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Absolutely Continuous Invariant Measures for Multidimensional Expanding Maps

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Abstract

We investigate the existence and statistical properties of absolutely continuous invariant measures for multidimensional expanding maps with singularities. The key point is the establishment of a spectral gap in the spectrum of the transfer operator. Our assumptions appear quite naturally for maps with singularities. We allow maps that are discontinuous on some extremely wild sets, the shape of the discontinuities being completely ignored with our approach.

Key-Words: Expanding maps with singularities, Invariant measure, Decay of correlations, Perron-Frobenius operator.

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Contents

1 Introduction

We consider piecewise invertible expanding maps T on some compact subset Ω of $\mathbb{R}^N$. The transformation T is locally uniformly expanding. However, due to the presence of persistent singularities, it may not satisfy some nice combinatorial behavior (like Markov partition, finite range structure, etc.). Therefore abstract dynamical coding like symbolic dynamics will not be considered.

The main focus of this paper is to prove the existence of a T-invariant probability measure absolutely continuous with respect to the Lebesgue measure m (in short ACIM). One way to find such a measure (and its properties) is to study the spectrum of the Perron-Frobenius (PF) or transfer operator, defined by the dynamic.

This approach has been successfully carried out in the one dimensional case. It is now proven that any piecewise monotonic map of the interval which is eventually expanding[1] —provided that T has some smoothness— possesses an ACIM and the dynamic decomposes very simply into "chaotic" elements.

Despite several attempts, the same question is still open in the multidimensional case. The unexpected difficulties are essentially due to the following facts :

(1) One has to find a functional space rich enough to contain the density (which may be discontinuous) but not too wide (to give PF a nice spectral decomposition).

[1]A map T is called eventually expanding when the inverse of the Jacobian matrix of some iterate T^n of the map has a norm strictly bounded by one. For maps of the interval, this property reduces to the fact that for some $n > 0$, DT^n is bounded from below by some constant strictly larger than one.

(2) In more than one dimension, the geometry of the dynamical partition becomes a crucial ingredient.

Point (1) has been extensively solved with functions of bounded variation[2] and some adaptation of them. This method, although highly powerful in dimension one, led to difficult problems in our case, essentially coming from point (2).

The analysis of PF requires, in the non-Markovian case, a combination of an extension and a trace theorem. This is one of the main problem in the multidimensional case[3]. The main difficulty comes from the fact that the variation does not control the supremum of a function. This point is the source of most of the problems; the need to control the integral of the function (actually, its trace on the surface) along a codimension one smooth surface appears to be unavoidable. Here is the point where assumptions on the shape of the partition arise (in particular, codimension one, piecewise $\mathcal{C}^{(2)}$ smoothness of the boundary and absence of cusps). We have to mention [Za], where the author avoids this problem, essentially using the fact that from outside, cusps can be simply forgotten (see fig. 1). But the difficulty remains if "the outside" does not exist, like in the example by [BGP], where the persistence of the problem is clearly identified.[4]

We propose here for point (1) to work with a functional space introduced in this context by G. Keller [Ke] for one-dimensional maps. This space indeed is still relevant in the multidimensional case, at least for absolutely continuous measures. In [Bl] M. Blank worked with this space and gave similar results with an ad hoc hypothesis (unfortunately, this hypothesis implies the same kind of restrictions that one has when working with bounded variation functions) on the dynamical partition.

Using a different approach, we are able to state a theorem with highly relaxed hypotheses on the dynamical partition. This allows us to deal with maps that are discontinuous on some *very irregular sets*, even on fractal sets (see Example 2.1). Moreover, *general conditions* are given for a map with piecewise $\mathcal{C}^{(1)}$ domains to enjoy the required properties.

Spectral properties shown for the PF operator ensure immediately the well known decomposition into a finite number of ergodic components (of positive Lebesgue measure), decomposable into mixing components for some

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[2]In more than one dimension, the notion of bounded variation does not rely on any ordered structure. Bounded variation functions are functions for which the derivative, in the sense of distributions, is a measure whose total variation is finite.

[3]Also in dimension one if the Darboux property does not holds (a map satisfies the Darboux property whenever the image of an interval is again an interval).

[4]The title of the paper, "Inadequacy of the bounded variation technique..." does not leave any doubt !

iterates of the map.

As a byproduct, we obtain that on these components, Hölder observables are *mixed exponentially* fast by the map.

A great property of the function space involved in this paper gives, without much effort, a *constructive upper bound* on the number of positive Lebesgue measure *ergodic components* (see Theorem 5.2).

We also investigate the question of *computing an upper bound for the rate of decay* of the correlation function, for Hölder observables. A method is presented where such a bound is computed; the strategy, already involved in [L2] for expanding maps of the interval, in [LSV1] for one dimensional equilibrium states and in [L1] for multidimensional symplectic maps, is to define a projective metric space, on which PF is a contraction, the contracting factor being computable.

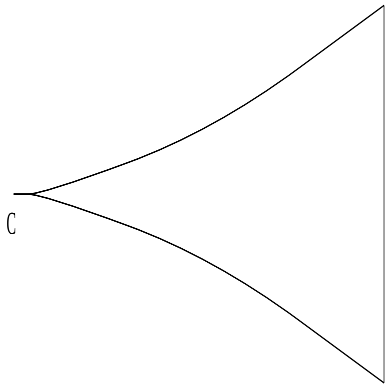

Figure 1 : An example of a domain containing a cusp in C.

Some notations

Given a Borel subset S of $\mathbb{R}^N$, we denote respectively its closure, its interior and its boundary by $\mathrm{clos}(S), \mathrm{int}(S)$ and $\partial S = \mathrm{clos}(S) \backslash \mathrm{int}(S)$.
For a function $f \in L^\infty$, $\operatorname*{Esup}_S f$ and $\operatorname*{Einf}_S f$ will represent the essential supremum and infimum of f on B with respect to the Lebesgue measure. We denote by $B_\varepsilon(x)$ the open ball of radius ε and center x, in the Euclidean metric d.
Moreover, given a real number $a > 0$, we write $B_a(S) = \{x \in \mathbb{R}^N | d(x, S) \leq a\}$.
In addition, we write indifferently $m(f) = \int f(x)dx = \int f dm$ for all $f \in L^1_m$.
$\gamma_N = \dfrac{\pi^{N/2}}{(N/2)!}$ denotes the N-volume of the N-dimensional unit ball of $\mathbb{R}^N$.

We now introduce the maps under consideration in this paper

2 Piecewise expanding maps

Let Ω be a compact subset of $\mathbb{R}^N$, with $\operatorname{clos}(\operatorname{int}(\Omega)) = \Omega$, and $T : \Omega \to \Omega$. Let us give our definition of a piecewise expanding map. We assume that there exists an at most countable family of disjoint open sets $U_i \subset \Omega$ and V_i such that $\operatorname{clos}(U_i) \subset V_i$, and maps $T_i : V_i \to \mathbb{R}^N$ satisfying for some $0 < \alpha \leq 1$ and some small enough $\varepsilon_0 > 0$:

(PE1) for all i, $T_{|U_i} = T_{i|U_i}$ and $T_i(V_i) \supset B_{\varepsilon_0}(TU_i)$,

(PE2) for all i, $T_i \in \mathcal{C}^{(1)}(V_i)$, T_i injective and $T_i^{-1} \in \mathcal{C}^{(1)}(T_iV_i)$. Moreover, the determinant is uniformly Hölder: for all i, $\varepsilon \leq \varepsilon_0$, $z \in T_iV_i$ and $x, y \in B_\varepsilon(z) \cap T_iV_i$ hold

$$|\det D_xT_i^{-1} - \det D_yT_i^{-1}| \leq c|\det D_zT_i^{-1}|\varepsilon^\alpha. \tag{1}$$

(PE3) $m(\Omega \backslash \bigcup_i U_i) = 0$,

(PE4) there exists $s = s(T) < 1$ such that for all $u, v \in TV_i$ such that $d(u, v) \leq \varepsilon_0$ we have $d(T_i^{-1}u, T_i^{-1}v) \leq sd(u, v)$.

(PE5) Let $G(\varepsilon, \varepsilon_0) := \sup_x G(x, \varepsilon, \varepsilon_0)$ where

$$G(x, \varepsilon, \varepsilon_0) := \sum_i \frac{m\big(T_i^{-1}B_\varepsilon(\partial TU_i) \cap B_{(1-s)\varepsilon_0}(x)\big)}{m\big(B_{(1-s)\varepsilon_0}(x)\big)},$$

and assume that η defined by $\eta(\varepsilon_0) := s^\alpha + 2\sup_{\varepsilon \leq \varepsilon_0} \frac{G(\varepsilon)}{\varepsilon^\alpha}\varepsilon_0^\alpha$ is such that $\sup_{\delta \leq \varepsilon_0} \eta(\delta) < 1$.

