RECURRENCE AND LYAPUNOV EXPONENTS

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ABSTRACT. We prove two inequalities between the Lyapunov exponents of a diffeomorphism and its local recurrence properties. We give examples showing that each of the inequalities is optimal.

1. INTRODUCTION

Given an ergodic map f of a measure space $(M,\mathcal{B},\mu)$ which is also a metric space and a measurable partition $\mathcal{A}$ of M consider the ball $B(x,r)$ of radius r around the point x and the n-cylinder around $A_n(x) \in \mathcal{A}^n := \bigvee_{j=0}^{n-1} f^{-j}\mathcal{A}$. Define the first return of any set $A \in \mathcal{B}$ by

$$\tau(A) := \min\left\{k>0\colon\, f^k(A)\cap A \neq \emptyset\right\}.$$

This quantity is also called the Poincaré recurrence of the set A. It was previously introduced in two contexts:

- *The statistics of return times into small neighborhoods.* It was shown in [5], that for certain weakly hyperbolic maps $\tau(A_n(x))$ grows linearly with n when the metric entropy of μ is positive. For systems with good mixing properties, this allows to ignore the contribution of those points which come back early, the other points giving an asymptotic distribution of exponential type e^{-t}.
- *The recurrence (or* Afraimovich-Pesin*) dimension.* In [1, 9], the quantity $\tau(B(x,r))$ was used as a set function in the Caratheodory covering of a given space, thus providing a parametrized family of Borel measures with transition point (dimension) located at some dynamical characteristic of the space (usually the topological entropy).

Especially motivated by the first item, we proved a general result on *any* measurable dynamical systems with positive entropy h_μ. In [11] we showed

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If the partition $\mathcal{A}$ is finite or countable and the entropy $h_\mu(f|\mathcal{A})$ is strictly positive, then

$$\varliminf_{n\to\infty} \frac{\tau(A_n(x))}{n} \geq 1, \quad \textit{for } \mu\textit{-almost every } x. \tag{1}$$

Under the additional assumption that f is invertible the two sided case

$$\varliminf_{m,n\to\infty} \frac{\tau(A_{m,n}(x))}{m+n} \geq 1, \quad \text{for } \mu\text{-almost every } x. \tag{2}$$

was proven in [2]. Here $x \in A_m^n(x) \in \mathcal{A}_m^n := \bigvee_{j=-m}^n f^{-j}\mathcal{A}$. In both cases equality holds if we assume the specification property.

In [11], we proposed a new technique to compute the Lyapunov exponent for a large class of weakly hyperbolic maps of the interval. The method expressed the exponent in terms of local recurrence properties. This is one result in the spirit of what was recently called "the thermodynamics of return times": reformulating the statistical properties of dynamical systems in terms of returns of points or sets in the neighborhood of (or into) themselves.

Comparing cylinders with balls naturally relates r^{-1} with the growth (in the expanding case) of the derivative $D_x f^n$ around the center x and has two immediate consequences: it allows to compare a ball of radius r with a cylinder of order $n := n(r)$ and to compare r with (some) Lyapunov exponents Λ of the map *via* the identification: $r^{-1} \sim |D_x f^n| \sim e^{n\Lambda}$. In [11] this idea was worked out rigorously for piecewise monotone maps of an interval with a finite number of branches and with bounded derivative of p-bounded variation with an invariant measures with positive entropy. For this class we approximated a cylinder from the inside with a ball, used the monotonicity of the set function $\tau(\cdot)$ and invoked (1) to show that:

$$\varliminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \geq \frac{1}{\lambda_\mu} \qquad \mu - \text{a.e.} \tag{3}$$

where λ_μ is the positive Lyapunov exponent of the measure μ of positive metric entropy. When the map satisfies the specification property the existence of the limit in (1) implies the existence of the limit in (3), and its equality with the inverse of the Lyapunov exponent.

In this paper we generalize formula (3) for multidimensional transformations, providing examples which show that our inequalities are sharp and other examples where strict inequalities hold. We will work with $C^{1+\alpha}$ diffeomorphisms of a compact manifold M endowed with an invariant ergodic measure μ of positive metric entropy or invertible maps which are locally $C^{1+\alpha}$ diffeomorphisms with reasonable singularity sets such as in the monograph [7].

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the Lyapunov exponents of the measure μ. In this very general setting, we will prove that the left hand side in (3) will be bounded from below by $\frac{1}{\Lambda^u} - \frac{1}{\Lambda^s}$, where Λ^u (resp. Λ^s) is the largest ($= \lambda_n$) (resp. smallest ($= \lambda_1$)) Lyapunov exponent. The technique of the proof will consists in using (2) together with a comparison of balls and cylinders in local Lyapunov charts. In order to get an upper bound for $\overline{\lim} \frac{\tau(B(x,r))}{-\log r}$, one needs the additional properties that the measure is hyperbolic (all the Lyapunov exponents are non-zero) and what we call a "nonuniform specification property" which will insure the existence of periodic points, of a well specified period, following closely the orbit of any other point x in a big set up to a certain time n (see the next section for the precise definition). Using this assumption we will compare balls not with cylinders, but with Bowen (dynamical) balls, which will contain periodic points and which will return therefore into themselves with a controlled time. In this case we will get an upper bound of the form $\frac{1}{\lambda^u} - \frac{1}{\lambda^s}$, where λ^u is the smallest positive Lyapunov exponent and λ^s is the largest negative Lyapunov exponent. These bounds become particularly easy, and more accessible, for two-dimensional compact hyperbolic sets (or more generally for two-dimensional invariant sets enjoying the non-uniform specification property quoted above): in these cases the limit exists and gives the difference of the inverses of the positive and negative Lyapunov exponents.

