 t}$ *for some constant* $\beta_1 > 0$.

Remark 1.2. *The constant* β_0 *is equal to* $\frac{2\beta}{\sum_{i\in I}|\partial O_i|}$*, with* $\beta = \frac{1}{2\pi\sqrt{\det\Sigma^2}}$ *where* Σ^2 *is the asymptotic covariance matrix of the cell shift function* κ *for the billiard map* $(\bar{T}, \bar{\mu})$ *defined by* (13)*; See Section 4 for precisions. The constant* β_1 *is equal to* $\frac{2\pi\beta}{\sum_{i\in I}|\partial O_i|}$.

In Section 2 we define the billiard maps associated to our billiard flow. In Section 3 we investigate the behavior of return times for the billiard map. In Section 4 we pursue this analysis for the extended billiard map, and building on the previous section we prove some preparatory results. Section 5 is then devoted to the proof of the part of Theorem 1.1 relative to returns in the phase space. Finally, in Section 6 we prove the part relative to returns for the position.

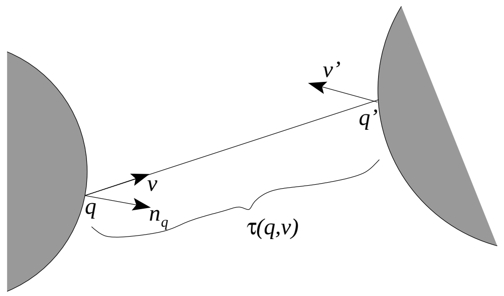

FIGURE 4. The Poincaré section at collisions times

2. Billiard maps

2.1. **Discrete time dynamics and new coordinates.** In order to study the statistical properties of the billiard flow, it is classical to make a *Poincaré section* at collisions times, i.e. when $\Phi_t(q,v) \in \partial Q \times S^1$. For definiteness, when $q \in \partial Q$ we choose the velocity v pointing outside the obstacle, that is right after the collision. Denote for such a $q \in \partial Q$ and $v \in S^1$ by $\tau(q,v)$ the time before the next collision: $\tau(q,v) = \min\{t > 0 \colon \Phi_t(q,v) \in \partial Q \times S^1\}$. Let ϕ be the Poincaré map: $\phi(q,v) = \Phi_{\tau(q,v)}(q,v) = (q',v')$ (see Figure 4).

Next, we make a *change of coordinates* for the base map. For each obstacle O_i we choose an arbitrary origin and parametrize its boundary ∂O_i by counter-clockwise arc-length. The position $q \in \partial Q$ is represented by (ℓ, i, r) if $q \in \ell + \partial O_i$ and r is the parametrization of the point q. The normal of the boundary at each point q is denoted by n_q and the velocity v is represented by its angle $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with n_q. Let

$$M = \bigcup_{\ell \in \mathbb{Z}^2} \bigcup_{i \in I} \left(\{(\ell, i)\} \times \mathbb{R}/_{|\partial O_i|\mathbb{Z}} \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right)$$

endowed with the product metric. Denote by $\psi \colon M \to \partial Q \times S^1$ the change of coordinate, such that $\psi(\ell, i, r, \varphi) = (q, v)$. The *extended billiard map* $T \colon M \to M$ is the Poincaré map ϕ in these new coordinates: $T = \psi^{-1} \circ \phi \circ \psi$. The flow Φ_t is conjugated to the special flow Ψ_t defined over the map T under the free flight function $\tau \circ \psi$. Let $M_\tau = \{(m, s) \in M \times \mathbb{R} \colon 0 \le s < \tau(\psi(m))\}$. We denote by $\pi \colon M_\tau \to M$ the projection onto the base defined by $\pi(m,s) = m$ and extends the conjugation ψ to M_τ by setting $\psi(m,s) = \Phi_s(\psi(m))$.

Let $\bar{M}$ be the subset of M corresponding to the cell $\ell = 0$. We define the *billiard map* $\bar{T} \colon \bar{M} \to \bar{M}$ corresponding to the quotient map of T by $\mathbb{Z}^2$; this is well defined by $\mathbb{Z}^2$-periodicity of the obstacles. The *cell shift* function $\kappa \colon M \to \mathbb{Z}^2$ is defined by $\kappa(\ell, i, r, \varphi) = \ell' - \ell$ if $T(\ell, i, r, \varphi) = (\ell', i', r', \varphi')$.

During the proof of our theorems on the billiard flow we will prove a version of the *local limit theorem* for the billiard map suitable for our purpose, as well as a property of recurrence called *exponential law for the return time* statistics.

2.2. **Different quantities related to recurrence.** The notion of recurrence in these billiard maps gives rise to the definition of the following different quantities. Let $m \in M$ and $\bar{m} \in \bar{M}$.

Let $W_A(m)$ be the first iterate $n \geq 1$ such that $T^n m \in A$ for some subset $A \subset M$.

Let $\bar{W}_B(\bar{m})$ be the first iterate $n \geq 1$ such that $\bar{T}^n \bar{m} \in B$ for some subset $B \subset \bar{M}$.

Let $W_\varepsilon(m)$ be the first iterate $n \geq 1$ such that $d(T^n m, m) < \varepsilon$ for some $\varepsilon > 0$.

Let $\bar{W}_\varepsilon(\bar{m})$ be the first iterate $n \geq 1$ such that $d(\bar{T}^n \bar{m}, \bar{m}) < \varepsilon$ for some $\varepsilon > 0$.

3. Recurrence for the billiard map

Recall that the billiard map $\bar{T}$ preserves a probability measure $\bar{\mu}$ equivalent to the Lebesgue measure on $\bar{M}$, whose density is given by

$$\rho(\ell, i, r, \varphi) = \frac{1}{2\Gamma} \cos\varphi, \quad \text{where} \quad \Gamma := \sum_{i \in I} |\partial O_i|.$$

The billiard system $(\bar{M}, \bar{T})$ is two dimensional with one negative and one positive Lyapunov exponent and the singularities are not too wild, therefore the result on recurrence rate [24] applies.

Theorem 3.1 ([24]). *The recurrence rate of the billiard map is equal to the dimension:*

$$\lim_{\varepsilon \to 0} \frac{\log \bar{W}_\varepsilon}{-\log \varepsilon} = 2 \quad \bar{\mu} \text{ a.e.}$$

Lemma 3.2. *For $\bar{\mu}$-almost every $m \in \bar{M}$, for all $c_1 > 0$, $c_2 > 0$, $\alpha > 0$ and for all family $(D_\varepsilon)_\varepsilon$ of sets containing m such that $D_\varepsilon \subseteq B(m, c_2\varepsilon)$ and $\bar{\mu}(D_\varepsilon) \geq c_1(\operatorname{diam}(D_\varepsilon))^2$, we have*

$$\bar{\mu}(\bar{W}_{B(m,\varepsilon)} \leq \varepsilon^{-2+\alpha} | D_\varepsilon) \to 0.$$

Proof. Let $\alpha > 0$, $c_1 > 0$ and $c_2 > 0$. Choose some $a \in (0, \alpha)$ and set for some $\varepsilon_0 > 0$

$$F_a = \{m \in \bar{M} \colon \forall \varepsilon \leq \varepsilon_0, \frac{\log \bar{W}_\varepsilon(m)}{-\log \varepsilon} \geq 2 - a\}.$$

By Theorem 3.1 we have $\bar{\mu}(F_a) \to 1$ as $\varepsilon_0 \to 0$. There exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$ we have the inclusions

$$D_\varepsilon \cap \{\bar{W}_{B(m,\varepsilon)} \leq \varepsilon^{-2+\alpha}\} \subset D_\varepsilon \cap \{\bar{W}_{(1+c_2)\varepsilon} \leq \varepsilon^{-2+\alpha}\} \subset D_\varepsilon \cap F_a^c.$$

Thus for any density point m of the set F_a relative to the Lebesgue basis given by $(B(\cdot, \varepsilon))_\varepsilon$ we obtain

$$\begin{aligned} \bar{\mu}(\bar{W}_{B(m,\varepsilon)} \leq \varepsilon^{-2+\alpha} | D_\varepsilon) &\leq \bar{\mu}(F_a^c | D_\varepsilon) \\ &\leq \bar{\mu}(F_a^c | B(m, \operatorname{diam} D_\varepsilon)) \frac{\bar{\mu}(B(m, \operatorname{diam} D_\varepsilon))}{\bar{\mu}(D_\varepsilon)} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. $\square$

We call *non-sticky* a point m satisfying the conclusion of Lemma 3.2 and we denote by $\mathcal{NS}$ the set of non-sticky points. We emphasize that $\bar\mu(\mathcal{NS}) = 1$

Next theorem says that the return times and entrance times in balls are exponentially distributed for the billiard map.

Theorem 3.3. *Let $m \in \mathcal{NS}$ be a non-sticky point. We have*

$$\bar\mu(\bar\mu(B(m,\varepsilon))\bar W_{B(m,\varepsilon)}(\cdot) > t | B(m,\varepsilon)) \to e^{-t},$$
$$\bar\mu(\bar\mu(B(m,\varepsilon))\bar W_{B(m,\varepsilon)}(\cdot) > t) \to e^{-t},$$

uniformly in $t \ge 0$, as $\varepsilon \to 0$.

We denote by $A^{[\eta]}$ the η-neighborhood of a set A.

Proof. We use an approximation by cylinders, the exponential mixing and the method developed in [15] for exponential return times and entrance times. We write $A = B(m,\varepsilon)$ for convenience. According to Theorem 2.1 in [15], it suffices to show that

$$\sup_n \left|\bar\mu(\bar W_A > n|A) - \bar\mu(\bar W_A > n)\right| = o_\varepsilon(1),$$

since it will imply that the limiting distributions exist and are both exponential.

Let $c_3 > 0$ be such that $\mu(\partial A^{[\eta]}) \le c_3\eta$ independently of ε. Let k be an integer such that $\delta^k \approx \varepsilon^3$. Let g be an integer such that $\theta^{g-2k} \approx \varepsilon^3$, where θ is the constant appearing in Theorem A.3.

If m is a non-sticky point, observing that g is logarithmic in ε, we have for any integer n,

$$\left|\bar\mu(\bar W_A > n|A) - \bar\mu(\bar W_A \circ \bar T^g > n-g|A)\right| \le \bar\mu(\bar W_A \le g|A) = o_\varepsilon(1).$$

Set $E = \{\bar W_A > n-g\}$. We approach A and E by a union of cylinder sets:

Let A' be the union of all the cylinders (see Appendix A.1 for the precise definition) $Z \in \xi^k_{-k}$ such that $Z \subset A$. We have $A' \subset A$ and $A \setminus A' \subset \partial A^{[c_0\delta^k]}$ by Lemma A.1. Thus we get $\bar\mu(A \setminus A') \le c_3c_0\delta^k$.

Let

$$E' = \bigcap_{j=1}^{n-g} \bar T^{-j}(\cup_{Z\in\xi^{k+j}_{-k-j}, Z\cap A\neq\emptyset} Z)^c.$$

We have $E' \subset E$ and by Lemma A.1 again

$$E \setminus E' \subset (\partial A)^{[c_0\delta^k]} \cup \bigcup_{j=1}^{n-g} \bar T^{-j}(\partial A)^{[c_0\delta^{k+j}]}.$$

Thus by the invariance of $\bar\mu$ we get $\bar\mu(E \setminus E') \le c_3c_0\frac{\delta^k}{1-\delta}$. Using the decay of correlations (for cylinders, see Theorem A.3 in Appendix A.1) we get that

$$\left|\bar\mu(A' \cap \bar T^{-g}E') - \bar\mu(A')\bar\mu(E')\right| \le C\theta^{g-2k} = o(\bar\mu(A)).$$

Furthermore,

$$\left|\bar\mu(\bar W_A > n) - \bar\mu(E)\right| \le \bar\mu(\bar W_A \le g) \le g\bar\mu(A) = o(1).$$

Putting together all these estimates gives

$$\left|\bar\mu(\bar W_A > n|A) - \bar\mu(\bar W_A > n)\right| = o(1),$$

uniformly in $n \in \mathbb{N}$. □

Next, using the mixing property again we can condition on a smaller set and still get the same limiting law.

