HAUSDORFF DIMENSION OF MEASURES VIA POINCARÉ RECURRENCE

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Abstract. We study the quantitative behavior of Poincaré recurrence. In particular, for an equilibrium measure on a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism f, we show that the recurrence rate to each point coincides almost everywhere with the Hausdorff dimension d of the measure, that is, $\inf\{k > 0 : f^k x \in B(x,r)\} \sim r^{-d}$. This result is a non-trivial generalization of work of Boshernitzan concerning the quantitative behavior of recurrence, and is a dimensional version of work of Ornstein and Weiss for the entropy. We stress that our approach uses different techniques. Furthermore, our results motivate the introduction of a new method to compute the Hausdorff dimension of measures.

1. Introduction

One of the basic but fundamental results of the theory of dynamical systems is the Poincaré Recurrence Theorem. Essentially it states that any dynamical system preserving a finite invariant measure exhibits a non-trivial recurrence to each set of positive measure. More precisely, let $T \colon X \to X$ be a measurable transformation, and μ a T-invariant probability measure in X. The Poincaré Recurrence Theorem says that if $A \subset X$ is a measurable set of positive measure, then $\operatorname{card}\{n > 0 : T^n x \in A\} = \infty$ for μ-almost every point $x \in A$.

Unfortunately this information is only of qualitative nature. In particular it does not address the following natural problems:

1. with which frequency an orbit visits a given set of positive measure;
2. with which rate a given point returns to an arbitrarily small neighborhood of itself.

The Birkhoff Ergodic Theorem provides a comprehensive answer to the first problem. The second problem has been given considerable growing interest during the last decade, also in connection with other fields, including compression algorithms, numerical study of dynamical systems, and applications in linguistics. In particular, there exist several results towards a partial answer of this problem, including the noteworthy work of Boshernitzan [3] and Ornstein and Weiss [8] (see Sections 2 and 4 for details).

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The purpose of this paper is to provide a comprehensive answer to the above-mentioned problem 2, concerning the quantitative behavior of recurrence. In particular, our results are non-trivial generalizations of the above-mentioned results of Boshernitzan, and provide a dimensional version of the work of Ornstein and Weiss for the entropy (see Sections 2 and 4 for explanations and examples). We emphasize that our approach uses different techniques. In particular we obtain a new proof of one of the main results of Boshernitzan in [3].

We now illustrate our results with a rigorous statement; see Section 4 for details. We shall prove that if μ is an ergodic Gibbs measure of a $C^{1+\alpha}$ diffeomorphism f on a locally maximal hyperbolic set, then

$$\lim_{r\to 0}\frac{\log\inf\{k>0 : f^k x\in B(x,r)\}}{-\log r}=\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \tag{1}$$

for μ-almost every point x, where $B(x,r)$ is the ball of radius r centered at x. Note that the identity (1) relates two quantities of very different nature, called respectively recurrence rate and pointwise dimension. In particular, only the first quantity depends on the diffeomorphism, while only the second quantity depends on the measure.

Furthermore, our results motivate the introduction of a new method to compute the Hausdorff dimension of a measure.

The structure of the paper is as follows. The main statements and inequalities relating the lower and upper pointwise dimensions, and the lower and upper recurrence rates are formulated and discussed in Sections 2 and 3. We also present examples which indicate that the hypotheses in our results are optimal. In Section 4 we apply those results to the case of equilibrium measures supported on locally maximal hyperbolic sets, and establish the identity (1) for μ-almost every point. Section 5 contains an application to suspension flows. The proofs are collected in Section 6.

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2. Lower bounds for the pointwise dimension

Let $T\colon X\to X$ be a Borel measurable transformation on the separable metric space (X,d). Note that T is not necessarily invertible. We define the *return time* of a point $x\in X$ into the open ball $B(x,r)$ by

$$\tau_r(x)\stackrel{\text{def}}{=}\inf\{k\in\mathbb{N} : T^k x\in B(x,r)\}=\inf\{k\in\mathbb{N} : d(T^k x,x)<r\},$$

where $\mathbb{N}$ denotes the set of positive integers. We also define the *lower* and *upper recurrence rates* of x by

$$\underline{R}(x)=\varliminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\quad\text{and}\quad\overline{R}(x)=\varlimsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r}.$$

Furthermore, the *lower* and *upper pointwise dimensions* of μ at a point $x\in X$ are given by

$$\underline{d}_\mu(x)=\varliminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\quad\text{and}\quad\overline{d}_\mu(x)=\varlimsup_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}, \tag{2}$$

where $B(x,r)$ is the open ball of radius r centered at the point x.

The following statement provides upper bounds for each of these quantities in terms of the lower and upper pointwise dimensions (see (2) for the definitions).

Theorem 1. *If $T\colon X \to X$ is a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, and μ is a T-invariant probability measure on X, then*

$$\underline{R}(x) \le \underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) \le \overline{d}_\mu(x) \tag{3}$$

for μ-almost every $x \in X$.

It follows from Whitney's embedding theorem that if X is an arbitrary subset of a finite-dimensional smooth manifold, then it can be smoothly embedded into $\mathbb{R}^d$ for some $d \in \mathbb{N}$, and thus Theorem 1 applies.

Example 3 in Section 4 illustrates that the inequalities in (3) may be strict on a set of positive measure.

Boshernitzan proved in [3] that if the α-dimensional Hausdorff measure is σ-finite on X (that is, if X can be written as a countable union of sets X_i for $i = 1, 2, \ldots$ such that $m_\alpha(X_i) < \infty$ for all i), and μ is an invariant probability measure on X, then

$$\varliminf_{n\to\infty} [n^{1/\alpha} \cdot d(T^n x, x)] < \infty \tag{4}$$

for μ-almost every $x \in X$. He also showed that if, in addition, $m_\alpha(X) = 0$, then

$$\varliminf_{n\to\infty} [n^{1/\alpha} \cdot d(T^n x, x)] = 0. \tag{5}$$

for μ-almost every $x \in X$.

Recall that the *Hausdorff dimension* of a probability measure μ on X is given by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(Z) = 1\},$$

where $\dim_H Z$ denotes the Hausdorff dimension of the set Z. The measure μ is called *exact dimensional* if there exists a constant d such that

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d \text{ for } \mu\text{-almost every } x \in X. \tag{6}$$

It follows from Young's criteria (see [13] for details) that if (6) holds, then $\dim_H \mu = d$.

In our setting Boshernitzan's result can be reformulated in the following manner (for details, see Section 6 and in particular Lemma 4).

Theorem 2 ([3]). *If T is a Borel measurable transformation on the separable metric space X, and μ is a T-invariant probability measure on X, then $\underline{R}(x) \le \dim_H \mu$ for μ-almost every $x \in X$.*

We can also rephrase the first inequality in (3) in a form similar to (5).

Theorem 3. *If $T\colon X \to X$ is a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, and μ is a T-invariant probability measure on X, then* (5) *holds for μ-almost every $x \in X$ such that $\underline{d}_\mu(x) < \alpha$.*

Using Young's criteria (see [13]), one can show that $\dim_H \mu \geq \underline{d}_\mu(x)$ for μ-almost every $x \in X$. Therefore, Theorem 3 may in general provide a stronger statement than that in (5), and the first inequality in (3) may be sharper than that in Theorem 2. This possibility indeed occurs in the following example.

Example 1. In [10], Pesin and Weiss presented an example of a Hölder homeomorphism in a closed subset X of $[0,1]$, whose unique (and thus ergodic) measure of maximal entropy μ is not exact dimensional. More precisely, there exist disjoint sets A_1, $A_2 \subset [0,1]$ with positive μ-measure whose union is equal to X, and there exist positive constants c_1 and c_2 with $c_1 \neq c_2$ such that $\mu|A_i$ is exact dimensional and $\underline{d}_\mu(x) = \overline{d}_\mu(x) = c_i$ for μ-almost every $x \in A_i$ and $i = 1, 2$. Clearly $\dim_H \mu = \max\{c_1, c_2\}$ and thus $\underline{d}_\mu(x) < \dim_H \mu$ on a set of positive μ-measure (on the set A_i with i such that $c_i = \min\{c_1, c_2\}$). $\square$

This example illustrates that in general Theorem 3 provides a stronger statement then that in (5). Therefore, one can see the first inequality in Theorem 1 as a non-trivial generalization of one of Boshernitzan's main results in [3]. Furthermore, we are able to give an estimate for the upper recurrence rate, and we shall see (in Sections 3 and 4) that for several classes of maps and measures the inequalities in (3) are in fact identities on a full measure set. Therefore, the inequalities in Theorem 1 and 3 are optimal.