We point out that the product of this difference with the metric entropy gives the Hausdorff dimension of the measure μ (Young's formula). Furthermore if the map has a constant Jacobian, the knowledge of one Lyapunov exponent will immediately gives the other. The numerical computation of the quantity $\frac{\tau(B(x,r))}{-\log r}$ seems very efficient and reliable when applied to plane attractors and conformal repellers; the detailed numerical analysis will be published elsewhere.

In the last section we provide several examples. First we provide examples where the limit of $\frac{\tau(B(x,r))}{-\log r}$ exists and equals the bounds of Theorem 1 showing that each of the inequalities is optimal. Furthermore we provide an example where both inequalities are strict. In this example the limit equals a mixture of the λ_i.

As a byproduct of these results on return times we obtain some information on the spectrum of Poincaré recurrence of an invariant measure introduced in [2]. It turns out that the quantity $\lim_{r\to 0} \frac{\tau(B(x,r))}{-\log r}$ constitutes one of the component of the pointwise dimension associated to the Caratheodory structure attached to the recurrence dimension. See Section 5 for details.

2. Basic definitions

Throughout the article we suppose that M is an n–dimensional Riemannian manifold and $f : M \to M$ a $C^{1+\alpha}$ diffeomorphism [1]. A f–invariant measure μ is called *hyperbolic* if all its Lyapunov exponents are non-null and *partially hyperbolic* if at least one of its positive Lyapunov exponents and at least one of its negative Lyapunov exponents are non-null. Suppose μ is ergodic and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the corresponding Lyapunov exponents. The Ruelle inequality implies that if the entropy $h_\mu(f)$ is positive then μ is hyperbolic or partially hyperbolic. Thus if μ is ergodic and of positive entropy then we can set $\Lambda^s := \lambda_1, \Lambda^u := \lambda_n, \lambda^s := \max\{\lambda_i : \lambda_i < 0\}$ and $\lambda^u := \min\{\lambda_i : \lambda_i > 0\}$.

Next we define Lyapunov charts on the support of an ergodic measure μ. Let $\mathbb{R}^j$ be the j-dimensional Euclidean space. Let $s := \#\{\lambda_i : \lambda_i < 0\}$, $c := \#\{\lambda_i : \lambda_i = 0\}$ and $u := \#\{\lambda_i : \lambda_i > 0\}$. For $x := (x^u, x^c, x^s) \in \mathbb{R}^u \times \mathbb{R}^c \times \mathbb{R}^s$ let $|x| := |x^u|_u + |x^c|_c + |x^s|_s$ where $|\cdot|_u$, $|\cdot|_c$ and $|\cdot|_s$ are the Euclidean norms on $\mathbb{R}^u$, $\mathbb{R}^u$ and $\mathbb{R}^s$. Let $|x|_u := |x^u|_u$, $|x|_c := |x^c|_c$ and $|x|_s := |x^s|_s$. We define a distance function $\rho(z, z') := |z - z'|$. The closed disk of radius ρ centered at x in $\mathbb{R}^u$ is denoted by $R^u(x, \rho)$ and $R(\rho) := R^u(0, \rho) \times R^c(0, \rho) \times R^s(0, \rho)$. Fix $\eta > 0$. For μ–almost every $x \in M$ there is a Borel function $q(x) := q(x, \eta) : M \to [1, \infty)$ and an embedding $\Phi_x(R(q^{-1}(x))) \to M$ such that the following hold:

i) $q(f^{\pm 1}x) \leq q(x)e^\eta$, i.e. $q(x)$ is η-slowly varying,

ii) $\Phi_x 0 = x$, $D\Phi_x(0)$ takes $\mathbb{R}^u$, $\mathbb{R}^c$ and $\mathbb{R}^s$ to $E^u(x)$, $E^c(x)$ and $E^s(x)$ respectively,

iii) let $f_x := \Phi_{fx}^{-1} \circ f \circ \Phi_x$ then

$$\begin{array}{rll} e^{\Lambda^s - \eta}|v| \leq & |Df_x(0)v| & \leq e^{\lambda^s + \eta}|v| \quad \text{for } v \in \mathbb{R}^s, \\ e^{\lambda^u - \eta}|v| \leq & |Df_x(0)v| & \leq e^{\Lambda^u + \eta}|v| \quad \text{for } v \in \mathbb{R}^u, \\ e^{-\eta}|v| \leq & |Df_x(0)v| & \leq e^{\eta}|v| \quad \text{for } v \in \mathbb{R}^c, \end{array}$$

iv) $\mathrm{Lip}(f_x - Df_x(0)) < \eta$, $\mathrm{Lip}(f_x^{-1} - Df_x^{-1}(0)) < \eta$, in $R(q^{-1}(x))$,

v) $C^{-1}d(\Phi_x z, \Phi_x z') \leq \rho(z, z') \leq q(x)d(\Phi_x z, \Phi_x z')$ for $z, z' \in R(q^{-1}(x))$ for some universal positive constant C.

