

Application of the convergence of the spatio-temporal processes for visits to small sets

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Abstract The goal of this article is to point out the importance of spatio-temporal processes in different questions of quantitative recurrence. We focus on applications to the study of the number of visits to a small set before the first visit to another set (question arising from a previous work by Kifer and Rapaport), the study of high records, the study of line processes, the study of the time spent by a flow in a small set. We illustrate these applications by results on billiards or geodesic flows. This paper contains in particular new result of convergence in distribution of the spatio temporal processes associated to visits by the Sinai billiard flow to a small neighbourhood of arbitrary points in the billiard domain.

1 Introduction

Let $(\Omega, \mathcal{F}, \mu, T)$ or $(\Omega, \mathcal{F}, \mu, Y = (Y_t)_{t \geq 0})$ be a probability preserving dynamical system in discrete or continuous times. Let $(A_\varepsilon)_{\varepsilon > 0}$ be a family of measurable subsets of Ω with $\mu(A_\varepsilon) \rightarrow 0+$ as $\varepsilon \rightarrow 0$. Given a family $(h_\varepsilon)_{\varepsilon > 0}$ of positive real numbers and a family $(H_\varepsilon)_{\varepsilon > 0}$ of measurable normalization functions $H_\varepsilon : A_\varepsilon \rightarrow V$ where V is a locally compact metric space endowed with its Borel σ -algebra \mathcal{V} , we study the family of spatio-temporal point processes $(\mathcal{N}_\varepsilon)_{\varepsilon > 0}$ on $[0, +\infty) \times V$ given by

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$$\mathcal{N}_\varepsilon(x) := \mathcal{N}(T, A_\varepsilon, h_\varepsilon, H_\varepsilon) := \sum_{n \geq 1 : T^n(x) \in A_\varepsilon} \delta_{(nh_\varepsilon, H_\varepsilon(T^n(x)))} \quad \text{for a map } T \quad (1)$$

or

$$\mathcal{N}_\varepsilon(x) := \mathcal{N}(Y, A_\varepsilon, h_\varepsilon, H_\varepsilon) = \sum_{t > 0 : Y_t \text{ enters } A_\varepsilon} \delta_{(th_\varepsilon, H_\varepsilon(Y_t(x)))} \quad \text{for a flow } Y. \quad (2)$$

We are interested in results of convergence in distribution of $(\mathcal{N}_\varepsilon)_{\varepsilon > 0}$ to a point process \mathcal{P} as $\varepsilon \rightarrow 0$ with a particular focus on applications of results of such kind. Various results of convergence of such processes to Poisson point processes have been proved in [21, 13] for billiard maps and flows.

Let us point out the fact that these spatio-temporal processes contain a lot of information: they do not only contain information on the visit time but they also contain informations on the spatial position at these visit times. For these reasons, one may extract further information from results of convergence of these processes. Among the applications that have already been studied, let us mention:

- Study of the visits to a small neighborhood of an hyperbolic periodic point of a transformation (see [21, Section 5], with application to Anosov maps). Such visits occurs by clusters (once a point visits such a neighbourhood, it stays close to the periodic point during an unbounded time before living this area). The idea we used to study these clusters was to consider a process \mathcal{N}_ε corresponding to the last (or first) position of the clusters.
- Convergence of a normalized Birkhoff sum processes

$$\left(\left(n^{-\frac{1}{\alpha}} \sum_{k=0}^{\lfloor nt \rfloor - 1} f \circ T^k \right)_{t \geq 0} \right)_{n \geq 1}$$

to an α -stable process. In [25] Tyran-Kamińska provided criteria ensuring such a result. One of the conditions is the convergence of

$$\mathcal{N}_{1/n} = \mathcal{N}(T, \{|f| > \gamma n^{\frac{1}{\alpha}}\}, 1/n, n^{-\frac{1}{\alpha}} f(\cdot))$$

(for every $\gamma > 0$) to some Poisson point process. The general results of [21] combined with the criteria of [25] have been used in [13] to prove convergence to a Lévy process for the Birkhoff sum process of Hölder observable of billiards in dispersing domains with cusps.

We won't detail again the above applications. Our goal here is to emphasize on further ones.

After recalling in Section 2 below the general results of convergence of spatio-temporal point processes to Poisson point processes established in [21], we present in the remaining sections four other important applications of such convergence results:

- The number of visits to (or of the time spent in) a small set before the first visit to a second small set (motivated by Kifer and Rapaport [16]), with application to the Sinai billiard flow with finite horizon,
- The evolution of the number of records larger than some threshold, with an application to billiards with corners and cusps of order larger than 2,
- The Line process of random geodesics (motivated by Athreya, Lalley, Sapir and Wroten [2]),
- The time spent by a flow in a small set, with application to the Sinai billiard flow with finite horizon.

Appendix 6 contains a new theorem of convergence of point processes for the Sinai billiard flow and for neighborhoods of arbitrary positions in the billiard domain, which is used in the examples that illustrate the applications above. Finally we also present an application to the closest approach by the billiard flow.

2 Convergence results for transformations and special flows

We set $E := [0, +\infty) \times V$ and we endow it with its Borel σ -algebra $\mathcal{E} = \mathcal{B}([0, +\infty)) \otimes \mathcal{V}$. We also consider the family of measures $(m_\varepsilon)_{\varepsilon>0}$ on (V, \mathcal{V}) defined by

$$m_\varepsilon := \mu(H_\varepsilon^{-1}(\cdot)|A_\varepsilon) \quad (3)$$

and a family \mathcal{W} closed under finite unions and intersections of relatively compact open subsets of V , that generates the σ -algebra \mathcal{V} . Let λ be the Lebesgue measure on $[0, \infty)$.

We will approximate the point process defined by (1) or (2) by a Poisson point process on E . Given a σ -finite measure η on (E, \mathcal{E}) , recall that a process \mathcal{N} is a Poisson point process on E of intensity η if

- \mathcal{N} is a point process (i.e. $\mathcal{N} = \sum_i \delta_{x_i}$ with x_i being E -valued random variables),
- For every pairwise disjoint Borel sets $B_1, \dots, B_n \subset E$, the random variables $\mathcal{N}(B_1), \dots, \mathcal{N}(B_n)$ are independent Poisson random variables with respective parameters $\eta(B_1), \dots, \eta(B_n)$.

Let $M_p(E)$ be the space of all point measures defined on E , endowed with the topology of vague convergence; it is metrizable as a complete separable metric space. A family of point processes $(\mathcal{N}_\varepsilon)_{\varepsilon>0}$ converges in distribution to \mathcal{N} if for any bounded continuous function $f: M_p(E) \rightarrow \mathbb{R}$ the following convergence holds true

$$\mathbb{E}(f(\mathcal{N}_\varepsilon)) \rightarrow \mathbb{E}(f(\mathcal{N})), \quad \text{as } \varepsilon \rightarrow 0. \quad (4)$$

For a collection \mathcal{A} of measurable subsets of Ω , we define the following quantity:

$$\Delta(\mathcal{A}) := \sup_{A \in \mathcal{A}, B \in \sigma(\cup_{n=1}^{\infty} T^{-n}\mathcal{A})} |\mu(A \cap B) - \mu(A)\mu(B)|. \quad (5)$$

We set λ for the Lebesgue measure on $[0, \infty)$.

Theorem 1 (Convergence result for transformations [21, Theorem 2.1]) *We assume that*

- (i) *for any finite subset \mathcal{W}_0 of \mathcal{W} we have $\Delta(H_\varepsilon^{-1}\mathcal{W}_0) = o(\mu(A_\varepsilon))$,*
- (ii) *there exists a measure m on (V, \mathcal{V}) such that for every $F \in \mathcal{W}$, $m(\partial F) = 0$ and $\lim_{\varepsilon \rightarrow 0} \mu(H_\varepsilon^{-1}(F)|A_\varepsilon)$ converges to $m(F)$.*

Then the family of point processes $(\mathcal{N}_\varepsilon)_{\varepsilon > 0}$ converges strongly¹ in distribution, as $\varepsilon \rightarrow 0$, to a Poisson point process \mathcal{P} of intensity $\lambda \times m$.

In particular, for every relatively compact open $B \subset E$ such that $(\lambda \times m)(\partial B) = 0$, $(\mathcal{N}_\varepsilon(B))_{\varepsilon > 0}$ converges in distribution, as $\varepsilon \rightarrow 0$, to a Poisson random variable with the parameter $(\lambda \times m)(B)$.

Let us explain roughly the strategy used in [21] to apply Theorem 1. First the measure m appears as the limit of $(\mu(H_\varepsilon^{-1}(\cdot)|A_\varepsilon))_{\varepsilon > 0}$. Second, we construct \mathcal{W} as the union of finer and finer finite partitions of \mathcal{V} with boundary neglectable with respect to m . Finally we obtain (i) as a consequence of some decorrelation result combined with the neglectability of fast returns.

Theorem 2 (Convergence result for special flows [21, Theorem 2.3]) *Assume $(\Omega, \mu, Y = (Y_t)_t)$ can be represented as a special flow over a probability preserving dynamical system (M, ν, F) with a roof function $\tau : M \rightarrow (0, +\infty)$ with $M \subset \Omega$ and set $\Pi : \Omega \rightarrow M$ for the projection such that $\Pi(Y_s(x)) = x$ for all $x \in M$ and all $s \in [0, \tau(x))$.*

Assume moreover that Y enters A_ε at most once between two consecutive visits to M and that there exists a family of measurable normalization functions $G_\varepsilon : M \rightarrow V$ such that the family of point processes $(\mathcal{N}(F, \Pi(A_\varepsilon), h_\varepsilon, G_\varepsilon))_{\varepsilon > 0}$ converges in distribution, as $\varepsilon \rightarrow 0$ and with respect to some probability measure $\tilde{\nu} \ll \nu$, to a Poisson point process of intensity $\lambda \times m$, where m is some measure on (V, \mathcal{V}) . Then the family of point processes $(\mathcal{N}(Y, A_\varepsilon, h_\varepsilon/\mathbb{E}_\nu[\tau], G_\varepsilon \circ \Pi))_{\varepsilon > 0}$ converges in distribution, as $\varepsilon \rightarrow 0$ (with respect to any probability measure absolutely continuous with respect to μ), to a Poisson process \mathcal{P} of intensity $\lambda \times m$.

¹ i.e. with respect to any probability measure absolutely continuous w.r.t. μ

3 Number of visits to a small set before the first visit to a second small set

Suppose B_ε^0 and B_ε^1 are two disjoint sets. We define the spatio-temporal process \mathcal{N}_ε with $A_\varepsilon = B_\varepsilon^0 \cup B_\varepsilon^1$, $H_\varepsilon(x) = \ell$ if $x \in B_\varepsilon^\ell$, $\ell = 0, 1$, that is on $[0, +\infty) \times \{0, 1\}$

$$\mathcal{N}_\varepsilon(x) = \sum_{n=1}^{\infty} \sum_{\ell=0}^1 \delta_{(n\mu(A_\varepsilon), \ell)} 1_{B_\varepsilon^\ell}(T^n x) \quad (6)$$

in the case of a transformation T or

$$\mathcal{N}_\varepsilon(x) = \sum_{t>0} \sum_{\ell=0}^1 \delta_{(th_\varepsilon, \ell)} 1_{Y_t \text{ enters } B_\varepsilon^\ell} \quad (7)$$

in the case of a flow Y . In [16] Kifer and Rapaport studied the distribution of a (multiple) event $T^n x \in B_\varepsilon^1$ until a (multiple) hazard $T^n(x) \in B_\varepsilon^0$. We stick here to single event and hazard and define, in the case of a transformation T ,

$$\mathcal{M}_\varepsilon(x) := \sum_{n=1}^{\tau_{B_\varepsilon^0}(x)} 1_{B_\varepsilon^1}(T^n x), \quad (8)$$

where we set $\tau_B(x) := \inf\{n \geq 1 : T^n(x) \in B\}$ or, in the case of a flow Y :

$$\mathcal{M}_\varepsilon(x) := \sum_{t \in (0, \tau_{B_\varepsilon^0}(x))} 1_{Y_t \text{ enters } B_\varepsilon^1}, \quad (9)$$

where we set $\tau_B(x) := \inf\{t > 0 : Y_t(x) \in B\}$. The process \mathcal{M}_ε counts the number of entrances of the flow in the 1-set before its first visit to the 0-set. In the case of a flow, it is also natural to consider the following process \mathcal{M}'_ε measuring the time spent by the flow in the 1-set before its first visit to the 0-set:

$$\mathcal{M}'_\varepsilon(x) := \int_0^{\tau_{B_\varepsilon^0}(x)} 1_{B_\varepsilon^1} \circ Y_s(x) ds. \quad (10)$$

In view of the study of this last process, we will consider the following process measuring the time spent by the flow in each set:

$$\left(\mathcal{L}_\varepsilon := \sum_{j=0}^1 \sum_{t: Y_t \text{ enters } B_\varepsilon^j} \delta_{th_\varepsilon, j, a_\varepsilon D_{B_\varepsilon^j} \circ Y_t} \right)_{\varepsilon > 0}$$

with $D_A := \tau_{\Omega \setminus A}$.