Example 1 also illustrates that for an arbitrary transformation the functions $\underline{d}_\mu$ and $\overline{d}_\mu$ need not be invariant μ-almost everywhere. Assume now that T is a Lipschitz map with Lipschitz inverse, and let $c > 1$ be a Lipschitz constant for T and T^{-1}. It is easy to verify that $\tau_{cr}(Tx) \leq \tau_r(x) \leq \tau_{r/c}(Tx)$ for every $x \in X$ and every $r > 0$. Thus

$$\underline{R}(Tx) = \underline{R}(x) \quad \text{and} \quad \overline{R}(Tx) = \overline{R}(x)$$

for every $x \in X$. One can also verify that for every $x \in X$ we have

$$\underline{d}_\mu(Tx) = \underline{d}_\mu(x) \quad \text{and} \quad \overline{d}_\mu(Tx) = \overline{d}_\mu(x).$$

3. Coincidence of recurrence rate and pointwise dimension

3.1. **Formulation of the main result.**

In this section we investigate conditions under which the inequalities in (3) become identities (see also Section 4). These conditions will be shown to hold for a large class of invariant measures.

The *return time* of the point $y \in B(x,r)$ into $B(x,r)$ is defined by

$$\tau_r(y,x) \stackrel{\text{def}}{=} \inf\{k > 0 : d(T^k y, x) < r\}. \tag{7}$$

One can easily verify that if $d(x,y) < r$ then

$$\tau_{4r}(y) \leq \tau_{2r}(y,x) \leq \tau_r(y). \tag{8}$$

For each $x \in X$ and r, $\varepsilon > 0$, we consider the set

$$A_\varepsilon(x,r) = \{y \in B(x,r) : \tau_r(y,x) \leq \mu(B(x,r))^{-1+\varepsilon}\}.$$

We shall say that the measure μ has *long return time* (with respect to T) if

$$\varliminf_{r \to 0} \frac{\log \mu(A_\varepsilon(x,r))}{\log \mu(B(x,r))} > 1 \tag{9}$$

for μ-almost every $x \in X$ and every sufficiently small $\varepsilon > 0$.

The class of measures μ with long return time includes equilibrium measures supported on locally maximal hyperbolic sets (see Theorem 5 below). On the other hand, Example 2 below illustrates that a T-invariant measure μ may have long return time even if T is not uniformly hyperbolic. See also Section 3.2 for a discussion of the relation of the notion of long return time with return time statistics.

The following is a considerably strengthen version of Theorem 1 for measures with long return time.

Theorem 4. *Let $T\colon X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, and μ a T-invariant probability measure on X. If μ has long return time, and $\underline{d}_\mu(x) > 0$ for μ-almost every $x \in X$, then*

$$\underline{R}(x) = \underline{d}_\mu(x) \quad \textit{and} \quad \overline{R}(x) = \overline{d}_\mu(x) \tag{10}$$

for μ-almost every $x \in X$.

See Sections 3.2 and 4 for applications of Theorem 4.

Remark that the recurrence rates $\underline{R}(x)$ and $\overline{R}(x)$ are essentially quantities of topological nature, which are defined independently of any measure. Therefore, the identities in (10) provide non-trivial relations between topological and measure-theoretic quantities.

3.2. **Relation with return time statistics.** We define the *distribution of return time* of T (with respect to μ) on the ball $B(x,r)$ by

$$\mathcal{F}_{x,r}(t) \stackrel{\text{def}}{=} \frac{\mu(\{y \in B(x,r) : \tau_r(y,x) > t/\mu(B(x,r))\})}{\mu(B(x,r))}$$

for each $t \geq 0$. In a variety of systems with some kind of hyperbolicity it has been established that $\mathcal{F}_{x,r}(t) \to \mathrm{e}^{-t}$ as $r \to 0$, for μ-almost every $x \in X$. This behavior is known as the *exponential statistic of return time*, and is becoming an important ingredient in the analysis of recurrence in dynamical systems. This study is closely related to the above-introduced notion of long return time. This can be readily seen from the equation

$$\begin{aligned}\mu(A_\varepsilon(x,r)) &= [1 - \mathcal{F}_{x,r}(\mu(B(x,r))^\varepsilon)]\mu(B(x,r)) \\ &\leq \mu(B(x,r))^{1+\varepsilon} + \mu(B(x,r)) \sup_{t\geq 0} |\mathcal{F}_{x,r}(t) - \mathrm{e}^{-t}|,\end{aligned} \tag{11}$$

which holds for all sufficiently small $r > 0$. This implies the following criterion for long return time.

Proposition 1. *Let T be a Borel measurable transformation on the separable metric space X, and μ a probability measure on X. Assume that for μ-almost every $x \in X$ there exists $\gamma = \gamma(x) > 0$ such that*

$$\sup\{|\mathcal{F}_{x,r}(t) - \mathrm{e}^{-t}| : t \geq 0\} \leq \mu(B(x,r))^\gamma$$

for all sufficiently small $r > 0$. Then μ has long return time.

In fact, it is not necessary to have an exponential statistic of return time. For example, assume that for μ-almost every $x \in X$ there exist $\gamma = \gamma(x) > 0$

and a function $\mathcal{F}_x\colon [0,+\infty) \to [0,1]$ which is Hölder continuous in an open neighborhood of 0 such that

$$\sup\{|\mathcal{F}_{x,r}(t) - \mathcal{F}_x(t)| : t \ge 0\} \le \mu(B(x,r))^\gamma$$

for all sufficiently small $r > 0$. Note that one must have $\mathcal{F}_x(0) = 1$. Then it follows from (11) that the measure μ has long return time.

The following example illustrates that an invariant measure may have long return time even if the map is not hyperbolic.

Example 2. Let $\alpha \in (0,1)$ and consider the map $T\colon [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0,1/2] \\ 2x-1 & \text{otherwise} \end{cases}. \tag{12}$$

Note that due to the presence of the neutral fixed point 0 (since $T'(0^+) = 1$), the map is not uniformly hyperbolic. We recall that there exists a unique invariant probability measure μ which is ergodic and absolutely continuous with respect to Lebesgue measure (see, for example, [6] for references). Since the set of points $x \in [0,1]$ such that $|T'(x)| > 1$ has full μ-measure, the measure μ is hyperbolic (see Section 4 for the definition).

Denote by a_n the left preimages of $a_0 = 1$. Let ξ be the countable partition of $[0,1]$ defined by $\xi = \{(a_{n+1}, a_n] : n \ge 0\}$. It is proved in [6] that T has exponential statistic of return time for cylinders of the partition ξ. More precisely, the following sharp estimate is established: there exists $\gamma > 0$ such that for μ-almost every $x \in [0,1]$ and all sufficiently large $m \in \mathbb{N}$, we have

$$\sup_{t\ge 0} \left| \frac{\mu(\{y \in \xi_m(x) : \tau_{\xi_m(x)}(y) > t/\mu(\xi_m(x))\})}{\mu(\xi_m(x))} - \mathrm{e}^{-t} \right| \le \mu(\xi_m(x))^\gamma,$$

where $\xi_m = \bigvee_{k=0}^{m-1} T^{-k}\xi$, and

$$\tau_A(y) = \inf\{k \in \mathbb{N} : T^k y \in A\}.$$

Proceeding as in (11) we conclude that for μ-almost every $x \in [0,1]$ we have

$$\mu(\{y \in \xi_m(x) : \tau_{\xi_m(x)} \le \mu(\xi_m(x))^{-1+\varepsilon}\}) \le 2\mu(\xi_m(x))^{1+\varepsilon}$$

for every $\varepsilon \le \gamma$ and every sufficiently large m.