Next we define a nonuniform hyperbolic version of the specification property which we call *nonuniform specification* for (f, μ). For μ-almost every x, any integer m, n, and any $\varepsilon > 0$ there exists $K := K(\eta, \varepsilon, x, m, n)$ such that:

[1] Again, the results hold also for piecewise differentiable maps with reasonable singularity sets such as those considered in [7].

i) the *nonuniform* Bowen ball

$$\widetilde{B}_m^n(x,\varepsilon) := \bigcap_{k=-m}^{n} f^{-k}B(f^k x, \varepsilon q(f^k x,\eta)^{-2})$$

contains a periodic point with period $p \le n+m+K$;

ii) the dependence of K on m,n satisfies

$$\lim_{\eta\to 0}\overline{\lim_{m,n\to\infty}} K(\eta,\varepsilon,x,m,n)/(m+n) = 0. \tag{4}$$

We remark that even if $\operatorname{supp}(\mu)$ is a compact (uniformly) hyperbolic set for f, then in general $q(\cdot)$ cannot be chosen constant, so that there is no simple relation between Bowen balls and nonuniform Bowen balls, except the obvious one

$$\widetilde{B}_m^n(x,\varepsilon) \subset B_m^n(x,\varepsilon).$$

However we have the following relation.

Proposition 2.1. *Assume that* $\operatorname{supp}(\mu)$ *is a hyperbolic set for* f *and that* $f|_{\operatorname{supp}(\mu)}$ *satisfies the usual definition of specification (see for example* [6]*). Then* (f,μ) *is non-uniformly specified as well.*

Recall that by the Spectral Decomposition Theorem $\operatorname{supp}\mu$ can always be decomposed into disjoint closed sets, where for some iterate of the map the specification property holds.

Proof. Denote the Bowen balls by

$$B_m^n(z,\varepsilon) := \bigcap_{k=-m}^{n} f^{-k}B(f^k z,\varepsilon).$$

Remark that by uniform hyperbolicity, if $\varepsilon > 0$ is sufficiently small there exists $a \in (0,1)$ such that for all $z \in \operatorname{supp}\mu$ and integer q we have

$$B_q^q(z,\varepsilon) \subset B(z,a^q). \tag{5}$$

Let x be such that for any $\eta > 0$ we have $q(x,\eta) < \infty$. Note that this concerns μ-almost every $x \in M$.

Let $\nu > 0$. Fix $\eta > 0$ so small that $a^\nu e^{2\eta} < 1$. We write for simplicity $q(\cdot) := q(\cdot,\eta)$. Assume that m and n are such that $m+n$ is so large that

$$(a^\nu e^{2\eta})^{n+m} < a\varepsilon q(x)^{-2}.$$

Consider a "nonuniform" Bowen ball

$$\widetilde{B}_m^n(x,\varepsilon) := \bigcap_{k=-m}^{n} f^{-k}B(f^k x,\varepsilon q(f^k x)^{-2}).$$

Our aim is to show that if m and n are sufficiently large then the nonuniform Bowen ball $\widetilde{B}_m^n(x,\varepsilon)$ contains a periodic point with a sufficiently small period. Observe that for any $k = -m, \dots, n$ we have, by Equation (i) of Lyapunov chart

$$\varepsilon q(f^k x)^{-2} \geq \varepsilon q(x)^{-2} e^{-2\eta|k|} \geq \varepsilon q(x)^{-2} e^{-2\eta(m+n)}.$$

Let $q := [\nu(m+n)]$. By the specification property the Bowen ball $B_{m+q}^{n+q}(x,\varepsilon)$ contains a periodic point, say y, with a period $p \leq (m+n)(1+2\nu)+c$, with the constant c depending only on ε.

For every $k = -m, \dots, n$ we have $f^k y \in B_q^q(f^k x, \varepsilon)$ hence by (5)

$$d(f^k x, f^k y) < a^q \leq a^{\nu(n+m)-1} < \varepsilon q(x)^{-2} e^{-2\eta(n+m)},$$

the last inequality being satisfied since $m+n$ is sufficiently large. Thus the periodic point y belongs to $\widetilde{B}_m^n(x,\varepsilon)$. This shows that the function K in (4) satisfies

$$\overline{\lim_{m+n\to\infty}} K(\eta,\ell,\varepsilon,x,m,n)/(m+n) \leq 2\nu.$$

The proposition follows from the arbitrariness of ν. □

3. Statement of results

Theorem 1. *If μ is an f–invariant, ergodic probability measure with entropy $h_\mu(f) > 0$ then*

$$\frac{1}{\Lambda_\mu^u} - \frac{1}{\Lambda_\mu^s} \leq \underline{\lim_{r\to 0}} \frac{\tau(B(x,r))}{\log 1/r}$$

for μ–almost every x.

If μ is an f–invariant, ergodic probability measure which is hyperbolic and $f|_{\mathrm{supp}(\mu)}$ satisfies the nonuniform specification property then

$$\overline{\lim_{r\to 0}} \frac{\tau(B(x,r))}{-\log r} \leq \frac{1}{\lambda_\mu^u} - \frac{1}{\lambda_\mu^s}$$

for μ–almost every x.

The following corollaries show that inequalities in the theorem may be optimal in some situation. If M is two dimensional we have $\Lambda_\mu^u = \lambda_\mu^u$ and $\Lambda_\mu^s = \lambda_\mu^s$. Furthermore whenever $h_\mu(f) > 0$ the Ruelle inequality implies that μ is hyperbolic thus the following corollary follows

Corollary 1. *If M is two dimensional, $h_\mu(f) > 0$ and $f|_{\mathrm{supp}(\mu)}$ satisfies the nonuniform specification property then*

$$\lim_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \frac{1}{\lambda_\mu^u} - \frac{1}{\lambda_\mu^s} = \frac{1}{\Lambda_\mu^u} - \frac{1}{\Lambda_\mu^s}$$

for μ–almost every x.

Corollary 2. *If f is a diffeomorphism and* $\mathrm{supp}(\mu)$ *is a compact locally maximal hyperbolic set for f then*

$$\overline{\lim_{r\to 0}} \frac{\tau(B(x,r))}{-\log r} \leq \frac{1}{\lambda^u_\mu} - \frac{1}{\lambda^s_\mu}$$

for μ–almost every x.