Proposition 3.4. *For any $m \in \mathcal{NS}$ there exists a function f_m such that $\lim_{\varepsilon\to 0} f_m(\varepsilon) = 0$ and such that the following holds:*
For any $\varepsilon > 0$ and any balls D_ε, A_ε of $\bar M$ such that
(i) $m \in D_\varepsilon \subset A_\varepsilon = B(m,\varepsilon)$,
(ii) $\bar\mu(D_\varepsilon) \ge \varepsilon^{2.25}$,
we have for any n

$$\left|\bar\mu(\bar W_{A_\varepsilon}(\cdot) > n|D_\varepsilon) - e^{-n\bar\mu(A_\varepsilon)}\right| \le f_m(\varepsilon).$$

Proof. We approximate the sets D and $E = \{\bar W_A > n\}$ from the inside by sets D' and E' as we approximated the sets A and E in the proof of Theorem 3.3. With the same g we get

$$\left|\bar\mu(\bar W_A > n|D) - \bar\mu(\bar W_A \circ \bar T^g > n-g|D)\right| \le \bar\mu(\bar W_A \le g|D) = o(1).$$

for non-sticky points. Using the exponential decay of correlations for cylinders given by Theorem A.3 we get that

$$\begin{aligned}\bar\mu(\bar W_A \circ \bar T^g > n-g|D) &= \bar\mu(\bar W_A \circ \bar T^g > n-g) + o(1)\\ &= e^{-n\bar\mu(A)} + o(1)\end{aligned}$$

by Theorem 3.3. □

The following result of independent interest will not be used in the sequel an can be derived from Proposition 3.4 as Proposition 4.7 would be derived from Proposition 4.6. Therefore we omit its proof.

Proposition 3.5. *The random variable $4\varepsilon^2\rho(\cdot)\bar W_\varepsilon(\cdot)$ converges, in the strong distribution sense, to the exponential law with parameter one.*

The random variable $\varepsilon^2\bar W_\varepsilon(\cdot)$ converges, under the law of $\bar\mu$, to a random variable Y which is a continuous mixture of exponentials. More precisely Y has distribution

$$\mathbb{P}(Y > t) = \int_{\bar M} e^{-4t\rho}\, d\bar\mu.$$

4. Recurrence for the extended billiard map

Recall that the extended billiard map (M, T) preserves the σ-finite measure μ equivalent to the Lebesgue measure on M, which is the image of the Lebesgue measure on $Q \times S^1$, whose density is equal to $\cos\varphi$. Note that

$$\mu|_{\bar{M}} = 2\Gamma\bar{\mu}. \tag{3}$$

4.1. **Preliminary results on the extended billiard map.** We will use the following extension of Szász and Varjú's local limit theorem [28]. For simplicity we use the notation $\bar{\mu}(A_1; \dots; A_n) = \bar{\mu}(A_1 \cap \cdots \cap A_n)$.

Proposition 4.1. *Let $p > 1$. There exists $c > 0$ such that, for any $k \geq 1$, if $A \subset \bar{M}$ is a union of components of ξ_{-k}^{k} and $B \subset \bar{M}$ is a union of ξ_{-k}^{∞} then for any $n > 2k$ and $\ell \in \mathbb{Z}^2$*

$$\left| \bar{\mu}(A \cap \{S_n\kappa = \ell\} \cap \bar{T}^{-n}(B)) - \frac{\beta e^{-\frac{1}{2(n-2k)}(\Sigma^2)^{-1}\ell\cdot\ell}}{(n-2k)} \bar{\mu}(A)\bar{\mu}(B) \right| \leq \frac{ck\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}$$

where $\beta = \frac{1}{2\pi\sqrt{\det \Sigma^2}}$.

The proof of Proposition 4.1 is in Appendix A.2.

Proposition 4.2. *Let c_1, c_2, c_3 and c_4 be some positive constants. For any $m \in \mathcal{NS}$ there exists a function f_m such that $\lim_{\varepsilon\to 0} f_m(\varepsilon) = 0$ and such that the following holds:*

For any $\varepsilon > 0$ and any subsets D_ε, A_ε of $\bar{M}$ such that
(i) $m \in D_\varepsilon \subset A_\varepsilon$,
(ii) $c_1\varepsilon^2 \leq \bar{\mu}(A_\varepsilon)$ and $A_\varepsilon \subset B(m, c_2\varepsilon)$,
(iii) for any $\eta > 0$, $\bar{\mu}(\partial A_\varepsilon^{[\eta]}) \leq c_3\eta$, and also $\bar{\mu}(\partial D_\varepsilon^{[\eta]}) \leq c_3\eta$,
(iv) $\bar{\mu}(D_\varepsilon) \geq c_1(\mathrm{diam}(D_\varepsilon))^2$ and $\bar{\mu}(D_\varepsilon) \geq c_4\varepsilon^{2.25}$,
uniformly in $N \in (e^{\log^2\varepsilon}, e^{\frac{1}{\varepsilon^{2.5}}})$ we have

$$\bar{\mu}(W_{A_\varepsilon}(\cdot) > N | A_\varepsilon) = \frac{1 + o_\varepsilon(1)}{1 + \log(N)\bar{\mu}(A_\varepsilon)\beta}$$

and

$$\bar{\mu}(W_{A_\varepsilon}(\cdot) > N | D_\varepsilon) = \frac{1}{1 + \log(N)\bar{\mu}(A_\varepsilon)\beta} + o_\varepsilon(1)$$

where the error terms $o_\varepsilon(1)$ is bounded by $f_m(\varepsilon)$.

Lemma 4.3. *Under the hypothesis of Proposition 4.2, for all $m \in \bar{M}$ (even those not belonging to $\mathcal{NS}$), we have*

$$\bar{\mu}(W_A > N|D) + \beta\log(N)\bar{\mu}(A)\bar{\mu}(W_A > N|A) \leq 1 + o_\varepsilon(1),$$

where the error term only depends on the positive constants c_i.

Proof. As used by Dvoretzky and Erdös in [11], a partition of D with respect to the last entrance time q into the set A in the time interval $[0,\dots,N]$ gives

$$\begin{aligned}\bar\mu(D) &= \sum_{q=0}^{N} \bar\mu(D; S_q\kappa = 0; \bar T^{-q}(A\cap\{W_A > N-q\}))\\ &\geq \sum_{q=0}^{N} \bar\mu(D; S_q\kappa = 0; \bar T^{-q}(E))\end{aligned} \tag{4}$$

with $E = A\cap\{W_A > N\}$.

Let k be such that $\delta^k \approx \varepsilon^3$. We approach D and E by cylindrical sets:

Let D' be the union of cylinders $Z\in\xi_{-k}^k$ such that $Z\subset D$. We have $D'\subset D$ and $D\setminus D'\subset \partial D^{[c_0\delta^k]}$ by Lemma A.1, thus by the hypothesis (iii) we get $\bar\mu(D\setminus D')\leq c_3c_0\delta^k$.

Let A' be the corresponding cylindrical approximation for A and set

$$E' = A'\cap(\bigcap_{j=1}^{N}\left[\{S_j\kappa\neq 0\}\cup \bar T^{-j}(\cup_{Z\in\xi_{-k-j}^{k+j}, Z\cap A\neq\emptyset} Z)^c\right]).$$

We have $E'\subset E$ and by Lemma A.1

$$E\setminus E'\subset (\partial A)^{[c_0\delta^k]}\cup\bigcup_{j=1}^{N}\bar T^{-j}(\partial A)^{[c_0\delta^{k+j}]}.$$

Thus by the hypothesis (iii) and the invariance of $\bar\mu$ we get $\bar\mu(E\setminus E')\leq c_3c_0\frac{\delta^k}{1-\delta}$.

Set $p_0\approx\varepsilon^{-a}$ with $a = 4.6 > 2\times 2.25$. By (4) and the inclusions we get

$$\bar\mu(D)\geq\bar\mu(D\cap E) + \sum_{q=p_0}^{N}\bar\mu(D'; S_q\kappa = 0; \bar T^{-q}E').$$

It follows from Proposition 4.1 that

$$\bar\mu(D)\geq\bar\mu(D\cap E) + \sum_{q=p_0}^{N}\beta\frac{\bar\mu(D')\bar\mu(E')}{q-2k} - \sum_{q=p_0}^{N}\frac{ck}{(q-2k)^{\frac32}}.$$

The error term is bounded by $\frac{ck}{\sqrt{p_0-2k}} = O(\log(\varepsilon)\varepsilon^{a/2}) \ll c_4\varepsilon^{2.25}\leq\bar\mu(D)$. Thus, since $\log p_0 = o(\log N)$,

$$\bar\mu(D\cap E) + \beta\log(N)\bar\mu(D')\bar\mu(E')\leq\bar\mu(D)(1+o(1)).$$

Therefore, using $\bar\mu(D\setminus D')\leq c_3c_0\delta^k = c_3c_0\varepsilon^3 \ll c_4\varepsilon^{2.25} - c_3c_0\varepsilon^3\leq\bar\mu(D')$, we get

$$\bar\mu(D\cap E) + \beta\log(N)\bar\mu(D)\bar\mu(E')\leq\bar\mu(D)(1+o(1)).$$

Notice that $\bar\mu(E\setminus E')\log N\leq\frac{c_3c_0}{1-\delta}\delta^k\log N = o(1)$, from which it follows that

$$\bar\mu(D\cap E) + \beta\log(N)\bar\mu(D)\bar\mu(E)\leq\bar\mu(D)(1+o(1)).$$

A division by $\bar\mu(D)$ yields, since $E = A \cap \{W_A > N\}$ and $D \subset A$,

$$\bar\mu(W_A > N|D) + \beta \log(N)\bar\mu(A)\bar\mu(W_A > N|A) \le 1 + o(1)$$

$\square$

Lemma 4.4. *Under the hypotheses of Proposition 4.2 we have*

$$\bar\mu(W_A > N|D) + \beta \log(N)\bar\mu(A)\bar\mu(W_A > N|A) = 1 + o_\varepsilon(1).$$

Proof. Let $\alpha \in (0, 0.25)$ and set $M_\varepsilon = \varepsilon^{2(-1+\alpha)}$. We use the same decomposition as in Equation (4) again, with $n_N = N \log(N)$ and $m_N = n_N - N$:

$$\bar\mu(D) = \sum_{q=0}^{n_N} \bar\mu(D; S_q\kappa = 0; \bar T^{-q}(A \cap \{W_A > n_N - q\})).$$

We divide this sum into four blocks: S_0 is the term for $q = 0$, S_1 is the sum for q in the range $1, \dots, M_\varepsilon$, S_2 in the range $M_\varepsilon + 1, \dots, m_N$ and S_3 in the range $m_N + 1, \dots, n_N$.

The value of S_0 is simply

$$S_0 = \bar\mu(D; W_A > n_N) \le \bar\mu(D; W_A > N).$$

By assumption (conclusion of Lemma 3.2), we have

$$S_1 = \bar\mu(D; W_A \le M_\varepsilon) \le \bar\mu(D; \bar W_{B(m, c_2\varepsilon)} \le M_\varepsilon) = o(\bar\mu(D)).$$

When $q \le m_N$ we have $n_N - q \ge N$, therefore we have

$$S_2 \le \sum_{q=M_\varepsilon+1}^{m_N} \bar\mu(D; S_q\kappa = 0; \bar T^{-q}(E))$$

with $E = A \cap \{W_A > N\}$. Let k be such that $\delta^k \approx \varepsilon^3$. We approximate the sets D and E by cylinders: let D'' be the union of cylinders $Z \in \xi_{-k}^k$ such that $Z \cap D \ne \emptyset$. Let A'' be the corresponding enlargement for A and let

$$E'' = A'' \cap \bigcap_{j=1}^{N} \left[\{S_j\kappa \ne 0\} \cup \bar T^{-j}(\cup_{Z \in \xi_{-k-j}^{k+j}, Z \subseteq A} Z)^c \right].$$

We have $D \subset D''$ and by Lemma A.1, $D'' \setminus D \subset (\partial D)^{[c_0\delta^k]}$. Thus by Hypothesis (iii) we get that $\bar\mu(D'' \setminus D) \le c_3c_0\delta^k$. Similarly, $E \subset E''$ and $E'' \setminus E \subset (\partial A)^{[c_0\delta^k]} \cup \bigcup_{j=1}^N \bar T^{-j}(\partial A)^{[c_0\delta^{k+j}]}$. Thus by hypothesis (iii) we get that $\bar\mu(E'' \setminus E) \le c_3c_0\frac{\delta^k}{1-\delta}$ and so $\log(m_N)\bar\mu(E'' \setminus E) = o(1)$. By Proposition 4.1

with p such that $1+\frac{2}{p}>2.5$, we get

$$\begin{aligned}
S_2 &\le \sum_{q=M_\varepsilon+1}^{m_N} \bar\mu(D''; S_q\kappa=0; \bar T^{-q}(E''))\\
&\le \sum_{q=M_\varepsilon+1}^{m_N} \left[\beta\frac{\bar\mu(D'')\bar\mu(E'')}{q-2k} + \frac{ck\bar\mu(E'')^{\frac{1}{p}}}{(q-2k)^{\frac{3}{2}}}\right]\\
&\le \log(m_N)\beta\bar\mu(D'')\bar\mu(E'') + \frac{ck\bar\mu(A'')^{\frac{1}{p}}}{\sqrt{M_\varepsilon-2k}}\\
&\le \log(m_N)\beta\bar\mu(D)\bar\mu(E)(1+o(1)) + o(\bar\mu(D)) + O(\log(\varepsilon)\varepsilon^{1-\alpha}\varepsilon^{2/p})
\end{aligned}$$

