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POINTWISE DIMENSIONS FOR POINCARÉ RECURRENCES ASSOCIATED WITH MAPS AND SPECIAL FLOWS

Valentin Afraimovich
IICO-UASLP
A. Obregón 64, 78000 San Luis Potosí
SLP, México

Jean René Chazottes
CNRS-CPHT
École polytechnique
91128 Palaiseau, France

Benoît Saussol
CNRS-LAMFA
Université de Picardie Jules Verne
80039 Amiens, France

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Abstract. We introduce pointwise dimensions and spectra associated with Poincaré recurrences. These quantities are then calculated for any ergodic measure of positive entropy on a weakly specified subshift. We show that they satisfy a relation comparable to Young's formula for the Hausdorff dimension of measures invariant under surface diffeomorphisms. A key-result in establishing these formula is to prove that the Poincaré recurrence for a 'typical' cylinder is asymptotically its length. Examples are provided which show that this is not true for some systems with zero entropy. Similar results are obtained for special flows and we get a formula relating spectra for measures of the base to the ones of the flow.

1. **Introduction.** Poincaré recurrences are main indicators and characteristics of the repetition of behavior of dynamical systems in time. A traditional approach is to study statistical properties of the quantity $\tau(x, U)$, the first return time of the orbit through x into a set U, [22] and references therein. These investigations led to a series of deep results. But they have the disadvantage that one does not get control on the sets of zero measure. As it was shown in [6], the remaining zero set can be very large in terms of topological entropy or dimension. We adopt another point of view: instead of looking at the mean return time or at the return time of points, we are going to study $\tau(U)$, the smallest possible return time into U, that is we define the **Poincaré recurrence for a set**, as the infimum over all return times of the points inside the set [2]. Poincaré recurrences for a set U can be very different from return times $\tau(x, U)$. If $U = \xi^n(x)$ is a cylinder of length n, then the return time $\tau(x, \xi^n(x))$ of μ-generic point behaves like $\exp(n\mathrm{h}_\mu(T,\xi))$ [14] (where T is map generating the dynamical system, $\mathrm{h}_\mu(T,\xi)$ is the entropy of μ, w.r.t. T and ξ), whereas the Poincaré recurrence for $\xi^n(x)$ is typically of order n, provided that μ is ergodic, T is weakly specified and $\mathrm{h}_\mu(T,\xi) > 0$ (Theorem 4.3

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below). Roughly speaking, n-periodic points are uniformly distributed among n-cylinders about generic points. Let us emphasize that this result does not depend on a particular choice of a map T, a partition ξ and an ergodic measure μ.

Since we deal with a function of sets (namely $U \mapsto \tau(U)$), then it is natural to use ideas and methods from dimension theory [17]. We define and calculate **pointwise dimensions for Poincaré recurrences** (Theorem 5.1) to obtain **spectra for measures for Poincaré recurrences** (Theorem 5.2). These quantities reflect the balance between times needed for the return to the set $\xi^n(x)$ and diam $\xi^n(x)$ for μ-almost every point x, provided that n is big enough. Let us remark that positiveness of entropy is an unavoidable assumption (Theorem 4.4).

We also study in the present work special flows over weakly specified subshifts. We introduce and study pointwise dimensions (Theorem 6.3) and spectra for measures (Theorem 6.4). The distribution of periodic orbits (zeta functions) in this situation is well-known [15]. Nevertheless asymptotic properties of Poincaré recurrences remain not so well understood. Therefore our results provide a new insight into the nature of recurrences for special flows. For example, we show that for almost every point of the special space, Poincaré recurrence for the ball about the point asymptotically behaves logarithmically with respect to the diameter of the ball (Proposition 6.1).

The article is organized as follows. In Section 2, we define dynamical systems we deal with, and Poincaré recurrences for balls. In Section 3, we introduce definitions of dimensions for Poincaré recurrences for maps, flows and measures. Section 4 is devoted to the study of local rates of return times for cylinders. We show that positiveness of entropy is an unavoidable assumption by considering two systems with zero entropy. In Section 5, we prove the existence almost everywhere of pointwise dimension and show that they coincide with spectra for measures. Similar results are obtained in Section 6 for special flows. Special flows over subshifts of finite type are involved in the construction of the symbolic dynamics of hyperbolic and Anosov flows [7].