In particular if M is two dimensional and $h_\mu(f) > 0$ the limit exists and equals the right hand side of this inequality.

Proof of Corollary 2. Using the Spectral Decomposition Theorem (see for example [6]) $\mathrm{supp}(\mu)$ is decomposed into disjoint closed sets $\Omega_1, \dots, \Omega_m$ which are permuted by f. If k_i denotes the smallest integer such that $f^{k_i}\Omega_i = \Omega_i$ then $f^{k_i}|_{\Omega_i}$ is topologically mixing. In this situation it is well known that topological mixing is equivalent to the specification property, hence by Proposition 2.1 this implies that the nonuniform specification property is satisfied. The proof follows from Theorem 1 if we notice that whenever $x \in \Omega_i$ we have $k_i\tau(B(x,r),f) \leq k_i\tau(B(x,r), f|_{\mathrm{supp}\,(\mu)}) = \tau(B(x,r), f^{k_i}|_{\Omega_i})$ and $k_i\lambda^u(f) = \lambda^u(f^{k_i})$ and $k_i\lambda^s(f) = \lambda^s(f^{k_i})$. □

Proof of Theorem 1. The proof of the first statement compares the ball $B(x,r)$ to a cylinder set of an appropriate partition. We need a finite partition $\mathcal{Z} := \{\zeta_1, \zeta_2, \dots, \zeta_m\}$ of positive entropy which satisfies the following property:

$$\mu(\{x \in M : d(x, M\backslash\zeta(x)) < r\}) < cr \tag{6}$$

for all $r > 0$ for some positive constant c. Here $\zeta(x)$ is the element of $\mathcal{Z}$ containing x. The existence of such a partition was shown in [3].

Fix x such that for any $\eta > 0$, $q(x,\eta) < \infty$. Let $y \in M\backslash\zeta_0^n(x)$ satisfying

$$d(x,y) \leq q^{-2}(x)e^{-(\Lambda^u+3\eta)n}.$$

Let $\hat{x} := \Phi_x^{-1}(x)$ and $\hat{y} := \Phi_x^{-1}(y)$. We remark that $\hat{x} = 0$ by definition of Lyapunov charts. The points $\hat{x}$ and $\hat{y}$ are close enough that we can apply Equation v) of Lyapunov charts, it yields

$$\rho(\hat{x},\hat{y}) \leq \frac{e^{-(\Lambda^u+3\eta)n}}{q(x)}.$$

For brevity we set $f_x^{(k)} := f_{f^{k-1}x} \circ \cdots \circ f_x$. For $0 \leq k \leq n-1$ we apply Equations ii)-iv) of Lyapunov charts to show that

$$\rho\left(f_x^{(k)}\hat{x}, f_x^{(k)}\hat{y}\right) \leq \frac{e^{-(\Lambda^u+3\eta)n}e^{(\Lambda^u+2\eta)k}}{q(x)}.$$

We remark that a simple induction shows that Equation iv) can be used, this induction follows from the fact that the right hand side of the above equation is majorized by $e^{-k\eta}q^{-1}(x) \leq q^{-1}(f^k x)$.

Again Equation v) of Lyapunov charts is applicable, it implies that

$$d(f^k x, f^k y) \le C \frac{e^{-(\Lambda^u+3\eta)n} e^{(\Lambda^u+2\eta)k}}{q(x)}.$$

Since $q(x) \ge 1$ this yields

$$d(f^k x, f^k y) \le C e^{-\eta n}.$$

Since $y \notin \zeta_0^n(x)$ this implies that for some $k \in \{0, \dots, n-1\}$ the distance between $f^k x$ and $M \setminus \zeta(f^k x)$ is less than $Ce^{-\eta n}$. This yields

$$\begin{aligned} &\mu\left(\left\{x \in M : d(x, M \backslash \zeta_0^n(x)) \le q^{-2}(x) e^{-((\Lambda^u+3\eta)n}\right\}\right) \\ &\le n \cdot \max_{0 \le k \le n-1} \mu\left(\left\{x \in M : d(f^k x, M \backslash \zeta(f^k x)) \le C e^{-\eta n}\right\}\right) \\ &\le n \cdot c \cdot C e^{-\eta n}. \end{aligned}$$

Here we used Equation (6) and the f–invariance of μ in the last inequality.

By the Borel–Cantelli lemma we conclude that

$$B(x, q^{-2}(x) e^{-(\Lambda^u+3\eta)n}) \subset \zeta_0^n(x) \tag{7}$$

for μ–almost every x for sufficiently large $n := n(x)$. By considering f^{-1} we have a similar statement for the backwards direction

$$B(x, q^{-2}(x) e^{(\Lambda^s-3\eta)m}) \subset \zeta_m^0(x). \tag{8}$$

Fix $r > 0$. Let m_r be the largest integer such that $q^{-2}(x)e^{(\Lambda^s-3\eta)m} > r$ and n_r the largest integer such that $q^{-2}(x)e^{-(\Lambda^u+3\eta)n} > r$.

Combining Equations (7) and (8) with the definitions of m_r and n_r yields

$$B(x, r) \subset \zeta_{m_r}^{n_r}(x). \tag{9}$$

Thus

$$\left(\frac{1}{\Lambda^u+3\eta} - \frac{1}{\Lambda^s-3\eta}\right)^{-1} \varliminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \ge \varliminf_{m,n\to\infty} \frac{\tau(\zeta_m^n(x))}{m+n}.$$

In [2] it was shown that the right hand side is greater or equal to one. Since $\eta > 0$ is arbitrary this concludes the proof of the first statement of the theorem.