Remark 2.1. *1. We emphasize the fact that Ω nor the U_i's do not have to be connected.*

2. When the family U_i is finite and (PE2) is true, (PE4) is equivalent to the following property:

$$\text{There exists } s < 1 \text{ such that } \sup_i \sup_{x \in TV_i} \|D_xT_i^{-1}\| < s.$$

where $\|\cdot\|$ stands for the Euclidean norm.

3. (PE5) may look very hard to obtain. Basically, it implies that typical trajectories will not feel the discontinuities, so that the dynamic is essentially a smooth one. In example 2.1 we will compute this quantity for some maps discontinuous on a fractal set. Note that this condition implies that the boundaries are at least of co(box)dimension α. In Lemma 2.1, we will give explicit conditions to get (PE5) in the case of piecewise $\mathcal{C}^{(1)}$ boundaries. Fig. 2 gives an example of some domains U_i with piecewise smooth boundaries ($G(x,\varepsilon)$ is the sum of the three areas between the dotted lines, divided by the area of the disk $B(x)$).

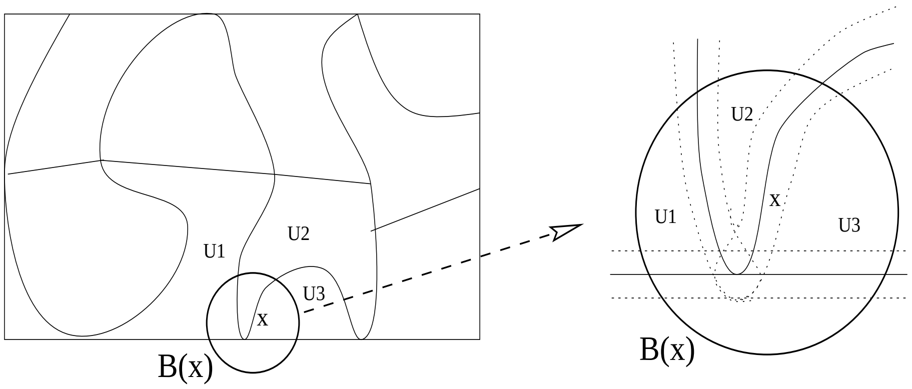

Figure 2 : Typical domains U_i with neighborhood of boundaries.

With these assumptions, it is easy to see that the Perron-Frobenius operator is well defined on L^1_m, and reads like :

$$Ph = \sum_i (gh) \circ T_i^{-1} 1\!\!1_{TU_i},$$

where the weight g is given by $g = \dfrac{1}{|\det DT|}$. We recall that P satisfies for all $h \in L^1, f \in L^\infty$,

$$\int_\Omega f \circ Thdm = \int_\Omega fPh\, dm.$$

Moreover, $h \in L^1$ is an invariant density of an ACIM if and only if h is a positive eigenvector of P with eigenvalue 1.

The following lemma concerns the case of a finite partition $\mathcal{Z} = \{U_i\}$ with piecewise smooth boundaries. If this is so, the computation of $G(\varepsilon)$ can be made without much trouble and the hypothesis (PE5) becomes extremely clear.

Lemma 2.1. *Let T be a map which satisfies (PE1)–(PE4) with a finite family of open sets U_i. Suppose further that the boundary of the U_i are included* *in piecewise C^1 codimension one embedded compact submanifolds, and let Y be the maximal number of these smooth components that can meet in one point, i.e.*

$$Y(T) := \sup_{x\in\mathbb{R}^N} \sum_i \#\left\{\textit{smooth pieces intersecting } \partial U_i \textit{ and containing } x\right\}.$$

Setting $\eta_0(T) := s(T)^\alpha + \dfrac{4s(T)}{1-s(T)} Y(T) \dfrac{\gamma_{N-1}}{\gamma_N}$, *we have the following:*

If $\eta_0(T) < 1$ then (PE5) holds.

Proof. Let $x \in \mathbb{R}^N$ and consider an element U_i of the partition. We need to compute the Lebesgue measure of the set

$$T_i^{-1}(B_\varepsilon(\partial TU_i)) \cap B_{(1-s)\varepsilon_0}(x). \tag{2}$$

By (PE4), we know that $T_i^{-1}(B_\varepsilon(\partial TU_i)) \subset B_{s\varepsilon}(\partial U_i)$. Moreover, $\partial U_i \subset \cup_j \Gamma_{ij}$ where Γij are compact $C^{(1)}$ embedded submanifolds, hence $T_i^{-1}(B_\varepsilon(\partial TU_i)) \subset \cup_j B_{s\varepsilon}(\Gamma_{ij})$. We can now estimate (2) by looking at each smooth component Γ_{ij} of the boundary separately. The aim is then to compute the following quantity :

$$m(B_\nu(\Gamma) \cap B_\delta(x)), \tag{3}$$

for $\delta = (1-s)\varepsilon_0$, $\Gamma = \Gamma_{ij}$ and $\nu = s\varepsilon < \delta$, where ε_0 will be fixed later. If x does not belong to the ν-neighborhood of Γ then nothing has to be done. Otherwise, let us consider the preferred local coordinate map Φ of the embedded submanifold Γ. We can suppose $\Phi : B_\delta(x) \to \mathbb{R}^N$ for some δ independent of x (remember that Γ is compact). We recall that $\Phi(\Gamma)$ is contained in a hyperplane H of $\mathbb{R}^N$. To simplify, let us choose Φ such that $D_x\Phi = I$.

The following estimates hold for $y \in B_\delta(x)$ ($|\cdot|$ represents the Euclidean norm in $\mathbb{R}^N$) :

$$|\Phi(y) - \Phi(x)| \le |D_x\Phi(y-x)| + o(\delta) \le \delta(1 + o(1)),$$

hence ($o(1)$ stands for a function which vanishes with δ)

$$\Phi(B_\delta(x)) \subset B_{\delta(1+o(1))}(\Phi(x)).$$

Moreover, given $z \in \Gamma$ and $y \in B_\nu(z) \cap B_\delta(x)$ we have

$$|\Phi(y) - \Phi(z)| \le |D_z\Phi(y-z)| + o(\nu) \le (1 + o(\delta))\nu + o(\nu),$$

which implies

$$\Phi(B_\nu(\Gamma)) \subset B_{\nu(1+o(\delta))+o(\nu)}(H) \subset B_{\nu(1+o(1))}(H).$$

Since the coordinate map is a $\mathcal{C}^{(1)}$ perturbation of a translation, it changes volumes by a factor $1+o(1)$. This means that (3) is bounded by $1+o(1)$ times the measure of $B_{\delta(1+o(1))}(\Phi(x)) \cap B_{\nu(1+o(1))}(H)$. It is obviously maximal when the hyperplane H crosses the center of the ball. In this case, the quantity can be easily estimated by $2\nu(1+o(1))$ times the $(N-1)$ volume of the $(N-1)$-dimensional ball of radius $\delta(1+o(1))$. Hence

$$(3) \leq 2\nu\gamma_{N-1}\delta^{N-1}(1+o(1)). \tag{4}$$

This inequality is sufficient for each smooth piece.

Since there are only finitely many Γ_{ij}, it is possible to get all the $o(1)$ uniform in x and Γ_{ij}. Moreover, if δ is chosen small enough, then any ball of radius δ intersects at most Y smooth pieces of the boundary. Consequently, if ε_0 is small enough

$$G(\varepsilon, \varepsilon_0) \leq Y2\frac{\gamma_{N-1}}{\gamma_N}\frac{s\varepsilon}{(1-s)\varepsilon_0}(1+o(1)).$$

Finally, (PE5) will hold provided $\eta_0(T) < 1$ and ε_0 is small enough. □

The hypothesis (PE5) can be stated in a very nice way, showed to us by J.Buzzi, if one is not interested in computing the exact values of the constants entering into the inequalities. We first define the multiplicity entropy $\mathrm{H}_{mult}(\mathcal{Z}, T)$ of the partition $\mathcal{Z} = \{U_i\}$. Denote by $\mathcal{Z}^{(n)} = \mathcal{Z} \bigvee T^{-1}\mathcal{Z} \bigvee \cdots \bigvee T^{-n+1}\mathcal{Z}$ the dynamical partition.

$$\mathrm{H}_{mult}(\mathcal{Z}, T) := \lim_{n\to\infty} \frac{1}{n} \log \sup_x \# \left\{ Z \in \mathcal{Z}^{(n)} \;\middle|\; x \in \mathrm{clos}(Z) \right\}$$

Next, define the dilatation coefficient $\delta(T)$ of the map T by

$$\delta(T) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x \in T^n(\Omega)} \|D_x T^{-n}\|,$$

where the norm of the derivative is taken along all the smooth branches of T^{-n}. Then we have the following criterion for the hypothesis (PE5):

Lemma 2.2. *Let T be a piecewise invertible $C^{(1)}$ map with a partition into smooth components $\mathcal{Z}$ such that (PE1)–(PE3) hold for some $0 < \alpha \leq 1$. Suppose that the boundary of the partition is included in a finite number of $C^{(1)}$ compact embedded submanifolds.*

If $H_{mult}(\mathcal{Z}, T) + \delta(T) < 0$ then some iterate of the map satisfies (PE1)–(PE5).