Theorem 3 *Let $p \in (0, 1)$ and \mathbb{P} be a probability measure on Ω . Assume, in the case of a flow, that $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(B_\varepsilon^0 \cup B_\varepsilon^1) = 0$.*

If the spatio-temporal process \mathcal{N}_ε defined as in (6) or (7) converges, with respect to \mathbb{P} , to a PPP of intensity $\lambda \times \mathcal{B}(p)$ where $\mathcal{B}(p)$ denotes the Bernoulli measure with parameter p (for a transformation we expect $p = \lim_{\varepsilon \rightarrow 0} \mu(B_\varepsilon^1)/\mu(A_\varepsilon)$), then the process $(\mathcal{M}_\varepsilon)_{\varepsilon > 0}$ has asymptotically geometric distribution, more precisely it converges in distribution to \mathcal{M} with $\mathbb{P}(\mathcal{M} = k) = p^k(1 - p)$ for any $k \geq 0$; in particular the asymptotic value for the commutor function is

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\tau_{B_\varepsilon^0} < \tau_{B_\varepsilon^1}) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}(\mathcal{M}_\varepsilon = 0) = 1 - p.$$

In the case of a flow, if $(a_\varepsilon \tau_{\Omega \setminus B_\varepsilon^1})_{\varepsilon > 0}$ converges in probability \mathbb{P} to 0 and if $(\mathcal{L}_\varepsilon)_{\varepsilon > 0}$ supported on $[0, +\infty) \times \{0, 1\} \times \mathbb{R}_+$ converges in distribution with respect to \mathbb{P} to a PPP \mathcal{L}_0 with intensity $\lambda \times \sum_{j=0}^1 p_j(\delta_j \times m'_j)$ where the m'_j are probability measures, then $(a_\varepsilon \mathcal{M}'_\varepsilon)_{\varepsilon > 0}$ converges to $\sum_{i=1}^{\mathcal{M}} X_i$ where $(X_i)_i$ is a sequence of i.i.d. random variables with distribution m'_1 and independent of \mathcal{M} where \mathcal{M} is as above.

Proof We first observe that the mapping

$$J : \xi \in M_p([0, +\infty) \times \{0, 1\}) \mapsto \xi([0, \tau^0] \times \{1\})$$

is continuous where $\tau^0 = \sup\{t \geq 0 : \xi([0, t] \times \{0\}) = 0\}$ is continuous at a.e. realization ξ of $\chi := PPP(\lambda \times \mathcal{B}(p))$. Indeed, $\xi(\cdot \times \{0\})$ and $\xi(\cdot \times \{1\})$ are the realization of two homogeneous independent Poisson process hence τ^0 is a.s. not an atom of $\xi(\cdot \times \{1\})$. Observe that, in the case of a transformation, $\mathcal{M}_\varepsilon = J(\mathcal{N}_\varepsilon)$ and in the case of a flow $\mathbb{P}(\mathcal{M}_\varepsilon \neq J(\mathcal{N}_\varepsilon)) = \mathbb{P}(Y_0 \in B_\varepsilon^0 \cup B_\varepsilon^1) \rightarrow 0$. Therefore, by the continuous mapping theorem, $(\mathcal{M}_\varepsilon)_{\varepsilon > 0}$ converges in distribution to $G := J(\chi)$ as ε goes to 0.

We now compute the law of G . The first hazard τ^0 has an exponential distribution with parameter $1 - p$, while $\chi^1(\cdot) := \chi(\cdot \times \{1\})$ is a Poisson point process with intensity $p\lambda$, and the two are independent. Therefore, for any $k \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(G = k) &= \mathbb{P}(\chi^1([0, \tau^0]) = k) \\ &= \int_0^\infty e^{-pt} \frac{(pt)^k}{k!} (1 - p) e^{-(1-p)t} dt = (1 - p)p^k. \end{aligned}$$

This ends the proof of the first part of the Theorem. Let us now prove the last one. We use the fact that the mapping $J : \xi \in M_p([0, +\infty) \times \{0, 1\} \times \mathbb{R}_+) \mapsto \int_{[0, \tau^0] \times \{1\} \times [0, K_0]} z d\xi(t, j, z)$ is continuous at a.e. realization ξ of χ and conclude the proof as above by the continuous mapping theorem and the Slutsky theorem since $a_\varepsilon \mathcal{M}'_\varepsilon = \mathbf{1}_{\{Y_0 \notin B_\varepsilon^0\}} (J(\mathcal{L}_\varepsilon) + a_\varepsilon \tau_{\Omega \setminus B_\varepsilon^1})$. \square

Example 1 Consider the billiard flow $(Y_t)_t$ associated to a Sinai billiard with finite horizon in a domain $Q \subset \mathbb{T}^2$ (see Appendix for details). Let \mathbb{P} be any probability measure on $\Omega := Q \times S^1$ absolutely continuous with respect to Lebesgue. We fix two distinct point positions $q_0, q_1 \in Q$ and two positive real numbers $r_0, r_1 > 0$. Set $B_\varepsilon^i := B(q_i, r_i \varepsilon) \times S^1$ and $d_i = 2 - \mathbf{1}_{q_i \in \partial Q}$.

Then $(\mathcal{M}_\varepsilon)_{\varepsilon>0}$ converges in distribution with respect to \mathbb{P} to \mathcal{M} with $\mathbb{P}(\mathcal{M} = k) = p^k(1-p)$ for any $k \geq 0$ and with $p = \frac{d_1 r_1}{d_0 r_0 + d_1 r_1}$.

Moreover $(\varepsilon^{-1} \mathcal{M}'_\varepsilon)_{\varepsilon>0}$ converges in distribution with respect to \mathbb{P} to $r_1 \sum_{i=1}^{\mathcal{M}} Y_i$ where $(Y_i)_{i \geq 1}$ is a sequence of i.i.d. random variables with density $y \mapsto \frac{y}{\sqrt{1-y^2}} \mathbf{1}_{[0,1]}(y)$ independent of \mathcal{M} , with \mathcal{M} as above.

Proof Recall that the billiard flow Y preserves the normalized Lebesgue measure μ on $Q \times S^1$. In view of applying Theorem 3, observe first that $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(B_\varepsilon^0 \cup B_\varepsilon^1) = 0$ and $\mathbb{E}[\varepsilon^{-1} \tau_{\Omega \setminus B_\varepsilon^1}] \leq 2r_1 \mathbb{P}(B_\varepsilon^1)$, thus $(\varepsilon \tau_{\Omega \setminus B_\varepsilon^1})_{\varepsilon>0}$ converges in probability \mathbb{P} to 0.

As a direct consequence of Theorem 6, the family of spatio-temporal processes $(\mathcal{N}_\varepsilon)_{\varepsilon>0}$ given by (7), with $h_\varepsilon = \frac{(d_0 r_0 + d_1 r_1) \pi \varepsilon}{\text{Area}(Q)}$, converges in distribution to a PPP of intensity $\lambda \times \mathcal{B}(\frac{d_1 r_1}{d_0 r_0 + d_1 r_1})$ and so the first conclusion of Theorem 3 holds true with $p = \frac{d_1 r_1}{d_0 r_0 + d_1 r_1}$. This ends the proof of the convergence of $(\mathcal{M}_\varepsilon)_{\varepsilon>0}$.

Due to Theorem 5, $(\mathcal{L}_\varepsilon)_{\varepsilon>0}$ (with $a_\varepsilon = \varepsilon$ and h_ε as before) converges in distribution to a PPP with intensity $\lambda \times \sum_{j=0}^1 p_j (\delta_j \times m'_j)$ where $p_j := \frac{d_j r_j}{d_0 r_0 + d_1 r_1}$ and where m'_j has density $y \mapsto \frac{y}{2r_j \sqrt{4r_j^2 - y^2}} \mathbf{1}_{[0, 2r_j]}(y)$. Thus the last conclusion of Theorem 3 holds also true with these notations. We conclude by taking $Y_i = X_i/(2r_1)$. \square

4 Number of high records

We define the high records point process by

$$\mathcal{R}_f(u, \ell) = \sum_{k=1}^{\infty} \delta_{ku} \mathbf{1}_{\{f \circ T^k > \max(\ell, f, \dots, f \circ T^{k-1})\}}.$$

The successive times of records of an observable along an orbit are obviously tractable from the time and values of the observations along this orbit. The following proposition states that this is still the case for the corresponding asymptotic distributions. This has already been noticed in [12], in particular in the context of Extremal events. Our result is similar to the proof of [12, Theorem 3.1] from [12, Theorem 5.1].

Proposition 1 *Let $(\Omega, \mathcal{F}, \mu, T)$ be a probability preserving dynamical system and $f : \Omega \rightarrow [0, +\infty)$ be a measurable function. Assume the family*

$(\mathcal{N}_\varepsilon = \mathcal{N}(T, \{f > \varepsilon^{-1}\}, h_\varepsilon, 1/(\varepsilon f)))_{\varepsilon > 0}$ of point processes on $[0, +\infty) \times [0, 1]$ converges in distribution with respect to \mathcal{P} to a Poisson point process of intensity $\lambda \times m$ with m a probability measure on $[0, 1]$ without any atom. Then $(\mathcal{R}_f(h_\varepsilon, \varepsilon^{-1}))_{\varepsilon > 0}$ converges in distribution, as $\varepsilon \rightarrow 0$ to a Point process $\mathcal{R} = \sum_{\ell=1}^{\infty} Z_\ell \delta_{T_\ell}$ where $T_\ell = \sum_{i=1}^{\ell} X_i$, the X_i are independent standard exponential random variable and the Z_ℓ are independent random variable having Bernoulli distribution with respective parameters ℓ^{-1} , and the two sequences are independent.

Proof Define the mapping

$$F: \xi = \sum_i \delta_{(t_i, v_i)} \in M_p([0, \infty) \times [0, 1]) \mapsto \sum_{i \in I(\xi)} \delta_{t_i},$$

where $I(\xi)$ are the records of ξ , defined by those i such that for any j one has $t_j < t_i \implies v_j > v_i$. The map F is continuous at each ξ such that the t_i 's, and the v_i 's, are distincts. This is the case for a.e. realization ξ of a Poisson process of intensity $\lambda \times m$. Therefore by the continuous mapping theorem $\mathcal{R}_f(h_\varepsilon, \varepsilon^{-1}) = F(\mathcal{N}_\varepsilon)$ converges to $\chi = F(\text{PPP}(\lambda \times m))$.