The last error term is $o(\bar\mu(D))$ provided $1-\alpha+2/p>2.25$. In addition $\log m_N \sim \log N$, hence

$$S_2 \le \beta\log(N)\bar\mu(D)\bar\mu(E)(1+o(1)) + o(\bar\mu(D)).$$

Finally, by Proposition 4.1 we get

$$\begin{aligned}
S_3 &\le \sum_{q=m_N}^{n_N} \bar\mu(D''; S_q\kappa=0; \bar T^{-q}A'')\\
&\le \sum_{q=m_N}^{n_N} \left[\beta\frac{\bar\mu(D'')\bar\mu(A'')}{q-2k} + \frac{ck}{(q-2k)^{\frac{3}{2}}}\right]\\
&\le \beta\log\left(\frac{n_N}{m_N}\right)\bar\mu(D)\bar\mu(A)(1+o(1)) + \frac{k}{\sqrt{m_N-2k}}
\end{aligned}$$

Moreover $\log(\frac{n_N}{m_N}) = o(1)$, and the last error term is again $o(\bar\mu(D))$ since $m_N \ge N$.

We conclude that

$$(1+o(1))\bar\mu(D) \le \bar\mu(D\cap E) + \beta\log(N)\bar\mu(D)\bar\mu(E).$$

A division by $\bar\mu(D)$ yields, since $E = A\cap\{W_A>N\}$ and $D\subset A$,

$$\bar\mu(W_A>N|D) + \beta\log(N)\bar\mu(A)\bar\mu(W_A>N|A) \ge 1-o(1).$$

The reverse inequality also holds by Lemma 4.3, finishing the proof. □

Proof of Proposition 4.2. Lemma 4.4 with $D=A$ gives us

$$\bar\mu(W_A>N|A) = \frac{1+o(1)}{1+\beta\log(N)\bar\mu(A)}. \tag{5}$$

This proves the proposition in the special case $D=A$. We turn now to the general case. Applying Lemma 4.4 again, together with (5) we get,

$$\bar\mu(W_A>N|D) + \beta\log(N)\bar\mu(A)\frac{1+o(1)}{1+\beta\log(N)\bar\mu(A)} = 1+o(1)$$

which proves the proposition. □

4.2. **Recurrence results for the extended billiard map.**

Proposition 4.5. *The recurrence rate for the extended billiard map is given by*

$$\lim_{\varepsilon\to 0}\frac{\log\log W_\varepsilon}{-\log\varepsilon}=2\quad \mu\text{-a.e.}$$

Proof. Note that by $\mathbb{Z}^2$-periodicity it suffices to prove the statement $\bar\mu$ a.e. in $\bar M$.

Upper bound : Let $\delta>0$ and set

$$\bar M_\delta=\{m\in\mathcal{NS}\colon \rho(m)>\delta \text{ and } \sup_{\varepsilon\le\delta} f_m(\varepsilon)\le 1\},$$

where the function $f_m(\varepsilon)$ appears in Proposition 4.2. Let us notice that there exist constants c_i for which the hypotheses of Proposition 4.2 are satisfied for any $D_\varepsilon=A_\varepsilon=B(m,\varepsilon/2)$, with $m\in\bar M_\delta$. Let $\alpha\in(0,\frac12)$, $n\ge 1$ and $\varepsilon_n=\log^{-\alpha} n$. Take a cover of $\bar M_\delta$ by some sets $B(m,\varepsilon_n/2)$, $m\in\mathcal{P}_n\subset\bar M_\delta$ such that $\#\mathcal{P}_n=O((\varepsilon_n)^{-2})$. According to Proposition 4.2, we have

$$\begin{aligned}\bar\mu(\{W_{\varepsilon_n}\ge n\}\cap\bar M_\delta) &\le \sum_{m\in\mathcal{P}_n}\bar\mu\left(W_{B(m,\frac{\varepsilon_n}{2})}\ge n\,\Big|B(m,\frac{\varepsilon_n}{2})\right)\bar\mu\left(B(m,\frac{\varepsilon_n}{2})\right)\\ &\le O((1+\beta c_1\log^{1-2\alpha}n)^{-1}).\end{aligned}$$

Now, by taking $n_k=\exp(k^{2/(1-\alpha)})$ and according to the Borel-Cantelli lemma, we get that, for almost all m in $\bar M_\delta$, there exists N_m such that, for any $k\ge N_m$, $W_{\varepsilon_{n_k}}(m)<n_k$ and hence

$$\overline{\lim_{k\to+\infty}}\frac{\log\log W_{\varepsilon_{n_k}}(m)}{-\log\varepsilon_{n_k}}\le\frac{1}{\alpha}.$$

Since $\log\varepsilon_{n_k}\sim\log\varepsilon_{n_{k+1}}$, we get that $\bar\mu$-a.e. on $\bar M_\delta$

$$\overline{\lim_{\varepsilon\to 0}}\frac{\log\log W_\varepsilon}{-\log\varepsilon}\le\frac{1}{\alpha}.$$

We conclude that almost everywhere in $\bar M$, we have

$$\overline{\lim_{\varepsilon\to 0}}\frac{\log\log W_\varepsilon}{-\log\varepsilon}\le 2.$$

Lower bound : Let $\alpha>1/2$. Let $n\ge 1$ and $\varepsilon_n=\log^{-\alpha}n$. We consider a cover of $\bar M$ by balls $B(m,\varepsilon_n)$ for $m\in\mathcal{P}'_n$ such that $\#\mathcal{P}'_n=O(\varepsilon_n^{-2})$. Let k be such that $\delta^k\approx\varepsilon_n^5$. For each $m\in\mathcal{P}'_n$ we consider the sets B''_m and C''_m constructed from $B(m,\varepsilon_n)$ and $B(m,2\varepsilon_n)$ (respectively) like A'' was constructed from A in the proof of Lemma 4.4.

Applying Proposition 4.1 we get

$$\begin{aligned}\bar\mu(d(\cdot,T^n(\cdot))<\varepsilon_n) &\le \sum_{m\mathcal{P}'_n}\bar\mu(B(m,\varepsilon_n)\cap\{S_n=0\}\cap\bar T^{-n}(B(m,2\varepsilon_n)))\\ &\le \sum_m \bar\mu(B''_m\cap\{S_n=0\}\cap\bar T^{-n}(C''_m))+O(\delta^k)\\ &\le \sum_m \frac{\bar\mu(B''_m)\bar\mu(C''_m)}{n-2k}+\frac{ck}{(n-2k)^{3/2}}+O(\delta^k)\\ &\le O\left(n^{-1}\log^{-2\alpha} n\right).\end{aligned}$$

Hence, according to the first Borel Cantelli lemma, for almost every $m\in\bar M$, there exists N_m such that, for all $n\ge N_m$, we have

$$d(m,T^n(m))\ge\varepsilon_n.$$

Let $u=\min(d(m,T^n(m)),\ n=1,...,N_m)$. Note that $u>0$, otherwise we would have $m=T^p(m)$ for some p and hence $m=T^n(m)$ infinitely often, which would contradict $d(m,T^n(m))\ge\varepsilon_n$. For all $n\ge N_m$ such that $\varepsilon_n<u$ we have $W_{\varepsilon_n}(m)\ge n$. Hence

$$\varliminf_{n\to+\infty}\frac{\log\log W_{\varepsilon_n}(m)}{-\log\varepsilon_n}\ge\frac{1}{\alpha}.$$

Since $\log\varepsilon_n\sim\log\varepsilon_{n+1}$ we get $\varliminf_{\varepsilon\to0}\frac{\log\log W_\varepsilon}{-\log\varepsilon}\ge\frac{1}{\alpha}$ almost everywhere on $\bar M$. Therefore $\varliminf_{\varepsilon\to0}\frac{\log\log W_\varepsilon}{-\log\varepsilon}\ge 2$ $\bar\mu$-a.e. □

Proposition 4.6. *For a.e. $m\in\bar M$, and sequences of sets (A_ε) and (D_ε) such that the hypotheses (i)–(iv) of Proposition 4.2 are satisfied we have*

$$\mu(W_{A_\varepsilon}>\exp(\frac{t}{\bar\mu(A_\varepsilon)})|D_\varepsilon)\to\frac{1}{1+t\beta}\quad \textit{as } \varepsilon\to0.$$

Proof. Proposition 4.2 with $N=\exp(\frac{t}{\bar\mu(A_\varepsilon)})$ immediately gives the result. □

Note that in particular the proposition applies to the sequence of balls $A_\varepsilon=D_\varepsilon=B(m,\varepsilon)$. This is the corresponding result to that of Theorem 3.3 in the case of the extended billiard map.

Proposition 4.7. *The random variable $4\varepsilon^2\rho(\cdot)\log W_\varepsilon(\cdot)$ converges in the strong distribution sense, to a random variable with law $P(Y>t)=\frac{1}{1+\beta t}$.*

Proof. The proof is similar to that of Theorem 1.1-(ii), without the flow direction; See Section 5 for details. Since it is an obvious modification of it and since this result will not be used in the sequel, we omit its proof. □

5. Proof of the main theorem: recurrence in the phase space

We prove in this section Theorem 1.1-(i)and (ii) about the return times in the phase space Z_ε defined by (1).

5.1. **Almost sure convergence: the first statement.** By $\mathbb{Z}^2$-periodicity it is sufficient to prove the result on $\bar{M}$. Let $m \in \bar{M}$ be a point which is not on a singular orbit of T and such that $W_\varepsilon(m)$ follows the limit given by Proposition 4.5. By regularity of the change of variable ψ (away from the singular set) there exist two constants $0 < a < b$ such that, for any $0 \le s \le \tau(m)$, we have

$$(\min\tau)(W_{b\varepsilon}(m)-1) \le Z_\varepsilon(\Phi_s\psi(m)) \le (\max\tau)W_{a\varepsilon}(m) \tag{6}$$

since the free flight function τ is bounded from above and from below. This implies the result for all the points $\Phi_s\psi(m)$. By Fubini's theorem this concerns a.e. points in $Q \times S^1$, which proves the first statement.

5.2. **Convergence in distribution: the second statement.** Unfortunately we cannot exploit the relation (6) above anymore. The problem is not with the multiplicative factor coming from τ, but the fluctuations are sensible to the constants a and b and a direct method could only lead to rough bounds in terms of these constants.

The following lemma gives the measure of the projection of a ball $B(x,\varepsilon)$ onto M.

Lemma 5.1. *For any $x \in X$ and $\varepsilon > 0$ such that the ball $B(x,\varepsilon)$ does not intersect the boundary $\partial Q \times S^1$, we have*

$$\mu(\pi\psi^{-1}B(x,\varepsilon)) = 4\varepsilon^2.$$

Proof. Let $x = (q_0, v_0) \in X$. We consider the ball $B(q_0,\varepsilon)$ as a new obstacle added in our billiard domain. Let

$$\Delta_\varepsilon := \left\{(q,v) \in Q \times S^1 \colon q \in \partial B(q_0,\varepsilon),\, |\angle(v_0,v)| < \varepsilon,\, \langle n_q, v\rangle > 0\right\}.$$

Since the billiard map preserves the measure $\cos\varphi dr d\varphi$, we have

$$\mu(\pi\psi^{-1}B(x,\varepsilon)) = \int_{\Delta_\varepsilon} \cos\angle(n_q,v)\, dq dv.$$

For any v such that $|\angle(v_0,v)| < \varepsilon$ a classical computation gives

$$\int_{\{q:(q,v)\in\Delta_\varepsilon\}} \cos\angle(n_q,v)\, dq = 2\varepsilon,$$

whence the result. □

Let $\mathbb{P} = h d\mathcal{L}$ be the probability measure on X under which we will compute the law of Z_ε. Let $\bar{X} = \psi(\pi^{-1}\bar{M})$. By $\mathbb{Z}^2$-periodicity, Z_ε has the same distribution under $\mathbb{P}$ as under $\bar{\mathbb{P}} = \bar{h} d\mathcal{L}$ where $\bar{h}(\cdot) = \sum_{\ell\in\mathbb{Z}^2} h(\cdot+\ell)1_{\bar{X}}$. Therefore we suppose that $\operatorname{supp} h \subset \bar{X}$.