Proof. Clearly properties (PE1)–(PE3) are still true for the iterates of the map T. Moreover, since $\delta(T) < 0$, also (PE4) will be true for n big enough and ε_0 sufficiently small.

Let us consider now (PE5). Let S be the number of smooth pieces containing the boundary of the partition $\mathcal{Z}$. Since for each i the map T_i is smooth in some neighborhood of U_i, a simple induction shows that the number of smooth pieces containing the boundary of an element Z of the partition $\mathcal{Z}^{(n)}$ is bounded by Sn.

It follows that the constant $Y(T^n)$ in Theorem 2.1 is bounded by

$$Sn \times \sup_x \# \left\{ Z \in \mathcal{Z}^{(n)} \;\middle|\; x \in \mathrm{clos}(Z) \right\}.$$

Moreover, the contraction $s(T^n)$ goes to zero like $\exp(n\delta(T))$. This implies that for n big enough $\eta_0(T^n) < 1$. Lemma 2.1 applies and concludes the proof. $\square$

Remark 2.2. *We want to stress that there is no need to control the angles between smooth elements of the partition, contrary to other methods. In the case of an affine mod $\mathbb{Z}^N$ map of the unit cube, $Y(T^n)$ grows only polynomially with n (see [B1]), ensuring that $H_{mult}(\mathcal{Z}, T) = 0$, hence (PE5) is true for some iterate. This result has been also used in [B2] to ensure the existence of* ACIM *for piecewise analytic expanding maps of the unit square.*

Remark 2.3. *The fact that the pieces of the boundaries are embedded submanifolds is not essential to get (PE5), but it is unclear how to give a general statement which allows singular points (like edge of cones, etc...).*

Let us give now an example of a fractal boundary which could satisfy the hypotheses.

Example 2.1. *Let us consider a map T on $\Omega = [0,1]^2$ which is discontinuous on a von Koch's fractal set Γ. The construction of Γ is done as follows: start from the diagonal D of the unit square. Then, cut the diagonal into three segments of equal length, and replace the one in the middle by two segments of the same length, as shown in Fig. 3. Repeat this procedure with each small segments, and so on... Call Γ the set so obtained.*

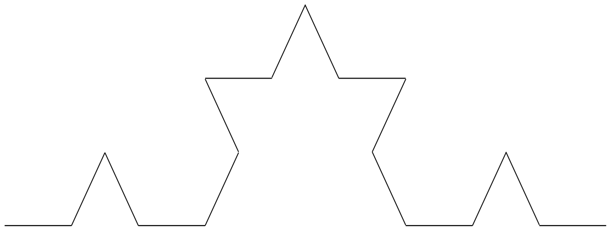

Figure 3 : Third step of the construction of Γ.

The contribution of the boundary Γ *to* $G(\varepsilon,\varepsilon_0)$ *(in property (PE5)) is such that for all* $\alpha \leq \operatorname{codim}(\Gamma) = 2 - \dfrac{\log 4}{\log 3}$ *we have*

$$\sup_{\varepsilon\leq\varepsilon_0}\sup_x \frac{m(B_{s\varepsilon}(\Gamma)\cap B_{(1-s)\varepsilon_0}(x))}{m(B_{(1-s)\varepsilon_0}(x))}\frac{\varepsilon_0^\alpha}{\varepsilon^\alpha} \leq \frac{2^{\beta+1}(1+2/\pi)}{(1-s)^{2-\beta}}s^{2-\beta}.$$

Proof. Let $\beta = \frac{\log 4}{\log 3}$ (it is the fractal dimension of Γ). Let $\delta > 0$ be small enough, $\nu < \delta$ and $\alpha \leq 2-\beta$. We want to compute the quantity (3) (see Lemma 2.1). Let $n = [\frac{\log \nu^{-1}}{\log 3}] + 1$. Since our accuracy is ν, it suffices to consider the n-step approximation of the fractal Γ. At the order n, each segment is of length $(1/3)^n$, and there is at most $4^n(2\delta)^\beta$ segments in a ball of size δ. It follows that

$$(3) \leq 2^\beta[(1/3)^n 2\nu + \pi\nu^2]4^n\delta^\beta \leq 2^\beta(2+\pi)\nu^{2-\beta}\delta^\beta.$$

So the measure of the intersection divided by the measure of the ball of radius δ is bounded by

$$2^\beta(1+2/\pi)\delta^{\beta-2}\nu^{2-\beta}.$$

Hence the contribution to $G(\varepsilon,\varepsilon_0)$ from the boundary Γ is bounded by

$$2^\beta(1+2/\pi)(\frac{s}{1-s})^{2-\beta}(\frac{\varepsilon}{\varepsilon_0})^{2-\beta}, \tag{5}$$

which finishes the proof. □

This shows that for such a map T *it is possible that (PE5) holds, provided that* s *is small enough.*

3 Quasi-Hölder space

We now introduce the functional space on which we will study the spectrum of the operator PF.

Let $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$ be real numbers, and $f \in L^1_m(\mathbb{R}^N, \mathbb{R})$. For a Borel subset S of $\mathbb{R}^N$, we define the oscillation of f on S by

$$\operatorname{osc}\big(f,S\big) = \operatorname*{Esup}_S f - \operatorname*{Einf}_S f.$$

By definition of the oscillation, it is easy to prove :

Proposition 3.1. *For each $f \in L^1_m(\mathbb{R}^N, \mathbb{R})$, the function $x \to \operatorname{osc}\big(f, B_\varepsilon(x)\big)$ is lower semi-continuous, hence measurable.*

Remark 3.1. *Although functions in L^1 are only defined almost everywhere, the oscillation is a real positive function defined everywhere on $\mathbb{R}^N$. In addition,* $\operatorname{supp} \operatorname{osc}\big(f, B_\varepsilon(\cdot)\big) \subset B_\varepsilon(\operatorname{supp} f)$.

By proposition 3.1, we can define

$$|f|_\alpha = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{R}^N} \operatorname{osc}\big(f, B_\varepsilon(x)\big)\, dx.$$

Remark 3.2. *Although it is not explicitly written, it is important to remember that $|f|_\alpha$ may depend on ε_0. However, the sets V_α do not depend on ε_0.*

We define now

$$V_\alpha = \left\{ f \in L^1_m(\mathbb{R}^N, \mathbb{R}) \;\middle|\; |f|_\alpha < \infty \right\}.$$

It is clear that any compactly supported α-Hölder continuous function belongs to V_α, but this space is bigger, since functions in V_α may have discontinuities[5].

Proposition 3.2. *Let $f, f_i, g \in L^\infty(\mathbb{R}^N, \mathbb{R})$, g positive, $0 < a, b, c$ and S a Borel subset of $\mathbb{R}^N$. The oscillation has the following properties :*

(i) $\operatorname{osc}\big(\sum_i f_i, B_a(\cdot)\big) \leq \sum_i \operatorname{osc}\big(f_i, B_a(\cdot)\big)$,

(ii) $\operatorname{osc}\big(f 1\!\!1_S, B_a(\cdot)\big) \leq \operatorname{osc}\big(f, S \cap B_a(\cdot)\big) 1\!\!1_S(\cdot) + 2 \left[\operatorname*{Esup}_{B_a(\cdot)\cap S} |f| \right] 1\!\!1_{B_a(S)\cap B_a(S^c)}(\cdot)$,

(iii)

$$\operatorname{osc}\big(fg, S\big) \leq \operatorname{osc}\big(f, S\big) \operatorname*{Esup}_S g + \operatorname{osc}\big(g, S\big) \operatorname*{Einf}_S |f|,$$

(iv) if $a + b \leq c$ then for all $x \in \mathbb{R}^N$ we have

$$\operatorname*{Esup}_{B_a(x)} f \leq \frac{1}{m(B_b(x))} \int_{B_b(x)} \big[f(y) + \operatorname{osc}\big(f, B_c(y)\big) \big]\, dy.$$

[5] We recall that we cannot avoid discontinuities, since we do not ask any Markovian property on T.

Proof. (i) is trivial.