We now present a simple argument showing that one can replace cylinders by balls in the last inequality. For each sufficiently small $r > 0$, it is possible to choose integers $m_r \ge n_r$ such that $\xi_{m_r}(x) \subset B(x,r) \subset \xi_{n_r}(x)$ with $m_r/n_r \to 1$ as $r \to 0$. Since

$$\tau_{\xi_{n_r}(x)}(y) \le \tau_r(y) \quad \text{and} \quad \mu(B(x,r)) \ge \mu(\xi_{m_r}(x)),$$

we obtain

$$\mu(A_\varepsilon(x,r)) \le \mu(\{y \in \xi_{n_r}(x) : \tau_{\xi_{n_r}(x)}(y) \le \mu(\xi_{m_r}(x))^{-1+\varepsilon}\}).$$

In view of the inequalities

$$\mu(\xi_{m_r}(x))^{-1+\varepsilon} \le \mu(\xi_{n_r}(x))^{-1+\varepsilon/2} \quad \text{and} \quad \mu(\xi_{n_r}(x)) \le \mu(B(x,r))^{1-\varepsilon/4},$$

which are valid for all sufficiently small r, we conclude that for μ-almost every $x \in [0,1]$ we have

$$\mu(A_\varepsilon(x,r)) \le 2\mu(\xi_{n_r}(x))^{1+\varepsilon/2} \le 2\mu(B(x,r))^{(1+\varepsilon/2)(1-\varepsilon/4)}$$

for all sufficiently small r. By taking $\varepsilon > 0$ sufficiently small, this inequality implies that the measure μ has long return time.

It follows from Theorem 4 that the identities in (10) hold for Lebesgue almost every point. □

We remark that it is an open problem to decide whether all hyperbolic measures (not necessarily supported on uniformly hyperbolic sets; see Section 4) have long return time.

4. Hyperbolic Gibbs measures

We now consider equilibrium measures supported on a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism. The following result shows that in this situation the identities in (10) hold on a set of full measure.

Theorem 5. *Let X be a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism on a compact smooth manifold, for some $\alpha > 0$. If μ is an ergodic equilibrium measure of a Hölder continuous potential on X then:*

1. *μ has long return time;*
2. *$\underline{R}(x) = \overline{R}(x) = \dim_H \mu$ for μ-almost every $x \in X$.*

When μ is not ergodic one can consider the finite ergodic decomposition of μ on (relatively) open subsets $X_i \subset X$ such that $(X_i, T, \mu|X_i)$ is ergodic. Since each X_i is (relatively) open it follows immediately from Theorem 5 that $\underline{R}(x) = \overline{R}(x) = \dim_H \mu_{X_i}$ for μ-almost every $x \in X_i$.

The following example illustrates that for invariant measures which are not hyperbolic the second statement in Theorem 5 may not hold.

Example 3. Consider a rotation of the circle by an irrational number ω which is well approximable by rational numbers. We recall that ω is said to be *well approximable by rational numbers* if $\nu(\omega) > 1$, where $\nu(\omega)$ is the supremum of all $\nu > 0$ such that $|\omega - p/q| < 1/q^{\nu+1}$ for infinitely many relatively prime integers p and q. The unique invariant measure of the rotation is the Lebesgue measure m, which is clearly exact dimensional. Furthermore, it is easy to verify that if $0 < q_1 < q_2 < \cdots$ is a sequence of positive integers such that $|q_n\omega - p_n| < 1/{q_n}^\nu$ for some integer p_n, then

$$\tau_{1/{q_n}^\nu}(x) = \inf\{k \in \mathbb{N} : k\omega (\mathrm{mod} 1) < 1/{q_n}^\nu\} \leq q_n$$

for every x in the circle, and thus

$$\underline{R}(x) \leq \varliminf_{n\to\infty} \frac{\log \tau_{1/{q_n}^\nu}(x)}{-\log(1/{q_n}^\nu)} \leq \frac{1}{\nu(\omega)} < 1 = \dim_H m.$$

Note that the Lebesgue measure is not hyperbolic in this example. □

In view of Theorem 4, Example 3 also illustrates that an exact dimensional measure may not have long return time.

We now show that the above Theorems 4 and 5 can be seen as generalizations of work of Ornstein and Weiss for the measure-theoretic entropy.

Let $T : X \to X$ be a measurable transformation (note that X need not be a metric space), and $\mathcal{Z}$ a measurable partition of X. Consider the partitions $\mathcal{Z}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{Z}$ for each n. We shall denote by $h_\mu(T, \mathcal{Z})$ the μ-entropy of T with respect to $\mathcal{Z}$. Then Theorem 1 in [8] can be reformulated in the following manner.

Proposition 2 ([8]). *Let $T\colon X \to X$ be a measurable transformation, $\mathcal{Z}$ a measurable partition of X, and μ an ergodic T-invariant probability measure on X. If we endow X with the (pseudo) metric $d_{\mathcal{Z}}(x,y) = \mathrm{e}^{-n}$, where n is the smallest positive integer such that $\mathcal{Z}_n(x) \neq \mathcal{Z}_n(y)$, then $\underline{R}(x) = \overline{R}(x) = h_\mu(T,\mathcal{Z})$ for μ-almost every $x \in X$.*

We stress that with the special metric $d_{\mathcal{Z}}$, the measure-theoretic entropy $h_\mu(T,\mathcal{Z})$ coincides with the Hausdorff dimension $\dim_H \mu$ of the measure μ. Theorems 4 and 5 provide versions of Proposition 2 in metric spaces which may have "non-homogeneous" distances.

Let now $f\colon M \to M$ be a diffeomorphism of a compact smooth manifold. Given $x \in M$ and $v \in T_xM$ the *Lyapunov exponent* of v at x is defined by

$$\lambda(x,v) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|d_x f^n v\|.$$

The measure μ is said to be *hyperbolic* if there exists a set $Y \subset M$ of full μ-measure such that $\lambda(x,v) \neq 0$ for every $x \in Y$ and every $v \in T_xM$.

One should notice that the relation between Proposition 2 and Theorem 5 is similar to the relation between the Shannon–McMillan–Breiman theorem, and the following statement established by Barreira, Pesin, and Schmeling.

Proposition 3 ([1]). *If f is a $C^{1+\alpha}$ diffeomorphism on a compact smooth manifold M, for some $\alpha > 0$, and μ is a hyperbolic f-invariant probability measure on M, then $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for μ-almost every $x \in M$.*

The role of Proposition 3 in dimension theory of dynamical systems is similar to the role of the Shannon–McMillan–Breiman theorem in the entropy theory. While the first ensures the coincidence of many characteristics of dimension type of the measure (such as the Hausdorff dimension, lower and upper box dimensions, and lower and upper correlation dimensions), the later ensures the coincidence of various definitions of the entropy (such as those due to Kolmogorov and Sinai, Katok, Brin and Katok, and Pesin). See [1] for details.

In a similar fashion, while Proposition 2 relates the measure-theoretic entropy with recurrence, Theorem 5 relates dimension-like characteristics with recurrence. Both results provide a non-trivial insight concerning the quantitative behavior of recurrence.

A similar observation can be made about the hypotheses under which the results are established. Namely, the assumptions in Proposition 3 are known to be optimal (see [1, 9] for details), while the Shannon–McMillan–Breiman theorem only assumes the invariance of the probability measure. Similarly, while Proposition 2 only requires the measure to be ergodic, Theorem 5 requires more from the dynamical system and the invariant measure. Example 3 illustrates that the assumption that the measure is hyperbolic is essential in Theorem 5.

We believe that the other assumption in Theorem 5, concerning the regularity of the map, is also essential, although to the best of our knowledge no counterexample is known with weaker regularity.

Moreover we would like to formulate the following plausible conjecture.

Conjecture. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact smooth manifold M, for some $\alpha > 0$. If μ is an ergodic hyperbolic f-invariant probability measure on M, then $\underline{R}(x) = \overline{R}(x) = \dim_H \mu$ for μ-almost every $x \in M$.