Assume for the moment that the density h is continuous and compactly supported in the set $\bar{X}' = \bar{X}\setminus(\psi(\bar{M}\times\{0\}\cup\pi^{-1}R_0))$, where $R_0 = \{\varphi = \pm\frac{\pi}{2}\}$. Then for any $r > 0$ sufficiently small we have

$$\operatorname{supp} h \subset \bar{X}_r := \{\Phi_s(\psi(m)) \colon m \in \bar{M}, r < d(m,R_0), r \le s \le \tau(m) - r\}.$$

Let $K \subset \mathcal{NS}$ be a set of points where the convergence in Proposition 4.6 is uniform and such that

$$\mathbb{P}(\{\Phi_s(\psi(m))\colon m \in K, 0 \le s < \tau(m)\}) > 1 - r.$$

For any $\varepsilon \in (0, r)$ sufficiently small, the ε-neighborhood of $\bar{X}_r$ is contained in $\bar{X}$.

Let $\nu_\varepsilon = \varepsilon^{5/4}$. Choose a family of pairwise disjoint open balls of radius ν_ε in $\bar{M}$ such that their union has $\bar{\mu}$-measure larger than $1 - 4\nu_\varepsilon$. We drop all the balls not intersecting K and call $\{D_i\}$ the remaining family. For each i we choose a point $m_i \in D_i \cap K$. For each i, we take the family of times $s_{ij} = j\nu_\varepsilon \in (0, \min_{D_i} \tau)$. Let

$$P_{ij} = \{\Phi_s \psi(D_i)\colon s_{ij} \le s \le s_{ij} + \nu_\varepsilon\}.$$

We finally drop the P_{ij}'s not intersecting $\bar{X}' \cap \psi\pi^{-1}K$. Set $y_{ij} = \Phi_{s_{ij}}(\psi(m_i))$. We have

$$\begin{aligned}\mathbb{P}\left(Z_\varepsilon > \exp\left(\frac{t}{4\varepsilon^2}\right)\right) &\approx \pm r + \sum_{i,j} \mathbb{P}\left(Z_\varepsilon > \exp\left(\frac{t}{4\varepsilon^2}\right); P_{ij}\right) \\ &\approx \pm r + \sum_{i,j} h(y_{ij})\mathcal{L}\left(Z_\varepsilon > \exp\left(\frac{t}{4\varepsilon^2}\right); P_{ij}\right)\end{aligned} \tag{7}$$

by uniform continuity of h. Let

$$\begin{aligned}A_{ij}^{\pm} &= \left\{m \in \bar{M}\colon \exists 0 \le s < \tau(m) \text{ s.t. } \Phi_s(\psi(m)) \in B(y_{ij}, \varepsilon \pm \nu_\varepsilon)\right\} \\ &= \pi\psi^{-1}B(y_{ij}, \varepsilon \pm \nu_\varepsilon)\end{aligned}$$

denotes the projection onto the base of the balls. Let $\tau_- = \min \tau$ and $\tau_+ = \max \tau$. For any $x \in P_{ij}$, setting $m = \pi\psi^{-1}x \in \bar{M}$ its projection, we have

$$(\tau_-)(W_{A_{ij}^+}(m) - 1) \le Z_\varepsilon(x) \le (\tau_+)W_{A_{ij}^-}(m). \tag{8}$$

Hence we have for any real $t > 0$

$$\nu_\varepsilon\mu((\tau_-)(W_{A_{ij}^+} - 1) > t; D_i) \le \mathcal{L}(Z_\varepsilon > t; P_{ij}) \le \nu_\varepsilon\mu((\tau_+)W_{A_{ij}^-} > t; D_i) \tag{9}$$

Using the regularity of the projection π on X_r, we see that the sets $A_{ij}^{\pm}$ fulfill the hypotheses of Proposition 4.6 with uniform constants. Moreover, by Lemma 5.1 and the relation (3), we have

$$\bar{\mu}(A_{ij}^{\pm}) = \frac{4(\varepsilon \pm \nu_\varepsilon)^2}{2\Gamma}.$$

Therefore by our choice of the m_i's, the difference

$$\left|\mu\left((\tau_\mp)W_{A_{ij}^{\pm}} > \exp\left(\frac{t\Gamma}{2\varepsilon^2}\right) \Big| D_i\right) - \frac{1}{1 + \beta t}\right|$$

tends to zero uniformly as $\varepsilon \to 0$. Putting it together with (9) in the computation (7) yields to

$$\overline{\lim_{\varepsilon\to 0}} \left| \mathbb{P}\left(Z_\varepsilon > \exp\left(\frac{t\Gamma}{2\varepsilon^2}\right)\right) - \frac{1}{1+\beta t} \right| \leq r.$$

Letting $r \to 0$ gives the conclusion for a continuous density compactly supported on $\bar{X}'$. The conclusion follows by an approximation argument, since any density $h \in L^1(\bar{X}, \mathcal{L})$ may be approximated by a sequence h_n of such densities.

6. Proof of the main theorem: recurrence for the position

In this section we prove Theorem 1.1-(iii) and (iv) about the return times $\mathcal{Z}_\varepsilon$ defined by (2). The proof follows the scheme of the previous section but has additional arguments. We will detail the differences and indicate the common points.

We recall that Π_Q is the canonical projection from $X = Q \times S^1$ onto Q. We will use the first return time $\overline{\mathcal{Z}}_\varepsilon$ in the ε-neighborhood of the initial position modulo $\mathbb{Z}^2$ defined by

$$\overline{\mathcal{Z}}_\varepsilon(x) = \min\left\{ t > \varepsilon \colon \Phi_t(x) \in \bigcup_{\ell\in\mathbb{Z}^2} B(\Pi_Q(x)+\ell, \varepsilon) \times S^1 \right\}.$$

For any q in Q and any $\varepsilon > 0$, we define the backward projection of $B_\varepsilon(q)\times S^1$ on M and on $\bar{M}$ by

$$\mathcal{A}_\varepsilon(q) = \left\{ m \in M \colon \exists s \in [0, \tau(\psi(m))),\ \Phi_s(\psi(m)) \in B(q,\varepsilon)\times S^1 \right\},$$

$$\bar{\mathcal{A}}_\varepsilon(q) = \left\{ m \in \bar{M} \colon \exists s \in [0, \tau(\psi(m))),\ \Phi_s(\psi(m)) \in \bigcup_{\ell\in\mathbb{Z}^2} B(q+\ell,\varepsilon)\times S^1 \right\}.$$

Lemma 6.1. *For any $q \in Q$ and any $\varepsilon \in (0, d(q, \partial Q))$, we have $\mu(\mathcal{A}_\varepsilon(q)) = 4\pi\varepsilon$ and so $\bar{\mu}(\bar{\mathcal{A}}_\varepsilon(q)) = \frac{2\pi\varepsilon}{\Gamma}$.*

Proof. Indeed, since the measure $\cos(\varphi)dr d\varphi$ is preserved by billiard maps, $\mu(\mathcal{A}_\varepsilon(q))$ is equal to the measure of the outgoing vectors based on $\partial B_\varepsilon(q)$ (for the measure $\cos(\varphi)dr d\varphi$), which is equal to $2 \times 2\pi\varepsilon$. The second assertion follows from (3). □

We first need a result similar to Theorem 3.1.

Lemma 6.2. *Lebesgue almost everywhere we have* $\underline{\lim}_{\varepsilon\to 0} \dfrac{\log \overline{\mathcal{Z}}_\varepsilon}{-\log\varepsilon} \geq 1.$

Proof. We consider again the set $\bar{X} = \psi(\pi^{-1}\bar{M})$ of points in X with previous reflection in $\bar{M}$. Let $\alpha > 0$ and set

$$\bar{X}'_\alpha = \{x = (q, v) \in \bar{X} \colon d(q, \partial Q) > \alpha\}.$$

Let $n \geq 1$ be an integer and set $r_n := \frac{1}{n(\log n)^2}$. We define the set G_n of points in $\bar{X}'_\alpha$ coming back (modulo $\mathbb{Z}^2$) in the r_n-neighborhood of the initial position between the n-th and the $(n+1)$-th reflections by

$$G_n := \{x \in \bar{X}'_\alpha \colon T^{n-1}(\Phi_{\tau(x)}(x)) \in \bigcup_{\ell \in \mathbb{Z}^2} \mathcal{A}_{r_n}(\Pi_Q(x) + \ell)\}.$$

We take a family of pairwise disjoint open balls $D_i \subset \bar{M}$ of radius r_n such that their union has $\bar{\mu}$-measure larger than $1 - 4r_n$. As in Section 5, we then construct the family P_{ij} following the same procedure. We drop those P_{ij}'s not intersecting $\bar{X}'_\alpha$. For each i, j we fix a point $y_{ij} \in P_{ij} \cap \bar{X}'_\alpha$. There exists $L_0 > 0$ such that for all $x \in X'_\alpha$ we have $\bar{\mathcal{A}}_{r_n}(\Pi_Q(x)) \subset \bar{\mathcal{A}}_{L_0 r_n}(\Pi_Q(y))$ whenever $d(x, y) < r_n$. Thus

$$\begin{aligned} Leb(G_n) &\leq \sum_{i,j} Leb\left(x \in P_{ij} \colon \bar{T}^{n-1}(\Phi_{\tau(x)}(x)) \in \bar{\mathcal{A}}_{r_n}(\Pi_Q(x))\right) \\ &\leq \sum_{i,j} r_n \bar{\mu}\left(D_i \cap \bar{T}^{-n}\left(\bar{\mathcal{A}}_{L_0 r_n}(\Pi_Q(y_{ij}))\right)\right). \end{aligned}$$

Now, we approximate the indicator function of D_i by the Lipschitz function $f_i = \max(1 - \frac{d(\cdot, D_i)}{r_n}, 0)$. We approximate in the same way the indicator function of $\bar{\mathcal{A}}_{L_0 r_n}(\Pi_Q(y_{ij}))$ by a Lipschitz function g_{ij}. Using the exponential decay of covariance for Lipschitz functions (Theorem A.3) we get

$$\bar{\mu}(D_i \cap \bar{T}^{-n} \bar{\mathcal{A}}_{L_0 r_n}(\Pi_Q(y_{ij}))) \leq C\theta^n r_n^{-2} + \int f_i d\bar{\mu} \int g_{ij} d\bar{\mu}.$$

Therefore

$$Leb(G_n) \leq C\theta^n r_n^{-5} + \sum_{i,j} r_n 4\bar{\mu}(D_i)\bar{\mu}(\bar{\mathcal{A}}_{L_1 r_n}(\Pi_Q(y_{ij}))),$$

for some constant L_1 (since $\Pi_Q \circ \Phi_s \circ \psi$ is Lipschitz for any $0 \leq s \leq \tau_+$). According to Lemma 6.1 we get $\sum_{n \geq 1} Leb(G_n) < +\infty$. Therefore, by the first Borel-Cantelli lemma, for almost every $x \in \bar{X}'_\alpha$, there exists N_x such that, for all $n \geq N_x$, $T^{n-1}(\Phi_{\tau(x)}(x)) \notin \bigcup_{\ell \in \mathbb{Z}^2} \mathcal{A}_{r_n}(\Pi_Q(x) + \ell)$. Let

$$\varepsilon_0 = \min\{d(\Pi_Q(\Phi_s(x)), \Pi_Q(x) + \mathbb{Z}^2) \colon s \in [\alpha, N_x \tau_+]\}.$$

We admit temporarily the following result :

Sub-Lemma 6.3. *The set* $\{x \in X \colon \exists s > 0,\ \Pi_Q(\Phi_s(x)) - \Pi_Q(x) \in \mathbb{Z}^2\}$ *has zero Lebesgue measure.*