(ii) Let $x \in \mathbb{R}^N$. if x is such that $d(x, S) < a$ and $d(x, S^c) < a$ then

$$\operatorname{osc}\big(f 1\!\!I_S, B_a(x)\big) \leq 2 \sup_{B_a(x) \cap S} |f|.$$

Otherwise, the ball $B_a(x)$ and S are disjoint, hence the oscillation of $f 1\!\!I_S$ is null, or the ball is included in S and then $f 1\!\!I_S = f$ on the ball. Whence the result.

(iii) If the sign of f does not change on S, we can suppose that $f \geq 0$,

$$\begin{aligned} \operatorname{osc}\big(fg, S\big) &\leq \operatorname*{Esup}_S f \operatorname*{Esup}_S g - \operatorname*{Einf}_S f \operatorname*{Einf}_S g \\ &\leq \operatorname*{Esup}_S g(e\sup_S f - \operatorname*{Einf}_S f) + \operatorname*{Einf}_S f(\operatorname*{Esup}_S g - \operatorname*{Einf}_S g). \end{aligned}$$

If the sign of f does change then

$$\operatorname*{Esup}_S fg - \operatorname*{Einf}_S fg = \operatorname*{Esup}_S fg + \operatorname*{Esup}_S -fg \leq \operatorname*{Esup}_S g(\operatorname*{Esup}_S f + \operatorname*{Esup}_S -f).$$

(iv) Let $x \in \mathbb{R}^N$ be fixed. For all $y \in B_b(x)$ we have $B_a(x) \subset B_c(y)$, hence almost everywhere,

$$\operatorname*{Esup}_{B_a(x)} f \leq \operatorname*{Esup}_{B_c(y)} f \leq f(y) + \operatorname{osc}\big(f, B_c(y)\big).$$

Which yields the result by integration over $y \in B_b(x)$. $\square$

Using standard functional analysis, the following can be shown (see [Ke] for more properties).

Proposition 3.3. *If we put the norm $\|\cdot\|_\alpha = \|\cdot\|_{L^1_m} + |\cdot|_\alpha$ then V_α becomes a Banach space. In addition, since Ω is compact, the intersection of the unit ball of V_α with the set of functions supported on Ω is compact in $L^1_m(\mathbb{R}^N)$.*

Proposition 3.4. *V_α is continuously injected in L^∞_m,*

$$\forall f \in V_\alpha, \quad \|f\|_{L^\infty} \leq \frac{\max(1, \varepsilon_0^\alpha)}{\gamma_N \varepsilon_0^N} \|f\|_\alpha. \tag{6}$$

Moreover, V_α is an algebra with the usual sum and product of essentially bounded functions

$$\forall f, g \in V_\alpha, \quad \|fg\|_\alpha \leq \frac{2\max(1, \varepsilon_0^\alpha)}{\gamma_N \varepsilon_0^N} \|f\|_\alpha \|g\|_\alpha. \tag{7}$$

Proof. Proposition 3.2.iv gives for all $a > 0$ and $b = \varepsilon_0 - a$

$$\|f\|_{L^\infty} \leq \frac{\max(1, \varepsilon_0^\alpha)}{\gamma_N(\varepsilon_0 - a)^N}\|f\|_\alpha.$$

which shows inequality (6) letting $a \to 0$.

By Proposition 3.2.iii and Hölder inequality we get that

$$\|fg\|_\alpha \leq |f|_\alpha \|g\|_{L^\infty} + \|f\|_{L^1}|g|_\alpha + \|f\|_{L^1}\|g\|_{L^\infty}$$

which gives by inequality (6)

$$\|fg\|_\alpha \leq \frac{2\max(1, \varepsilon_0^\alpha)}{\gamma_N \varepsilon_0^N}.$$

□

Lemma 3.1. *For every positive $h \in V_\alpha$, $h \neq 0$, there exists a ball on which the infimum of h is strictly positive. The radius ε of the ball can be taken as*

$$\varepsilon = \min\left(\varepsilon_0, \left(\frac{\int h dm}{|h|_\alpha}\right)^{\frac{1}{\alpha}}\right).$$

Proof. Let $0 \leq h \in V_\alpha$, $h \neq 0$ and ε as in the Lemma. We suppose that h is not constant (otherwise the lemma is proven) We claim that the infimum of h is strictly positive on some ball of radius ε. Otherwise, the infimum of h on every ball would be null, and this would imply (the first inequality being strict, for h is not constant)

$$\int h(x)dx < \int \operatorname*{Esup}_{B_\varepsilon(x)} h dx = \int \operatorname{osc}\big(h, B_\varepsilon(x)\big)dx \leq |h|_\alpha \varepsilon^\alpha,$$

which is contradictory by our choice of ε. □

4 A Lasota-Yorke type Inequality

In this section we will establish the key inequality, which says that not only does P act continuously on V_α, but also that P is a quasi-compact operator, hence spectral results follow. Notice that we consider functions defined on all of $\mathbb{R}^N$, despite the fact that a density is always supported on Ω. Whereas we could have done the same computations on Ω, we would have found very bad estimates with respect to the dimension or the shape of Ω.[6]

[6] It is just because on $\mathbb{R}^N$, the Lebesgue measure of a ball depends only on its radius, while the volume of $B_\varepsilon(x) \cap \Omega$ may vary a lot when x is near the boundary of Ω.

Lemma 4.1. *Let us assume (PE1)–(PE5). Provided ε_0 is small enough, there exists $\eta < 1$ and $D < \infty$ such that, for $f \in V_\alpha$, $Pf \in V_\alpha$ with the following*

$$|Pf|_\alpha \leq \eta |f|_\alpha + D \int_{\mathbb{R}^N} |f|.$$

Proof. Let $f \in V_\alpha$ and $\varepsilon \leq \varepsilon_0$. For almost every $x \in \mathbb{R}^N$, Properties (PE3) and (PE1) together with Proposition 3.2 yields

$$\begin{aligned}
\operatorname{osc}\big(Pf, B_\varepsilon(x)\big) &\leq \sum_i \operatorname{osc}\big((fg)\circ T_i^{-1} 1\!\!1_{TU_i}, B_\varepsilon(x)\big) \\
&\leq \sum_i \operatorname{osc}\big((fg)\circ T_i^{-1}, TU_i \cap B_\varepsilon(x)\big) 1\!\!1_{TU_i}(x) + \\
&\qquad +2\left[\operatorname*{Esup}_{TU_i\cap B_\varepsilon(x)} |fg|\circ T_i^{-1}\right] 1\!\!1_{B_\varepsilon(\partial TU_i)}(x) \\
&\leq \sum_i \operatorname{osc}\big(fg, U_i \cap T^{-1}B_\varepsilon(x)\big) 1\!\!1_{TU_i}(x) + \\
&\qquad +2\left[\operatorname*{Esup}_{U_i\cap T_i^{-1}B_\varepsilon(x)} |fg|\right] 1\!\!1_{B_\varepsilon(\partial TU_i)}(x).
\end{aligned}$$

We will show now that the right hand side of that inequality has an integral bounded by $\eta|f|_\alpha\varepsilon^\alpha + D\|f\|_{L^1_m}\varepsilon^\alpha$ for some constant $\eta < 1$ and D arbitrarily large.

Let us begin with the first term of the right hand side; For $x \in TU_i$, setting $y_i = T_i^{-1}x$ gives by (PE4)

$$R_i^{(1)}(x) := \operatorname{osc}\big(fg, U_i \cap T_i^{-1}B_\varepsilon(x)\big) \leq \operatorname{osc}\big(fg, U_i \cap B_{s\varepsilon}(y_i)\big),$$

which is by Proposition 3.2.iii, for almost all $x \in TU_i$, less or equal to

$$\begin{aligned}
R_i^{(1)}(x) &\leq \operatorname{osc}\big(f, B_{s\varepsilon}(y_i)\big) \operatorname*{Esup}_{U_i\cap B_{s\varepsilon}(y_i)} g + \operatorname{osc}\big(g, U_i\cap B_{s\varepsilon}(y_i)\big) \operatorname*{Einf}_{B_{s\varepsilon}(y_i)} |f| \\
&\leq (1 + cs^\alpha\varepsilon^\alpha)\operatorname{osc}\big(f, B_{s\varepsilon}(y_i)\big) g(y_i) + |f|(y_i)g(y_i)cs^\alpha\varepsilon^\alpha.
\end{aligned}$$

Here we have used inequality (1). Hence an estimation of the first term can be

$$\sum_i R_i^{(1)} 1\!\!1_{TU_i} \leq (1 + cs^\alpha\varepsilon^\alpha) P\Big(\operatorname{osc}\big(f, B_{s\varepsilon}(\cdot)\big)\Big) + cs^\alpha\varepsilon^\alpha P|f|. \tag{8}$$