Theorem 5 establishes this statement when μ is supported on a *uniformly* hyperbolic set. By Theorem 1 and Proposition 3, observe that in order to establish the conjecture in the affirmative, one must show that $\underline{R}(x) \geq \dim_H \mu$ for μ-almost every $x \in X$.

Our results motivate the introduction of a new method to compute the Hausdorff dimension of measures. More precisely, one can use Statement 2 in Theorem 5 to compute the Hausdorff dimension of an equilibrium measure μ supported on a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism. Namely, for μ-almost every point we have

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_H \mu.$$

Therefore, one can use the following algorithm:

1. choose "μ-randomly" a point $x \in X$;
2. iterate the point x and determine the successive "best" return times to a neighborhood of x, i.e., the smallest possible positive integers $m_1 < m_2 < \cdots$ such that $d(T^{m_1}x, x) > d(T^{m_2}x, x) > \cdots$;
3. plot the points $(\log m_n, -\log d(T^{m_n}x, x))$ in a plane;
4. estimate $\dim_H \mu$ from the asymptotic slope defined by these points.

5. Application to suspension flows

We assume that $T\colon X \to X$ is a bi-Lipschitz transformation on the separable metric space (X, d). Let $\varphi\colon X \to (0, \infty)$ be a Lipschitz function. Consider the space

$$Y = \{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq \varphi(x)\},$$

with the points $(x, \varphi(x))$ and $(Tx, 0)$ identified for each $x \in X$. The *suspension flow over T with height function φ* is the flow $\Psi = \{\psi_t\}_t$ on Y where each transformation $\psi_t\colon Y \to Y$ is defined by $\psi_t(x, s) = (x, s + t)$.

We equip the space Y with the Bowen–Walters distance d_Y introduced in [4], and define the *return time* of a point $y \in Y$ (with respect to the flow Ψ) into the open ball $B_Y(y, r)$ by

$$\tau_r^\Psi(y) \stackrel{\text{def}}{=} \inf\{t > \rho_r(y) : \psi_t y \in B_Y(y, r)\} = \inf\{t > \rho_r(y) : d_Y(\psi_t y, y) < r\},$$

where

$$\rho_r(y) = \inf\{t > 0 : \psi_t y \notin B_Y(y, r)\}$$

is the *escape time* of y from the ball $B_Y(y, r)$. Observe that $\psi_t y \in B_Y(y, r)$ for all sufficiently small t, and thus we need to ensure that the orbit $\psi_t y$ has escaped from $B_Y(y, r)$ when defining $\tau_r^\Psi(y)$.

We also define the *lower* and *upper recurrence rates* of y by

$$\underline{R}^\Psi(y) = \varliminf_{r\to 0} \frac{\log \tau_r^\Psi(y)}{-\log r} \quad \text{and} \quad \overline{R}^\Psi(y) = \varlimsup_{r\to 0} \frac{\log \tau_r^\Psi(y)}{-\log r}.$$

Let μ a T-invariant Borel probability measure in X. It is well known that μ induces a Ψ-invariant probability measure ν in Y such that

$$\int_Y g\,d\nu = \int_X \int_0^{\varphi(x)} g(x,s)\,ds d\mu(x) \Big/ \int_X \varphi\,d\mu$$

for every continuous function $g\colon Y \to \mathbb{R}$ (where ds refers to Lebesgue measure in the line), and that any Ψ-invariant measure ν in Y is of this form for some T-invariant Borel probability measure μ in X.

Theorem 5 can be used to establish the following result for suspension flows.

Theorem 6. *Let X be a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism T on a compact smooth manifold, for some $\alpha > 0$, and μ an equilibrium measure of a Hölder continuous potential on X. If Ψ is a suspension flow over $T|X$ then*

$$\underline{R}^{\Psi}(y) = \overline{R}^{\Psi}(y) = \dim_H \nu - 1$$

for ν-almost every $y \in Y$.

6. Proofs

Following Federer [5], a measure μ is called *diametrically regular* if there exist constants $\eta > 1$ and $c > 0$ such that $\mu(B(x,\eta r)) \le c\mu(B(x,r))$ for every $x \in X$ and $r > 0$. Examples include equilibrium measures with a Hölder continuous potential for several classes of topologically mixing hyperbolic systems, and namely subshifts of finite type, conformal expanding maps, surface axiom A diffeomorphisms, and, more generally, conformal axiom A diffeomorphisms. See [9] for full details.

We shall say that a measure μ is *weakly diametrically regular* on a set $Z \subset X$ if there is a constant $\eta > 1$ such that for μ-almost every $x \in Z$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $r < \delta$ then

$$\mu(B(x,\eta r)) \le \mu(B(x,r))r^{-\varepsilon}. \tag{13}$$

It is easy to verify that if μ is a weakly diametrically regular measure on a set Z, then for *each* fixed constant $\eta > 1$, there exists $\delta = \delta(x,\varepsilon,\eta) > 0$ for μ-almost every $x \in Z$ and every $\varepsilon > 0$, such that (13) holds for every $r < \delta$. Clearly, diametrically regular measures are weakly diametrically regular on X.

Lemma 1. *Any Borel probability measure on $\mathbb{R}^d$ is weakly diametrically regular.*

Proof of Lemma 1. Let μ be a Borel probability measure on $\mathbb{R}^d$. Clearly, it is sufficient to show that for μ-almost every $x \in \mathbb{R}^d$ we have

$$\mu(B(x,2^{-n})) \le n^2\mu(B(x,2^{-n-1})) \tag{14}$$

for all sufficiently large $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $\delta > 0$ let

$$K_n(\delta) \overset{\text{def}}{=} \left\{x \in \operatorname{supp}\mu : \mu(B(x,2^{-n-1})) < \delta\mu(B(x,2^{-n}))\right\}.$$

Taking a maximal 2^{-n-2}-separated set $E \subset K_n(\delta)$ we obtain

$$\mu(K_n(\delta)) \le \sum_{x\in E} \mu(B(x,2^{-n-1})) \le \sum_{x\in E} \delta\mu(B(x,2^{-n})).$$

Since E is 2^{-n-2}-separated, there exists a constant M (depending only on d) such that E can be decomposed into the union $E = \bigcup_{i=1}^{M} E_i$ where each set E_i is 2^{-n}-separated. Thus for each $i = 1, \ldots, M$ the union $\bigcup_{x \in E_i} B(x, 2^{-n})$ is disjoint. Therefore

$$\mu(K_n(\delta)) \leq \sum_{i=1}^{M} \sum_{x \in E_i} \delta \mu(B(x, 2^{-n})) \leq M\delta.$$

Since

$$\sum_{n>0} \mu(K_n(n^{-2})) \leq M \sum_{n>0} n^{-2} < \infty,$$

we conclude from the the Borel–Cantelli lemma that (14) holds for μ-almost every $x \in X$ and all sufficiently large $n \in \mathbb{N}$. This completes the proof. $\square$

This shows that the class of weakly diametrically regular measures is very broad. In particular, due to Whitney's embedding theorem, this class contains *all* probability measures supported in a finite-dimensional smooth manifold. Further weakly diametrically regular measures on arbitrary metric spaces include any measure μ on a separable metric space X restricted to the set

$$\{x \in X : \underline{d}_\mu(x) = \overline{d}_\mu(x)\}.$$

This readily follows from the definition of pointwise dimension.

Note that the property of $\mathbb{R}^d$ that we used in the proof of Lemma 1 is that the maximal cardinality, say $M(r)$, of a $\frac{1}{4}r$-separated subset of balls of radius r is bounded by some constant M. Our proof readily extends to separable metric spaces X with the property that $M(r) = o(r^{-\varepsilon})$ for any $\varepsilon > 0$. In this case, instead of (14) one can show that for each $\delta > 0$ we have

$$\mu(B(x, 2^{-n})) \leq 2^{n\delta} \mu(B(x, 2^{-n-1}))$$

for μ-almost every $x \in X$ and all sufficiently large $n \in \mathbb{N}$. This readily implies the weak regularity of μ.

We now provide an example of very different nature.