We turn to the proof of the second statement. For this proof it is more convenient to use Bowen balls. Let

$$\widetilde{B}_m^n(x, \varepsilon) := \{y \in M : d(f^k x, f^k y) < \varepsilon q^{-2}(f^k x) \quad \forall -m \le k \le n\}.$$

We need to "reverse" the inclusion in Equation (9) using nonuniform Bowen balls instead of partition elements.

Fix $\varepsilon > 0$ sufficiently small. We choose $m := m(r)$ and $n := n(r)$ the smallest possible integers such that

$$Cq^{-1}(x)\varepsilon \max(e^{(\lambda^s+3\eta)m}, e^{-(\lambda^u-3\eta)n}) \leq \frac{r}{2}. \tag{10}$$

Consider $y \in \widetilde{B}_m^n(x,\varepsilon)$. As before let $\hat{x} := \Phi_x^{-1}(x)$ and $\hat{y} := \Phi_x^{-1}(y)$. In coordinates we write $\hat{x} := (\hat{x}^u, \hat{x}^s)$ and $\hat{y} := (\hat{y}^u, \hat{y}^s)$.

For the definition of Bowen balls it is clear that Equations i)-v) of Lyapunov charts can be applied along the orbit segment for $f^{-m}x$ to $f^n x$. From Equation v) of Lyapunov charts we have $\rho(f_x^{(k)}\hat{x}, f_x^{(k)}\hat{y}) \leq q^{-1}(f^k x)\varepsilon$ for any $-m \leq k \leq n$. Equations ii)–iv) of Lyapunov charts imply that

$$\begin{cases} \rho(\hat{x}^u, \hat{y}^u) \leq q^{-1}(f^n x)\varepsilon e^{(-\lambda^u+2\eta)n}, \\ \rho(\hat{x}^s, \hat{y}^s) \leq q^{-1}(f^{-m}x)\varepsilon e^{(\lambda^s+2\eta)m}. \end{cases}$$

The definition of ρ along with these two inequalities yields

$$\rho(\hat{x}, \hat{y}) \leq q^{-1}(f^{-m}x)\varepsilon e^{(\lambda^s+2\eta)m} + q^{-1}(f^n x)\varepsilon e^{-(\lambda^u-2\eta)n}.$$

Equation v) of Lyapunov charts implies that

$$\begin{aligned} d(x,y) &\leq C\varepsilon(q^{-1}(f^{-m}x)e^{(\lambda^s+2\eta)m} + q^{-1}(f^n x)e^{-(\lambda^u-2\eta)n}) \\ &\leq Cq^{-1}(x)\varepsilon(e^{(\lambda^s+3\eta)m} + e^{-(\lambda^u-3\eta)n}) \\ &\leq r \end{aligned}$$

where the middle inequality use Equation i) of Lyapunov charts. Thus we conclude that

$$\widetilde{B}_m^n(x,\varepsilon) \subset B(x,r). \tag{11}$$

Equations (10) and (11) imply that

$$\left(\frac{1}{\lambda^u - 3\eta} - \frac{1}{\lambda^s + 3\eta}\right)^{-1} \overline{\lim_{r\to 0}} \frac{\tau(B(x,r))}{-\log r} \leq \overline{\lim_{m,n\to\infty}} \frac{\tau(\widetilde{B}_m^n(x,\varepsilon))}{m+n}. \tag{12}$$

By the nonuniform specification property we have

$$\overline{\lim_{n,m\to\infty}} \frac{\tau(\widetilde{B}_m^n(x,\varepsilon))}{m+n} \leq 1 + \overline{\lim_{n,m\to\infty}} \frac{K(\eta,\varepsilon,x,n,m)}{m+n} \leq 1 + \delta(\eta),$$

for some function δ such that $\lim_{\eta\to 0}\delta(\eta) = 0$. We then combine this inequality with (12), and the conclusion follows from the arbitrariness of η. $\square$

Remark: In the case of endomorphisms, after the simplification consisting in ignoring possible stable directions, the proof of Theorem 1 can be carried out essentially in the same way. Therefore for ergodic measure μ with entropy $h_\mu(f) > 0$ it holds

$$\frac{1}{\Lambda^u_\mu} \le \varliminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \quad \text{for } \mu\text{–almost every } x. \tag{13}$$

Additionally, with the appropriate change to the notion of specification, namely considering "forward" Bowen balls $\widetilde{B}^n_0$, we get that if μ is an ergodic measure with all exponents positive and $f|_{\mathrm{supp}(\mu)}$ satisfies the nonuniform specification property then

$$\varlimsup_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \le \frac{1}{\lambda^u_\mu} \quad \text{for } \mu\text{–almost every } x. \tag{14}$$

4. Examples

These examples show that the inequalities presented in the previous section are sometimes attained, but that still they may be strict.

The first example gives the optimality of the upper bound without the upper and lower bounds coinciding.

Proposition 4.1. *Suppose that the linear automorphism $A := A_1 \times A_2$ of $\mathbb{T}^4$ is a direct product of two linear hyperbolic automorphisms A_1, A_2 of $\mathbb{T}^2$.*

If μ_i, $i = 1, 2$ are A_i–invariant ergodic probability measures with positive entropy then the A–invariant ergodic probability measure $\mu := \mu_1 \times \mu_2$ satisfies

$$\lim_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \frac{1}{\lambda^u} - \frac{1}{\lambda^s}$$

for μ–almost every x.