Which gives after integration

$$\int_{\mathbb{R}^N} \sum_i R_i^{(1)} 1\!\!1_{TU_i} \leq (1 + cs^\alpha\varepsilon^\alpha) \int \operatorname{osc}\big(f, B_{s\varepsilon}(\cdot)\big) + cs^\alpha\varepsilon^\alpha \int_{\mathbb{R}^N} |f|.$$

By definition of $|f|_\alpha$ we end up with

$$\int_{\mathbb{R}^N} \sum_i R_i^{(1)} 1\!\!I_{TU_i} \leq (1 + cs^\alpha \varepsilon^\alpha)|f|_\alpha s^\alpha \varepsilon^\alpha + cs^\alpha \varepsilon^\alpha \int_{\mathbb{R}^N} |f|. \tag{9}$$

For the second term the computations are more delicate, and need some deeper knowledge about the structure of the partition $\{U_i\}$. This is where the geometrical hypotheses on the domains are used.

$$R_i^{(2)}(x) := \left[\operatorname*{Esup}_{U_i \cap T_i^{-1} B_\varepsilon(x)} |fg| \right] 1\!\!I_{B_\varepsilon(\partial TU_i)}(x).$$

If $x \notin B_\varepsilon(TU_i)$ then $R_i^{(2)}(x) = 0$. Otherwise, by definition of g, (PE4) and inequality (1) we obtain (still y_i is $T_i^{-1}x$)

$$R_i^{(2)}(x) \leq \left[\operatorname*{Esup}_{B_\varepsilon(y_i)} |f| \right] |\det D_x T_i^{-1}|(1 + cs^\alpha \varepsilon^\alpha) 1\!\!I_{B_\varepsilon(\partial TU_i))}(x).$$

An integration over $\mathbb{R}^N$ followed by a change of variable gives

$$\frac{1}{(1 + cs^\alpha \varepsilon^\alpha)} \int_{\mathbb{R}^N} R_i^{(2)}(x)dx \leq \int_{\mathbb{R}^N} 1\!\!I_{B_\varepsilon(\partial TU_i)}(T_i y_i) \operatorname*{Esup}_{B_{s\varepsilon}(y_i)} |f| dy_i. \tag{10}$$

Proposition 3.2.iv (with $a = s\varepsilon$, $b = (1-s)\varepsilon_0$ and $c = \varepsilon_0$) tells that the supremum of $|f|$ is bounded by its oscillation plus its average. This yields (10) less or equal to

$$\int_{\mathbb{R}^N} dy \frac{1\!\!I_{B_\varepsilon(\partial TU_i)}(T_i y)}{m(B_{(1-s)\varepsilon_0}(y))} \int_{B_{(1-s)\varepsilon_0}(y)} \left[|f|(z) + \operatorname{osc}\big(f, B_{\varepsilon_0}(z)\big) \right] dz,$$

which becomes, after changing the order of integration

$$\int dz \left[|f|(z) + \operatorname{osc}\big(f, B_{\varepsilon_0}(z)\big) \right] \int dy \frac{1\!\!I_{T_i^{-1} B_\varepsilon(\partial TU_i)}(y) 1\!\!I_{B_{(1-s)\varepsilon_0}(z)}(y)}{m(B_{(1-s)\varepsilon_0}(y))}.$$

Finally, since the measure of a ball depends only on its radius, we can replace the second integral by $\dfrac{m(T_i^{-1} B_\varepsilon(\partial TU_i) \cap B_{(1-s)\varepsilon_0}(z))}{m(B_{(1-s)\varepsilon_0}(z))}$, and by definition of $G(\varepsilon, \varepsilon_0)$ we get

$$\frac{1}{(1 + cs^\alpha \varepsilon^\alpha)} \int_{\mathbb{R}^N} \sum_i R_i^{(2)}(x)dx \leq G(\varepsilon, \varepsilon_0) \left(\int_{\mathbb{R}^N} \operatorname{osc}\big(f, B_{\varepsilon_0}(\cdot)\big) dm + \int_{\mathbb{R}^N} |f| dm \right).$$

This gives by definition of $|f|_\alpha$,

$$\frac{1}{(1+cs^\alpha\varepsilon^\alpha)}\int_{\mathbb{R}^N}\sum_i R_i^{(2)}(x)dx \le G(\varepsilon,\varepsilon_0)|f|_\alpha\varepsilon_0^\alpha + G(\varepsilon,\varepsilon_0)\int_{\mathbb{R}^N}|f|dm. \quad (11)$$

To conclude, it is sufficient to put together estimates (9) and (11)

$$\begin{aligned}\int_{\mathbb{R}^N}\operatorname{osc}\big(Pf,B_\varepsilon(x)\big)dx &\le (1+c(s\varepsilon)^\alpha)\left[s^\alpha\varepsilon^\alpha+2G(\varepsilon,\varepsilon_0)\varepsilon_0^\alpha\right]|f|_\alpha +\\ &\quad + \left[c(s\varepsilon)^\alpha+2(1+c(s\varepsilon)^\alpha)G(\varepsilon,\varepsilon_0)\right]\int_{\mathbb{R}^N}|f|dm\\ &\le \left(\eta|f|_\alpha + D\int_{\mathbb{R}^N}|f|dm\right)\varepsilon^\alpha.\end{aligned}$$

Where the constant η and D are given by

$$\eta = (1+cs^\alpha\varepsilon_0^\alpha)\left(s^\alpha+2\sup_{\varepsilon\le\varepsilon_0}\frac{G(\varepsilon,\varepsilon_0)\varepsilon_0^\alpha}{\varepsilon^\alpha}\right) = (1+cs^\alpha\varepsilon_0^\alpha)\eta(\varepsilon_0),$$

and

$$D = cs^\alpha + 2(1+cs^\alpha\varepsilon_0^\alpha)\sup_{\varepsilon\le\varepsilon_0}G(\varepsilon)\varepsilon^{-\alpha} = cs^\alpha + (1+cs^\alpha\varepsilon_0^\alpha)\eta(\varepsilon_0)\varepsilon_0^{-\alpha} < \infty.$$

The Lemma is proven by definition of $|\cdot|_\alpha$, provided that the constant ε_0 is so small that $\eta < 1$, which can be achieved because $\limsup_{\varepsilon_0\to 0}\eta(\varepsilon_0) < 1$ by (PE5). $\square$

5 Spectral results

In this section we will present the main result, which is the spectral decomposition of PF for piecewise expanding maps satisfying our hypotheses. This theorem follows from Lemma 4.1 and an ergodic theorem of Ionescu-Tulcea and Marinescu (see [IM]). Its statement, adapted here for our special case, is borrowed from [Ke].

Theorem 5.1. *Under assumptions (PE1)–(PE5) holds :*

(i) $P : L^1_m \to L^1_m$ *has a finite number of eigenvalues* $\lambda_1, \dots, \lambda_r$ *of modulus one.*

(ii) $E_i = \{f \in L^1_m | Pf = \lambda_i f\} \subset V_\alpha$ *and* $\dim(E_i) < \infty$ *for* $i = 1, \dots, r$.

(iii) $P = \sum_{i=1}^{r} \lambda_i P_i + Q$, *where the* P_i *are projections onto the eigenspaces* E_i, $\|P_i\|_1 \le 1$, *and* Q *is a linear operator on* L^1_m *with* $Q(V_\alpha) \subset V_\alpha$, $\sup_{n\in\mathbb{N}} \|Q^n\|_1 < \infty$, *and* $\|Q^n\|_\alpha = O(q^n)$ *for some* $0 < q < 1$. *Furthermore* $P_i P_j = 0$ *if* $i \ne j$ *and* $P_i Q = Q P_i = 0$ *for all* i. *(This means that* P *is quasi-compact as an operator on* $(V_\alpha, \|\cdot\|_\alpha)$*).*

(iv) 1 *is an eigenvalue of* P, *and assuming* $\lambda_1 = 1$ *and* $h_* = P_1 1$, $\mu = h_* m$ *is the greatest* ACIM *for* T, *i.e. if* $\tilde{\mu}$ *is* T*-invariant and* $\tilde{\mu} \ll m$ *then* $\tilde{\mu} \ll \mu$.

(v) The support of $\tilde{\mu}$ *can be decomposed into a finite number of disjoint measurable sets* $W_{j,l}$ *for* $j = 1, ..., \dim(E_1)$ *and* $l = 1, ..., L_j$ *such that* $T(W_{j,l}) = W_{j,l+1 \bmod L_i}$ *and* T^{L_j} *is mixing on each* $W_{j,l}$.