Example 4. Let $\alpha \in (\frac{1}{2}, 1)$ and define the sequence $\beta_n = n^\alpha$. Consider the space of sequences $X = \{0,1\}^{\mathbb{N}}$ and define a metric on X by requiring that $\operatorname{diam} C_n(x) = e^{-\beta_n}$, where $C_n(x)$ is any cylinder of length n. Consider also the Bernoulli measure μ on X such that $\mu(C_n(x)) = 2^{-n}$. One can easily verify that μ is weakly diametrically regular (by checking that any ball of radius r contains at most $r^{-\varepsilon(r)}$ balls of radius $\frac{1}{4}r$, where $\varepsilon(r) \to 0$ as $r \to 0$), and also that $\dim_H \mu = +\infty$. In particular $X = \operatorname{supp} \mu$ cannot be smoothly embedded into $\mathbb{R}^d$ for any $d \in \mathbb{N}$.

The same measure μ may not be weakly diametrically regular if the sequence β_n increases in a slower fashion. This is the case for example when $\beta_n = \log n$. $\square$

We recall that for any $a > 0$ the following identities hold:

$$\underline{d}_\mu(x) = \varliminf_{n\to\infty} \frac{\log \mu(B(x, ae^{-n}))}{-n}, \quad \overline{d}_\mu(x) = \varlimsup_{n\to\infty} \frac{\log \mu(B(x, ae^{-n}))}{-n}, \quad (15)$$

$$\underline{R}(x) = \varliminf_{n\to\infty} \frac{\log \tau_{a\mathrm{e}^{-n}}(x)}{n}, \quad \overline{R}(x) = \varlimsup_{n\to\infty} \frac{\log \tau_{a\mathrm{e}^{-n}}(x)}{n}. \tag{16}$$

Lemma 2. *Let T be a Borel measurable transformation on the separable metric space X, and μ a T-invariant probability measure on X. If μ is weakly diametrically regular on a measurable set $Z \subset X$ with $\mu(Z) > 0$, then (3) holds for μ-almost every $x \in Z$.*

Proof of Lemma 2. Observe that the function $\delta(x, \cdot)$ in the definition of a weakly regular measure (see (13)) can be made measurable for each fixed x. Fix $\varepsilon > 0$, and choose $\delta > 0$ sufficiently small such that the set

$$G = \{x \in Z : \delta(x, \varepsilon) > \delta\}$$

has measure $\mu(G) > \mu(Z) - \varepsilon$. For any r, $\lambda > 0$ and $x \in X$ consider the set

$$A_{r,x} = \{y \in B(x, 4r) : \tau_{4r}(y, x) \geq \lambda^{-1}\mu(B(x, 4r))^{-1}\},$$

where $\tau_{4r}(y, x)$ is defined in (7). Chebychev's inequality implies that

$$\mu(A_{r,x}) \leq \lambda\mu(B(x, 4r)) \int_{B(x,4r)} \tau_{4r}(y, x)\, d\mu(y).$$

Since μ is invariant, Kac's lemma tells us that

$$\int_{B(x,4r)} \tau_{4r}(y, x)\, d\mu(y) = \mu(\{y \in X : \tau_{4r}(y, x) < \infty\}) \leq 1.$$

Since $B(x, 2r) \subset B(x, 4r)$, we obtain

$$\mu(\{y \in B(x, 2r) : \tau_{4r}(y, x)\mu(B(x, 4r)) \geq \lambda^{-1}\}) \leq \lambda\mu(B(x, 4r)).$$

Furthermore

$$\tau_{4r}(y, x)\mu(B(x, 4r)) \geq \tau_{8r}(y)\mu(B(y, 2r))$$

whenever $d(x, y) < 2r$ (see (8)), and thus

$$\mu(\{y \in B(x, 2r) : \tau_{8r}(y)\mu(B(y, 2r)) \geq \lambda^{-1}\}) \leq \lambda\mu(B(x, 4r)). \tag{17}$$

By Lemma 3 we can find an at most countable maximal r-separated set $E \subset G$. Using (17) with $\lambda = r^{2\varepsilon}$ and (13) with $\eta = 4$ (see also the discussion after (13)), we obtain

$$\begin{aligned} D_\varepsilon(r) &\stackrel{\text{def}}{=} \mu(\{y \in G : \tau_{8r}(y)\mu(B(y, 2r)) \geq r^{-2\varepsilon}\}) \\ &\leq \sum_{x\in E} \mu(\{y \in B(x, 2r) : \tau_{8r}(y)\mu(B(y, 2r)) \geq r^{-2\varepsilon}\}) \\ &\leq r^{2\varepsilon} \sum_{x\in E} \mu(B(x, 4r)) \\ &\leq r^{\varepsilon} \sum_{x\in E} \mu(B(x, r)) \leq r^{\varepsilon}. \end{aligned}$$

We conclude that

$$\sum_{n>-\log\delta} D_\varepsilon(\mathrm{e}^{-n}) \leq \sum_{n>-\log\delta} \mathrm{e}^{-\varepsilon n} < \infty.$$

By the Borel–Cantelli lemma we find that for μ-almost any $x \in G$, we have

$$\frac{\log \tau_{8\mathrm{e}^{-n}}(x)}{n} \leq 2\varepsilon + \frac{\log \mu(B(x, 2\mathrm{e}^{-n}))}{-n}$$

for all sufficiently large n. The desired result now follows from the identities in (15) and (16), and the arbitrariness of ε. $\square$

In particular, if μ is an exact dimensional probability measure on a separable metric space X, then by Lemma 2 we have $\overline{R}(x) \le \dim_H \mu$ for μ-almost every $x \in X$. On the other hand, if μ is not exact dimensional then in general one can only show that $\underline{R}(x) \le \dim_H \mu$ for μ-almost every $x \in X$ (see Theorem 2 and Example 1).

We notice that as in Lemma 2, the condition that $X \subset \mathbb{R}^d$ in Theorem 4 may also be replaced by the hypothesis that μ is weakly diametrically regular on X.

We continue with an auxiliary statement.

Lemma 3. *Let μ be a finite Borel measure on the separable metric space X, and $G \subset \operatorname{supp}\mu$ a measurable set. Given $r > 0$, there exists a countable set $E \subset G$ such that:*

1. *$B(x,r) \cap B(y,r) = \varnothing$ for any two distinct points x, $y \in E$;*
2. *$\mu(G \setminus \bigcup_{x\in E} B(x,2r)) = 0$.*

Proof of Lemma 3. The existence of the set E can be obtained using Zorn's lemma on the non-empty family of subsets of G which satisfy the first property, ordered by inclusion. Then the second property is satisfied for any maximal element. Since $\mu(B(x,r)) > 0$ for each $x \in E \subset \operatorname{supp}\mu$, the set E is at most countable. $\square$

We shall call the set E in Lemma 3 a *maximal r-separated set* for G.

We now start establishing the results in the former sections.

Proof of Theorem 1. The desired statement follows from Lemma 1 and Theorem 2. $\square$

Lemma 4. *Given $x \in X$, we have $\underline{R}(x) \le d$ if and only if for every $\varepsilon > 0$, we have*

$$\varliminf_{n\to\infty} [n^{1/(d+\varepsilon)} d(T^n x, x)] = 0. \tag{18}$$

Proof of Lemma 4. Assume first that $\underline{R}(x) \le d$. Given $\varepsilon > 0$ there exists a sequence of numbers r_n such that $r_n \to 0$, and $\tau_{r_n}(x) < {r_n}^{-(d+\varepsilon)}$ for all n. Let $m_n = \tau_{r_n}(x)$. If the sequence m_n is bounded, then x is periodic and (18) holds. Assume now that m_n is unbounded. Note that $d(T^{m_n}x, x) < r_n$ and

$$\begin{aligned} {m_n}^{1/(d+2\varepsilon)} d(T^{m_n}x, x) &< \tau_{r_n}(x)^{1/(d+2\varepsilon)} r_n \\ &< {r_n}^{-(d+\varepsilon)/(d+2\varepsilon)} r_n = {r_n}^{\varepsilon/(d+\varepsilon)}. \end{aligned}$$

Therefore

$$\varliminf_{n\to\infty} [n^{1/(d+2\varepsilon)} d(T^n x, x)] \le \varliminf_{n\to\infty} [{m_n}^{1/(d+2\varepsilon)} d(T^{m_n}x, x)] = 0.$$

This establishes (18) for each $\varepsilon > 0$.