We are left to compute the distribution. Observe that $\text{PPP}(\lambda \times m)$ is distributed as $\sum_{\ell=1}^{\infty} \delta_{(T_\ell, W_\ell)}$ with (T_ℓ) as in the statement and the W_ℓ are i.i.d. with distribution m , the two sequences being independent. Let $Z_\ell = 1_{\{W_\ell \text{ is a record}\}}$. By [23, Proposition 4.3] the Z_ℓ are independent, have probability $1/\ell$, and when $Z_\ell = 1$ we keep the point T_ℓ . \square

In particular, for every $t > 0$ the number of records exceeding the value ε^{-1} before the time th_ε^{-1} corresponds to $\mathcal{R}_f(h_\varepsilon, \varepsilon^{-1})([0, t])$ and the conclusion of Proposition 1 implies that it converges to $\sum_{\ell=1}^{N_t} Z_\ell$ where Z_ℓ are as in Proposition 1 and where $(N_s)_{s \geq 0}$ is a standard Poisson Process independent of $(Z_\ell)_{\ell \geq 1}$.

Example 2 Consider a dispersive billiard with corners and cusps of maximal order $\beta_* > 2$ as in [13]. Consider the induced system (Ω, μ, T) corresponding to the successive reflection times outside a neighbourhood \mathcal{U} of cusps and write $R(x)$ for the number of reflections in \mathcal{U} starting from x . Set $\alpha = \frac{\beta_*}{\beta_* - 1} \in (1, 2)$.

Setting $A_\varepsilon := \{R \circ T^{-1} > \varepsilon^{-1}\}$, it has been proved in [13, Lemma 4.5] that there exists an explicit $c_0 > 0$ such that $\mu(A_\varepsilon) \sim c_0 \varepsilon^\alpha$ as $\varepsilon \rightarrow 0$.

The assumptions of Proposition 1 hold true with $f = R \circ T^{-1}$ and $h_\varepsilon = \mu(A_\varepsilon) \sim c_0 \varepsilon^\alpha$. So the same assumptions hold true with $h_\varepsilon = c_0 \varepsilon^\alpha$.

Furthermore the number R_n of records of R higher than $n^{1/\alpha}$ before the n -th reflection outside cusps converges to $\sum_{\ell=1}^N Z_\ell$ where Z_ℓ are as in Proposition 1 and where N is a Poisson random variable of parameter c_0 and independent of $(Z_\ell)_{\ell \geq 1}$.

Proof It follows from the proof of [13, Lemma 4.8] that² the family of point processes $(\mathcal{N}(T, A_\varepsilon, \mu(A_\varepsilon), \varepsilon R \circ T^{-1}))_{\varepsilon > 0}$ on $[0, +\infty) \times [1, +\infty]$ converges in distribution to a PPP with intensity of density $(t, y) \mapsto \alpha y^{-\alpha-1} \mathbf{1}_{y>0}$ with respect to the Lebesgue measure.

Therefore the assumptions of Proposition 1 hold true with $f = R \circ T^{-1}$ and $h_\varepsilon = \mu(A_\varepsilon) \sim c_0 \varepsilon^\alpha$. So the same assumptions hold true with $h_\varepsilon = c_0 \varepsilon^\alpha$. This ends the proof of the first part.

For the second one, we apply Proposition 1 with $\varepsilon = n^{-\frac{1}{\alpha}}$. \square

5 Line process of random geodesics

We study the line process generated by a geodesic as in [2] and recover their main result. Let N be a compact Riemannian surface of negative curvature. The geodesic flow $(Y_t)_t$ on the unit tangent bundle $\Omega = T^1 N$ preserves the Liouville measure μ . Let $\pi_N: T^1 N \rightarrow N$ be the canonical projection $(q, v) \mapsto q$. We denote by $D(q, \varepsilon)$ the ball in N of radius ε . We now state the main theorem, postponing the details and precise definitions thereafter.

Theorem 4 *Fix $q_0 \in N$. For any $a > 0$, the intersection of the neighborhood $D(q_0, \varepsilon)$ with the geodesic segment $\pi_N(\{Y_t(x), 0 \leq t \leq a\varepsilon^{-1}\})$, where x is taken at random on (Ω, μ) , converges in distribution, after normalization, as $\varepsilon \rightarrow 0$, to a Homogeneous Poisson line process in the unit disk of intensity $a/\text{Area}(N)$.*

A Poisson line process in the unit disk D of the plane, of intensity $\kappa \in (0, \infty)$, is a probabilistic process which draw lines in the disk. Each line L is parametrized by $(r, \theta) \in [-1, 1] \times [0, \pi]$ where

$$L = \{(x, y) \in D: r = x \cos \theta + y \sin \theta\},$$

and the parameters (r, θ) are produced by a Poisson point process of intensity $\frac{\kappa}{\pi} dr d\theta$ on $[-1, 1] \times [0, \pi]$. Equivalently, changing the parametrization to (s, φ) where $s \in \partial D =: S$ is one point of intersection of the line with the unit circle and φ is the angle between the line L (directed into the disk) and the normal at s pointing inside the disk (see Figure 1), gives a Poisson point process of intensity $\frac{\kappa \cos \varphi}{2\pi} ds d\varphi$ (the jacobian is $\cos \varphi$ and each line has two representations in this parametrization). The intensity κ in the theorem is equal to $a/\text{Area}(N)$, therefore the intensity in this parametrization will be $\frac{a}{2\pi \text{Area}(N)} \cos \varphi ds d\varphi = \frac{a}{\text{Vol}(T^1 N)} \cos \varphi ds d\varphi$. The convergence of a point process in this parametrization implies it in the original one (by continuity of the change of parameter; see [23, Proposition 3.18]).

² [13, Lemma 4.8] states that this convergence is true in the set of point processes on $[0, +\infty) \times [1, +\infty)$, but its proof can be adapted in a straightforward way to obtain our purpose by considering not only intervals of the form (c, c') but also intervals of the form $(c, +\infty]$.

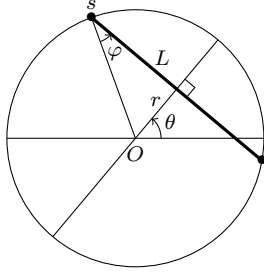


Fig. 1 Parametrization of the line L by (r, θ) or (s, φ) .

The exponential map \exp_{q_0} is a local diffeomorphism on a neighborhood $U \subset T_{q_0}N$ of 0. Thus its inverse is well defined on $D(q_0, \varepsilon)$ for ε small enough so that $B(0, \varepsilon) \subset U$. We identify $T_{q_0}N$ with \mathbb{R}^2 . Set $V = S \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. For $q \in D(q_0, \varepsilon)$ we let $s_\varepsilon(q) = \varepsilon^{-1} \exp_{q_0}^{-1}(q)$ and for $q \in \partial D(q_0, \varepsilon)$ and $v \in T_q N$ we denote by $\phi_q(v)$ the angle between the normal at q pointing inside the disk and v (see Figure 2).

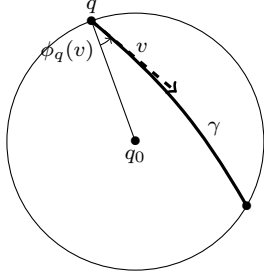


Fig. 2 A geodesic arc γ entering the ball $D(q_0, \varepsilon)$.

The intersection $\mathcal{I}_\varepsilon^a(x) := \pi_N(Y_{[0, a\varepsilon^{-1}]}(x)) \cap D(q_0, \varepsilon)$ consists of finitely many geodesic arcs $\gamma_i := \pi_N(Y_{[t_i, t_i + \ell_i]}(x))$, where ℓ_i is the length of the arc; we drop the dependence on x and ε for simplicity. The arcs γ_i are fully crossing the ball, except possibly for the two extremities (at $t = 0$ or $t = a\varepsilon^{-1}$) which could give an incomplete arc. The latter happens with a vanishing probability as $\varepsilon \rightarrow 0$, therefore we will ignore this eventuality. The arc γ_i enters the ball at the position q_i with direction v_i where $(q_i, v_i) := Y_{t_i}(x)$.

When $\varepsilon \rightarrow 0$, the geodesic arcs γ_i which compose the intersection $\mathcal{I}_\varepsilon^a$ become more and more straight. This justifies the definition of the convergence in distribution of $\mathcal{I}_\varepsilon^a$ as the convergence in distribution of the point process

$$\sum_i \delta_{(s_\varepsilon(q_i), \phi_q(v_i))}. \quad (11)$$

Loosely speaking, we identify the images $s_\varepsilon(\gamma_i)$ with the chord of the unit disk D originated in $s_\varepsilon(q_i)$ and direction v_i .

We now proceed with the proof of the theorem. Let $\mathcal{A}_\varepsilon \subset T^1N$ be the set of points (q, v) such that $q \in \partial D(q_0, \varepsilon)$ and v is pointing inside the ball. We define on \mathcal{A}_ε

$$\mathcal{H}_\varepsilon(q, v) = (s_\varepsilon(q), \phi_q(v)) \in V. \quad (12)$$

The theorem is a byproduct of the following result for the geodesic flow.

Proposition 2 *The process of entrances in the ball for the position for the geodesic flow $\mathcal{N}(Y, \mathcal{A}_\varepsilon, 2\varepsilon/\text{Area}(N), \mathcal{H}_\varepsilon)$ on $[0, +\infty[\times V$ converges to a Poisson point process with intensity $\frac{1}{4\pi} \cos \varphi dt ds d\varphi$.*

Proof (Proof of Theorem 4) The counting process

$$\mathcal{L}_\varepsilon^a(\cdot) := \mathcal{N}(Y, \mathcal{A}_\varepsilon, 2\varepsilon/\text{Area}(N), \mathcal{H}_\varepsilon)([0, 2a/\text{Area}(N)] \times \cdot) \quad (13)$$

produces a point (s, φ) each time that the geodesic flow Y_t enters in $D(q_0, \varepsilon)$ for some t such that $2\varepsilon t/\text{Area}(N) \leq 2a/\text{Area}(N)$, that is $t \leq a\varepsilon^{-1}$. By Proposition 2 and the continuous mapping theorem the point process $\mathcal{L}_\varepsilon^a$ converges to a Poisson point process of intensity $\frac{2a}{\text{Area}(N)} \frac{1}{4\pi} \cos \varphi ds d\varphi$. By the above discussion, in particular (11), this completes the proof of the theorem. \square

We emphasize that this proof only uses the convergence stated in Proposition 2, therefore it applies for more general 'geodesic-like flows', for instance the argument applies immediately to billiards systems, using Theorem 6 in place of Proposition 2.

Proof (Proof of Proposition 2) The first step is to construct a Markov section for the geodesic flow, subordinated to a finite family of disks $D_i \subset T^1N$. Fix some $\delta > 0$ sufficiently small. By Bowen [3] there exists a Markov section $(X_i)_i$ of size δ , in particular $\text{diam } X_i < \delta$ and $T^1N = \cup_i Y_{[-\delta, 0]}(X_i)$. One can choose the disks $D_i \supset X_i$ in such a way that

$$D_i \subset \{(q, v) : q \in Q_i, |\angle(n_q, v)| > \frac{\pi}{2} - \delta\}$$

where Q_i are C^2 curve in N and n_q is the normal vector to Q_i at q (with $q \mapsto n_q$ continuous). Without loss of generality we assume that $q_0 \notin \cup_i Q_i$.