Proof. Since $\left\{ f \in V_\alpha \,\middle|\, \operatorname{supp} f \subset \Omega,\ \|f\|_\alpha \le 1 \right\}$ is compact in $L^1(\mathbb{R}^N)$ by proposition 3.3, by Ionescu-Tulcea and Marinescu Theorem and Lemma 4.1 we get the usual spectral decomposition of P. □

Theorem 5.2. *Let* T *be a map which satisfies (PE1)–(PE5). The ergodic decomposition of any* ACIM *is finite, and the number* $E(T)$ *of ergodic* ACIM*s is bounded by*

$$E(T) \le \frac{m(\Omega)}{\gamma_N} \left(\frac{D}{1-\eta} \right)^{N/\alpha}.$$

Where the constant η *and* D *are given by Lemma 4.1.*

Proof. Theorem 5.1 shows that the ergodic decomposition of the maximal ACIM is finite. Hence each ergodic element of the decomposition is again an ACIM. But for all ACIMs, there exists a positive eigenfunction h associated to the eigenvalue 1 on L^1_m.

From Theorem 5.1.(ii) we know that $h \in V_\alpha$, so Lemma 4.1 yields $|Ph|_\alpha \le \eta |h|_\alpha + D$. Hence $|h|_\alpha \le \frac{D}{1-\eta}$. By Lemma 3.1 the infimum of h on some ball of radius $\varepsilon = \left(\frac{1-\eta}{D}\right)^{\frac{1}{\alpha}}$ is strictly positive. One can also check that $\varepsilon < \varepsilon_0$.

It follows immediately that the number of ergodic ACIMs is limited by the maximal number of balls of radius ε contained in Ω, which is roughly[7] bounded by $\dfrac{m(\Omega)}{\gamma_N \varepsilon^N}$. □

[7] Actually, if we do not put any restrictions on Ω, this bound is optimal. Sphere packing is a very hard problem, and it is an open question to calculate the real bound.

Proposition 5.1. *Let T be a map which satisfies (PE1)–(PE5). The interior of the support of any* ACIM μ *is of full μ-measure, hence two ergodic* ACIM*s are different if and only their supports have disjoint interior.*

Proof. We will prove something stronger, that for almost all points in the support of an ACIM μ there exists a ball on which the density is strictly positive.

Let us consider the sets $N_\varepsilon = \{x \in \Omega |\, \mathrm{Einf}_{B_\varepsilon(x)}\, h = 0\}$ where $0 < \varepsilon < \varepsilon_0$. Since the density $h \in V_\alpha$,

$$\mu(N_\varepsilon) = \int_{U_\varepsilon} h(x)dx \le \int_{U_\varepsilon} \mathrm{osc}\big(h, B_\varepsilon(x)\big)dx \le |h|_\alpha \varepsilon^\alpha.$$

This proves that $\mu(N_\varepsilon) \to 0$ when $\varepsilon \to 0$. Hence $N = \cap N_\varepsilon$ is a null set, consequently for almost all points $x \in \Omega$, there exists a ball on which the infimum of h is strictly positive. Such points are clearly in the support of μ. □

6 Decay of correlations

Up to now we only dealt with theoretical results about expanding maps. We would like then to give a concrete estimate on the rate of decay of correlations. The spectral gap ensures that this rate is exponential for V_α observables, and in particular for α-Hölder ones. But the Ionescu-Tulcea and Marinescu theorem does not give any estimate about the rate of decay. We will use for that the cone technique introduced by P.Ferrero, B.Schmitt and C.Liverani, which is becoming an important tool to investigate such questions. We will explicitely compute a bound for the rate of decay of correlations in terms of the constants η and D of Lemma 4.1 and some speed of mixing of any finite partition of sufficiently small diameter. To be more precise, let us fix the setting:

We suppose that the map T satisfies (PE1)–(PE5) and we denote by μ the mixing[8] ACIM.

Step 1. *We first fix two parameters $0 < \delta < 1$ and $0 < \sigma < 1$, the choice is arbitrary or may be dictated by some properties of the map. We then define*

[8] If μ is not mixing, one can always consider the restriction of μ to a subset of Ω and some mixing iterate of T, given by Theorem 5.1.

the following constants

$$\begin{aligned} B &= \frac{D}{1-\eta}, & (12)\\ K_0 &= \frac{\log \frac{\sigma\delta}{2}}{\log \eta}, & (13)\\ a &= \frac{B+2}{\sigma\delta}, & (14)\\ \varepsilon &= \left(\frac{\sigma\delta^2(1-\delta)}{2(B+2)}\right)^{\frac{1}{\alpha}} < \varepsilon_0 & (15)\\ w(\delta) &= (1-\delta)^2. & (16) \end{aligned}$$

Definition 6.1. *We say that a finite partition $\mathcal{A}$ is mixed with an accuracy $w \geq 0$ by K iterations of the map T if for all $(A, A') \in \mathcal{A}^2$ we have*

$$1 - w \leq \frac{m(A' \cap T^{-K}A)}{m(A')\mu(A)} \leq 1 + w.$$

Remark 6.1. *We remark that once the spectral gap is proved on V_α for the transfer operator, it implies immediately that for any real $w > 0$ there exists K such that the partition $\mathcal{A}$ is mixed with an accuracy w after K iterations, provided that each element of the partition $\mathcal{A}$ has a characteristic function in V_α.*

Step 2. *Let us choose now a partition $\mathcal{A}$ of diameter less than ε, and find a $K = K(\mathcal{A}, \delta) > K_0$ such that the partition $\mathcal{A}$ is mixed with an accuracy $w(\delta)$ by the map T^K.*

Remark 6.2. *The number of iterates for which the partition will be mixed with the proper accuracy may also be estimated by computer experiments, if the measure μ is (sufficiently) known.*

Theorem 6.1. *The rate of decay of the correlation is exponential on V_α, and the following constructive estimate holds:*

$$\left|\int_\Omega f \circ T^n h d\mu\right| \leq C\|f\|_{L^1}\|h\|_\alpha \Lambda^n, \qquad \forall f \in L^1, \mu(f) = 0.$$

Where $C < \infty$ and $\Lambda < 1$ are given by:

$$\begin{aligned} C &= 2(1+B)(2+3a)\frac{\Delta}{\Lambda^{2K}}\left(1 + \frac{a^{-1} + 1/\inf_{A\in\mathcal{A}}\mu(A)}{1-\sigma}\right)\left(\frac{\max(1,\varepsilon_0^\alpha)}{\gamma_N \varepsilon_0^N}\right)^2, \\ \Lambda &= \left[\tanh\left(2\log\frac{1+\sigma}{1-\sigma} - 4\log\delta\right)\right]^{1/K}. \end{aligned}$$

Let us introduce the normalized Perron-Frobenius operator $\widetilde{P}$, defined on L^1_μ by $\widetilde{P}f = \dfrac{P(fh_*)}{h_*}$, where $h_* = \dfrac{d\mu}{dm}$. Let us recall that $\widetilde{P} = T^*_\mu$, hence $\widetilde{P}\mathbb{1} = \mathbb{1}$.

Theorem 6.2. *$\widetilde{P}^n h$ converges exponentially fast towards $\mu(h)\mathbb{1}$ for $h \in V_\alpha$. The speed is at least*

$$|||[\widetilde{P}^n(h) - \mu(h)]h_*\|_\alpha \leq C'\|h\|_\alpha \Lambda^n.$$

Where $C' < \infty$ and $\Lambda < 1$ are given by:

$$\begin{aligned} C' &= 2(1+B)(2+3a)\frac{\Delta}{\Lambda^{2K}}\left(1+\frac{a^{-1}+1/\inf_{A\in\mathcal{A}}\mu(A)}{1-\sigma}\right)\frac{\max(1,\varepsilon_0^\alpha)}{\gamma_N\varepsilon_0^N}, \\ \Lambda &= \left[\tanh\left(2\log\frac{1+\sigma}{1-\sigma} - 4\log\delta\right)\right]^{1/K}. \end{aligned}$$

Proof of Theorem 6.1. For $\widetilde{P} = T^*_\mu$ and $\mu(f) = 0$ we have

$$\begin{aligned} \left|\int f\circ T^n h d\mu\right| &= \left|\int f(\widetilde{P}^n h - \mu(h))d\mu\right| \\ &\leq \|f\|_{L^1_\mu}\|\widetilde{P}^n h - \mu(h)\|_{L^\infty_\mu} \\ &\leq \|f\|_{L^1_\mu}\frac{\max(1,\varepsilon_0^\alpha)}{\gamma_N\varepsilon_0^N}|||[\widetilde{P}^n h - \mu(h)]h_*\|_\alpha. \end{aligned}$$

By inequality (6). Theorem 6.2 gives the conclusion, with $C = \dfrac{\max(1,\varepsilon_0^\alpha)}{\gamma_N\varepsilon_0^N}C'$.

□

The idea of the proof of Theorem 6.2 is to link the speed of mixing of smooth (V_α) observables to the one of the finite partition $\mathcal{A}$. We will establish some preparatory lemmas which show step by step that PF is a contraction for a suitable metric (Hilbert-Birkhoff projective metric on a convex cone) defined on V_α.