Hence ε_0 is almost surely non-null. Therefore, for almost every point x in $\bar{X}'_\alpha$, for all $n \geq N_x$ such that $r_n < \varepsilon_0$, and all $k = 0, ..., n$, the point $T^{k-1}(\Phi_{\tau(x)}(x)) \notin \bigcup_{\ell \in \mathbb{Z}^2} \mathcal{A}_{r_n}(\Pi_Q(x) + \ell)$ and so $\overline{\mathcal{Z}}_{r_n}(x) \geq (n-1)\tau_-$. Hence $\varliminf_{n \to +\infty} \dfrac{\log \overline{\mathcal{Z}}_{r_n}(x)}{-\log r_n} \geq 1$. Since $\log r_n \sim \log r_{n+1}$, we end up with $\varliminf_{\varepsilon \to 0} \dfrac{\log \overline{\mathcal{Z}}_\varepsilon}{-\log \varepsilon} \geq 1$ μ-a.e. on X'_α. The conclusion follows from $\mu(X'_\alpha) \to 1$ as $\alpha \to 0$. $\square$

Proof of Sub-lemma 6.3. Let x be a point in X such that, for some $s > 0$, we have $\Pi_Q(\Phi_s(x)) - \Pi_Q(x) \in \mathbb{Z}^2$. Then either $s < \tau(x)$ which implies that x has a rational direction, or there exists $n \geq 1$ such that a particle with configuration $T^{n-1}(\Phi_{\tau(x)}(x))$ will visit $\Pi_Q(x)+\mathbb{Z}^2$ before the next reflection. We have to prove that the set C of points x satisfying the second condition has zero Lebesgue measure. For any q in $Q \setminus \partial Q$, we denote by C_q the set of points of C with position q. We have

$$Leb_X(C|\Pi_Q = q) = Leb_Q(C_q) = \int_{\bar{T}(\overline{\mathcal{A}}_0(q))\cap \bar{T}^{-(n-1)}(\overline{\mathcal{A}}_0(q))} f_q(r)dr,$$

(for some positive measurable function f_q) where $\overline{\mathcal{A}}_0(q)$ is the set of points $m \in \bar{M}$ that visits $q + \mathbb{Z}^2$ before the next reflection. The set $T(\overline{\mathcal{A}}_0(q))$ is a finite union of curves γ_1 given by $\varphi = \varphi_1(r)$. Analogously, the set $\bar{T}^{-(n-1)}(\overline{\mathcal{A}}_0(q))$ is a finite union of curves $\gamma_{-(n-1)}$ given by $\varphi = \varphi_{-(n-1)}(r)$. Moreover, each γ_1 is transversal to each $\gamma_{-(n-1)}$ (φ_1 is stricly increasing and $\varphi_{-(n-1)}$ is strictly decreasing). Hence the intersection of $\bar{T}(\mathcal{A}_0(q))$ and of $\bar{T}^{-(n-1)}(\overline{\mathcal{A}}_0(q))$ is finite. □

Lemma 6.2 enables to prove the following lemma analogous to Lemma 3.2. We call

$$\bar{M}_\tau := \{(m, s) \in \bar{M} \times \mathbb{R} \colon 0 \leq s < \tau(\psi(m))\}.$$

Lemma 6.4. *For $\bar{\mu}$-almost every $(m, s) \in \bar{M}_\tau$ the following holds:*
For any families $(q_\varepsilon)_\varepsilon$ of Q, $(D_\varepsilon)_\varepsilon$ of subsets of $\bar{M}$ such that
(i) $m \in D_\varepsilon \subset \bar{\mathcal{A}}_\varepsilon(q_\varepsilon)$
(ii) $\Phi_s(\psi(m)) \in \bigcup_{\ell\in\mathbb{Z}^2} B(q_\varepsilon + \ell, \varepsilon) \times S^1$
(iii) D_ε is either a ball or the set $\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)$
we have for all $\alpha > 0$

$$\bar{\mu}(\bar{W}_{\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)} \leq \varepsilon^{-1+\alpha}|D_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. We do not detail the proof when D_ε is a ball since it is a direct adaptation of the proof of Lemma 3.2 with the use of Lemma 6.2 instead of Theorem 3.1.

We suppose that $D_\varepsilon = \bar{\mathcal{A}}_\varepsilon(q_\varepsilon)$. The idea is to consider the billiard flow modulo $\mathbb{Z}^2$ and to adapt the proof of Lemma 3.2 thanks to the Fubini theorem.

Let $\alpha > 0$ and let $a \in (0, \alpha)$. Let $\eta > 0$ and $\varepsilon_0 > 0$. We set for all q' in Q

$$Bad(q') = \left\{ v \in S^1 \colon \exists \varepsilon \leq \varepsilon_0,\ \frac{\log \overline{\mathcal{Z}}_\varepsilon(q', v)}{-\log \varepsilon} < 1 - a \right\}$$

and

$$F_\eta(\varepsilon_0) = \{q' \in Q \colon Leb_{S^1}(Bad(q')) \leq \eta\}.$$

Let m and s be such that $\Pi_Q(\Phi_s(\psi(m)))$ is a density point in Q of the set $F_\eta(\varepsilon_0)$ with respect to the Lebesgue basis of balls in Q. We have

$$\lim_{\varepsilon\to 0} \sup_{q_\varepsilon} Leb_Q(F_\eta(\varepsilon_0)^c | B(q_\varepsilon, 2\varepsilon)) = 0 \tag{10}$$

where the supremum is taken among all the q_ε satisfying the hypothesis. We observe that $F_\eta(\varepsilon_0)$ is stable by $\mathbb{Z}^2$-translations and that, for all $\eta > 0$, $\lim_{\varepsilon_0\to 0} Leb_Q((Q\cap[0;1)^2)\setminus F_\eta(\varepsilon_0)) = 0$. Therefore, for a.e. (m,s) and any $\eta>0$ there exists a choice of ε_0 such that (10) holds. Let

$$H_\varepsilon := (B(q_\varepsilon,2\varepsilon)\times S^1)\cap \bigcup_{s\in(6\varepsilon(\tau_+)\varepsilon^{-1+\alpha})} \Phi_{-s}\left(\bigcup_{\ell\in\mathbb{Z}^2} B(q_\varepsilon+\ell,2\varepsilon)\times S^1\right).$$

There exists $\varepsilon_1\in(0,\varepsilon_0)$ such that, for all $\varepsilon\in(0,\varepsilon_1)$, we have

$$\begin{aligned} H_\varepsilon &\subset (B(q_\varepsilon,2\varepsilon)\times S^1)\cap\{\overline{\mathcal{Z}}_{4\varepsilon}\le \tau_+\varepsilon^{-1+\alpha}\}\\ &\subset \{(q',v)\in B(q_\varepsilon,2\varepsilon)\times S^1\colon v\in Bad(q')\}. \end{aligned}$$

Therefore

$$\begin{aligned} Leb_X(H_\varepsilon) &= Leb_X(\Pi_Q^{-1}(F_\eta(\varepsilon_0))\cap H_\varepsilon) + Leb_X(H_\varepsilon\setminus \Pi_Q^{-1}(F_\eta(\varepsilon_0)))\\ &\le \eta Leb_Q(B(q_\varepsilon,2\varepsilon)) + 2\pi Leb_Q(B(q_\varepsilon,2\varepsilon)\setminus F_\eta(\varepsilon_0)). \end{aligned}$$

This together with (10) yields to

$$\overline{\lim_{\varepsilon\to 0}}\, Leb_X(H_\varepsilon|B(q_\varepsilon,2\varepsilon)\times S^1) \le \frac{\eta}{2\pi}.$$

Since $\eta>0$ is arbitrary, for almost every (m,s), we get

$$\lim_{\varepsilon\to 0} Leb_X(H_\varepsilon|B(q_\varepsilon,2\varepsilon)\times S^1) = 0.$$

Hence

$$Leb_X(H_\varepsilon\cap(B(q_\varepsilon,2\varepsilon)\times S^1)) = o(\varepsilon^2).$$

Moreover, setting $I_s(m) = \text{length}\{s\in(0;\tau(m))\colon \Phi_s(m)\in B(q_\varepsilon,2\varepsilon)\times S^1\}$ and using the representation of Φ_s as a special flow over T gives

$$\begin{aligned} Leb_X(H_\varepsilon\cap(B(q_\varepsilon,2\varepsilon)\times S^1)) &\ge \int_{\bar{\mathcal{A}}_{2\varepsilon}(q_\varepsilon)\cap\{\bar W_{\bar{\mathcal{A}}_{2\varepsilon}(q_\varepsilon)}\le\varepsilon^{-1+\alpha}\}} I_s(m)\,d\mu(m)\\ &\ge \int_{\bar{\mathcal{A}}_{\varepsilon}(q_\varepsilon)\cap\{\bar W_{\bar{\mathcal{A}}_{\varepsilon}(q_\varepsilon)}\le\varepsilon^{-1+\alpha}\}} I_s(m)\,d\mu(m)\\ &\ge \varepsilon\mu(\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)\cap\{\bar W_{\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)}\le\varepsilon^{-1+\alpha}\}) \end{aligned}$$

This finally gives

$$\bar\mu(\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)\cap\{\bar W_{\bar{\mathcal{A}}_\varepsilon(q_\varepsilon)}\le\varepsilon^{-1+\alpha}\}) = o(\varepsilon) = o(\bar\mu(\bar{\mathcal{A}}_\varepsilon(q_\varepsilon))).$$

□

We denote by $\mathcal{N}S'$ the set of couples $(m,s)\in\bar M_\tau$ satisfying the conclusion of Lemma 6.4. This is essential for the following lemma analogous to Proposition 4.2

Lemma 6.5. *For all $(m,s)\in\mathcal{NS}'$, there exists a function $f_{m,s}$ such that $\lim_{\varepsilon\to 0} f_{m,s}(\varepsilon)=0$ and such that, for any families $(q_\varepsilon)_\varepsilon$ of Q and $(D_\varepsilon)_\varepsilon$ of subsets of M such that :*

(i) *$m\in D_\varepsilon\subset\mathcal{A}_\varepsilon(q_\varepsilon)$;*

(ii) $\Phi_s\psi(m) \in B(q_\varepsilon, \varepsilon) \times S^1$;
(iii) D_ε *is a ball of radius larger than* $\varepsilon^{1.2}$ *or is the set* $\mathcal{A}_\varepsilon(q_\varepsilon)$;

for all $N \in (e^{\log^2 \varepsilon}, e^{\frac{1}{\varepsilon^{2.5}}})$, *we have :*

$$\left| \mu(W_{\mathcal{A}_\varepsilon(q_\varepsilon)}(\cdot) > N | D_\varepsilon) - \frac{1}{1 + \log(N)\bar{\mu}(\bar{\mathcal{A}}_\varepsilon(q_\varepsilon))\beta} \right| \le f_{m,s}(\varepsilon)$$

and

$$\mu(W_{\mathcal{A}_\varepsilon(q_\varepsilon)}(\cdot) > N | \mathcal{A}_\varepsilon(q_\varepsilon)) = \frac{1 + o_\varepsilon(1)}{1 + \log(N)\bar{\mu}(\bar{\mathcal{A}}_\varepsilon(q_\varepsilon))\beta},$$

where the error term $o_\varepsilon(1)$ *is bounded by* $f_{m,s}(\varepsilon)$.

Proof. To simplify the proof, we use the notations $A = \mathcal{A}_\varepsilon(q_\varepsilon)$ and $\bar{A} = \bar{\mathcal{A}}_\varepsilon(q_\varepsilon)$.