The results presented here were announced in [3], though in a slightly different form and in a less general setting. Let us mention the papers [16, 8, 13] dealing with a dimension also based on Poincaré recurrences for sets but different from ours. The present work should rather be considered as a companion of the paper [4] in which was defined and studied spectra of *sets* whereas here we deal with *measures* responsible for the statistical properties (long-term behavior) of the system.

2. **Set-up and definitions for maps and special flows.**

2.1. **Maps.** We shall deal in this work with dynamical systems (X, T) which are weakly specified subshifts. This means that there exists a finite set Σ, called the alphabet, and X is a closed subset of $\Sigma^{\mathbb{N}}$ (non-invertible case) or $\Sigma^{\mathbb{Z}}$ (invertible case) which is invariant by the shift map T defined by $(Tx)_i = x_{i+1}$. We endow X with the product topology, which makes X a compact metrizable space. Weak specification means that given $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that for any two points $x^{(1)}, x^{(2)} \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2$ with $a_i - b_{i-1} \geq N(\epsilon)$, $i = 1, 2$, and for any integer p with $p \geq N(\epsilon) + b_2 - a_1$, there exists $z \in X$ with $T^p z = z$ such that $d(T^n z, x^{(i)}) \leq \epsilon$, for $a_i \leq n \leq b_i$, $i = 1, 2$ (d is any distance compatible with the product topology). Fundamental examples of specified subshifts are subshifts of finite type and sofic subshifts (for which in fact the above property holds given any k arbitrary points $x^{(i)}$, $1 \leq i \leq k$, $k \geq 1$).

Given $a \in \Sigma$ let $[a] = \{x \in X : x_0 = a\}$ and let $\xi = \{[a] : a \in \Sigma\}$ denote the partition into 1-cylinders. Our results only concern measures with positive entropy, so we will assume that (X, T) has positive topological entropy (which is implied by weak specification if there exists at least two periodic points in X) ; see [11]. We now define a metric on X equivalent to the product topology (see Lemma 2.1).

Case when T is not invertible. Denote by ξ^n the dynamical partition, that is: $\xi^n \overset{\text{def}}{=} \bigvee_{j=0}^{n-1} T^{-j}\xi$, $\xi^0 \overset{\text{def}}{=} \{X, \emptyset\}$. Then $\xi^n(x)$ will be the atom of the refined partition ξ^n that contains x and will be referred to as the n-cylinder about x. Given a continuous function $\mathrm{u} : X \to (0, \infty)$ we endow X with the metric d_X defined by $\mathrm{d}_X(x, y) \overset{\text{def}}{=} \mathrm{e}^{-\mathrm{u}(\xi^n(x))}$ whenever $y \in \xi^n(x)$ and $y \notin \xi^{n+1}(x)$, where

$$\mathrm{u}(\xi^n(x)) = \sup_{k \le n} \sup_{z \in \xi^k(x)} \left(\mathrm{u}(z) + \mathrm{u}(Tz) + \cdots + \mathrm{u}(T^{k-1}z)\right), n = 1, 2, ...$$

Remark that the standard metric is recovered by choosing $\mathrm{u} \equiv 1$. Choosing $\mathrm{u}(x) = -\log \lambda(x_0)$, which is a constant on each atom of ξ, gives

$$\mathrm{d}_X(x, y) = \prod_{\ell=0}^{n-1} \lambda(x_\ell), \quad \text{and} \quad \operatorname{diam} \xi^n(x) = \prod_{\ell=0}^{n-1} \lambda(x_\ell) ,$$

i.e. we have a situation familiar to that encountered in Moran-like geometric construction. More generally, a Hölder continuous function u will gives the distance used to generate Cantor-like sets in $\mathbb{R}^d$ [4, 17] modeled by subshifts.

Case when T is invertible. Denote by ξ_m^n the dynamical partition, that is: $\xi_m^n \overset{\text{def}}{=} T^m\xi \vee T^{m-1}\xi \vee \cdots \vee T^{-n+1}\xi$, $\xi_0^0 \overset{\text{def}}{=} \{X, \emptyset\}$, where $m \ge 0$, $n \ge 0$. Then $\xi_m^n(x)$ will be the atom of the refined partition ξ_m^n that contains x and will be referred to as the (m, n)-cylinder about x. Given two continuous functions $\mathrm{u}, \mathrm{v} : X \to (0, \infty)$ such that $\mathrm{u}(x) = \mathrm{u}(y)$ whenever $\xi_0^n(x) = \xi_0^n(y)$ for every $n \ge 0$ and $\mathrm{v}(x) = \mathrm{v}(y)$ whenever $\xi_m^0(x) = \xi_m^0(y)$ for every $m \ge 0$, we endow X with the metric d_X defined by (2.1) below. For an arbitrary pair $x, y \in X$, there is a unique pair (m, n) such that $y \in \xi_m^n(x)$ and $y \notin (\xi_{m+1}^n(x) \cup \xi_m^{n+1}(x))$. Then