The flexibility of this method, shown to us by C.Liverani, relies on the fact that the partition can be chosen arbitrarily. In particular, it does not have to be any dynamical partition. Moreover we can consider cones of non necessarily positive functions, and this allows us to forget the "covering hypothesis". Especially in higher dimensions, this assumption is usually hard to prove, and probably not satisfied in many interesting cases.

Before entering into the proof, let us see some preliminary definitions and properties of this metric. For a more detailed review of these properties, see for example [LSV1].

Definition 6.2. *We define a partial order relation on a convex cone $\mathcal{C}$ by*

$$f \preceq h \Leftrightarrow h - f \in \mathcal{C}.$$

The distance $\Theta(f,h)$ between two points $f,h \in \mathcal{C}$ is given by

$$\begin{aligned} \alpha(f,h) &= \sup\{\lambda > 0 | \lambda f \preceq h\}, \\ \beta(f,h) &= \inf\{\mu > 0 | h \preceq \mu f\}, \\ \Theta(f,h) &= \log \frac{\beta(f,h)}{\alpha(f,h)}. \end{aligned}$$

Where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

The distance Θ is a pseudo-metric, because two elements can be at an infinite distance from each other, and it is a projective metric because any two proportional elements have a null distance.
The next theorem, due to G. Birkhoff [Bi] shows that every positive linear operator is a contraction, provided that the diameter of the image is finite.

Theorem 6.3. *Let $\mathcal{V}$ be a vector space, $\mathcal{C} \subset \mathcal{V}$ a convex cone[9] and $L : \mathcal{V} \to \mathcal{V}$ a positive linear operator (which means $L(\mathcal{C}) \subset \mathcal{C}$). Let Θ be the Hilbert metric associated to the cone $\mathcal{C}$. If we denote*

$$\Delta = \sup_{f,h \in L(\mathcal{C})} \Theta(f,h) \quad ,$$

then

$$\Theta(Lf, Lh) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta(f,h) \ \ \forall f,h \in \mathcal{C},$$

($\tanh(\infty) = 1$).

We denote by $\mathbb{E}_\mu(f|\mathcal{A})$ the conditional expectation of a L^1_μ function f with respect to the partition $\mathcal{A}$. Let us define the cone of functions

$$\mathcal{C}_a(\mathcal{A}) = \left\{ 0 \neq f \in L^1_\mu(\Omega, \mathbb{R}) \big| \ |f h_*|_\alpha \leq a \mathbb{E}_\mu(f|\mathcal{A}) \right\}.$$

We will show that $\mathcal{C}_a(\mathcal{A})$ is left invariant by the normalized operator $\widetilde{P}$.

Lemma 6.1.

$$\forall f \in \mathcal{C}_a(\mathcal{A}) \quad \delta\mu(f) \leq \mathbb{E}_\mu(\widetilde{P}^K f | \mathcal{A}) \leq \frac{1}{\delta}\mu(f).$$

[9] To be completely honest, $\mathcal{C}$ cannot be any convex cone, weak properties have to be satisfied : (i) $\mathcal{C} \cap -\mathcal{C} = \emptyset$, (ii) $\forall \lambda > 0, \lambda\mathcal{C} = \mathcal{C}$, (iii) $\mathcal{C}$ is convex and (iv) $\forall f,h \in \mathcal{C}, \forall t_n \in \mathbb{R}$ such that $t_n \to t$, $h - t_n f \in \mathcal{C}$ imply $h - tf \in \mathcal{C} \cup \{0\}$.

Proof. For all $x \in \Omega$, denote by $A = A(x)$ the element of $\mathcal{A}$ which contains x. For all $f \in \mathcal{C}_a(\mathcal{A})$,

$$\begin{aligned}
\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A})(x) &= \frac{1}{\mu(A(x))} \int_{A(x)} \widetilde{P}^K f d\mu \\
&= \frac{1}{\mu(A(x))} \int_{T^{-K}A(x)} f d\mu \\
&= \sum_{A' \in \mathcal{A}} \frac{1}{\mu(A(x))} \int_{A' \cap T^{-K}A(x)} f d\mu.
\end{aligned}$$

But m-almost everywhere on $A' \cap T^{-K}A$ we have

$$\begin{aligned}
fh_* &\geq \frac{1}{m(A')} \int_{A'} f d\mu - \operatorname{osc}\big(fh_*, A'\big) \\
&\geq \frac{1}{m(A')} \left(\int_{A'} f d\mu - \int_{A'} \operatorname{osc}\big(fh_*, B_\varepsilon(y)\big) dy \right).
\end{aligned}$$

Now an integration over $A' \cap T^{-K}A$ with respect to m leads to

$$\begin{aligned}
\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A})(x) &\geq \sum_{A' \in \mathcal{A}} \frac{m(A' \cap T^{-K}A)}{m(A')\mu(A(x))} \left(\int_{A'} f d\mu - \int_{A'} \operatorname{osc}\big(fh_*, B_\varepsilon(y)\big) dy \right) \\
&\geq (1 - w(\delta))\mu(f) - (1 + w(\delta))|fh_*|_\alpha \varepsilon^\alpha \\
&\geq (1 - w(\delta) - (1 + w(\delta))a\varepsilon^\alpha)\mu(f)
\end{aligned}$$

Thanks to our choice of a and ε, one can check that

$$1 - w(\delta) - (1 + w(\delta))a\varepsilon^\alpha \geq \delta,$$

which proves the first inequality.

The second one is obtained in a similar way, because

$$\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A})(x) \leq (1 + w(\delta))\left(\mu(f) + |fh_*|_\alpha \varepsilon^\alpha\right) \leq (1 + w(\delta))(1 + a\varepsilon^\alpha)\mu(f).$$

The Lemma follows by checking that $(1 + w(\delta))(1 + a\varepsilon^\alpha) \leq 1/\delta$. $\square$

Lemma 6.2. *The cone is mapped strictly inside itself by the operator $\widetilde{P}^K$:*

$$\widetilde{P}^K \mathcal{C}_a(\mathcal{A}) \subset \mathcal{C}_{\sigma a}(\mathcal{A}).$$

Proof. For $f \in \mathcal{C}_a(\mathcal{A})$, Let us show that $\widetilde{P}^K f \in \mathcal{C}_a(\mathcal{A})$, i.e.

$$|P^K(fh_*)|_\alpha \leq a\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A}).$$

The positiveness of $\mathbb{E}_\mu(f|\mathcal{A})$ implies that

$$\mu(|f|) \leq \mu(f) + |fh_*|_\alpha \varepsilon^\alpha. \tag{17}$$

By Lemma 4.1 (applied K times) we get

$$\begin{aligned} |P^K(fh_*)|_\alpha &\leq \eta^K |fh_*|_\alpha + B\mu(|f|) \\ &\leq \eta^K |fh_*|_\alpha + B[|fh_*|_\alpha \varepsilon^\alpha + \mu(f)] \\ &\leq \left[a(\eta^K + B\varepsilon^\alpha) + 1\right] \mu(f). \end{aligned}$$

Moreover, Lemma 6.1 gives the lower bound

$$\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A}) \geq \delta\mu(f).$$

Combining the two estimates, $\widetilde{P}^K f$ will belongs to the cone $\mathcal{C}_{\sigma a}(\mathcal{A})$ provided

$$\eta^K + B\varepsilon^\alpha + \frac{1}{a} \leq \sigma\delta.$$

Which is true because $\eta^K \leq \eta^{K_0} = \dfrac{\sigma\delta}{2}$ and $B\varepsilon^\alpha + \dfrac{1}{a} \leq \dfrac{\sigma\delta}{2}$. □

We need then to show that the hyperbolic diameter of the image of the cone is finite, which will give the contraction in the Hilbert metric.