The flow (Y_t) is represented by a special flow over the Poincaré section $M := \cup_i X_i$, with a C^2 roof function τ . Let Π be the projection onto M along the flow in backward time. The flow $(T^1N, (Y_t), \mu)$ projects down to a system (M, F, ν) , conjugated to a subshift of finite type with a Gibbs measure of a Hölder potential. In order to apply Theorem 2 we need to check that the set $A_\varepsilon := \Pi \mathcal{A}_\varepsilon$ and $H_\varepsilon(x) := \mathcal{H}_\varepsilon(Y_s(x))$ where $s > 0$ is the minimal time such that $Y_s(x) \in \mathcal{A}_\varepsilon$ fulfills the hypotheses of Theorem 1. For that we will apply [21, Proposition 3.2]. The Poincaré map F has a hyperbolic structure with an exponential rate, thus it satisfies the setting of [21, Proposition 3.2] with any polynomial rate α , in particular $\alpha = 4$ works. Here the boundary is meant in

the induced topology on M . It suffices to prove that for some $p_\varepsilon = o(\nu(A_\varepsilon))$ one has (i) $\nu(\tau_{A_\varepsilon} \leq p_\varepsilon) = o(1)$ and (ii) $\nu((\partial A_\varepsilon)^{[p_\varepsilon^{-1}]}) = o(\nu(A_\varepsilon))$, the two other assumptions being trivially satisfied in our situation.

Measure of A_ε : The Liouville measure μ is the product of the normalized surface on N times the Haar measure on T^1N . Its projection ν to the Poincaré section satisfies $d\nu = c_\nu \cos \varphi dr d\varphi$ for some normalizing constant $c_\nu = \left(\sum_i \int_{X_i} \cos \varphi dr d\varphi\right)^{-1}$, where r is the curvilinear abscissa on Q_i and φ the angle between the velocity and the normal to Q_i . Moreover we have $d\mu = (\int_M \tau d\nu)^{-1} d\nu \times dt|_{M_\tau}$ where $M_\tau = \{(x, t) : x \in M, 0 \leq t < \tau(x)\}$.

The geodesic flow preserves the measure $\cos \varphi dr d\varphi$ from $A_\varepsilon \subset M$ to \mathcal{A}_ε , therefore

$$\begin{aligned} \nu(A_\varepsilon) &= c_\nu \int_{A_\varepsilon} \cos \varphi dr d\varphi = c_\nu \int_{\mathcal{A}_\varepsilon} \cos \varphi dr d\varphi = c_\nu \int_{\partial D(q_0, \varepsilon)} dr \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \\ &\sim c_\nu 4\pi\varepsilon. \end{aligned}$$

Short returns: For any $q \in D(q_0, \varepsilon)$, let $R_\varepsilon(q)$ be the set of $v \in T_q^1N$ such that the geodesic segment $\gamma_{[0, \varepsilon^{-1/2}]}(q, v)$ enters again $D(q_0, \varepsilon)$ after leaving $D(q_0, 2\varepsilon)$. The result of [2, Lemma 5.3] ensures the existence of $K > 0$ such that for any $q \in D(q_0, \varepsilon)$

$$\text{Leb}(R_\varepsilon(q)) \leq K\varepsilon^{1-1/2} = K\sqrt{\varepsilon}.$$

Therefore, setting $\hat{\mathcal{A}}_\varepsilon = \{(q, v) \in \mathcal{A}_\varepsilon : v \in R_\varepsilon(q)\}$ we get that the two dimensional Lebesgue measure of $\hat{\mathcal{A}}_\varepsilon$ is $O(\varepsilon^{3/2})$. A fortiori since the projection Π preserves the measure $\cos \varphi dr d\varphi$ we get

$$\nu(\Pi \hat{\mathcal{A}}_\varepsilon) = c_\nu \int_{\Pi \hat{\mathcal{A}}_\varepsilon} \cos \varphi dr d\varphi = c_\nu \int_{\hat{\mathcal{A}}_\varepsilon} \cos \varphi dr d\varphi = O(\varepsilon^{3/2}).$$

Let $p_\varepsilon = \lfloor (\max \tau)^{-1} \varepsilon^{-1/2} \rfloor$ and notice that $A_\varepsilon \cap \{\tau_{A_\varepsilon} \leq p_\varepsilon\} \subset \Pi \hat{\mathcal{A}}_\varepsilon$. By the previous estimates we get

$$\nu(A_\varepsilon \cap \{\tau_{A_\varepsilon} \leq p_\varepsilon\}) = O(\varepsilon^{3/2}).$$

Hence

$$\nu(\tau_{A_\varepsilon} \leq p_\varepsilon | A_\varepsilon) = o(1).$$

This is the assumption (i).

We now prove (ii). The boundary of A_ε in the induced topology of M is included in the set of $\Pi(q, v)$ where v is tangent to the boundary of $\partial D(q_0, \varepsilon)$. This defines for each i such that $X_i \cap A_\varepsilon$ is nonempty and contains at most two C^2 curves in D_i of finite length (by transversality), therefore its ε^2 -neighborhood has a measure $O(\varepsilon^2)$.

Finally, the measure $dm_\varepsilon = (H_\varepsilon)_* \nu(\cdot | A_\varepsilon)$ is equal to the measure $dm := \frac{1}{4\pi} \cos \varphi ds d\varphi$, since the measure $\cos \varphi dr d\varphi$ is preserved by the inverse of

the projection Π from A_ε to \mathcal{A}_ε and \mathcal{H}_ε has constant jacobian ε in these coordinates. By Theorem 1 the point process $\mathcal{N}(F, A_\varepsilon, \nu(A_\varepsilon), H_\varepsilon)$ converges to a Poisson point process of intensity $\lambda \times m$. Applying Theorem 2 with $h_\varepsilon = c_\nu 4\pi\varepsilon$ and $h'_\varepsilon = h_\varepsilon/E_\nu(\tau)$ we get that $\mathcal{N}(Y, \mathcal{A}_\varepsilon, h'_\varepsilon, \mathcal{H}_\varepsilon)$ converges to a Poisson point process of intensity $\lambda \times m$. In addition,

$$\int_M \tau d\nu = c_\nu \int_M \tau \cos \varphi dr d\varphi = c_\nu \int_{M_\tau} \cos \varphi dt dr d\varphi = c_\nu \text{Vol}(T^1 N).$$

Thus, since $\text{Vol}(T^1 N) = 2\pi \text{Area}(N)$ we get that $h'_\varepsilon = \frac{2\varepsilon}{\text{Area}(N)}$, proving the proposition. \square

6 Time spent by a flow in a small set

Given a flow $Y = (Y_t)_t$ defined on Ω and a set $A \subset \Omega$, a very natural question is to study the time spent by the flow in the set A , that is the local time $L_T(A)$ given by following quantity :

$$L_T(A) := \lambda(\{t \in [0, T] : Y_t \in A\}).$$

This quantity measures the time spent by the flow Y in the set A between time 0 and time T (the symbol L refers to the local time). We also write $D_A := \inf\{t > 0 : Y_t \notin A\}$ for the duration of the present visit to the set A .

Proposition 3 *Let $J \geq 1$ and $Y = (Y_t)_{t \geq 0}$ be a flow defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $(\mathcal{N}_\varepsilon = \mathcal{N}(Y, A_\varepsilon, h_\varepsilon, H_\varepsilon))_{\varepsilon > 0}$ converges in distribution (with respect to \mathbb{P}) to a PPP \mathcal{N}_0 of intensity $\lambda \times m$ with $H_\varepsilon(A_\varepsilon) \subset V = \{1, \dots, J\} \times W$ where $m = \sum_{j=1}^J (p_j \delta_j \times m_j)$, with $\sum_{j=1}^J p_j = 1$ and where m_j are probability measure on some separable metric space W . Suppose in addition that, for some a_ε and each x entering in A_ε , $a_\varepsilon D_{A_\varepsilon}(x) = \mathfrak{D}_\varepsilon(H_\varepsilon(x))$ with $\lim_{\varepsilon \rightarrow 0} \mathfrak{D}_\varepsilon(j, w) =: \mathfrak{D}_j(w)$ uniformly in $w \in W$, where $\mathfrak{D}_j : W \rightarrow \mathbb{R}_+$ is continuous.*

Then

$$\left(\mathcal{L}_\varepsilon = \sum_{t : Y_t(x) \text{ enters } A_\varepsilon} \delta_{th_\varepsilon, H_\varepsilon^{(1)}(Y_t(x)), a_\varepsilon D_{A_\varepsilon} \circ Y_t(x)} \right)_{\varepsilon > 0}$$

converges in distribution with respect to \mathbb{P} to a PPP \mathcal{L}_0 on $[0, +\infty) \times \{1, \dots, J\} \times \mathbb{R}_+$ with intensity $\lambda \times \sum_{j=1}^J (p_j \delta_j \times (\mathfrak{D}_j)_(m_j))$.*

If moreover $a_\varepsilon D_{A_\varepsilon} \xrightarrow{\mathbb{P}} 0$, setting $L_T^{(i)}(A_\varepsilon) := L_T(A_\varepsilon \cap H_\varepsilon^{-1}(\{i\} \times W))$, then, for every $T > 0$, $((a_\varepsilon L_{[t/h_\varepsilon]}^{(j)}(A_\varepsilon))_{t \in [0, T], j=1, \dots, J})_{\varepsilon > 0}$ converges in distribution to $\left(\sum_{k=1}^{N_t^{(j)}} X_k^{(j)} \right)_{t \in [0, T], j=1, \dots, J}$ as $\varepsilon \rightarrow 0$, where $(N_t^{(j)})_{t > 0}$ are independent

Poisson process with parameter p_j and where $(X_k^{(j)})_{k \geq 1}$ are independent sequences of independent identically distributed random variables with distribution $\sum_{j=1}^J p_j (\mathfrak{D}_j)_*(m_j)$ independent of $(N_t^{(j)})_{t > 0}$

Proof Observe that, for every $\epsilon \geq 0$, $\mathcal{L}_\epsilon = (\psi_\epsilon)_*(\mathcal{N}_\epsilon)$ with $\psi_\epsilon : (t, j, w) \mapsto (t, j, \mathfrak{D}_\epsilon(j, w))$ if $\epsilon > 0$ and with $\psi_0 : (t, j, w) \mapsto (t, j, \mathfrak{D}_j(w))$. Using [23, Proposition 3.13] we prove the first statement.

Assume now that $a_\epsilon D_{A_\epsilon} \xrightarrow{\mathbb{P}} 0$. Then

$$\left(a_\epsilon L_{t/h_\epsilon}^{(j)} = a_\epsilon \min(t/h_\epsilon, D_{A_\epsilon \cap H_\epsilon^{-1}(\{j\} \times W)}) + \int_{[0, t] \times \{j\} \times \mathbb{R}_+} z d\mathcal{L}_\epsilon(s, i, z) \right)_{t \in [0, T], j=1, \dots, J}$$

which converges to $\left(a_\epsilon L_{t/h_\epsilon}^{(j)} = \int_{[0, t] \times \{j\} \times \mathbb{R}_+} z d\mathcal{L}_0(s, i, z) \right)_{t \in [0, T], j=1, \dots, J}$. \square

We apply the previous result to the dispersive billiard flow in a Sinai billiard with finite horizon.

Theorem 5 (Time spent by the billiard flow in a shrinking ball for the position)

Consider the billiard flow associated to a Sinai billiard with finite horizon in a domain $Q \subset \mathbb{T}^2$ (see Appendix for details). Recall that this flow preserves the normalized Lebesgue measure on $Q \times S^1$. Let J be a positive integer. Let $q_1, \dots, q_J \in Q$ be a J pairwise distinct fixed position in the billiard domain and r_1, \dots, r_J be J positive real numbers. We set $d_j = 2$ if $q_j \notin \partial Q$ and $d_j = 1$ if $q_j \in \partial Q$ and $d := \sum_{j=1}^J d_j r_j$ and also

$$\left(\mathcal{L}_\epsilon = \sum_{j=1}^J \sum_{t: Y_t(x) \text{ enters } B(q_j, r_j \epsilon) \times S^1} \delta_{\frac{d\pi\epsilon t}{\text{Area}(Q)}, j, \epsilon^{-1} D_{B(q_j, r_j \epsilon) \times S^1} \circ Y_t(x)} \right)_{\epsilon > 0}$$

and

$$L_{t/\epsilon}^{(j)} := \int_0^{\frac{t}{\epsilon}} \mathbf{1}_{\{Y_s(\cdot) \in B(q_j, r_j \epsilon) \times S^1\}} ds.$$

Then, $(\mathcal{L}_\epsilon)_{\epsilon > 0}$ converges strongly in distribution to a PPP \mathcal{L}_0 with intensity $\lambda \times \sum_{j=1}^J \frac{d_j r_j}{d} (\delta_j \times m'_j)$ where m'_j is the distribution of $r_j X$ with X a random variable of density $y \mapsto \frac{y}{4} \cdot \arccos'(\frac{y}{2}) \mathbf{1}_{[0, 2]}(y) = \frac{y}{2\sqrt{4-y^2}} \mathbf{1}_{[0, 2]}(y)$.