$$\mathrm{d}_X(x, y) \overset{\text{def}}{=} \max\left\{\mathrm{e}^{-\mathrm{u}(\xi_0^n(x))}, \mathrm{e}^{-\mathrm{v}(\xi_m^0(x))}\right\} \tag{2.1}$$

where

$$\mathrm{u}(\xi_0^n(x)) = \sup_{k \le n} \sup_{z \in \xi_0^k(x)} \left(\mathrm{u}(z) + \mathrm{u}(Tz) + \cdots + \mathrm{u}(T^{k-1}z)\right), n = 1, 2, ...,$$

$$\mathrm{v}(\xi_m^0(x)) = \sup_{k \le m} \sup_{z \in \xi_k^0(x)} \left(\mathrm{v}(z) + \mathrm{v}(T^{-1}z) + \cdots + \mathrm{v}(T^{-k+1}z)\right), m = 1, 2, ...$$

Taking $\mathrm{u}(x) = -\log \lambda(x_0)$, $\mathrm{v}(x) = -\log \gamma(x_{-1})$, which is constant on each atom of ξ_0^1 and ξ_1^0, respectively, gives

$$\operatorname{diam} \xi_m^n(x) = \max\left\{ \prod_{\ell=1}^{m} \gamma(x_{-\ell}), \prod_{\ell=0}^{n-1} \lambda(x_\ell)\right\}.$$

Such a situation occurs, for example, in the case of a piecewise linear Smale horseshoe. In the general case of basic axiom A sets on surfaces, there exists an associated subshift of finite type (X, T) and some functions u, v satisfying the above assumptions (i.e. depending only on forward, respectively backward, itineraries) giving rise to a metric d_X which is "adapted" to the initial system.

Although X is not a smooth manifold, by analogy with the smooth case it makes sense to call the numbers $\mathrm{e}^{\mathrm{u}(x)}$ and $\mathrm{e}^{-\mathrm{v}(x)}$ the derivative of the map T at the point

x in the unstable and stable directions. If μ is an invariant probability measure it is natural to call the numbers

$$\chi_\mu^+ \stackrel{\text{def}}{=} \int \mathrm{u}(x) d\mu(x), \quad \chi_\mu^- \stackrel{\text{def}}{=} \int -\mathrm{v}(x) d\mu(x) \tag{2.2}$$

the Lyapunov exponents of the map.

Given $x \in X$ and $\varepsilon > 0$ we denote by $B(x,\varepsilon)$ the open ball of radius ε centered at x in the metric d_X.

Lemma 2.1. *(X, d_X) is an ultra-metric space, and for any $x \in X$ and $\varepsilon > 0$ we have*

1. *$B(x,\varepsilon) = \xi^{n_{x,\varepsilon}}(x)$ where we set $n_{x,\varepsilon} = \min\{n \in \mathbb{N} : \mathrm{e}^{-\mathrm{u}(\xi^n(x))} < \varepsilon\}$ in the non-invertible case;*
2. *$B(x,\varepsilon) = \xi_{m_{x,\varepsilon}}^{n_{x,\varepsilon}}(x)$ where we set $n_{x,\varepsilon} = \min\{n \in \mathbb{N} : \mathrm{e}^{-\mathrm{u}(\xi_0^n(x))} < \varepsilon\}$ and $m_{x,\varepsilon} = \min\{m \in \mathbb{N} : \mathrm{e}^{-\mathrm{v}(\xi_0^m(x))} < \varepsilon\}$ in the invertible case;*
3. *The topology generated by d_X is equivalent to the product topology.*

Proof. We write the proof in the non-invertible case, the invertible case can be done along the same lines.