Assume now that (18) holds for every $\varepsilon > 0$. Setting $r_n = 2d(T^n x, x)$, we conclude that $\tau_{r_n}(x) \le n$, and it follows from (18) that

$$\varliminf_{n\to\infty} [\tau_{r_n}(x)^{1/(d+\varepsilon)} r_n] = 0.$$

Thus there exists a diverging sequence of positive integers k_n such that $\tau_{r_{k_n}}(x)^{1/(d+\varepsilon)} r_{k_n} < 1$ for each n. Therefore

$$\underline{R}(x) \leq \varliminf_{n\to\infty} \frac{\log \tau_{r_n}(x)}{-\log r_n} \leq \varliminf_{n\to\infty} \frac{\log({r_{k_n}}^{d+\varepsilon})}{-\log r_{k_n}} = d + \varepsilon.$$

The arbitrariness of ε implies the desired result. □

Proof of Theorem 2. We remind that the statement in Theorem 2 is a reformulation of a result of Boshernitzan.

By the definition of the Hausdorff dimension of a measure, for any $\alpha > \dim_H \mu$ and all sufficiently small $\delta > 0$ there exists a set $Z \subset X$ of full μ-measure such that $\alpha > \dim_H Z > \dim_H \mu - \delta$, and hence $m_\alpha(Z) = 0$. It follows from (5) and Lemma 4 that $\underline{R}(x) \leq \alpha$ for μ-almost every $x \in Z$. Letting $\alpha \to \dim_H Z$ and $\delta \to 0$ we obtain $\underline{R}(x) \leq \dim_H \mu$ for μ-almost every $x \in X$. □

Proof of Theorem 3. The desired statement follows from Theorem 1 and Lemma 4. □

Proof of Theorem 4. By Theorem 1 we have $\underline{R}(x) \leq \underline{d}_\mu(x)$ and $\overline{R}(x) \leq \overline{d}_\mu(x)$ for μ-almost every $x \in X$. We shall now establish the reverse inequalities.

By Lemma 1 the measure μ is weakly diametrically regular on X. Since μ has long return time (see (9)), μ is weakly diametrically regular on X (see (13)), and $\underline{d}_\mu(x) > 0$ for μ-almost every $x \in X$, if $\varepsilon > 0$ is sufficiently small we conclude that there exist numbers a, γ, $\rho > 0$ and a set $G \subset X$ with $\mu(G) > 1 - \varepsilon$ such that if $x \in G$ and $r \in (0, \rho)$ then

$$\mu(A_\varepsilon(x, 2r)) \leq \mu(B(x, 2r))^{1+\gamma}, \tag{19}$$

$$\mu(B(x, 2r)) \leq \mu(B(x, r/2)) r^{-a\gamma/2}, \tag{20}$$

$$\mu(B(x, r)) \leq r^a. \tag{21}$$

Consider the set

$$A_\varepsilon(r) \stackrel{\text{def}}{=} \{y \in G : \tau_r(y) \leq \mu(B(y, 3r))^{-1+\varepsilon}\}.$$

Whenever $d(x, y) < r$ we have $\tau_r(y) \geq \tau_{2r}(y, x)$ (see (8)). Since $B(x, 2r) \subset B(y, 3r)$, if $x \in G$ then using (19), (20), and (21), we obtain

$$\begin{aligned}
\mu(B(x, r) \cap A_\varepsilon(r)) &\leq \mu(\{y \in B(x, r) : \tau_{2r}(y, x) \leq \mu(B(x, 3r))^{-1+\varepsilon}\}) \\
&\leq \mu(A_\varepsilon(x, 2r)) \\
&\leq \mu(B(x, 2r))^{1+\gamma} \\
&\leq \mu(B(x, r/2)) r^{-a\gamma/2} (2r)^{a\gamma}.
\end{aligned}$$

If $E \subset G$ is a maximal $\frac{r}{2}$-separated set given by Lemma 3, then

$$\begin{aligned}
\mu(A_\varepsilon(r)) &\leq \sum_{x\in E} \mu(B(x, r) \cap A_\varepsilon(r)) \\
&\leq \sum_{x\in E} \mu(B(x, r/2)) r^{-a\gamma/2} (2r)^{a\gamma} \\
&\leq 2^{a\gamma} r^{a\gamma/2}.
\end{aligned}$$

We conclude that

$$\sum_{n=1}^{\infty} \mu(A_\varepsilon(\mathrm{e}^{-n})) < \infty.$$

The Borel–Cantelli lemma implies that for μ-almost every $x \in G$ we have

$$\tau_{\mathrm{e}^{-n}}(x) > \mu(B(x, 3\mathrm{e}^{-n}))^{-1+\varepsilon}$$

for all sufficiently large n. The identities in (15) and (16) imply that

$$\underline{R}(x) \geq (1-\varepsilon)\underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) \geq (1-\varepsilon)\overline{d}_\mu(x)$$

for μ-almost every $x \in G$. The desired statement follows from the arbitrariness of ε. □

For a non-uniformly hyperbolic set X in the manifold M, and a Lyapunov regular point $x \in X$, let $\Phi_x \colon U_x \to M$ be the Lyapunov chart at x for some sufficiently small open neighborhood $U_x \subset \mathbb{R}^{\dim M}$ of 0, which satisfies $\Phi_x 0 = x$, and

$$d_0\Phi_x(\mathbb{R}^{\dim E^s(x)} \times \{0\}) = E^s(x) \quad \text{and} \quad d_0\Phi_x(\{0\} \times \mathbb{R}^{\dim E^u(x)}) = E^u(x).$$

See the appendix to [1] for details. We denote by $D^s(x,r) \subset \mathbb{R}^{\dim E^s(x)}$ and $D^u(x,r) \subset \mathbb{R}^{\dim E^u(x)}$ the balls of radius r at the origin. We also denote by $B^s(x,r) \subset V^s(x)$ and $B^u(x,r) \subset V^u(x)$ the balls of radius r centered at the point x, with respect to the distances induced in the local stable and unstable manifolds $V^s(x)$ and $V^u(x)$.

In [7], Ledrappier and Young constructed two measurable partitions ξ^s and ξ^u of M such that for μ-almost every $x \in M$ we have:

1. $\xi^s(x) \subset V^s(x)$ and $\xi^u(x) \subset V^u(x)$;
2. $\xi^s(x) \supset V^s(x) \cap B(x,\gamma)$ and $\xi^u(x) \supset V^s(x) \cap B(x,\gamma)$ for some $\gamma = \gamma(x) > 0$.

We denote by μ_x^s and μ_x^u the conditional measures associated respectively to the partitions ξ^s and ξ^u. Recall that any measurable partition ξ of M has associated a family of conditional measures: for μ-almost every $x \in M$ there exists a probability measure μ_x defined on the element $\xi(x)$ of ξ containing x. The conditional measures are characterized completely by the following property: if $\mathcal{B}_\xi$ is the σ-subalgebra of the Borel σ-algebra generated by unions of elements of ξ then for each Borel set $A \subset M$, the function $x \mapsto \mu_x^s(A \cap \xi(x))$ is $\mathcal{B}_\xi$-measurable and

$$\mu(A) = \int_A \mu_x^s(A \cap \xi(x))\, d\mu.$$

We will later need the following result concerning the product structure of hyperbolic measures. Instead of formulating the result in all its generality we state it in a form adapted to our purposes.

Proposition 4 ([12], after [1]). *Let X be a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism f on a compact smooth manifold, for some $\alpha > 0$. If μ is an equilibrium measure of a Hölder continuous potential on X and*

a, b, $c > 0$, then for μ-almost every $x \in X$, there exists $\varepsilon(r) > 0$ for each each $r > 0$ such that $\varepsilon(r) \to 0$ as $r \to 0$ and

$$r^{\varepsilon(r)} \leq \frac{\mu_x^s(B^s(x, r^a))\mu_x^u(B^u(x, r^b))}{\mu(\Phi_x(D^s(x, cr^a) \times D^u(x, cr^b)))} \leq r^{-\varepsilon(r)}$$

for all sufficiently small $r > 0$.