Moreover, for every $T > 0$, $((\epsilon^{-1} L_{t/\epsilon}^{(j)})_{t \in [0, T], j=1, \dots, J})_{\epsilon > 0}$ converges strongly in distribution to $\left(r_j \sum_{k=1}^{N_t^{(j)}} X_k^{(j)} \right)_{t \in [0, T], j=1, \dots, J}$ as $\epsilon \rightarrow 0$, where $(N_t^{(j)})_{t > 0}$ are

independent Poisson process with parameter $\frac{d_j \text{Area}(Q)}{d^2 \pi}$, where $(X_k)_{k \geq 1}$ is a sequence of independent identically distributed random variables with density $x \mapsto \frac{x}{2\sqrt{4-x^2}} \mathbf{1}_{[0, 2]}(x)$ independent of $(N_t)_{t > 0}$.

Proof (Proof of Theorem 5) Due to Theorem 6, we know that the family of processes

$$\sum_{j=1}^J \sum_{t : (Y_s(y))_s \text{ enters } B(q_j, \varepsilon) \times S^1 \text{ at time } t} \delta_{\left(\frac{d\pi_{\varepsilon} t}{\text{Area}(Q)}, \frac{\Pi_Q(Y_t(y)) - q_j}{\varepsilon}, \Pi_V(Y_t(y))\right)}$$

converges in distribution (when y is distributed with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on \mathcal{M}) as $\varepsilon \rightarrow 0$ to a Poisson Point Process with intensity $\lambda \times \tilde{m}_0$ where \tilde{m}_0 is the probability measure on $\{1, \dots, J\} \times S^1 \times S^1$ with density $(j, p, \mathbf{u}) \mapsto \sum_{j=1}^J \frac{r_j d_j}{d} \frac{1}{2d_j \pi} \langle (-p), \mathbf{u} \rangle^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_j} \rangle \geq 0\}}$ with $d := \sum_{j=1}^J d_j r_j$.

We will apply Proposition 3 with $A_\varepsilon := \bigcup_{j=1}^J B(q_j, \varepsilon r_j) \times S^1$ and $H_\varepsilon(q, \mathbf{v}) = \left(j, \frac{\overrightarrow{q_j q}}{r_j \varepsilon}, \mathbf{v}\right)$ if $q \in \partial B(q_j, r_j \varepsilon)$.

Let $x = (q, \mathbf{v})$ entering in $B(q_j, \varepsilon) \times S^1$. If the billiard flow crosses $B(q_j, \varepsilon) \times S^1$ before any collision with ∂Q , then

$$\varepsilon^{-1} D_{B(q_j, r_j \varepsilon) \times S^1}(q, \mathbf{v}) = 2\varepsilon^{-1} \widehat{(\overrightarrow{q q_j}, \mathbf{v})} = D_0(H_\varepsilon(x)),$$

with $D_0(j, p, \mathbf{u}) = 2r_j \widehat{(-p, \mathbf{u})}$. This is always the case if $q_j \notin \partial Q$. But, if $q_j \in \partial Q$, it can also happen that the billiard flow collides with ∂Q at a point $q' \in B(q_j, \varepsilon)$ before exiting $B(q_j, \varepsilon) \times S^1$. Then the point q' is at distance in $\mathcal{O}(\varepsilon^2)$ of the tangent line to ∂Q at q_j , and the tangent line of ∂Q at q' makes an angle in $\mathcal{O}(\varepsilon)$ with the tangent line of ∂Q at q_j . In this case

$$\varepsilon^{-1} D_{B(q_j, r_j \varepsilon) \times S^1}(q, \mathbf{v}) = 2\varepsilon^{-1} \widehat{(\overrightarrow{q q_j}, \mathbf{v})} + \mathcal{O}(\varepsilon) = D_0(H_\varepsilon(x)) + \mathcal{O}(\varepsilon),$$

uniformly in $x = (q, \mathbf{v})$ and ε . In any case, we set $a_\varepsilon = \varepsilon^{-1}$ and $\mathfrak{D}_\varepsilon = D_0 + \mathcal{O}(\varepsilon)$.

Applying now Proposition 3, we infer that $(\mathcal{L}_\varepsilon)_{\varepsilon > 0}$ converges strongly in distribution to a PPP \mathcal{L}_0 with intensity $\lambda \times \sum_{j=1}^J \frac{d_j r_j}{\sum_{j'=1}^J d_{j'} r_{j'}} (\delta_j \times (D_j)_*(m_j))$,

with $D_j(p, \mathbf{u}) = 2r_j \widehat{(-p, \mathbf{u})}$ and m_j the probability measure on $S^1 \times S^1$ with density

$$(p, \mathbf{u}) \mapsto \frac{d_j}{2\pi} \langle (-p), \mathbf{u} \rangle^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_j} \rangle \geq 0\}}.$$

It remains to identify the distribution $(D_j)_*(m_j)$. By the transfer formula, we obtain

$$\begin{aligned}
\int_0^\infty h(D_j(p, \mathbf{u})) dm_j(p, \mathbf{u}) &= \frac{1}{2d_j\pi} \int_{S^1 \times S^1} h(r_j \langle -2p, \mathbf{u} \rangle) \langle (-p), \mathbf{u} \rangle^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_j} \rangle \geq 0\}} dp d\mathbf{u} \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(2r_j \cos \varphi) \cos \varphi d\varphi \\
&= \int_0^{\frac{\pi}{2}} h(2r_j \cos \varphi) \cos \varphi d\varphi \\
&= \int_0^2 h(r_j y) (\arccos(\cdot/2))'(y) \frac{y}{2} dy.
\end{aligned}$$

Thus we have proved that the probability distribution $(D_j)_* \tilde{m}_j$ is the distribution of $r_j X$ with X a random variable of density $y \mapsto \frac{y}{4} \cdot \arccos'(\frac{y}{2}) \mathbf{1}_{[0,2]}(y) = \frac{y}{2\sqrt{4-y^2}} \mathbf{1}_{[0,2]}(y)$.

We can apply the last point of Proposition 3 since $\varepsilon^{-1} D_{A_\varepsilon} \leq 2 \max_j r_j \mathbf{1}_{A_\varepsilon} \xrightarrow{\mathbb{P}} 0$ for any probability measure \mathbb{P} absolutely continuous with respect to the Lebesgue measure on $Q \times S^1$. \square

Appendix. Visits by the Sinai flow to a finite union of balls in the billiard domain

In this appendix we are interested in spatio temporal processes for the Sinai billiard flow with finite horizon.

Let us start by recalling the model and introducing notations. We consider a finite family $\{O_i, i = 1, \dots, I\}$ of convex open sets of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We consider the billiard domain $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^I O_i$ and call the O_i obstacles. We assume that these obstacles have C^3 -smooth boundary with non null curvature and that their closures are pairwise disjoint. We consider a point particle moving in Q in the following way: the point particle goes straight at unit speed in Q and obeys the classical Descartes reflexion law when it collides with an obstacle. We then define the billiard flow $(Y_t)_{t \in \mathbb{R}}$ as follows. $Y_t(q, \mathbf{v}) = (q_t, v_t)$ is the couple position-velocity of the point particle at time t if the particle has position q and velocity \mathbf{v} at time 0. To avoid any confusion, we consider the billiard flow being defined on the quotient $(Q \times S^1)/\mathcal{R}$, with \mathcal{R} is the equivalence relation corresponding to the identification of pre-collisional and post-collisional vectors at a reflection time:

$$(q, \mathbf{v}) \mathcal{R} (q', \mathbf{v}') \Leftrightarrow (q, \mathbf{v}) = (q', \mathbf{v}') \quad \text{or} \quad \mathbf{v}' = \mathbf{v} - 2\langle \mathbf{n}_q, \mathbf{v} \rangle \mathbf{n}_q,$$

where \mathbf{n}_q is the unit normal vector to ∂Q at q directed inward Q if $q \in \partial Q$, with convention $\mathbf{n}_q = 0$ if $q \notin \partial Q$. This flow preserves the normalized Lebesgue measure μ on $Q \times S^1$.

We assume moreover that every billiard trajectory meets ∂Q (finite horizon assumption).

Let us write $\Pi_Q : Q \times S^1 \rightarrow Q$ and $\Pi_V : Q \times S^1 \rightarrow S^1$ for the canonical projections given respectively by $\Pi_Q(q, \mathbf{v}) = q$ and $\Pi_V(q, \mathbf{v}) = \mathbf{v}$.

Theorem 6 (Visits of the billiard flow to a finite union of shrinking balls in the billiard domain)

Let $q_1, \dots, q_J \in Q$ be pairwise distinct positions in the billiard domain and r_1, \dots, r_J be positive real numbers. We set $d_j = 2$ if $q_j \notin \partial Q$ and $d_j = 1$ if $q_j \in \partial Q$ and $d = \sum_{j=1}^J d_j r_j$.

Then, the family of processes

$$\sum_{j=1}^J \sum_{t : (Y_s(y))_s \text{ enters } B(q_j, \varepsilon r_j) \times S^1 \text{ at time } t} \delta_{\left(\frac{d\pi\varepsilon t}{\text{Area}(Q)}, j, \frac{\Pi_Q(Y_t(y)) - q_j}{r_j \varepsilon}, \Pi_V(Y_t(y)) \right)}$$

converges in distribution (when y is distributed with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on \mathcal{M}) as $\varepsilon \rightarrow 0$ to a Poisson Point Process with intensity $\lambda \times \tilde{m}_0$ where \tilde{m}_0 is the probability measure on $V := \{1, \dots, J\} \times S^1 \times S^1$ with the density $(j, p, \mathbf{u}) \mapsto \frac{r_j}{2d\pi} \langle (-p), \mathbf{u} \rangle^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_j} \rangle \geq 0\}}$.

Observe that if $q_j \in \partial Q$, the set of $p \in S^1$ satisfying $\langle p, \mathbf{n}_{q_j} \rangle \geq 0$ is a semicircle, whereas it is the full circle S^1 when q_j is in the interior of Q .

This result has already been proved in [21, Theorem 4.4] for $J = 1$ and Lebesgue-almost every position q_1 . The extension to a finite number of points is relatively easy. The most difficult part is to treat all the possible positions in the billiard domain.

Along the paper we provided various applications of this theorem to different questions. We present here a result on the closest approaches to a given point in the billiard table by the orbit of the billiard flow.

Example 3 Consider the billiard flow associated with the Sinai billiard having finite horizon in a domain $Q \subset \mathbb{T}^2$. Consider a fixed position $q_0 \in Q$. Set $d = 2 - \mathbf{1}_{q_0 \in \partial Q}$. During each visit of the flow to $B(q_0, \varepsilon)$, the closest distance to q_0 is given by $L_0(q, \mathbf{v}) := \varepsilon |\sin \angle(\vec{qq_0}, \mathbf{v})|$ where (q, \mathbf{v}) is the entry point. Then the family of closest approach point process

$$(\mathcal{C}_\varepsilon := \mathcal{N}(Y, B(q_0, \varepsilon) \times S^1, d\varepsilon / \text{Area}(Q), \varepsilon^{-1} L_0))_{\varepsilon > 0}$$

on $[0, +\infty) \times [0, 1]$ converges in distribution (with respect to any probability measure absolutely continuous with respect to the Lebesgue measure on $Q \times S^1$) to a PPP with intensity 1.