First step : We adapt the proof of Lemma 4.3 to prove that

$$\mu(W_A > N|D) + \beta \log(N)\bar{\mu}(\bar{A})\mu(W_A > N|A) \le 1 + o_\varepsilon(1).$$

A slight difficulty comes from the fact that the set A can be divided into several cells. More precisely, there exist pairwise disjoint subsets A_ℓ of $\bar{M}$ such that (with obvious notations)

$$A = \bigcup_{|\ell| \le \tau_+} (A_\ell + \ell) \text{ and } \bar{A} = \bigcup_{|\ell| \le \tau_+} A_\ell.$$

Analogously, there exist pairwise disjoint subsets D_ℓ of $\bar{M}$ such that

$$D = \bigcup_{|\ell| \le \tau_+} (D_\ell + \ell).$$

Hence, we have

$$\begin{aligned}
\mu(D) &= \sum_{q=0}^{N} \mu(D;\ T^{-q}(A; W_A > N - q)) \\
&\ge \mu(D; W_A > N) + \sum_{q=p_0}^{N} \mu(D;\ T^{-q}(A; W_A > N)) \\
&\ge \mu(D; W_A > N) + \sum_{q=p_0}^{N} \sum_{\ell,\ell'} \mu(D_{\ell'} + \ell';\ T^{-q}(A_\ell + \ell; W_A > N)) \\
&\ge \mu(D; W_A > N) + \sum_{q=p_0}^{N} \sum_{\ell,\ell'} \mu(D_{\ell'}; S_q\kappa = \ell - \ell'; \bar{T}^{-q}(A_\ell; W_{A-\ell} > N)).
\end{aligned}$$

This together with (3), as in the proof of Lemma 4.3, give

$$\mu(D) \ge \mu(D; W_A > N) + \beta \log(N) \frac{\mu(D)}{2\Gamma} \mu(A; W_A > N) + o(\mu(D))$$

and so

$$1 \ge \mu(W_A > N|D) + \beta \log(N)\bar{\mu}(\bar{A})\mu(W_A > N|A) + o(1).$$

Second step : To prove the following lower bound

$$\mu(W_A > N|D) + \beta \log(N)\bar{\mu}(\bar{A})\mu(W_A > N|A) \geq 1 + o_\varepsilon(1),$$

we use the notations m_N and n_N of the proof of Lemma 4.4 and we write

$$\begin{aligned}\mu(D) &= \sum_{q=0}^{n_N} \mu(D;\ T^{-q}(A; W_A > n_N - q)) \\ &= \mu(D; W_A > N) + \sum_{q=1}^{n_N} \sum_{\ell,\ell'} \mu(D_{\ell'};\ S_q\kappa = \ell - \ell'; \bar{T}^{-q}(A_\ell; W_{A-\ell} > N)).\end{aligned}$$

A first difference with the proof of lemma 4.4 is that we work with $D_{\ell'}$ and A_ℓ instead of considering directly D and A. We approximate $D_{\ell'}$ by a set $D''_{\ell'}$ and A_ℓ by a set A''_ℓ as we approximate D by D'' in the proof of Lemma 4.4.

We fix $\alpha \in (0, 0.5)$ and we follow the scheme of the proof of Lemma 4.4 for the estimate of S_0 and S_3 (using $D''_{\ell'}$ and A''_ℓ). We take $M_\varepsilon = \varepsilon^{-1+\alpha}$ instead of $M_\varepsilon = \varepsilon^{2(-1+\alpha)}$. According to Lemma 6.4, this choice of M_ε gives the correct estimate of S_1. We introduce $M'_\varepsilon = \varepsilon^{-6}$. We decompose S_2 in two blocks : S'_2 is the sum for q in the range $M_\varepsilon + 1, ..., M'_\varepsilon$ and S''_2 in the range $M'_\varepsilon + 1, ..., m_N$.

To estimate S'_2 and S''_2, we approximate $E_\ell := A_\ell \cap \{W_{A-\ell} > N\}$ by a set E''_ℓ as we approximate E by E'' in the proof of Lemma 4.4.

We estimate S''_2 as we estimate S_2 in the proof of Lemma 4.4 with M'_ε instead of M_ε:

$$S''_2 \leq \log\left(\frac{m_N}{M'_\varepsilon}\right) \beta \frac{\mu(D)\mu(E)}{2\Gamma}(1 + o(1)) + \frac{ck\bar{\mu}(A)^{\frac{1}{p}}}{\sqrt{M'_\varepsilon - 2k}} + o(\mu(D))$$

and the error term is in $O(\log(\varepsilon)\varepsilon^{1/p}\varepsilon^3) = o(\mu(D))$ provided $3 + 1/p > 2.4$.

To estimate S'_2, we use the symmetry π_0 on M with respect to the normal n given by : $\pi_0(\psi(\ell, i, r, \varphi)) = \pi_0(\psi(\ell, i, r, -\varphi))$. Let us notice that π_0 preserves $\bar{\mu}$. Using this symmetry and applying Proposition 4.1 with p such that $1/4 > 2.4(1 - 1/p)$, we get

$$\begin{aligned} S'_2 &\leq 2\Gamma \sum_{q=M_\varepsilon}^{M'_\varepsilon} \sum_{\ell,\ell'} \bar{\mu}(D''_{\ell'}, S_q\kappa = \ell - \ell'; \bar{T}^{-q}(A''_\ell)) \\ &\leq 2\Gamma \sum_{q=M_\varepsilon}^{M'_\varepsilon} \sum_{\ell,\ell'} \bar{\mu}(\pi_0(A''_\ell); S_q\kappa = \ell' - \ell; \bar{T}^{-q}(\pi_0(D''_{\ell'}))) \\ &\leq 2\Gamma \sum_{q=M_\varepsilon}^{M'_\varepsilon} \sum_{\ell,\ell'} \left[\frac{\beta\bar{\mu}(A''_\ell)\bar{\mu}(D''_{\ell'})}{q - 2k} + \frac{ck\bar{\mu}(D''_{\ell'})^{1/p}}{(q - 2k)^{3/2}}\right] \\ &\leq \log\left(\frac{M'_\varepsilon}{M_\varepsilon}\right) \beta\bar{\mu}(\bar{A})\mu(D)(1 + o(1)) + \frac{c'k\mu(D)^{1/p}(1 + o(1))}{\sqrt{M_\varepsilon - 2k}}, \end{aligned}$$

the last error term being in $O\left(\log(\varepsilon)\bar\mu(D)^{1/p}\varepsilon^{(1-\alpha)/2}\right)=o(\bar\mu(D))$ since $(1-\alpha)/2>2.4(1-1/p)$. Hence, we have proved that, under the assumptions of Lemma 6.5, we have

$$\mu(W_A>N|D)+\beta\log(N)\bar\mu(\bar A)\mu(W_A>N|A)=1+o(1). \qquad (11)$$

In the special case $D=A$, we conclude that

$$\mu(W_A>N|A)=\frac{1+o(1)}{1+\beta\log(N)\bar\mu(\bar A)}. \qquad (12)$$

We turn now to the general case. Applying Equations (11) and (12) we get

$$\mu(W_A>N|D)=\frac{1}{1+\beta\log(N)\bar\mu(\bar A)}+o(1).$$

□

Proof of Theorem 1.1-(iii). Upper bound : Let $\bar X_0$ be a set of points of X with previous reflection in $\bar M$ and on which the estimate of Lemma 6.5 is uniform. Let $\alpha\in(0,1)$ and $\varepsilon_n=\log^{-\alpha}n$. Take a cover of $\bar X_0$ by some balls $B(q_n,\frac{\varepsilon_n}{2})\times S^1$ for $q_n\in\mathcal{Q}_n\subseteq Q$ such that $\#\mathcal{Q}_n=O(\varepsilon_n^{-2})$. We have

$$\begin{aligned}
Leb(\bar X_0;\mathcal{Z}_{\frac{\varepsilon_n}{2}}\ge n\tau_+) &\le \sum_{q_n} Leb(B(q_n,\frac{\varepsilon_n}{2});\mathcal{Z}_{\frac{\varepsilon_n}{2}}\ge n\tau_+)\\
&\le \sum_{q_n}\varepsilon_n\mu(W_{\mathcal{A}_{\frac{\varepsilon_n}{2}}(q_n)}>n;\mathcal{A}_{\frac{\varepsilon_n}{2}}(q_n))\\
&\le \varepsilon_n\sum_{q_n}\mu\left(W_{\mathcal{A}_{\frac{\varepsilon_n}{2}}(q_n)}>n\middle|\mathcal{A}_{\frac{\varepsilon_n}{2}}(q_n)\right)\mu(\mathcal{A}_{\frac{\varepsilon_n}{2}}(q_n))\\
&\le O((1+\beta c\log(n)\varepsilon_n)^{-1})\\
&\le O((1+\beta c\log^{1-\alpha}(n))^{-1}),
\end{aligned}$$

with $c=\frac{2\pi}{\Gamma}$ (according to Lemma 6.1). Now, by taking $n_k=\exp(k^{2/(1-\alpha)})$ and according to the Borel-Cantelli lemma, we get that, for almost all x in $\bar X_0$, there exists N_x such that, for any $k\ge N_x$, $\mathcal{Z}_{\frac{\varepsilon_{n_k}}{2}}(x)<n_k\tau_+$ and hence

$$\overline{\lim_{k\to+\infty}}\frac{\log\log\mathcal{Z}_{\varepsilon_{n_k}}(x)}{-\log\varepsilon_{n_k}}\le\frac{1}{\alpha}.$$

Since $\log\varepsilon_{n_k}\sim\log\varepsilon_{n_{k+1}}$, we conclude that almost everywhere in $\bar X_0$, we have :

$$\overline{\lim_{\varepsilon\to 0}}\frac{\log\log\mathcal{Z}_\varepsilon}{-\log\varepsilon}\le\frac{1}{\alpha}.$$

Therefore, almost everywhere in X, we have

$$\overline{\lim_{\varepsilon\to 0}}\frac{\log\log\mathcal{Z}_\varepsilon}{-\log\varepsilon}\le 1.$$

Lower bound : Let $\bar X$ be the set of points of X with previous reflection in $\bar M$. Let $\alpha>1$. For all $n\ge1$, we take $\varepsilon_n=\log^{-\alpha}n$ and we denote by K_n the set of points $x\in\bar X$ whose orbit (by the billiard flow) comes back to the ε_n-neighbourhood for the position between the n^{th} and the $(n+1)^{th}$ reflections :

$$K_n = \left\{x \in \bar{X} \colon \exists s \in I_n(x),\ d(\Pi_Q(x), \Pi_Q(\psi(T^n(\pi\psi^{-1}(x)), s))) < \varepsilon_n\right\},$$

with $I_n(x) := [0; \tau(T^n(\pi\psi^{-1}(x))))$. We consider a cover of $\bar{X}$ by sets $C_{\varepsilon_n}(q) = B(q, \varepsilon_n) \times S^1$ for $q \in \mathcal{Q}'_n \subseteq Q$ such that $\#\mathcal{Q}'_n = O(\varepsilon_n^{-2})$. Let $n \geq 1$. For any $q \in \mathcal{Q}_n$, there exist two families of pairwise disjoint subsets $(A_{1,\ell}(q))_\ell$ and $(A_{2,\ell}(q))_\ell$ of $\bar{M}$ such that :

$$\mathcal{A}_{\varepsilon_n}(q) = \bigcup \left(A_{1,\ell'}(q) + \ell'\right) \quad \text{and} \quad \mathcal{A}_{2\varepsilon_n}(q) = \bigcup \left(A_{2,\ell}(q) + \ell\right).$$

Let k be such that $\delta^k \approx \varepsilon_n^5$. Let $A''_{1,\ell'}(q)$ (resp. $A''_{2,\ell}(q)$) be the union of all the cylinders $Z \in \mathcal{Z}_{-k}^k$ intersecting $A_{1,\ell}(q)$ (resp. $A_{2,\ell}(q)$). We have :

$$\begin{aligned}
Leb(K_n) &\leq \sum_q Leb(x \in C_{\varepsilon_n}(q) \colon T^n(\pi\psi^{-1}(x)) \in \mathcal{A}_{2\varepsilon_n}(q)) \\
&\leq 2\varepsilon_n \sum_q \sum_{\ell,\ell'} \mu(A_{1,\ell'}(q) + \ell';\ T^{-n}(A_{2,\ell}(q) + \ell)) \\
&\leq 2\varepsilon_n \sum_q \sum_{\ell,\ell'} \mu(A_{1,\ell'}(q);\ S_n\kappa = \ell - \ell';\ \bar{T}^{-n}(A_{2,\ell}(q))) \\
&\leq 4\varepsilon_n \Gamma \sum_q \sum_{\ell,\ell'} \bar{\mu}(A''_{1,\ell'}(q);\ S_n\kappa = \ell - \ell';\ \bar{T}^{-n}(A''_{2,\ell}(q))) \\
&\leq 4\varepsilon_n \Gamma \sum_q \sum_{\ell,\ell'} \left[\beta \frac{\bar{\mu}(A''_{1,\ell'}(q))\bar{\mu}(A''_{2,\ell}(q))}{n-2k} + \frac{ck}{(n-2k)^{3/2}}\right] \\
&\leq \frac{\varepsilon_n}{\Gamma} \sum_q \left[\beta \frac{\mu(\mathcal{A}_{\varepsilon_n}(q))\mu(\mathcal{A}_{2\varepsilon_n}(q))}{n-2k}(1 + o(1))\right] + O(\varepsilon_n n^{-1}) \\
&\leq O(\varepsilon_n n^{-1}) = O(n^{-1} \log^{-\alpha} n).
\end{aligned}$$