Let $x, z \in X$, with $x \neq z$. There exists n such that $\xi^n(x) = \xi^n(z)$ but $\xi^{n+1}(x) \neq \xi^{n+1}(z)$. This implies that $\mathrm{d}_X(x,z) = \mathrm{e}^{-\mathrm{u}(\xi^n(x))} = \mathrm{e}^{-\mathrm{u}(\xi^n(z))}$. For any $y \in X$ either $y \notin \xi^{n+1}(x)$ or (and) $y \notin \xi^{n+1}(z)$. Suppose for simplicity that $y \notin \xi^{n+1}(x)$. Then there exists $k \leq n$ such that $y \in \xi^k(x)$ but $y \notin \xi^{k+1}(x)$, hence $\mathrm{d}_X(x,y) = \mathrm{e}^{-\mathrm{u}(\xi^k(x))}$. Since $\mathrm{u}(\xi^k(x))$ is increasing, we get that $\mathrm{d}_X(x,y) \geq \mathrm{d}_X(x,z)$. This proves that

$$\mathrm{d}_X(x,z) \leq \max\{\mathrm{d}_X(x,y), \mathrm{d}_X(y,z)\}.$$

Thus, d_X is a distance, and in addition the space (X, d_X) is ultra-metric (since the precedent strong inequality holds instead of the usual 'triangle inequality').

Let $x \in X$, $\varepsilon > 0$, and set $n_\varepsilon = \min\{n \in \mathbb{N} : \mathrm{e}^{-\mathrm{u}(\xi^n(x))} < \varepsilon\}$.

For any $y \in \xi^{n_\varepsilon}(x)$, $y \neq x$, there exists $n \geq n_\varepsilon$ such that $y \in \xi^n(x)$ but $y \notin \xi^{n+1}(x)$, and by definition $\mathrm{d}_X(x,y) = \mathrm{e}^{-\mathrm{u}(\xi^n(x))} \leq \mathrm{e}^{-\mathrm{u}(\xi^{n_\varepsilon}(x))} < \varepsilon$. Thus $y \in B(x,\varepsilon)$.

Let $y \in B(x,\varepsilon)$, $y \neq x$, and n such that $y \in \xi^n(x)$ but $y \notin \xi^{n+1}(x)$. By definition we have $\mathrm{e}^{-\mathrm{u}(\xi^n(x))} = \mathrm{d}_X(x,y) < \varepsilon$, hence, $n \geq n_\varepsilon$, and $y \in \xi^{n_\varepsilon}(x)$. This proves that any ball is indeed a cylinder, and Statement 3 is now an immediate consequence. □

2.2. **Poincaré recurrences for maps.** Assume that T preserves a Borel probability measure μ. For any measurable subset U of X, we define the first return time of a point $x \in U$ into U:

$$\tau_T(x,U) \stackrel{\text{def}}{=} \inf\{k \geq 1 : T^k x \in U\}.$$

By convention we put that this return time is infinite if the point x never comes back to U.

For any T-invariant measure μ, Poincaré's recurrence Theorem asserts that μ-almost every point returns to any measurable subset U of positive measure. In the case when μ is ergodic, Kac's Theorem tells us that the mean value of $\tau(\cdot, U)$ over U (suppose that $\mu(U) > 0$) is the inverse measure of U. (See [19] for instance for these two basic results.)

We adopt another point of view: instead of looking at the mean return time or at the return time of points, we are going to study the smallest possible return time into U, that is we define the *first return time of a set*, as the infimum over all return times of the points of the set. More precisely:

Definition 2.1 ([2]). *Let U be a subset of X. Then,*

$$\tau_T(U) \stackrel{\text{def}}{=} \inf\{\tau_T(x, U) : x \in U\} .$$

We collect in the following proposition various basic properties of the Poincaré recurrence of sets.

Proposition 2.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system, ξ a partition of X and $U \subset X$ any set. Then the following properties hold:*

1. $\tau_T(U) = \inf\{k > 0 : T^k U \cap U \neq \emptyset\} = \inf\{k > 0 : T^{-k} U \cap U \neq \emptyset\}$.
2. $\tau_T(U) = \tau_T(T^{-1}U)$. *If T is invertible then $\tau_T(TU) = \tau_T(U)$.*
3. *Monotonicity: $A \subset B \Rightarrow \tau_T(A) \geq \tau_T(B)$.*
4. *For any $n \geq 1$ and any $x \in X$, $\tau_T(\xi^{n-1}(Tx)) \leq \tau_T(\xi^n(x))$.*