Proof Due to Theorem 6, the family of spatio-temporal processes

$$(\mathcal{N}_\varepsilon := \mathcal{N}(Y, B(q_0, \varepsilon) \times S^1, d\varepsilon / \text{Area}(Q), H_\varepsilon))_{\varepsilon > 0}$$

with $H_\varepsilon(q, \mathbf{v}) = (\varepsilon^{-1} \overrightarrow{q_0 q}, \mathbf{v})$ converges in distribution (with respect to any probability measure absolutely continuous with respect to Lebesgue on $Q \times S^1$) to a PPP of intensity $\lambda \times \tilde{m}_0$ where \tilde{m}_0 is the probability measure on $S^1 \times S^1$ with the density $(p, \mathbf{u}) \mapsto \frac{1}{2d\pi} \langle (-p), \mathbf{u} \rangle^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_0} \rangle \geq 0\}}$ (where \mathbf{n}_{q_0} is the unit normal vector to ∂Q at q_0 directed inward Q if $q_0 \in \partial Q$, $\mathbf{n}_{q_0} = 0$ otherwise).

Observe that

$$\mathcal{C}_\varepsilon = \tilde{G}(\mathcal{N}_\varepsilon),$$

with $\tilde{G}(t, p, \mathbf{u}) = (t, G(p, \mathbf{u}))$ where $G(p, \mathbf{u}) = (t, |\sin \angle(-p, \mathbf{u})|)$. Thus $(\mathcal{C}_\varepsilon)_{\varepsilon>0}$ converges strongly in distribution to the PPP with intensity $\lambda \times G_*(\tilde{m}_0)$ and it remains to identify $\tilde{m}_1 = G_*(\tilde{m}_0)$. Due to the transfer formula, we obtain

$$\begin{aligned} \int_0^\infty h(G(p, \mathbf{u})) d\tilde{m}_0(p, \mathbf{u}) &= \frac{1}{2d\pi} \int_{S^1 \times S^1} h(|\sin \angle(-2p, \mathbf{u})|) (\cos \angle(-p, \mathbf{u}))^+ \mathbf{1}_{\{\langle p, \mathbf{n}_{q_j} \rangle \geq 0\}} dp d\mathbf{u} \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(|\sin \varphi|) \cos \varphi d\varphi \\ &= \int_0^{\frac{\pi}{2}} h(\sin \varphi) \cos \varphi d\varphi = \int_0^1 h(y) dy. \end{aligned}$$

Proof (Proof of Theorem 6) Due to [28, Theorem 1], it is enough to prove the result for the convergence in distribution with respect to μ . Assume $\varepsilon > \min_{j \neq j'} \frac{q_j q_{j'}}{4}$. We use the representation of the billiard flow as a special flow over the discrete time billiard system (M, ν, F) corresponding to collision times and with τ the length of the free flight before the next collision.

Set $\tilde{A}_\varepsilon = \bigcup_{j=1}^J \tilde{A}_\varepsilon^{(j)}$, where $\tilde{A}_\varepsilon^{(j)}$ is the set of the configurations entering in $A_\varepsilon^{(j)} := (Q \cap B(q_j, \varepsilon)) \times S^1$, i.e. $\tilde{A}_\varepsilon^{(j)}$ is the set of $(q, \mathbf{v}) \in (Q \cap \partial B(q_j, \varepsilon)) \times S^1$ s.t. $\langle \mathbf{q} \mathbf{q}_0, \mathbf{v} \rangle > 0$. Set also $A_\varepsilon := \bigcup_{j=1}^J A_\varepsilon^{(j)}$.

Set $h'_\varepsilon := d\pi\varepsilon / \text{Area}(Q)$ and $H_\varepsilon(q', \mathbf{v}) = (j, \frac{\overrightarrow{q_j q'}}{r_j \varepsilon}, \mathbf{v})$ if $q' \in \partial B(q_j, r_j \varepsilon)$. Here M is the set of reflected unit vectors based on ∂Q , ν is the probability measure with the density proportional to $(q, \mathbf{v}) \mapsto \langle \mathbf{n}(q), \mathbf{v} \rangle$, where $\mathbf{n}(q)$ is the unit vector normal to ∂Q at q directed towards Q and $F : M \rightarrow M$ is the transformation mapping a configuration at a collision time to the configuration corresponding to the next collision time.

The normalizing function G_ε is given by $G_\varepsilon(x) = H_\varepsilon(Y_{\tau_{\tilde{A}_\varepsilon}^{(Y)}(x)}(x))$ with

$$\tau_{\tilde{A}_\varepsilon}^{(Y)}(y) := \inf\{t > 0 : Y_t(y) \in \tilde{A}_\varepsilon\}.$$

As in the setting of Theorem 2, we write Π for the projection on M , that is $\Pi(q', \mathbf{v}) = (q, \mathbf{v})$ is the post-collision vector at the previous collision time. We take here $h_\varepsilon := \nu(\Pi(\tilde{A}_\varepsilon))$.

As for [21, Theorem 4.4], we will apply [21, Proposition 3.2] after checking its assumptions. We define $\tilde{A}_\varepsilon^{(j)} := \{(q, \mathbf{v}) \in \partial B(q_j, \varepsilon) \times S^1 : \langle \overrightarrow{qq_j}, \mathbf{v} \rangle \geq 0\}$.

- (i) Measure of the set. We have to adapt slightly the first item of the proof of [21, Theorem 4.4] which deals with the asymptotic behaviour of $\nu(B_\varepsilon)$ with $B_\varepsilon := \Pi(\tilde{A}_\varepsilon)$. Observe that $B_\varepsilon = \bigcup_{j=1}^J B_\varepsilon^{(j)}$ with $B_\varepsilon^{(j)} := \Pi(\tilde{A}_\varepsilon^{(j)})$, i.e. $B_\varepsilon^{(j)}$ is the set of configurations $(q, \mathbf{v}) \in M$ such that the billiard trajectory $(Y_t(q))_{t \geq 0}$ will enter $B(q_j, \varepsilon r_j)$ before touching ∂Q . As seen in [19, Lemma 5.1],

$$\text{if } q_j \in Q \setminus \partial Q, \quad \nu(B_\varepsilon^{(j)}) = \frac{|Q \cap \partial B(q_j, r_j \varepsilon)|}{|\partial Q|} = \frac{2\pi r_j \varepsilon}{|\partial Q|}.$$

With exactly the same proof, we obtain that

$$\text{if } q_j \in \partial Q, \quad \nu(B_\varepsilon^{(j)}) = \frac{|Q \cap \partial B(q_j, r_j \varepsilon)|}{|\partial Q|} \sim \frac{\pi r_j \varepsilon}{|\partial Q|}.$$

Moreover, for every distinct j, j' , $B_\varepsilon^{(j)} \cap B_\varepsilon^{(j')}$ is contained in $\Pi(B(x_{j,j'}, K_{j,j'} \varepsilon) \cup B(x_{j',j}, K_{j,j'} \varepsilon))$ where $x_{j,j'} = (q_j, \overrightarrow{q_j q_j'})$ and $K_{j,j'} = \max\left(1, \frac{3}{q_j q_{j'}}\right)$.

So, due to [19, Lemma 5.1], $\nu(B_\varepsilon^{(j)} \cap B_\varepsilon^{(j')}) = \mathcal{O}(\varepsilon^2) = o(\varepsilon)$. Hence we conclude that

$$\nu(B_\varepsilon) \sim \sum_{j=1}^J \nu(B_\varepsilon^{(j)}) \sim \frac{d\pi \varepsilon}{|\partial Q|},$$

as $\varepsilon \rightarrow 0$.

- (ii) Observe that

$$\mathcal{N}(Y, A_\varepsilon, h'_\varepsilon, H_\varepsilon) = \sum_{j=1}^J \mathcal{N}(Y, A_\varepsilon^{(j)}, h'_\varepsilon, H_\varepsilon) \geq \mathcal{N}(Y, A'_\varepsilon, h'_\varepsilon, H_\varepsilon),$$

where $A'_\varepsilon = \bigcup_{j=1}^J \Pi^{-1}(\Pi(\tilde{A}_\varepsilon^{(j)})) \setminus \bigcup_{j' \neq j} \Pi^{-1}(\Pi(\tilde{A}_\varepsilon^{(j')}))$ and that, for all $T > 0$,

$$\begin{aligned} & \mathbb{E}_\mu [(\mathcal{N}(Y, A_\varepsilon, h'_\varepsilon, H_\varepsilon) - \mathcal{N}(Y, A'_\varepsilon, h'_\varepsilon, H_\varepsilon))([0, T] \times V)] \\ & \leq \frac{T \max \tau}{2h_\varepsilon (\min \tau)^2} \sum_{j, j' : j \neq j'} \nu(\tilde{A}_\varepsilon^{(j)} \cap \tilde{A}_\varepsilon^{(j')}) = o(1), \end{aligned}$$

where we used the representation of Y as a special flow over (M, ν, F) due to the fact, proved in the previous item, that for any distinct labels j, j' , $\nu(\tilde{A}_\varepsilon^{(j)} \cap \tilde{A}_\varepsilon^{(j')}) = o(\varepsilon)$. Thus it is enough to prove the convergence in distribution of $\mathcal{N}(Y, A'_\varepsilon, h'_\varepsilon, H_\varepsilon)$ with respect to μ .

- (iii) The same argument ensures that, with respect to ν , the convergence in distribution of $\mathcal{N}(F, B_\varepsilon, h_\varepsilon, G_\varepsilon)$ to \mathcal{P} is equivalent to the convergence in distribution of $\mathcal{N}(F, B'_\varepsilon, h_\varepsilon, G_\varepsilon)$, with $B'_\varepsilon := \Pi(A'_\varepsilon)$.
- (iv) Note that $\nu((\partial B_\varepsilon)^{[\varepsilon^\delta]}) = o(\nu(B_\varepsilon))$, for every $\delta > 1$.
- (v) Due to Lemma 1, for every $\sigma > 1$, $\nu(\tau_{B_\varepsilon} \leq \varepsilon^{-\sigma} | B_\varepsilon) = o(1)$, where τ_B is here the first time $k \geq 1$ at which $F^k(\cdot) \in B$.
- (vi) Now let us prove that $(\nu(G_\varepsilon^{-1}(\cdot) | B_\varepsilon))_{\varepsilon > 0}$ converges to \tilde{m}_0 as $\varepsilon \rightarrow 0$.
Let us consider the measure $\tilde{\mu}$ on $\{1, \dots, J\} \times S^1 \times S^1$ with the density $(j, p, \mathbf{u}) \mapsto r_j \langle (-p), \mathbf{u} \rangle^+$.
Observe first that $\tilde{m}_0 = \tilde{\mu}(\cdot | A)$ with $A := \bigcup_{j=1}^J A^{(j)}$ and

$$A^{(j)} := \{(p, \mathbf{u}) \in S^1 \times S^1 : \langle (-p), \mathbf{u} \rangle \geq 0, \langle p, \mathbf{n}_{q_j} \rangle \geq 0\}$$

and second that $\nu(G_\varepsilon^{-1}(\cdot) | B_\varepsilon) = \tilde{\mu}(\cdot | G_\varepsilon(B_\varepsilon))$. But

$$\begin{aligned} \tilde{\mu}(A \setminus G_\varepsilon(B_\varepsilon)) &\leq \sum_{j=1}^J \tilde{\mu} \left(H_\varepsilon \left(Y_{\tau_{\tilde{A}_\varepsilon^{(j)}}(\cdot)} \left(\bigcup_{j' \neq j} (B_\varepsilon^{(j)} \cap B_\varepsilon^{(j')}) \right) \right) \right) \\ &\leq \sum_{j=1}^J 2 \max \tau |\partial Q| r_j \varepsilon \nu \left(\bigcup_{j \neq j'} (B_\varepsilon^{(j)} \cap B_\varepsilon^{(j')}) \right) = o(\nu(B_\varepsilon)) \end{aligned}$$

and $G_\varepsilon(B_\varepsilon) \setminus A$ corresponds to points $(p, \mathbf{u}) \in S^1 \times S^1$ with $q_j \in \partial Q$ with $0 < \langle p, \mathbf{u} \rangle \leq O(\varepsilon)$, thus

$$\tilde{\mu}(G_\varepsilon(B_\varepsilon) \setminus A) = O(\varepsilon).$$

This ends the proof of the convergence in distribution of the family of measures $(\nu(G_\varepsilon^{-1}(\cdot) | B_\varepsilon))_{\varepsilon > 0}$ to \tilde{m}_0 as $\varepsilon \rightarrow 0$.