Proof of Proposition 4. We use the notations of [12]. Take a Lyapunov regular point x, and let $\chi_1, \dots, \chi_d$ be the values of the Lyapunov exponent at x. Setting $a_i = -a$ for $\chi_i < 0$, and $a_i = -b$ for $\chi_i > 0$ we conclude that for any r sufficiently small there exist $n(r)$, $m(r) \in \mathbb{N}$ such that

$$R_{m(r)}(x) \subset \Phi_x(D^s(x, r^a) \times D^u(x, r^b)) \subset R_{n(r)}(x)$$

and $m(r)/n(r) \to 1$ as $r \to 0$. The desired result now follows immediately from Theorem 3.9 in [12]. $\square$

We define the *return time* of a set A into itself by

$$\tau(A) = \inf\{n \in \mathbb{N} : T^n A \cap A \neq \varnothing\}.$$

Given a partition ξ of X we consider a new partition $\xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$ for each n. Saussol, Troubetzkoy, and Vaienti show in [11] that the first return time of an element of ξ_n is typically large.

Proposition 5 ([11])**.** *Let $T\colon X \to X$ be a measurable transformation preserving an ergodic probability measure μ. If ξ is a finite or countable measurable partition with entropy $h_\mu(T, \xi) > 0$ then*

$$\varliminf_{n\to\infty} \frac{\tau(\xi_n(x))}{n} \geq 1$$

for μ-almost every $x \in X$.

Using this result, it can be shown that the first return time of a ball is also typically large.

Lemma 5. *Let $T\colon X \to X$ be a Lipschitz map with Lipschitz constant $L > 1$ on a compact metric space X. If μ is an ergodic T-invariant Borel probability measure with entropy $h_\mu(T) > 0$, then*

$$\varliminf_{r\to 0} \frac{\tau(B(x, r))}{-\log r} \geq \frac{1}{\log L}$$

for μ-almost every $x \in X$.

Proof of Lemma 5. We claim that for each $n > 0$ there exists a partition ζ_n of X with diameter $\operatorname{diam}\zeta_n \leq 2^{-n}$, such that if $r > 0$, then

$$\mu(\{x \in X : d(x, X \setminus \zeta_n(x)) < r\}) < c_n r, \tag{22}$$

where $\zeta_n(x)$ is the atom of ζ_n containing x, and c_n is some positive constant depending only on n.

Take a finite set $E_n \subset X$ such that $\bigcup_{x \in E_n} B(x, 2^{-n-1}) = X$. For each $x \in E_n$, we can find a sequence of real numbers $r_k = r_k(x) \in (2^{-n-1}, 2^{-n})$ satisfying $|r_{k+1} - r_k| \leq 2^{-k-1}$ and

$$\mu(B(x, r_k + 2^{-k-1}) \setminus B(x, r_k - 2^{-k-1})) \leq 2^{-k}.$$

Take $r(x) = \lim_{k\to\infty} r_k$ and consider the set

$$A_k = \{y \in X : r(x) - 2^{-k} \le d(x,y) \le r(x) + 2^{-k} \text{ for some } x \in E_n\}$$

for each $k \in \mathbb{N}$. Note that we obtain a cover of X satisfying

$$\mu(A_k) \le 2^{-k+1} \operatorname{card} E_n.$$

Writing $E_n = \{x_1, \dots, x_p\}$ we set

$$B_1 = B(x_1, r(x_1)) \quad \text{and} \quad B_\ell = B(x_\ell, r(x_\ell)) \setminus \bigcup_{j=1}^{\ell-1} B_j$$

for $\ell = 2, \dots, p$. Then the partition $\zeta_n = \{B_1, B_2, \dots, B_p\}$ has the desired properties, with $c_n = \operatorname{card} E_n$.

Since $\operatorname{diam} \zeta_n \to 0$ as $n \to \infty$, there exists $n_* \in \mathbb{N}$ such that $h_\mu(T, \zeta_{n_*}) > h_\mu(T)/2 > 0$. Let $\mathcal{Z} = \zeta_{n_*}$, $C = c_{n_*}$ and denote by $\mathcal{Z}_m(x)$ the unique element of $\bigvee_{k=0}^{m-1} T^{-k}\mathcal{Z}$ which contains $x \in X$.

Fix $\sigma < 1/L < 1$. Clearly, if $d(x, X \setminus \mathcal{Z}_m(x)) < \sigma^m$, then

$$d(T^k x, X \setminus \mathcal{Z}_1(T^k x)) < \sigma^m L^k$$

for some $k < m$. It follows from (22) that

$$\begin{aligned}
&\mu(\{x \in X : d(x, X \setminus \mathcal{Z}_m(x)) < \sigma^m\}) \\
&\quad \le m \max\{\mu(\{x : d(T^k x, X \setminus \mathcal{Z}_1(T^k x)) < \sigma^m L^k\}) : k = 0, \dots, m-1\} \\
&\quad \le Cm(\sigma L)^m,
\end{aligned}$$

using the invariance of the measure. By the Borel–Cantelli lemma we conclude that for μ-almost every $x \in X$ we have $B(x, \sigma^m) \subset \mathcal{Z}_m(x)$ for all sufficiently large m.

By Proposition 5, for μ-almost every $x \in X$ we have

$$1 \le \varliminf_{m\to\infty} \frac{\tau(\mathcal{Z}_m(x))}{m} \le -\log\sigma \varliminf_{m\to\infty} \frac{\tau(B(x,\sigma^m))}{-m\log\sigma} = -\log\sigma \varliminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r}.$$

Since σ can be made arbitrarily close to $1/L$, we obtain the desired statement. □

Proof of Theorem 5. Let $\mathcal{Z}$ be a Markov partition, and consider the new partitions $\mathcal{Z}_m^n = \bigvee_{k=m}^{k=n} f^{-k}\mathcal{Z}$ whenever $m \le n$. There exist constants $c > 0$ and $\lambda > 0$ such that for any cylinder $Z_m^n \subset \mathcal{Z}_m^n$

$$\operatorname{diam}_s Z_m^n \le c e^{+\lambda m} \quad \text{and} \quad \operatorname{diam}_u Z_m^n \le c e^{-\lambda n}, \tag{23}$$

where diam_s and diam_u denote the diameters along the stable and unstable manifolds.

For μ-almost every $x \in X$, we have $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d > 0$ and, by Lemma 5, there exists $\delta > 0$ and $\rho > 0$ such that

$$B(x,r) \cap f^{-k} B(x,r) = \varnothing$$

for every $r < \rho$ and $k \le t_r \stackrel{\text{def}}{=} -\delta \log r$.

Let $r < \rho$ and write $B = B(x,r)$. There exists a countable collection of points $\{x_a\}_{a\in A} \in B$ and integers $m_a < 0 < n_a$ for each $a \in A$ such that if $\mathcal{Z}_{m_a}^{n_a}(x_a) \in \mathcal{Z}_{m_a}^{n_a}$ denotes the unique element containing x_a, then

$$B = \bigcup_{a\in A} \mathcal{Z}_{m_a}^{n_a}(x_a) \pmod 0,$$

in such a way that the union is disjoint (mod 0). We may also assume, without loss of generality, that $\min\{-m_a, n_a\} \geq t_r$. Let $k \geq t_r$ and write $p = [k/2] > 0$ and $q = p + 1 - k < 0$. We have

$$B \subset \bigcup_{a\in A} \mathcal{Z}^p_{m_a}(x_a) \stackrel{\text{def}}{=} \mathcal{Z}^p_{-\infty}(x,r) \quad \text{and} \quad B \subset \bigcup_{b\in A} \mathcal{Z}^{n_b}_q(x_b) \stackrel{\text{def}}{=} \mathcal{Z}^{+\infty}_q(x,r).$$