Proof. Since the limit in question depends only on the equivalence class of metrics we suppose that the metric d on $\mathbb{T}^4$ is defined by $d := \max(d_1, d_2)$ where for $i = 1, 2$ d_i is the usual metric on $\mathbb{T}^2$. Any notation without a subscript pertains to the map A, while the corresponding notation with a subscript pertains to the maps A_i, i=1,2. For example λ^u_i and λ^s_i denote the Lyapunov exponents of A_i. Clearly $\lambda^u = \min_i \lambda^u_i$ and $\lambda^s = \max_i {\lambda^s}_i$. We remark that these equalities are realized by the same i since $\lambda^u_i + \lambda^s_i = 0$.

Let $x := (x_1, x_2)$ and notice that

$$\tau(B(x,r)) \ge \tau_i(B(x_i,r)). \tag{15}$$

Applying Corollary 2 to A_i yields

$$\lim_{r\to 0} \frac{\tau_i(B(x_i,r))}{-\log r} = \frac{1}{\lambda^u_i} - \frac{1}{\lambda^s_i}$$

for μ_i–almost every x_i. Combining this with Equation (15) yields

$$\underline{\lim}_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \geq \frac{1}{\lambda^u} - \frac{1}{\lambda^s}$$

for μ–almost every x. The conclusion follows by applying Corollary 2 to the map A. □

The second example yields the optimality of the lower bound without the upper and lower bounds coinciding.

Proposition 4.2. *Suppose that the linear automorphism $A := A_1 \times A_2$ of $\mathbb{T}^4$ is a direct product of two linear hyperbolic automorphisms A_1, A_2 of $\mathbb{T}^2$.*

There exist some A–invariant ergodic probability measures μ with positive entropy such that

$$\lim_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \frac{1}{\Lambda^u} - \frac{1}{\Lambda^s}$$

for μ–almost every x.

Proof. We choose the metric as in the proof of Proposition 4.1 and keep the same notation. Without loss of generality we can suppose that $(\Lambda^u, \Lambda^s) = (\lambda_1^u, \lambda_1^s)$. Let μ_1 be an ergodic A_1–invariant measure with positive entropy, and let μ_2 be a Dirac mass at some fixed point p of A_2. The direct product $\mu := \mu_1 \times \mu_2$ is an A–invariant ergodic probability measure with positive entropy. For μ-almost every x we have $x_2 = p$ which clearly implies that $\tau(B(x,r)) = \tau_1(B(x_1, r))$ for any $r > 0$. Applying Corollary 2 to A_1 finishes the proof. □

Notice that in this example the map has a zero entropy factor.

The third example is a map for which both inequalities in the theorem are strict, and furthermore more than one of the unstable Lyapunov exponents play a role. To simplify the exposition the following example is given by an expanding map of the torus, and not a diffeomorphism. Nevertheless we believe that the mechanism described there is widespread for automorphisms of the torus.

Proposition 4.3. *There exists a linear expanding map of the torus $\mathbb{T}^2$ such that*

$$\frac{1}{\Lambda^u} < \lim_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \frac{2}{\Lambda^u + \lambda^u} < \frac{1}{\lambda^u}$$

for Lebesgue-almost all x.

Proof. Consider the linear expanding map $f : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$f(x) = Ax, \qquad \text{where } A = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}$$

We first prove that for any x,

$$\overline{\lim_{r\to 0}} \frac{\log \tau(B(x,r))}{-\log r} \le \frac{2}{\Lambda^u + \lambda^u} \tag{16}$$

The eigenvalues of A are $e^{\lambda^u} = \frac{9-3\sqrt{5}}{2}$ and $e^{\Lambda^u} = \frac{9+3\sqrt{5}}{2}$ and the corresponding eigenvectors are

$$v^u = \begin{pmatrix} 1 \\ \frac{-\sqrt{5}-1}{2} \end{pmatrix} \text{ and } V^u = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Let $m = \|v^u\| + \|V^u\|$. Fix $r > 0$ and set

$$L(r) = [-r,r]v^u + [-r,r]V^u \subset \mathbb{T}^2.$$

We have $L(r) \subset B(0,mr)$. We want to find some integer n such that

$$f^n L(r) \supset \mathbb{T}^2. \tag{17}$$

Notice that the set

$$A^n L(r) = [-e^{\lambda^u n}r, e^{\lambda^u n}r]v^u + [-e^{\Lambda^u n}r, e^{\Lambda^u n}r]V^u$$

consists of a long and thin strip of width $2e^{\lambda^u n}r$ parallel to the direction V^u, which is wrapped around the torus $2e^{\Lambda^u n}r$ many times. Let $\theta = \frac{\sqrt{5}-1}{2}$, $c = 1/\sqrt{1+\theta^2}$ and let p_i/q_i be the convergents of the continued fraction approximation of θ.

Let $n = \left\lceil \dfrac{-2\log r + \log 4c/3}{\Lambda^u + \lambda^u} \right\rceil$.

We remark that θ has bounded quotients, more precisely $1 < q_i/q_{i-1} \le 2$, thus there exists i such that

$$2ce^{-n\lambda^u}/r \ge q_{i-1} > ce^{-n\lambda^u}/r.$$

For any $s \in \mathbb{T}^1$ there exists some integer $k = 0, \ldots, q_i$ such that

$$\|s - k\theta\| < 1/q_{i-1} < ce^{n\lambda^u} r. \tag{18}$$

Let R_θ be the rotation by angle θ on $\mathbb{T}^1$. Denote by

$$L^{uu}(r) = [-r,r]V^u \subset \mathbb{T}^2.$$

Since $V^u = (1,\theta)$ the following inclusion holds (see Figure 1)

$$\{(0, R_\theta^k(0)) \colon k \in \mathbb{N}, |k| \le e^{n\Lambda^u} r\} \subset A^n(L^{uu}(r)).$$

By our choice of n we have $q_i \le 2q_{i-1} \le 4ce^{-n\lambda^u}/r \le \lfloor e^{\Lambda^u n} r\rfloor - 1$, hence by Equation (18) the set $P = \{R_\theta^k(0) \colon k = 0, \ldots, \lfloor e^{\Lambda^u n} r\rfloor - 1\}$ is $ce^{n\lambda^u}r$-dense in $\mathbb{T}^1$. Therefore Equation (17) holds (see Figure 2). This certainly implies that $\tau(B(x,mr)) \le n$ for any $x \in \mathbb{T}^2$, and proves the upper bound (16) for any points $x \in \mathbb{T}^2$.