- (vii) For the construction of \mathcal{W} we use [21, Proposition 3.4]. \square

Thus, due to [21, Proposition 3.2], we conclude the convergence of distribution with respect to ν of $(\mathcal{N}(F, B_\varepsilon, h_\varepsilon, G_\varepsilon))_{\varepsilon > 0}$ and so, due to (ii), of $(\mathcal{N}(F, B'_\varepsilon, h_\varepsilon, G_\varepsilon))_{\varepsilon > 0}$ to a PPP \mathcal{P} with intensity $\lambda \times \tilde{m}_0$. Applying now Theorem 2, we deduce the strong convergence in distribution of $(\mathcal{N}(F, A'_\varepsilon, h_\varepsilon / \mathbb{E}_\nu[\tau], H_\varepsilon))_{\varepsilon > 0}$ to \mathcal{P} and so, due to (iii), the convergence in distribution with respect to μ of $(\mathcal{N}(F, A_\varepsilon, h_\varepsilon / \mathbb{E}_\nu[\tau], H_\varepsilon))_{\varepsilon > 0}$ to \mathcal{P} . Now we conclude by [28, Theorem 1] and by noticing that

$$\frac{h_\varepsilon}{\mathbb{E}_\nu[\tau]} = \frac{d\pi\varepsilon}{|\partial Q| \mathbb{E}_\nu[\tau]} = \frac{d\pi\varepsilon}{\text{Area}(Q)} = h'_\varepsilon.$$

Lemma 1

$$\forall \sigma \in (0, 1), \quad \nu(\tau_{B_\varepsilon} \leq \varepsilon^{-\sigma} | B_\varepsilon) = o(1) \quad (14)$$

Proof This point corresponds to the the second item of the proof of [21, Theorem 4.4], which for Lebesgue-almost every point came from [19, Lemma 6.4]. To prove (14), we write

$$\nu(\tau_{B_\varepsilon} \leq \varepsilon^{-\sigma} | B_\varepsilon) \leq \sum_{k=1}^{\lfloor \varepsilon^{-\sigma} \rfloor} \nu(F^{-n}(B_\varepsilon) | B_\varepsilon). \quad (15)$$

Thus our goal is to bound $\nu(F^{-n}(B_\varepsilon) | B_\varepsilon)$.

Step 1: Useful notations.

We parametrize M by $\bigcup_{i=1}^I \{i\} \times (\mathbb{R}/|\partial O_i|\mathbb{Z}) \times [-\frac{\pi}{2}; \frac{\pi}{2}]$. A reflected vector $(q, \mathbf{v}) \in M$ is represented by (i, r, φ) if $q \in \partial \Gamma_i$ as curvilinear absciss r ∂O_i and if φ is the angular measure in $[-\pi/2, \pi/2]$ of $(\mathbf{n}(q), \mathbf{v})$ where $\mathbf{n}(q)$ is the normal vector to ∂Q at q .

For any C^1 -curve γ in M , we write $\ell(\gamma)$ for the euclidean length in the (r, φ) coordinates of γ . If moreover γ is given in coordinates by $\varphi = \phi(r)$, then we also write $p(\gamma) := \int_\gamma \cos(\phi(r)) dr$. We define the time until the next reflection in the future by

$$\tau(q, \mathbf{v}) := \min\{s > 0 : q + s\mathbf{v} \in \partial Q\}.$$

It will be useful to define $\mathcal{S}_0 := \{\varphi = \pm\pi/2\}$. Recall that, for every $k \geq 1$, F^k defines a C^1 -diffeomorphism from $M \setminus \mathcal{S}_{-k}$ to $M \setminus \mathcal{S}_k$ with $\mathcal{S}_{-k} := \bigcup_{m=0}^k F^{-m}(\mathcal{S}_0)$ and $\mathcal{S}_k := \bigcup_{m=0}^k F^m(\mathcal{S}_0)$.

Step 2: Geometric study of B_ε and of $F(B_\varepsilon)$.

Moreover the boundary of each connected component of B_ε (resp. $F(B_\varepsilon)$) is made with a bounded number of C^1 curves of the following forms:

- curves of \mathcal{S}_0 , corresponding, in (r, φ) -coordinates, to $\{\varphi = \pm\frac{\pi}{2}\}$.
- C^1 curves of $F^{-1}(\mathcal{S}_0)$ (resp. $F(\mathcal{S}_0)$), which have the form $\varphi = \phi(r)$ with ϕ a C^1 decreasing (resp. increasing) function satisfying $\min \kappa \leq |\phi'(r)| \leq \max \kappa + \frac{1}{\min \tau}$, where $\kappa(q)$ is the curvature of ∂Q at $q \in \partial Q$ and where τ is the free flight length before the next collision time.
- if $q_0 \notin \partial Q$: C^1 curves, corresponding to the set of points $x = (q, \mathbf{v}) \in M$ (resp. $F(x)$) such that $[\Pi_Q(x), \Pi_Q(F(x))]$ is tangent to $\partial B(q_0, \varepsilon)$. These curves have the form $\varphi = \phi_\varepsilon(r)$ with ϕ_ε a decreasing (resp. increasing) function satisfying $\min \kappa \leq |\phi'_\varepsilon(r)| \leq \max \kappa + \frac{1}{d(q_0, \partial Q) - \varepsilon} \leq \max \kappa + \frac{2}{\tau_0}$, with $\tau_0 := d(q_0, \partial Q)$ as soon as $\varepsilon < \frac{\tau_0}{2}$.
- if $q_0 \in \partial Q$: C^1 curves, corresponding to the set of points $x = (q, \mathbf{v}) \in M$ (resp. $F(x)$) such that $[\Pi_Q(x), \Pi_Q(F(x))]$ is tangent to $\partial B(q_0, \varepsilon)$ or such that $\Pi_Q(F(x))$ is an extremity of $B(q_0, \varepsilon) \cap Q$ and $[\Pi_Q(x), \Pi_Q(F(x))]$ contains no other point of $B(q_0, \varepsilon)$. These curves have the form $\varphi = \phi_\varepsilon(r)$ with ϕ_ε a decreasing (resp. increasing) function satisfying $\min \kappa \leq |\phi'_\varepsilon(r)|$. The points $x = (q, \mathbf{v}) \in M$, with $d(q, q_0) \ll 1$ almost immediately entering

(resp. exiting) $B(q_0, \varepsilon) \times S^1$ are contained in a union R_ε of two rectangles of width $\mathcal{O}(\varepsilon^{1/2})$ for the position (around q_0) and of width $\mathcal{O}(\varepsilon)$ for the velocity direction (around the tangent vectors to ∂Q at q_0).

In $B_\varepsilon \setminus R_\varepsilon$ (resp. $F(B_\varepsilon) \setminus (R_\varepsilon \cup \Pi_Q^{-1}(B(q_0, \varepsilon)))$) we also have $|\phi'_\varepsilon(r)| \leq \max \kappa + \frac{2}{\tau_0}$ with $\tau_0 := \min \tau$ as soon as $\varepsilon < \frac{\tau_0}{2}$.

We say that a **curve γ of M satisfies assumption (C)** if it is given by $\varphi = \phi(r)$ with ϕ being C^1 -smooth, increasing and such that $\min \kappa \leq \phi' \leq \max \kappa + \frac{2}{\tau_0}$. We recall the following facts.

- There exist $C_0, C_1 > 0$ and $\lambda_1 > 1$ such that, for every γ satisfying Assumption (C) and every integer m such that $\gamma \cap \mathcal{S}_{-m} = \emptyset$, $F^m \gamma$ is a C^1 -smooth curve satisfying assumption (C) and $C_1 p(F^m \gamma) \geq \lambda_1^m p(\gamma)$ and $\ell(\gamma) \leq C_0 \sqrt{p(F \gamma)}$.
- There exist $C_2 > 0$ and $\lambda_2 > \lambda_1^{1/2}$ such that, for every integer m , the number of connected components of $M \setminus \mathcal{S}_{-m}$ is less than $C_2 \lambda_2^m$. Moreover \mathcal{S}_{-m} is made of curves $\varphi = \phi(r)$ with ϕ C^1 -smooth and strictly decreasing.
- If $\gamma \subset M \setminus \mathcal{S}_{-1}$ is given by $\varphi = \phi(r)$ or $r = \mathfrak{r}(\varphi)$ with ϕ or \mathfrak{r} increasing and C^1 smooth, then $F \gamma$ is C^1 , is given by $\varphi = \phi_1(r)$ with $\min \kappa \leq \phi'_1 \leq \max \kappa + \frac{1}{\min \tau}$. Moreover $\int_{F \gamma} d\varphi \geq \int_\gamma d\varphi$.

We observe that there exist $K'_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, $F(B_\varepsilon) \setminus R_\varepsilon$ is made of a bounded number of connected components $V_\varepsilon^{(i)}$ each of which is a strip of width at most $K'_0 \varepsilon$ of the following form in (r, φ) -coordinates:

- $\{(r, \varphi) : r \in J, \phi_1^{(i)}(r) \leq \varphi \leq \phi_2^{(i)}(r)\}$ (with J an interval) and is delimited by two continuous piecewise C^1 curves γ_j given by $\varphi = \phi_j(r)$ satisfying assumption (C) and $\|\phi_1^{(i)} - \phi_2^{(i)}\|_\infty \leq K'_0 \varepsilon$.
- or possibly, if $q_0 \in \partial Q$, $\{(r, \varphi) : r_{1,\varepsilon}^{(i)} \leq r \leq r_{2,\varepsilon}^{(i)}\}$ with $|r_{1,\varepsilon}^{(i)} - r_{2,\varepsilon}^{(i)}| \leq K'_0 \varepsilon$.

In particular, with the previous notations, any connected component $V_\varepsilon^{(i)}$ of $F(B_\varepsilon) \setminus R_\varepsilon$ has the form $\bigcup_{u \in [0,1]} \tilde{\gamma}_u^{(i)}$, where $\tilde{\gamma}_u^{(i)}$ corresponds to the graph $\{\psi^{(i)}(u, r) = (r, u\phi_1^{(i)}(r) + (1-u)\phi_2^{(i)}(r)) : r \in J_i\}$ (or possibly $\{\psi^{(i)}(u, \varphi) = (ur_{1,\varepsilon}^{(i)} + (1-u)r_{2,\varepsilon}^{(i)}, \varphi), \varphi \in J_i\}$ if $q_0 \in \partial Q$). Thus

$$\begin{aligned}
\forall E \in \mathcal{B}(M), \quad \nu(E \cap F(B_\varepsilon \setminus R_\varepsilon)) &\leq \frac{\text{Leb}(E \cap F(B_\varepsilon \setminus R_\varepsilon))}{2|Q|} \\
&\leq \sum_i \frac{1}{2|\partial Q|} \int_{J_i \times [0,1]} \mathbf{1}_{\psi^{(i)}(u,s) \in E} \left| \frac{\partial}{\partial u} \psi^{(i)}(u,s) \right| ds du \\
&\leq \frac{K'_0 \varepsilon}{2|\partial Q|} \sup_{[0,1]} \ell(E \cap \tilde{\gamma}_u). \tag{16}
\end{aligned}$$

Step 3: Scarcity of very quick returns.