Once can find sets $U \subset A$ and $V \subset A$ such that

$$\mathcal{Z}^p_{-\infty}(x,r) = \bigcup_{a\in U} \mathcal{Z}^p_{m_a}(x_a) \quad \text{and} \quad \mathcal{Z}^{+\infty}_q(x,r) = \bigcup_{b\in V} \mathcal{Z}^{n_b}_q(x_b),$$

with the two unions being disjoint (mod 0). Since

$$f^{-k}\mathcal{Z}^{n_b}_q(x_b) = \mathcal{Z}^{n_b+k}_{p+1}(f^{-k}x_b)$$

we obtain

$$B \cap f^{-k}B \subset \bigcup_{a\in U, b\in V} [\mathcal{Z}^p_{m_a}(x_a) \cap \mathcal{Z}^{n_b+k}_{p+1}(f^{-k}x_b)].$$

The Gibbs property of the measure implies that there exists a constant $\kappa > 0$ such that

$$\begin{aligned}
\mu(B \cap f^{-k}B) &\leq \sum_{a\in U, b\in V} \mu(\mathcal{Z}^p_{m_a}(x_a) \cap \mathcal{Z}^{n_b+k}_{p+1}(f^{-k}x_b)) \\
&\leq \kappa \sum_{a\in U, b\in V} \mu(\mathcal{Z}^p_{m_a}(x_a))\mu(\mathcal{Z}^{n_b+k}_{p+1}(f^{-k}x_b)) \\
&\leq \kappa \sum_{a\in U, b\in V} \mu(\mathcal{Z}^p_{m_a}(x_a))\mu(\mathcal{Z}^{n_b}_q(x_b)) \\
&\leq \kappa\mu(\mathcal{Z}^p_{-\infty}(x,r))\mu(\mathcal{Z}^{+\infty}_q(x,r)).
\end{aligned} \tag{24}$$

Setting

$$r_k = \max\{\operatorname{diam}_s \mathcal{Z}^{+\infty}_q(x,r), \operatorname{diam}_u \mathcal{Z}^p_{-\infty}(x,r)\}$$

it follows from (23) that $r_k \leq r + ce^{-\lambda k/2}$. Furthermore, for μ-almost every $x \in X$ (in fact for all Lyapunov regular points) we have

$$\mathcal{Z}^p_{-\infty}(x,r) \subset \Phi_x(D^s(x,2r) \times D^u(x,2r_k)) \tag{25}$$

and

$$\mathcal{Z}^{+\infty}_q(x,r) \subset \Phi_x(D^s(x,2r_k) \times D^u(x,2r)) \tag{26}$$

for all sufficiently small $r > 0$. Since $k \geq t_r$, we have $r_k \leq r + cr^{\lambda\delta/2} \leq r^g$ where $g = \lambda\delta/3 > 0$ (provided that r is sufficiently small). Proposition 4 together with (24), (25), and (26), and the main theorem in [1] yield

$$\begin{aligned}
\mu(B \cap f^{-k}B) &\leq \kappa\mu(\Phi_x(D^s(x,2r) \times D^u(x,2r^g))) \times \\
&\quad \times \mu(\Phi_x(D^s(x,2r^g) \times D^u(x,2r))) \\
&\leq \kappa r^{-2\varepsilon(r)}\mu^s_x(B^s(x,r))\mu^u_x(B^u(x,r^g)) \times \\
&\quad \times \mu^s_x(B^s(x,r^g))\mu^u_x(B^u(x,r)) \\
&\leq \kappa r^{-4\varepsilon(r)}\mu(B(x,r))\mu(B(x,r^g)) \\
&\leq \kappa\mu(B)r^{-5\varepsilon(r)}r^{gd},
\end{aligned}$$

where $\varepsilon(r) > 0$ and $\varepsilon(r) \to 0$ as $r \to 0$. Set now

$$k_r = \left[\frac{2}{L}(\log c - \log r)\right] + 1.$$

If $k \geq k_r$ then $r_k \leq 2r$, and a similar argument establishes that

$$\mu(B \cap f^{-k}B) \leq \kappa\mu(B)r^{-5\varepsilon(r)}r^d.$$

Then for any $s_r > 0$, summing the estimate above as k runs from t_r through s_r we obtain

$$\begin{aligned}\frac{\mu(\{y \in B(x,r) : \tau_r(y,x) \leq s_r\})}{\mu(B(x,r))} &\leq \sum_{k=t_r}^{k_r-1} \kappa r^{-5\varepsilon(r)}r^{gd} + \sum_{k=k_r}^{s_r} \kappa r^{-5\varepsilon(r)}r^d \\ &\leq \kappa r^{-5\varepsilon(r)}(k_r r^{gd} + s_r r^d).\end{aligned}$$

By rechoosing $\varepsilon(r)$ if necessary, we may assume that $\mu(B(x,r)) \geq r^{d+\varepsilon(r)}$ for all sufficiently small r. Setting $s_r = \mu(B(x,r))^{-1+\varepsilon}$ we obtain

$$\frac{\mu(\{y \in B(x,r) : \tau_r(y,x) \leq s_r\})}{\mu(B(x,r))} \leq \kappa r^{-5\varepsilon(r)}[(-\delta\log r)r^{gd} + r^{(d+\varepsilon(r))(-1+\varepsilon)}r^d],$$

provided that ε is sufficiently small. Since $gd > 0$, $\varepsilon d > 0$ and $\varepsilon(r) \to 0$ as $r \to 0$ we conclude that (9) holds. Since ε is arbitrarily small, the measure μ has long return time.

The second statement is now an immediate consequence of Theorem 4 and Proposition 3, together with Young's criteria (see [13]). □

Proof of Theorem 6. It follows immediately from Proposition 17 in [2] that there exists a constant $c > 1$ such that if $y = (x,s) \in Y \setminus (X \times \{0\})$ and $r > 0$ is sufficiently small (possibly depending on x) then

$$B(x,r/c) \times I(s,r/c) \subset B_Y(y,r) \subset B(x,cr) \times I(s,cr), \tag{27}$$

where $I(s,r) = (s-r, s+r) \subset \mathbb{R}$. Therefore

$$\mu(B(x,r/c)2r/c \leq \nu(B_Y(y,r)) \leq \mu(B(x,cr))2cr$$

for all sufficiently small r, and thus

$$\underline{d}_\nu(y) = \underline{d}_\mu(x) + 1 \quad \text{and} \quad \overline{d}_\nu(y) = \overline{d}_\mu(x) + 1.$$

It follows from Proposition 3 that $\underline{d}_\nu(y) = \overline{d}_\nu(y) = \dim_H \mu + 1$ for ν-almost every $y \in Y$, and applying Young's criteria (see [13]) we obtain $\dim_H \nu = \dim_H \mu + 1$.

By (27) we obtain

$$\tau_r^\Psi(y) \leq \inf\{t > \rho_r(y) : \psi_t y \in B(x,r/c) \times \{s\}\} + r/c,$$

$$\tau_r^\Psi(y) \geq \inf\{t > \rho_r(y) : \psi_t y \in B(x,cr) \times \{s\}\} - cr,$$

and hence

$$\sum_{k=0}^{\tau_{cr}(x)-1} \varphi(T^k x) - cr \leq \tau_r^\Psi(y) \leq \sum_{k=0}^{\tau_{r/c}(x)-1} \varphi(T^k x) + r/c \tag{28}$$

for all sufficiently small $r > 0$ (possibly depending on x). By Theorem 5, given $\varepsilon > 0$ we have $r^{-d+\varepsilon} < \tau_r(x) < r^{-d-\varepsilon}$ for all sufficiently small r, where $d = \dim_H \mu$. By (28) and the ergodicity of μ, we obtain

$$(cr)^{-d+\varepsilon}\left(\int_X \varphi\, d\mu - \varepsilon\right) - cr \le \tau_r^\Psi(y) \le (r/c)^{-d-\varepsilon}\left(\int_X \varphi\, d\mu + \varepsilon\right) + r/c$$

for all sufficiently small r. Therefore

$$\underline{R}^\Psi(y) = \overline{R}^\Psi(y) = \lim_{r\to 0} \frac{\log \tau_r^\Psi(y)}{-\log r} = d = \dim_H \mu$$

for ν-almost every $y \in Y$. This completes the proof of the theorem. □

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