Lemma 6.3. *The distance between two elements $f, g \in \mathcal{C}_{\sigma a}(\mathcal{A})$ in the cone $\mathcal{C}_a(\mathcal{A})$ is bounded by*

$$\Theta_a(f,g) \leq 2\log\frac{1+\sigma}{1-\sigma} + \log\left\|\frac{\mathbb{E}_\mu(g|\mathcal{A})}{\mathbb{E}_\mu(f|\mathcal{A})}\right\|_\infty \left\|\frac{\mathbb{E}_\mu(f|\mathcal{A})}{\mathbb{E}_\mu(g|\mathcal{A})}\right\|_\infty.$$

Proof. The distance between f and g is given by the supremum of r and the infimum on s such that $rf \preceq g \preceq sf$.

$$|(g - rf)h_*|_\alpha \leq |gh_*|_\alpha + r|fh_*|_\alpha \leq \sigma a\mathbb{E}_\mu(g|\mathcal{A}) + r\sigma a\mathbb{E}_\mu(f|\mathcal{A}).$$

It follows that $g - rf$ will belongs to the cone if

$$\sigma a\mathbb{E}_\mu(g|\mathcal{A}) + r\sigma a\mathbb{E}_\mu(f|\mathcal{A}) \leq a\mathbb{E}_\mu(g - rf|\mathcal{A}),$$

that is to say

$$r(1+\sigma)\mathbb{E}_\mu(f|\mathcal{A}) \leq (1-\sigma)\mathbb{E}_\mu(g|\mathcal{A}).$$

Hence the biggest r will be

$$\frac{1-\sigma}{1+\sigma}\,\mathrm{Einf}\,\frac{\mathbb{E}_\mu(g|\mathcal{A})}{\mathbb{E}_\mu(f|\mathcal{A})}.$$

The same computation gives s. □

Lemma 6.4. *The diameter of $\widetilde{P}^K\mathcal{C}_a(\mathcal{A})$ in $\mathcal{C}_a(\mathcal{A})$ is finite, bounded by*

$$\Delta = 2\log\frac{1+\sigma}{1-\sigma} + 4\log\delta^{-1} < \infty.$$

Proof. Let $f, g \in \mathcal{C}_a(\mathcal{A})$. Lemma 6.1 applied to f and g yields

$$\frac{\delta\mu(f)}{\frac{1}{\delta}\mu(g)} \leq \frac{\mathbb{E}_\mu(\widetilde{P}^K f|\mathcal{A})}{\mathbb{E}_\mu(\widetilde{P}^K g|\mathcal{A})} \leq \frac{\frac{1}{\delta}\mu(f)}{\delta\mu(g)}.$$

Moreover, Lemma 6.2 ensure that $\widetilde{P}^K f$ and $\widetilde{P}^K g$ belong to $\mathcal{C}_{\sigma a}(\mathcal{A})$. Hence we can apply Lemma 6.3 which gives the desired result. □

Lemma 6.5. *The convergence of $\widetilde{P}^{Kn} f$ for $f \in \mathcal{C}_a(\mathcal{A})$ towards $\mathbb{1}$ is exponential in the Hilbert metric of the cone $\mathcal{C}_a(\mathcal{A})$, the rate being at least Λ^{Kn}, where $\Lambda = \tanh(\frac{\Delta}{4})^{1/K} < 1$:*

$$\forall f \in \mathcal{C}_a(\mathcal{A}), \forall n > 0 \qquad \Theta_a(\widetilde{P}^{Kn} f, \mathbb{1}) \leq \frac{\Delta}{\Lambda^K}\Lambda^{Kn}.$$

Proof. Let us recall that $\widetilde{P}\mathbb{1} = \mathbb{1}$ and $\mathbb{1} \in \mathcal{C}_a(\mathcal{A})$. We know that the diameter of the image of the cone is finite, hence $\widetilde{P}^K$ is a strict contraction by Birkhoff's theorem 6.3, and the rate of contraction is at least $\Lambda^K := \tanh(\frac{\Delta}{4}) < 1$.

$$\begin{aligned}
\Theta_a(\widetilde{P}^{Kn} f, \mathbb{1}) &= \Theta_a(\widetilde{P}^{Kn} f, \widetilde{P}^{Kn}\mathbb{1}) \\
&\leq \Theta_a((\widetilde{P}^K)^{n-1}\widetilde{P}^K f, (\widetilde{P}^K)^{n-1}\widetilde{P}^K\mathbb{1}) \\
&\leq \Lambda^{K(n-1)}\Theta_a(\widetilde{P}^K f, \widetilde{P}^K\mathbb{1}).
\end{aligned}$$

Now the conclusion follows if we remark that both $\widetilde{P}^K f$ and $\widetilde{P}^K\mathbb{1}$ are at a distance smaller than Δ by Lemma 6.4. □

Proof of Theorem 6.2. Let $f \in \mathcal{C}_a(\mathcal{A})$ and suppose for simplicity that $\mu(f) = 1$. For all r, s such that $r\mathbb{1} \preceq f \preceq s\mathbb{1}$ we have the following bound for the L^1 norm

$$m\left(|\widetilde{P}^{Kn} f h_* - h_*|\right) \leq m\left(|\widetilde{P}^{Kn} f h_* - r h_*|\right) + (1-r)$$

Since $\widetilde{P}^{Kn} f - r \in \mathcal{C}_a(\mathcal{A})$ inequality (17) implies that

$$\begin{aligned}
m\left(|\widetilde{P}^{Kn} f h_* - r h_*|\right) &\leq |\widetilde{P}^{Kn} f h_* - r h_*|_\alpha \varepsilon^\alpha + m(\widetilde{P}^{Kn} f h_* - r h_*) + (1-r) \\
&\leq |\widetilde{P}^{Kn} f h_* - r h_*|_\alpha \varepsilon^\alpha + 2(1-r).
\end{aligned}$$

Moreover, the oscillation is bounded by

$$\begin{aligned}|\widetilde{P}^{Kn}fh_* - h_*|_\alpha &\le |\widetilde{P}^{Kn}fh_* - rh_*|_\alpha + (1-r)|h_*|_\alpha \\ &\le |\widetilde{P}^{Kn}fh_* - rh_*|_\alpha + a(1-r).\end{aligned}$$

These two estimates give us, since $\widetilde{P}^{Kn}f - r \in \mathcal{C}_a(\mathcal{A})$,

$$\begin{aligned}\|(\widetilde{P}^{Kn}f - 1)h_*\|_\alpha &\le 2a\mathbb{E}_\mu(\widetilde{P}^{Kn}f - r|\mathcal{A}) + (2+a)(1-r) \\ &\le (2+3a)(1-r).\end{aligned}$$

Since $1 - r \le -\log r \le \log \frac{s}{r}$, by definition of Θ_a we find

$$\|(\widetilde{P}^{Kn}f - 1)h_*\|_\alpha \le (2+3a)\Theta_a(\widetilde{P}^{Kn}, 1).$$

By Lemma 6.5, we know then that for each function f in $\mathcal{C}_a(\mathcal{A})$ we have

$$\|(\widetilde{P}^{Kn}f - \mu(f))h_*\|_\alpha \le (2+3a)\mu(f)\frac{\Delta}{\Lambda^K}\Lambda^{Kn}. \tag{18}$$

In order to get the convergence of any function $h \in V_\alpha$, it is enough to remark that $f_h := c_h + h$ will belong to the cone for some constant c_h sufficiently large. The condition for that is

$$|(h + c_h)h_*|_\alpha \le a\mathbb{E}_\mu(h + c_h|\mathcal{A})$$

which is implied by the following inequality:

$$|hh_*|_\alpha + c_h|h_*|_\alpha \le a\left(c_h - \|h\|_{L^1_\mu} / \inf_{A\in\mathcal{A}} \mu(A)\right). \tag{19}$$

Since $h_* \in \mathcal{C}_{\sigma a}(\mathcal{A})$ we have $|h_*|_\alpha \le \sigma a$, hence inequality (19) will be true with c_h chosen as $c_h = C_0\|hh_*\|_\alpha$ for C_0 defined by

$$C_0 = \frac{a^{-1} + 1/\inf_{A\in\mathcal{A}}\mu(A)}{1-\sigma}.$$

Then inequality (18) provides the estimate

$$\begin{aligned}\|(\widetilde{P}^{Kn}h - \mu(h))h_*\|_\alpha &= \|(\widetilde{P}^{Kn}f_h - \mu(f_h))h_*\|_\alpha \\ &\le (2+3a)\mu(f_h)\frac{\Delta}{\Lambda^K}\Lambda^{Kn} \\ &\le (2+3a)\frac{\Delta}{\Lambda^K}(1 + C_0)\|hh_*\|_\alpha.\end{aligned}$$

From Lemma 4.1 it is easy to deduce that for any integer p,

$$\|\widetilde{P}^p(g)h_*\|_\alpha \leq (1+B)\|gh_*\|_\alpha,$$

so finally, writing $n = mK + p$ with m integer and $0 \leq p < K$ we get

$$\begin{aligned}\|(\widetilde{P}^n(h) - \mu(h))h_*\|_\alpha &\leq (1+B)\|(\widetilde{P}^{Km}(h) - \mu(h))h_*\|_\alpha \\ &\leq (1+B)(2+3a)\frac{\Delta}{\Lambda^{2K}}(1+C_0)\Lambda^n\|hh_*\|_\alpha \\ &\leq C'\Lambda^n\|h\|_\alpha.\end{aligned}$$

Here we used inequality (7) in Proposition 3.4, and the constant C' is defined by $C' = (1+B)(2+3a)\dfrac{\Delta}{\Lambda^{2K}}(1+C_0)\dfrac{2\max(1,\varepsilon_0^\alpha)}{\gamma_N\varepsilon_0^N}$. □

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