Let us prove the existence of $K_1 > 0$ such that,

$$\forall s \geq 1, \forall \varepsilon < \frac{\tau_0}{2}, \quad \nu(F^{-s-1}(B_\varepsilon)|B_\varepsilon) \leq K_1(\lambda_2/\lambda_1^{\frac{1}{2}})^s \varepsilon^{\frac{1}{2}}. \quad (17)$$

Let $u \in (0, \varepsilon)$. We define γ to be a connected component of $\tilde{\gamma}_u \cap F(B_\varepsilon) \cap F^{-s}(B_\varepsilon)$. The curve γ satisfies Assumption (C) or is vertical. In any case, any connected component of $F(\gamma)$ satisfies Assumption (C) and $\ell(\gamma) \leq C_0 \sqrt{p(F(\gamma))}$ (indeed, if γ is vertical, then $\ell(\gamma) \leq \frac{1}{\min \tau} p(F(\gamma))$). It follows

$$\ell(\gamma) \leq C_0 \sqrt{p(F(\gamma))} \leq C_0 \sqrt{C_1 \lambda_1^{1-s} p(F^s \gamma)} \leq C'_0 \sqrt{C_1 \lambda_1^{1-s} K'_0 \varepsilon}$$

using first the fact that $F(\gamma)$ is an increasing curve contained in $M \setminus \mathcal{S}_{-s}$ and secondly, the fact that $F^s \gamma$ is an increasing curve satisfying Condition (C) and contained in B_ε . Since $F(\tilde{\gamma}_u) \setminus \mathcal{S}_s$ contains at most $C_2 \lambda_2^s$ connected components, using (16), we obtain

$$\nu(F^{-s-1}(B_\varepsilon) \cap B_\varepsilon \setminus R_\varepsilon) = \nu(F^{-s}(B_\varepsilon) \cap F(B_\varepsilon \setminus R_\varepsilon)) \leq \frac{K'_0 \varepsilon}{2|\partial Q|} \sup_{[0,1]} C_2 \lambda_2^s C'_0 \sqrt{C_1 \lambda_1^{\frac{s}{2}} \varepsilon^{\frac{1}{2}}}.$$

We conclude by using the fact that $\nu(B_\varepsilon) = \frac{2\pi\varepsilon}{|\partial Q|}$ and that $\nu(R_\varepsilon) = \mathcal{O}(\varepsilon^{\frac{3}{2}})$.

Step 4: Scarcity of intermediate quick returns.

We prove now that for any $a > 0$, there exists $s_a > 0$ such that

$$\sum_{n=-a \log \varepsilon}^{\varepsilon^{-s_a}} \nu(B_\varepsilon \cap F^{-n} B_\varepsilon) = o(\nu(B_\varepsilon)). \quad (18)$$

Since $\nu(B_\varepsilon) \approx \varepsilon$ and $\nu(R_\varepsilon) = \mathcal{O}(\varepsilon^{\frac{3}{2}})$, up to adding the condition $s_a < 1/2$, it remains to prove (18) with $\nu(B_\varepsilon \cap F^{-n} B_\varepsilon)$ replaced by $\nu((B_\varepsilon \setminus R_\varepsilon) \cap F^{-n} B_\varepsilon)$. If $q_0 \in \partial Q$ and if $\tilde{\gamma}_u$ is vertical, we replace it in the argument below by the connected components of $F(\tilde{\gamma}_u)$ and will conclude by noticing that; for any measurable set A , $\ell(\tilde{\gamma}_u \cap F^{-1}(A)) \leq C'_0 \ell(F(\tilde{\gamma}_u \cap A))$.

We denote the k th homogeneity strip³ by \mathbb{H}_k for $k \neq 0$ and set $\mathbb{H}_0 = \cup_{|k| < k_0} \mathbb{H}_k$ for some fixed k_0 . Set $s := \min(-a \log \theta, 1)/3$. Let $k_\varepsilon = \varepsilon^{-s}$ and $H^\varepsilon = \cup_{|k| \leq k_\varepsilon} \mathbb{H}_k$. For any $u \in [0, 1]$, we set $\tilde{\gamma}_{k,u} = \tilde{\gamma}_u \cap \mathbb{H}_k$. Each $\tilde{\gamma}_{k,u}$ is a weakly homogeneous unstable curve.

We cut each curve $\tilde{\gamma}_{k,u}$ into small pieces $\tilde{\gamma}_{k,u,i}$ such that each $F^j \tilde{\gamma}_{k,u,i}$, $j = 0, \dots, n$ is contained in a homogeneity strip and a connected component of $M \setminus \mathcal{S}_1$. For $x \in \tilde{\gamma}_{k,u,i}$ we denote by $r_n(x)$ the distance (in $F^n \tilde{\gamma}_u$) of $F^n(x)$ to the boundary of $F^n \tilde{\gamma}_{k,u,i}$.

Recall that the growth lemma [6, Theorem 5.52] ensures the existence of $\theta \in (0, 1)$, $c > 0$ such that, for any weakly homogeneous unstable curve γ one has

$$\ell(\gamma \cap \{r_n < \delta\}) \leq c\theta^n \delta + c\delta \ell(\gamma). \quad (19)$$

³ see [6] for notations and definitions.

Therefore,

$$\begin{aligned} & \ell(\tilde{\gamma}_u \cap F^{-n}(B_\varepsilon) \setminus \mathbb{H}_\varepsilon) \\ & \leq \sum_{|k| \leq k_\varepsilon} \ell(\cap\{r_n \geq \varepsilon^{1-s}\} \cap F^{-n}(B_\varepsilon)) + \ell(\tilde{\gamma}_{u,k} \cap \{r_n < \varepsilon^{1-s}\}). \end{aligned}$$

The first term inside the above sum is bounded by the sum $\sum_i \ell(\tilde{\gamma}_{u,k,i} \cap F^{-n}(B_\varepsilon))$ over those i 's such that $F^n(\tilde{\gamma}_{u,k,i})$ is of size larger than ε^{1-s} . In particular $\ell(\tilde{\gamma}_{u,k,i}) \geq \varepsilon^{1-s}$. On the other hand, by transversality

$$\ell(F^n(\tilde{\gamma}_{u,k,i}) \cap B_\varepsilon) \leq c\varepsilon.$$

By distortion (See Lemma 5.27 in [6]) we obtain

$$\ell(\tilde{\gamma}_{u,k,i} \cap F^{-n}(B_\varepsilon)) \leq c\varepsilon^s \ell(\tilde{\gamma}_{u,k,i}).$$

Summing up over these i gives the first term inside the sum is bounded by

$$\ell(\tilde{\gamma}_{u,k} \cap \{r_n \geq \varepsilon^{1-s}\} \cap F^{-n}(B_\varepsilon)) \leq c\varepsilon^s \ell(\tilde{\gamma}_{u,k,i}).$$

Thus

$$\ell(\tilde{\gamma}_{u,k} \cap \{r_n < \varepsilon^{1-s}\}) \leq c\theta^n \varepsilon^{1-s} + c\varepsilon^{1-s} \ell(\tilde{\gamma}_{u,k}).$$

A final summation over k gives

$$\ell(\tilde{\gamma}_u \cap F^{-n}(B_\varepsilon) \setminus \mathbb{H}_\varepsilon) \leq c(\varepsilon^s + \varepsilon^{1-s}) \ell(\tilde{\gamma}_u) + ck_\varepsilon \theta^n \varepsilon^{1-s}.$$

This combined with (16) leads to

$$\nu(F(B_\varepsilon \setminus R_\varepsilon) \cap F^{-n}(B_\varepsilon)) \leq \nu(F(B_\varepsilon \setminus R_\varepsilon) \cap \mathbb{H}_\varepsilon) + \mathcal{O}(\varepsilon^{1+s}) = \mathcal{O}(\varepsilon^s \nu(B_\varepsilon)).$$

where we use the fact that $B_\varepsilon \setminus \mathbb{H}_\varepsilon$ is contained in a uniformly bounded union of rectangles of horizontal width $\mathcal{O}(\varepsilon)$ and contained in the $k_\varepsilon^{-2} = \varepsilon^{2s}$ -neighbourhood of \mathcal{S}_0 . We take $s_a < \min(s, \frac{1}{2})$.

Step 5: End of the proof of (14).

Choose $a = 1/(4 \log(\lambda_2/\lambda_1^{1/2}))$. Observe that, due to (17), we have

$$\sum_{s=1}^{-a \log \varepsilon} \mu(F^{-s} A_\varepsilon | A_\varepsilon) \leq \frac{K_1}{\lambda_2/\lambda_1^{\frac{1}{2}} - 1} (\lambda_2/\lambda_1^{\frac{1}{2}})^{-a \log \varepsilon} \varepsilon^{1/2} \leq \frac{K_1}{\lambda_2/\lambda_1^{\frac{1}{2}} - 1} \varepsilon^{1/4}.$$

This combined with (18) leads to

$$\sum_{n=1}^{\varepsilon^{-sa}} \nu(F^{-n} B_\varepsilon | B_\varepsilon) = o(1). \quad (20)$$

Let $\sigma > 1$. In view of (15), it remains to control $\nu(F^{-n}B_\varepsilon|B_\varepsilon)$ for the intermediate integers n such that $\varepsilon^{-s_a} \leq n \leq \varepsilon^{-\sigma}$. We approximate the set B_ε by the union \tilde{B}_ε of connected components of $M \setminus (\mathcal{S}_{-k(\varepsilon)} \cup \mathcal{S}_{k(\varepsilon)})$ that intersects B_ε , with $k(\varepsilon) = \lfloor \log \varepsilon \rfloor^2$. There exists $\tilde{C} > 0$ and $\tilde{\theta} \in (0, 1)$ such that, for any positive integer k , the diameter of each connected component of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_k)$ is less than $\tilde{C}\tilde{\theta}^k$.

Thus $B_\varepsilon \subset \tilde{B}_\varepsilon$ and $\nu(\tilde{B}_\varepsilon \setminus B_\varepsilon) \leq \nu\left((\partial B_\varepsilon)^{[\tilde{C}\tilde{\theta}^{k(\varepsilon)}]}\right) = \mathcal{O}(\varepsilon\tilde{\theta}^{k(\varepsilon)})$. But, due to [20, Lemma 4.1], we also have

$$\forall m > 1, \forall n \geq 2k(\varepsilon), \quad \nu\left(\tilde{B}_\varepsilon \cap F^{-n}\tilde{B}_\varepsilon\right) = \nu(\tilde{B}_\varepsilon)^2 + \mathcal{O}(n^{-m}\nu(\tilde{B}_\varepsilon)).$$

Since $k(\varepsilon) = o(\varepsilon^{-s_a})$ and thus

$$\forall m > 1, \quad \sum_{n=\varepsilon^{-s_a}}^{\varepsilon^{-\sigma}} \nu(F^{-n}B_\varepsilon|B_\varepsilon) \leq \mathcal{O}\left(\varepsilon^{1-\sigma} + \varepsilon^{s_a(m-1)-\sigma} + \tilde{\theta}^{k(\varepsilon)}\right) = o(1),$$

as $\varepsilon \rightarrow 0$, since $\sigma < 1$, $\tilde{\theta} \in (0, 1)$, $k(\varepsilon) \rightarrow +\infty$ and by taking $m > 1 + \frac{\sigma}{s_a}$. This combined with (20) and (15) ends the proof of (14). \square

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