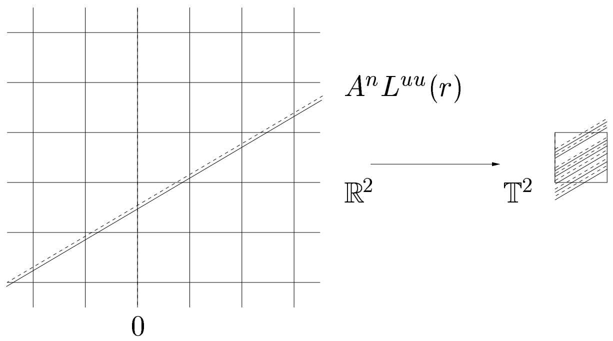

FIGURE 1. Intersection of the piece of strong unstable manifold $A^n L^{uu}(r)$ with $\{0\} \times \mathbb{T}^1$.

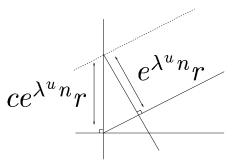

FIGURE 2. $ce^{\lambda^u n}r$-density in $\mathbb{T}^1$ of the set P imply $e^{\lambda^u n}r$-density of $A^n L^{uu}(r)$ in $\mathbb{T}^2$.

We turn now to the almost sure lower bound. Let $c = 1 - \|A^{-1}\| > 0$. Since the map f is locally expanding the Closing Lemma implies that for any $z \in \mathbb{T}^2$, real $r > 0$ and integer p, if $d(z, f^p z) < r$ then there exists a periodic point y with period p such that $d(y, z) < r/c$.

Observe that if $p = \tau(B(x, r))$ then there exists $z \in \mathbb{T}^2$ such that $d(z, x) < r$ and $d(f^n z, x) < r$, thus in particular $d(z, f^n z) < 2r$, so that by the previous paragraph there exists a periodic point y with period p such that $d(y, z) < 2r/c$. This implies that

$$d(x, F_p) \leq (1 + 2/c)r, \quad \text{where } F_p = \{y \in \mathbb{T}^2 \colon y = f^p y\}.$$

Let $a < (\det A)^{-1/2}$. Since the cardinality of F_p equals $\det(A^p - I) = (e^{p\lambda^u} - 1)(e^{p\Lambda^u} - 1) \leq \det A^p$ we have

$$\begin{aligned} m(x \in \mathbb{T} \colon \tau(B(x, a^n)) \leq n) &\leq \sum_{p=1}^{n} \sum_{y \in F_p} m(B(y, (1 + 2/c)a^n)) \\ &\leq \sum_{p=1}^{n} (\det A)^p (1 + 2/c)^2 a^{2n} \\ &\leq (1 + 2/c)^2 \frac{\det A}{\det A - 1} (a^2 \det A)^n. \end{aligned}$$

This is summable, hence by the Borel-Cantelli Lemma for Lebesgue almost all x

$$\underline{\lim}_{n\to\infty} \frac{\tau(B(x,a^n))}{n} \geq 1.$$

Taking a arbitrarily close to

$$(\det A)^{-1/2} = \exp\left(-\frac{\lambda^u + \Lambda^u}{2}\right)$$

gives that

$$\underline{\lim}_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \geq \frac{2}{\lambda^u + \Lambda^u}.$$

□

Notice that the second part of the proof may be followed in the more general case of hyperbolic basic set of diffeomorphisms to gives

$$\underline{\lim}_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \geq \frac{\dim_H \mu}{h_{\text{top}}(f)}.$$

This lower bound, although sometimes better than the one provided by Theorem 1 (in dimension larger than $1+1$), is not optimal even in dimension one. We conjecture that for any ergodic hyperbolic measure μ with nonzero entropy one has

$$\underline{\lim}_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \geq \frac{\dim_H \mu}{h_\mu(f)}.$$

It can be shown that this bound is never worse than the one given by Theorem 1.

5. Application to the spectrum of recurrence dimension

In this last section we propose to apply the main results to the recurrence dimension of measures introduced by Afraimovich *et al* in [2]. We briefly recall the construction of the spectrum for Poincaré recurrence of a measure.

Let (M, f, μ) be an ergodic measure preserving dynamical systems on a compact manifold M.

For any $A \subset X$, $\alpha \in \mathbb{R}$ and $q \in \mathbb{R}$ we define

$$\mathcal{M}^\tau(A, \alpha, q, \varepsilon) := \inf \sum_i \exp\big[-q\tau(B(x_i, \varepsilon_i))\big]\varepsilon_i^\alpha , \tag{19}$$

where the infimum is taken over all finite or countable collections of balls $B(x_i, \varepsilon_i)$ such that $\bigcup_i B(x_i, \varepsilon_i) \supseteq A$ and $\varepsilon_i \leq \varepsilon$.