Proof. 1. Let $z \in U$ such that $T^n z \in U$ and $n = \tau_T(U)$. Then $T^n z \in T^n U$ and therefore $U \cap T^n U \neq \emptyset$. 2. Take the same z as before. Then $z \in T^{-n}U$ by definition, so $T^{-n}U \cap U \neq \emptyset$. 3. $\tau_T(T^{-1}U) = \inf\{k > 0 : T^{-(k+1)}U \cap T^{-1}U \neq \emptyset\} = \inf\{k > 0 : T^{-1}(T^{-k}U \cap U) \neq \emptyset\} = \tau_T(U)$ because $T^{-1}A \neq \emptyset$ if and only if $A \neq \emptyset$. 4. This is a trivial consequence of the definition. 5. Let $x \in X$. For each integer $n > 0$, we have:

$$\xi^n(x) \cap T^k \xi^n(x) \neq \emptyset \Rightarrow \xi^{n-1}(Tx) \cap T^k \xi^{n-1}(Tx) \neq \emptyset ,$$

which implies the desired result. $\square$

2.3. **Special flows.** Let X be a compact metric space with a distance d_X, $T \colon X \to X$ a continuous map, and $\varphi : X \to \mathbb{R}$ a strictly positive Lipschitz continuous function.

Define the special space and the special flow as follows:

$$X^\varphi \stackrel{\text{def}}{=} \{\mathbf{x} = (x, t) : x \in X, 0 \leq t \leq \varphi(x)\} ,$$

where we identify the points $(x, \varphi(x))$ and $(Tx, 0)$ for each $x \in X$, and

$$\begin{cases} \Phi_s(x,t) = (x, t+s) \ \text{ if } \ t+s < \varphi(x) \\ \Phi_s(x,t) = (Tx, t+s-\varphi(x)) \ \text{ if } \ t+s \geq \varphi(x) . \end{cases}$$

We now recall the definition of the Bowen-Walters metric on X^φ as in [10]. Assume for a moment that $\varphi \equiv 1$. Let us recall the definition of the distance on X^1. Consider the subset $X \times \{t\}$ of $X \times [0,1]$ and let ρ_t denote the metric on $X \times \{t\}$ defined by $\rho_t((x,t),(y,t)) \stackrel{\text{def}}{=} (1-t)\mathrm{d}_X(x,y) + t\mathrm{d}_X(Tx,Ty)$, $x, y \in X$. Thus, $\rho_0((x,0),(y,0)) = \mathrm{d}_X(x,y)$ and $\rho_1((x,1),(y,1)) = \mathrm{d}_X(Tx,Ty)$. Consider a chain $w_0 = x, w_1, ..., w_n = y$ between x and y where for each i either w_i and w_{i+1} belong to $X \times \{t\}$ for some t ($[w_i, w_{i+1}]$ is said to be a horizontal segment and $length([w_i, w_{i+1}]) \stackrel{\text{def}}{=} \rho_t((w_i,t),(w_{i+1},t)))$ or w_i and w_{i+1} are on the same orbit ($[w_i, w_{i+1}]$ is said to be a vertical segment and $length([w_i, w_{i+1}])$ is the shortest temporal distance between w_i and w_{i+1} along the orbit). The length of the chain is defined to be the sum of the lengths of its segments. Then $d_{X^1}(\mathbf{x}, \mathbf{y})$ is defined to be the infimum of the lengths of all finite chains between $\mathbf{x}$ and $\mathbf{y}$.

Set $h(\mathbf{x}) \stackrel{\text{def}}{=} (x, s\varphi(x))$ for any $\mathbf{x} = (x, s) \in X^1$. This map is continuous and one-to-one ($h^{-1}(x,s) = (x, s/\varphi(x))$). We now introduce the following distance on X^φ:

$$d_{X^\varphi}(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} d_{X^1}(h^{-1}(\mathbf{x}), h^{-1}(\mathbf{y})) . \tag{2.3}$$

Given $(x,s),(y,t)\in X^\varphi$ we define as in [5]

$$d_\pi((x,s),(y,t)) = \min\left\{\begin{array}{l} d_X(x,y)+|s-t|,\\ d_X(Tx,y)+\varphi(x)-s+t,\\ d_X(x,Ty)+\varphi(y)-t+s \end{array}\right\}. \tag{2.4}$$

Note that d_π needs not to be a metric on X^φ. Nevertheless, we recall Proposition 17 in [5] which says that there exists some constant $\delta>0$ such that for each $\mathbf{x},\,\mathbf{y}\in X^\varphi$ the following property holds:

$$\delta^{-1}d_\pi(\mathbf{x},\mathbf{y})\le d_{X^\varphi}(\mathbf{x},\mathbf{y})\le \delta d_\pi(\mathbf{x},\mathbf{y}). \tag{2.5}$$

Now we define as usual by $B(\mathbf{x},\varepsilon)=\{\mathbf{y}\in X^\varphi : d_{X^\varphi}(\mathbf{x},\mathbf{y})<\varepsilon\}$ the open ball of radius ε centered at $\mathbf{x}$.