Hence, according to the first Borel Cantelli lemma, for almost every $x \in \bar{X}$, there exists N_x such that, for all $n \geq N_x$, for every $s \in I_n(x)$, we have

$$d(\Pi_Q(x), \Pi_Q(\psi(T^n(\pi\psi^{-1}(x)), s))) \geq \varepsilon_n.$$

According to Lemma 6.3,

$$u := \min\left(d(\Pi_Q(x), \Pi_Q(\psi(T^n(\pi\psi^{-1}(x)), s)),\ n = 1, ..., N_x,\ s \in I_n(x)\right)$$

is almost surely non-null. Therefore, for almost every point x in $\bar{X}$, for all $n \geq N_x$ such that $\varepsilon_n < u$, $\mathcal{Z}_{\varepsilon_n}(x) \geq (n-1)\tau_-$. Hence, almost everywhere in X, we have

$$\varliminf_{n \to +\infty} \frac{\log \log Z_{\varepsilon_n}}{-\log \varepsilon_n} \geq \alpha^{-1}.$$

Since $\log \varepsilon_n \sim \log \varepsilon_{n+1}$, we have $\underline{\lim}_{\varepsilon\to 0} \frac{\log\log \mathcal{Z}_\varepsilon}{-\log\varepsilon} \geq \alpha^{-1}$. Therefore, almost everywhere in X, we have

$$\underline{\lim}_{\varepsilon\to 0} \frac{\log\log \mathcal{Z}_\varepsilon}{-\log\varepsilon} \geq 1.$$

□

Sketch of proof of Theorem 1.1-(iv). This result is obtained by following the same scheme as in the proof of Theorem 1.1-(ii) in Section 5. We list the differences:

- The set $K \subset \mathcal{NS} \subset \bar{M}$ is replaced by a set $\mathcal{K} \subset \mathcal{NS}' \subset \bar{M}_\tau$ such that the convergence in Lemma 6.5 is uniform and such that
$$\mathbb{P}(\psi(\mathcal{K})) > 1 - r.$$
- The family P_{ij}: we first take a family of pairwise disjoint balls D_i of $\bar{M}$ of radius ν_ε such that their union has $\bar{\mu}$-measure larger than $1 - 4\nu_\varepsilon$. We construct the P_{ij}'s exactly as in Section 5. Finally we drop the P_{ij}'s not intersecting $\mathcal{K} \cap \bar{X}_r$. We choose $y_{ij} \in P_{ij} \cap \psi(\mathcal{NS}')$.
- The sets $A^{\pm}_{ij}$ are replaced by $\mathcal{A}^{\pm}_{ij} := \mathcal{A}_{\varepsilon\pm\nu_\varepsilon}(\Pi_Q(y_{ij}))$.
- We use the formula for the measure of the $\mathcal{A}^{\pm}_{ij}$ given by Lemma 6.1.

□

Appendix A. Transfer operator and local limit theorem

A.1. **Hyperbolicity, Young towers and spectral properties of the transfer operator.** We do not repeat the construction of stable and unstable manifolds but only emphasize the hyperbolic estimate that is used throughout the proofs. Recall that $R_0 = \{\varphi = \pm\frac{\pi}{2}\} \subset \bar{M}$ is the presingularity set. For any $k_1 \leq k_2$, let $\xi_{k_1}^{k_2}$ be the partition of $\bar{M} \setminus \bigcup_{j=k_1}^{k_2} \bar{T}^{-j}(R_0)$ into connected components. With a slight abuse of language we will call *cylinders* the elements of $\xi_{k_1}^{k_2}$.

Lemma A.1. *There exist some constants c_0 and $\delta > 0$ such that for every integer k, every set $Z \in \xi_{-k}^{k}$ has a diameter* $\operatorname{diam} Z \leq c_0\delta^k$.

Proof. We recall that there exists $C_0 > 0$ and $\Lambda_0 > 1$ such that, for any increasing curve contained in a same connected component of ξ_0^k, $T^n\gamma$ is an increasing curve satisfying

$$\operatorname{length}(T^n\gamma) \geq C_0\Lambda_0^n \operatorname{length}(\gamma)^2$$

and such that, for any decreasing curve contained in a same connected component of ξ_{-k}^0, $T^{-n}\gamma$ is a decreasing curve satisfying

$$\operatorname{length}(T^{-n}\gamma) \geq C_0\Lambda_0^n \operatorname{length}(\gamma)^2.$$

Let Z be in ξ_{-k}^k and be composed of points based on the same obstacle O_i. The set Z is delimitated by two increasing curves and two decreasing curves. Let m and m' be two points in Z. These two points can be joined by a monotonous curve γ in Z.

If the curve γ is increasing, then we have

$$\mathrm{length}(\gamma) \le \sqrt{\frac{\mathrm{length}(T^n\gamma)}{C_0\Lambda_0^n}} \le \sqrt{\frac{\pi + |\partial O_i|}{C_0\Lambda_0^n}}.$$

If the curve γ is decreasing, then, considering $\bar{T}^{-n}\gamma$, we get $\mathrm{length}(\gamma) \le \sqrt{\frac{\pi+|\partial O_i|}{C_0\Lambda_0^n}}$. □

We do not repeat the construction of the tower but only briefly recall its property and then introduce the Banach space suitable for the study of the transfer operator. Young constructed in [29] two dynamical systems $(\tilde{M}, \tilde{T}, \tilde{\mu})$ and $(\hat{M}, \hat{T}, \hat{\mu})$ such that there exist two measurable functions $\tilde{\pi}\colon \tilde{M} \to \bar{M}$ and $\hat{\pi}\colon \tilde{M} \to \hat{M}$ such that $\tilde{\pi} \circ \tilde{T} = \bar{T} \circ \tilde{\pi}$, $\tilde{\pi}_*\tilde{\mu} = \bar{\mu}$, $\hat{\pi} \circ \tilde{T} = \hat{T} \circ \hat{\pi}$, $\hat{\pi}_*\tilde{\mu} = \hat{\mu}$.

These dynamical systems are towers and are such for any measurable $f\colon \bar{M} \to \mathbb{C}$ constant on each stable manifold there exists $\hat{f}\colon \hat{M} \to \mathbb{C}$ such that $\hat{f} \circ \hat{\pi} = f \circ \tilde{\pi}$. For each $\ell \ge 0$, we denote by $\hat{\Delta}_\ell$ the ℓth floor of the tower $\hat{M}$. This ℓ-floor is partitioned in $\{\hat{\Delta}_{\ell,j}\colon j = 1, \dots, j_\ell\}$. The partition $\mathcal{D} = \{\hat{\Delta}_{\ell,j}\colon \ell \ge 0,\ j = 1, \dots, j_\ell\}$ is Markov. For any x, y belonging to the same atom of $\mathcal{D}$, we define

$$s(x, y) := \max\{n \ge 0\colon \forall i \le n,\ \mathcal{D}(\hat{T}^i x) = \mathcal{D}(\hat{T}^i y)\}.$$

For any such x, y, the sets $\tilde{\pi}\hat{\pi}^{-1}\{x\}$ and $\tilde{\pi}\hat{\pi}^{-1}\{y\}$ are contained in the same connected component of $\bar{M} \setminus \bigcup_{k=0}^{s(x,y)} \bar{T}^{-k} R_0$.

Let $p > 1$ and set q such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\varepsilon > 0$ and $\beta \in (0, 1)$ well chosen. Young defines for $\hat{f} \in L^q_{\mathbb{C}}(\hat{M}, \hat{\mu})$

$$\|\hat{f}\| = \sup_\ell \|\hat{f}_{|\hat{\Delta}_\ell}\|_\infty e^{-\ell\varepsilon} + \sup_{\ell,j} \mathrm{esssup}_{x,y\in\hat{\Delta}_{\ell,j}} \frac{|\hat{f}(x) - \hat{f}(y)|}{\beta^{s(x,y)}} e^{-\ell\varepsilon}.$$

Let $\mathcal{V} = \{\hat{f} \in L^q_{\mathbb{C}}(\hat{M}, \hat{\mu})\colon \|\hat{f}\| < \infty\}$. This defines a Banach space $(\mathcal{V}, \|\cdot\|)$, such that $\|\cdot\|_q \le \|\cdot\|$. Let P be the Perron-Frobenius operator on L^q defined as the adjoint of the composition by $\hat{T}$ on L^p. This operator P is quasicompact on $\mathcal{V}$. The construction of the tower can be adapted in such a way that its dominating eigenvalue on $\mathcal{V}$ is 1 and is simple. This choice will be convenient for our proof and we will adopt it, although it is not essential.

The cell shift function κ is centered in the sense that

$$\int \kappa \, d\bar{\mu} = 0$$

and its asymptotic covariance matrix

$$\Sigma^2 := \lim_{n\to\infty} \frac{1}{n} \mathrm{Cov}_{\bar{\mu}}(S_n\kappa) \tag{13}$$

is well defined and non-degenerated. Since $\kappa\colon \bar{M} \to \mathbb{Z}^2$ is constant on the local stable manifolds, there exists $\hat{\kappa}\colon \hat{M} \to \mathbb{Z}^2$ such that $\hat{\kappa} \circ \hat{\pi} = \kappa \circ \tilde{\pi}$. For any $u \in \mathbb{R}^2$, we define $P_u(\hat{f}) = P(e^{iu\cdot\hat{\kappa}}\hat{f})$. The method introduced by Nagaev [18, 19] and developed by Guivarc'h and Hardy [14] and many other authors has been applied in this context by Szász and Varjú [28] (see also [21]). They have established the following key result:

Proposition A.2. *There exist a real $a \in (0, \pi)$, a C^3 family of complex numbers $(\lambda_u)_{u\in[-a,a]^2}$, two C^3 families of linear operators on $\mathcal{V}$: $(\Pi_u)_{u\in[-a,a]^2}$ and $(N_u)_{u\in[-a,a]^2}$ such that*

(i) *for all $u \in [-a, a]^2$ we have $P_u^n = \lambda_u^n \Pi_u + N_u^n$; Moreover $\Pi_0 \hat{f} = \int_{\hat{M}} \hat{f}\, d\hat{\mu}$ for any $\hat{f} \in L^q$;*

(ii) *there exists $\nu \in (0, 1)$ such that*
$$\sup_{u\in[-a,a]^2} |||N_u^n||| = O(\nu^n) \quad \text{and} \quad \sup_{u\in[-\pi,\pi]^2\setminus[-a,a]^2} |||P_u^n||| = O(\nu^n);$$

(iii) *we have $\lambda_u = 1 - \frac{1}{2}\Sigma^2 u \cdot u = O(|u|^3)$;*

(iv) *there exists $\sigma > 0$ such that, for any $u \in [-a, a]^2$, $|\lambda_u| \le e^{-\sigma|u|^2}$ and $e^{-\frac{1}{2}\Sigma^2 u\cdot u} \le e^{-\sigma|u|^2}$.*

Note that by taking $u = 0$ in the proposition we recover the estimate on the rate of decay of correlations below. We state it here in a form suitable for our purpose, in particular to prove the results of Section 3.

Theorem A.3 ([29])**.** *There exist some constants $C > 0$ and $\theta \in (0, 1)$ such that for all Lipschitz functions f and g from $\bar{M}$ to $\mathbb{R}$,*
$$\int f \circ T^n g d\bar{\mu} - \int f d\bar{\mu} \int g d\bar{\mu} \le C\theta^n \|f\|_{Lip}\|g\|_{Lip}. \tag{14}$$
Moreover, if f is the indicator function of a union of components of ξ_{-k}^{k} and g is the indicator function of a union of components of $\xi_{-k}^{+\infty}$ then the covariance in (14) *is simply bounded by $C\theta^{n-2k}$.*

However this information is not sufficient to control the recurrence for the extended billiard map T, therefore we need a finer version.

A.2. **Conditional uniform local limit theorem.**

Here we prove the local limit theorem, Proposition 4.1, concerning the billiard map $\bar{T}$ and its $\mathbb{Z}^2$-cocycle $S_n\kappa$.

Proposition 4.1. *Let $p > 1$. There exists $c > 0$ such that, for any $k \ge 1$, if $A \subset \bar{M}$ is a union of components of ξ_{-k}^{k} and $B \subset \bar{M}$ is a union of ξ_{-k}^{∞} then for any $n > 2k$ and $\ell \in \mathbb{Z}^2$*
$$\left| \bar{\mu}(A \cap \{S_n\kappa = \ell\} \cap \bar{T}^{-n}(B)) - \frac{\beta e^{-\frac{1}{2(n-2k)}(\Sigma^2)^{-1}\ell\cdot\ell}}{(n-2k)} \bar{\mu}(A)\bar{\mu}(B) \right| \le \frac{ck\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}$$
where $\beta = \frac{1}{2\pi\sqrt{\det \Sigma^2}}$.