The limit $\mathcal{M}^\tau(A, \alpha, q) := \lim_{\varepsilon\to 0} \mathcal{M}^\tau(A, \alpha, q, \varepsilon)$ exists by monotonicity and we define for any non-empty $A \subset X$ and any $q \in \mathbb{R}$,

$$\alpha_\tau(A, q) := \inf\{\alpha\colon \mathcal{M}^\tau(A, \alpha, q) = 0\} = \sup\{\alpha\colon \mathcal{M}^\tau(A, \alpha, q) = \infty\} \tag{20}$$

with the convention that $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$. Notice that $\alpha_\tau(A,0)$ is equal to the Hausdorff dimension of the set A. Now we proceed to the definition of the spectra of measures, following [10]

$$\alpha_\tau^\mu(q) := \inf\{\alpha_\tau(Y,q)\colon Y \text{ measurable } \subset X,\, \mu(Y)=1\}\ .$$

This global quantity may be computed with the help of the corresponding pointwise dimension defined for any x by

$$d_{\mu,q}(x) = \varliminf_{r\to 0} \inf_{y\in B(x,r)} \frac{\log\mu(B(y,r)) + q\tau(B(y,r))}{\log r}.$$

A priori we cannot discard in general the infimum on $y\in B(x,r)$. However, when $q=0$ we have the equality

$$d_{\mu,0}(x) = \varliminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r}.$$

The well known equality $\dim_H\mu = \mu\text{-}\operatorname{esssup} d_{\mu,0}$ was generalized in [4] for $q\neq 0$:

$$\alpha_\tau^\mu(q) = \mu\text{-}\operatorname{esssup} d_{\mu,q}. \tag{21}$$

Corollary 3. *If μ is an f-invariant, ergodic measure with $h_\mu(f)>0$ then*

$$\begin{aligned} \alpha_\tau^\mu(q) &\leq \dim_H\mu - q\left(\tfrac{1}{\Lambda_\mu^u} - \tfrac{1}{\Lambda_\mu^s}\right) && \textit{if } q\geq 0,\\ \alpha_\tau^\mu(q) &\geq \dim_H\mu - q\left(\tfrac{1}{\Lambda_\mu^u} - \tfrac{1}{\Lambda_\mu^s}\right) && \textit{if } q\leq 0. \end{aligned}$$

If in addition μ is hyperbolic and $f|_{\operatorname{supp}\mu}$ satisfies the nonuniform specification property then

$$\alpha_\tau^\mu(q) \leq \dim_H\mu - q\left(\frac{1}{\lambda_\mu^u} - \frac{1}{\lambda_\mu^s}\right) \quad \textit{if } q\leq 0.$$

The immediate corollary follows

Corollary 4. *If f is a surface diffeomorphism, μ is an f-invariant ergodic measure with $h_\mu(f)>0$ and* $\operatorname{supp}\mu$ *is a compact locally maximal hyperbolic set for f then for any $q\leq 0$*

$$\begin{aligned} \alpha_\tau^\mu(q) &= \dim_H\mu - q\left(\frac{1}{\lambda_\mu^u} - \frac{1}{\lambda_\mu^s}\right)\\ &= \left(1 - \frac{q}{h_\mu(f)}\right)\dim_H\mu. \end{aligned}$$

Proof of Corollary 3. Whenever $q\geq 0$ we have

$$\begin{aligned} d_{\mu,q}(x) &\leq \varliminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} - q\varliminf_{r\to 0}\frac{\tau(B(x,r))}{-\log r}\\ &\leq d_{\mu,0}(x) - q\left(\frac{1}{\Lambda_\mu^u} - \frac{1}{\Lambda_\mu^s}\right), \end{aligned}$$

for μ-almost every x by Theorem 1. Using Equation (21) applied to $d_{\mu,0}$ and $d_{\mu,q}$ yields the upper bound for $\alpha_\tau^\mu(q)$ for $q \geq 0$.

Suppose $q \leq 0$. Notice that whenever $y \in B(x,r)$ we have $B(y,r) \subset B(x,2r)$, and thus

$$\begin{aligned} d_{\mu,q}(x) &\geq \underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} - q\, \underline{\lim}_{r\to 0} \frac{\tau(B(x,2r))}{-\log r} \\ &\geq d_{\mu,0}(x) - q\left(\frac{1}{\Lambda_\mu^u} - \frac{1}{\Lambda_\mu^s}\right), \end{aligned}$$

for μ-almost every x by Theorem 1. Again Equation (21) applied to $d_{\mu,0}$ and $d_{\mu,q}$ yields the lower bound for $\alpha_\tau^\mu(q)$ when $q \leq 0$.

We prove the last part of the corollary. If $q \leq 0$ we have

$$\begin{aligned} d_{\mu,q}(x) &\leq \underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} - q\, \overline{\lim}_{r\to 0} \frac{\tau(B(x,r))}{-\log r} \\ &\leq d_{\mu,0}(x) - q\left(\frac{1}{\lambda_\mu^u} - \frac{1}{\lambda_\mu^s}\right), \end{aligned}$$

for μ-almost every x, by the second statement of Theorem 1. Applying Equation (21) with $d_{\mu,0}$ and $d_{\mu,q}$ yields the result. □

Proof of Corollary 4. Under these conditions, $f|_{\operatorname{supp}\mu}$ satisfies the nonuniform specification property and the measure μ is hyperbolic. Since in addition $(\lambda_\mu^u, \lambda_\mu^s) = (\Lambda_\mu^u, \Lambda_\mu^s)$ the first equation follows from Corollary 3. The second equation is a consequence of Young's formula

$$\dim_H \mu = \frac{h_\mu(f)}{\lambda_\mu^u} - \frac{h_\mu(f)}{\lambda_\mu^s}$$

for surface diffeomorphisms [12]. □

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