Proof. The set $\bar T^{-k}A$ is a union of components of ξ_0^{2k} and $\bar T^{-k}B$ is a union of components of ξ_0^∞. Let $\hat A=\hat\pi(\tilde\pi^{-1}\bar T^{-k}A)$ and $\hat B=\hat\pi(\tilde\pi^{-1}\bar T^{-k}B)$. Note that $\tilde\pi^{-1}\bar T^{-k}A=\hat\pi^{-1}\hat A$ and $\tilde\pi^{-1}\bar T^{-k}B=\hat\pi^{-1}\hat B$. Setting

$$C_n(A,B,\ell):=\bar\mu(A;S_n\kappa=\ell;\bar T^{-n}B),$$

we have

$$\begin{aligned}
C_n(A,B,\ell)&=\int_{\hat M}1_{\hat A}1_{\{S_n\hat\kappa=\ell\}}\circ\hat T^k 1_{\hat B}\circ\hat T^n\,d\hat\mu\\
&=\int_{\hat M}P^k(1_{\hat A})1_{\{S_n\hat\kappa=\ell\}}1_{\hat B}\circ\hat T^{n-k}\,d\hat\mu\\
&=\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}e^{-iu\cdot\ell}\underbrace{\int_{\hat M}P^k(1_{\hat A})e^{iu\cdot S_n\kappa}1_{\hat B}\circ\hat T^{n-k}\,d\hat\mu}_{a(u)}\,du.
\end{aligned}$$

We have

$$\begin{aligned}
a(u)&=\int_{\hat M}P_u^n(P^k(1_{\hat A})1_{\hat B}\circ\hat T^{n-k})\,d\hat\mu\\
&=\int_{\hat M}P_u^k(1_{\hat B}P_u^{n-k}P^k(1_{\hat A}))\,d\hat\mu\\
&=\int_{\hat M}P_u^k(1_{\hat B}P_u^{n-2k}(b_u^k))\,d\hat\mu,
\end{aligned}$$

with $b_u^k:=P_u^kP^k(1_{\hat A})$. Set

$$a_1(u):=\int_{\hat M}P^k(1_{\hat B}P_u^{n-2k}(b_u^k))\,d\hat\mu.$$

We have, since $|||P_u^k-P^k|||_{L^1\to L^1}\le|u|k\|\kappa\|_\infty$,

$$\begin{aligned}
|a(u)-a_1(u)|&\le|||P_u^k-P^k|||_{L^1\to L^1}\int_{\hat M}1_{\hat B}|P_u^{n-2k}(b_u^k)|\,d\hat\mu\\
&\le\|\kappa\|_\infty k|u|\|P_u^{n-2k}b_u^k\|\hat\nu(\hat B)^{1/p}
\end{aligned}$$

by the Hölder inequality and since the norm $\|\cdot\|$ dominates the L^q norm. Let us notice that by the Markov property $\sup_{u\in[-\pi,\pi]^2}\|b_u^k\|=O(1)$, uniformly in A and k. We have by Proposition A.2 (i) and (ii)

$$\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}|u||||P_u^{n-2k}|||du=\int_{[-a,a]^2}|u||\lambda_u|^{n-2k}du+O(\nu^{n-2k}).$$

In addition, by Proposition A.2 (iv) we have

$$\int_{[-a,a]^2}|u||\lambda_u|^{n-2k}du\le\frac{1}{(n-2k)^{\frac32}}\int_{\mathbb R^2}|v|e^{-\sigma|v|^2}dv=O\left(\frac{1}{(n-2k)^{\frac32}}\right),\quad(15)$$

with the change of variable $v=\sqrt{n-2k}u$. Therefore

$$C_n(A,B,\ell)=\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}e^{-iu\cdot\ell}\int_{\hat M}1_{\hat B}P_u^{n-2k}(b_u^k)\,d\hat\mu\,du+O\left(\frac{k\bar\mu(B)^{\frac1p}}{(n-2k)^{\frac32}}\right)$$

By the Hölder inequality and since the norm $\|\cdot\|$ dominates the L^q norm and according to points (i) and (ii) of proposition A.2, we have :

$$C_n(A,B,\ell) = \frac{1}{(2\pi)^2}\int_{[-a,a]^2} e^{-iu\cdot\ell}\int_{\hat{M}} 1_{\hat{B}}\lambda_u^{n-2k}\Pi_u(b_u^k)\, d\hat{\mu}\, du + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right).$$

We will use here and thereafter the notation $f_u = O(g_u)$ to mean that there exists some constant c_* such that for all $u \in [-a,a]^2$, we have $|f_u| \le c_*|g_u|$. The differentiability of $u \mapsto \Pi_u$ gives $|||\Pi_u - \Pi_0||| = O(|u|)$. Hence using formula (15), we get

$$C_n(A,B,\ell) = \frac{1}{(2\pi)^2}\int_{[-a,a]^2} e^{-iu\cdot\ell}\lambda_u^{n-2k}\, du\hat{\mu}(\hat{B})\int_{\hat{M}} b_u^k\, d\hat{\mu} + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right).$$

For any $u \in [-a,a]^2$, we have

$$\int_{\hat{M}} b_u^k\, d\hat{\mu} = \int_{\hat{M}} e^{iu\cdot S_k\hat{\kappa}}P^k(\mathbf{1}_{\hat{A}})\, d\hat{\mu} = \int_{\hat{M}} e^{iu\cdot S_k\hat{\kappa}}\circ \hat{T}^k\mathbf{1}_{\hat{A}}\, d\hat{\mu} = \hat{\mu}(\hat{A}) + O(|u|).$$

Again, using formula (15) we have

$$C_n(A,B,\ell) = \frac{1}{(2\pi)^2}\int_{[-a,a]^2} e^{-iu\cdot\ell}\lambda_u^{n-2k}\, du\bar{\mu}(B)\bar{\mu}(A) + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right).$$

According to the point (iii) of proposition A.2, we have

$$\left|\lambda_u^{n-2k} - e^{-\frac{n-2k}{2}\Sigma^2 u\cdot u}\right| \le c_*(n-2k)e^{-\sigma|u|^2(n-2k-1)}O(|u|^3).$$

Hence, proceeding similarly as in formula (15), we get

$$\begin{aligned} C_n(A,B,\ell) &= \frac{\bar{\mu}(B)\bar{\mu}(A)}{(2\pi)^2}\int_{[-a,a]^2} e^{-iu\cdot\ell}e^{-\frac{n-2k}{2}\Sigma^2 u\cdot u}\, du + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right) \\ &= \frac{\bar{\mu}(B)\bar{\mu}(A)}{(2\pi)^2(n-2k)}\int_{\mathbb{R}^2} e^{-i\frac{v\cdot\ell}{\sqrt{n-2k}}}e^{-\frac{1}{2}\Sigma^2 v\cdot v}\, dv + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right) \end{aligned}$$

with the change of variable $v = u\sqrt{n-2k}$. Finally, using the formula of the characteristic function of a gaussian, we get

$$C_n(A,B,\ell) = \frac{\bar{\mu}(A)\bar{\mu}(B)}{(2\pi)^2(n-2k)}2\pi\sqrt{\det(\Sigma^2)^{-1}}e^{-\frac{(\Sigma^2)^{-1}\ell\cdot\ell}{2(n-2k)}} + O\left(\frac{k\bar{\mu}(B)^{\frac{1}{p}}}{(n-2k)^{\frac{3}{2}}}\right),$$

which proves the result after obvious simplifications. □

References

[1] M. Abadi et A. Galves, *Inequalities for the occurrence times of rare events in mixing processes. The state of art*, Markov Process. Related Fields **7** (2001) 97–112

[2] L. Barreira and B. Saussol, Hausdorff dimension of measures via Poincaré recurrence, *Comm. Math. Phys.* **219** (2001) 443–464.

[3] X. Bressaud and R. Zweimüller, , Non exponential law of entrance times in asymptotically rare events for intermittent maps with infinite invariant measure, *Ann. I.H.P. Phys. Th.* **2** (2001) 1–12

[4] L. Bunimovich and Y. Sinai, Markov partitions for dispersed billiards, *Commun. Math. Phys.* **78** (1980) 247–280

[5] L. Bunimovich and Y. Sinai, Statistical properties of Lorentz gas with periodic configuration of scatterers, *Commun. Math. Phys.* **78** (1981) 479–497

[6] L. Bunimovich, N. Chernov and Y. Sinai, Markov partitions for two-dimensional hyperbolic billiards, *Russ. Math. Surv.* **45**, No.3 (1990) 105–152; translation from Usp. Mat. Nauk 45, No.3(273) 97–134 (1990)

[7] L. Bunimovich, N. Chernov and Y. Sinai, Statistical properties of two-dimensional hyperbolic billiards, *Russ. Math. Surv.* **46**, No.4 (1991) 47–106; translation from Usp. Mat. Nauk 46, No.4(280) 43–92 (1991)

[8] P. Collet, A. Galves et B. Schmitt, Repetition time for gibsiann source, *Nonlinearity* **12** (1999) 1225–1237

[9] J.-P. Conze, Sur un critère de récurrence en dimension 2 pour les marches stationnaires, applications, *Erg. Th. Dyn. Syst.* **19**, No.5 (1999) 1233–1245

[10] D. Dolgopyat, D. Szász and T. Varjú, Recurrence properties of planar Lorentz process, *Duke Math. J.* 142 (2008) 241–281.

[11] A. Dvoretzky and P. Erdös, Some problems on random walk in space, *Proc. Berkeley Sympos. math. Statist. Probab.* (1951) 353–367.

[12] S. Galatolo, D.-H. Kim and K. Park; The recurrence time for ergodic systems with infinite invariant measures, *Nonlinearity* **19** (2006) 2567–2580

[13] G. Gallavotti and D. S. Ornstein, Billiards and Bernoulli schemes, *Commun. Math. Phys.* 38 (1974) 83–101

[14] Y. Guivarc'h & J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov, *Annales Inst. H. Poincaré (B), Probabilité et Statistiques* **24** (1988) 73–98

[15] M. Hirata, B. Saussol and S. Vaienti, Statistics of return times: a general framework and new applications, *Comm. Math. Phys.* **206** (1999), 33–55

[16] A. Katok and B. Hasselblatt *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, 1995

[17] A. Katok and J.-M. Strelcyn (with F. Ledrappier and F. Przytycki), *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Springer Lecture Notes in Mathematics **1222**, 1986

[18] S. V. Nagaev, Some limit theorems for stationary Markov chains, *Theor. Probab. Appl.* **2** (1957) 378–406; translation from *Teor. Veroyatn. Primen.* **2** (1958) 389–416

[19] S. V. Nagaev, More exact statement of limit theorems for homogeneous Markov chains, *Theor. Probab. Appl.* **6** (1961) 62–81; translation from *Teor. Veroyatn. Primen* **6** (1961) 67–86

[20] F. Pène, Applications des propriétés stochastiques du billard dispersif, *C. R. Acad. Sci., Paris, Sér. I, Math.* 330, No.12 (2000) 1103–1106

[21] F. Pène, Planar Lorentz process in a random scenery, *Ann. I.H.P. Prob. Stat.*, to appear

[22] F. Pène, Asymptotic of the number of obstacles visited by the planar Lorentz process, *Disc. Cont. Dyn. Sys.*, to appear

[23] F. Pène and B. Saussol, Quantitative recurrence in two-dimensional extended processes, *Ann. I.H.P. Prob. Stat.*, to appear

[24] B. Saussol, Recurrence rate in rapidly mixing dynamical systems, *Disc. and Cont. Dyn. Sys.* **15** (2006) 259–267

[25] K. Schmidt, On joint recurrence, *C. R. Acad. Sci., Paris, Sér. I, Math.* 327, No.9 (1998) 837–842

[26] N. Simányi, Towards a proof of recurrence for the Lorentz process, *Dyn. sys. and erg. th., 28th Sem. St. Banach Int. Math. Cent., Warsaw/Pol.* 1986 , Banach Cent. Publ. 23, 265-276 (1989)

[27] Y. Sinai, Dynamical systems with elastic reflections, *Russ. Math. Surv.* **25**, No.2 (1970) 137–189

[28] D. Szász and T. Varjú, Local limit theorem for the Lorentz process and its recurrence in the plane, *Erg. Th. Dyn. Syst.* **24**, No.1 (2004) 257–278

[29] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. of Math.* **147** (1998) 585–650.

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