For continuous time dynamical systems it does not make sense to define the return time of a point $\mathbf{x}\in\mathbf{U}$ into an open set $\mathbf{U}\subset X^\varphi$ as for maps, because the point has first to escape. Thus let us define for $\mathbf{x}\in\mathbf{U}$ the escape time

$$\mathrm{e}_\Phi(\mathbf{x},\mathbf{U})=\inf\{t>0:\Phi_t\mathbf{x}\notin\mathbf{U}\},$$

and then the return time

$$\tau_\Phi(\mathbf{x},\mathbf{U})\stackrel{\text{def}}{=}\inf\{t>\mathrm{e}_\Phi(\mathbf{x},\mathbf{U}):\Phi_t\mathbf{x}\in\mathbf{U}\}.$$

We can now define the first return time of the set $\mathbf{U}$ as usual:

Definition 2.2. *For any $\mathbf{x}\in X^\varphi$ the Poincaré recurrence for the set $\mathbf{U}$ is*

$$\tau_\Phi(\mathbf{U})\stackrel{\text{def}}{=}\inf\{\tau_\Phi(\mathbf{x},\mathbf{U}):\,\mathbf{x}\in\mathbf{U}\}.$$

If μ is a T-invariant probability measure on X, then we define a Φ-invariant probability measure $\overline{\mu}$ by

$$\int_{X^\varphi}F(\mathbf{x})\,d\overline{\mu}(\mathbf{x})\stackrel{\text{def}}{=}\frac{\int_X\left(\int_0^{\varphi(x)}F(x,s)\,ds\right)d\mu(x)}{\int_X\varphi(x)\,d\mu(x)}\ , \tag{2.6}$$

for any continuous function $F:X^\varphi\to\mathbb{R}$. That is, $\overline{\mu}$ is the normalization on X^φ obtained by taking the direct product of μ with Lebesgue measure on $\mathbb{R}$. Note that any Φ-invariant probability measure on X^φ can be obtained in this way from a T-invariant probability measure on X.

It has been shown by Abramov [1] that $\mathrm{h}_{\overline{\mu}}(\Phi_t)=|t|\mathrm{h}_{\overline{\mu}}(\Phi_1)$. This gives a natural definition of the entropy of the flow as $\mathrm{h}_{\overline{\mu}}(\Phi)\stackrel{\text{def}}{=}\mathrm{h}_{\overline{\mu}}(\Phi_1)$, similarly one has $\mathrm{h}_{\mathrm{top}}(\Phi)\stackrel{\text{def}}{=}\mathrm{h}_{\mathrm{top}}(\Phi_1)$. Abramov also showed that

$$\mathrm{h}_{\overline{\mu}}(\Phi)=\frac{\mathrm{h}_\mu(T)}{\int\varphi\,d\mu}\ \cdot \tag{2.7}$$

Let $\rho\colon[0,1]\to\mathbb{R}$ be a continuous function such that $\int_0^1\rho(t)dt=1$ and $\rho(0)=\rho(1)=0$. Define $\mathbf{u},\mathbf{v}\colon X^\varphi\to\mathbb{R}$ by

$$\mathbf{u}(x,s)=\frac{\mathrm{u}(x)}{\varphi(x)}\rho(s/\varphi(x)),\quad \mathbf{v}(x,s)=\frac{\mathrm{v}(x)}{\varphi(x)}\rho(s/\varphi(x)).$$

For instance we could take $\rho(t)=6t(1-t)$. Although the space X^φ is only a topological manifold, it makes sense to think of the two functions $\mathrm{e}^{\mathbf{u}}$ and $\mathrm{e}^{-\mathbf{v}}$ as the derivative of the special flow along the unstable and stable direction. In fact the "derivative" of the flow depends on the way we do the identification, i.e. depends on ρ. However, if $\overline{\mu}$ is an invariant measure then the quantities

$$\chi^+_{\overline{\mu}}(\Phi)=\int\mathbf{u}(\mathbf{x})d\overline{\mu}(\mathbf{x}),\quad \chi^-_{\overline{\mu}}(\Phi)=\int-\mathbf{v}(\mathbf{x})d\overline{\mu}(\mathbf{x})$$

will not depend on the particular choice of the smooth function ρ. This justifies calling these numbers the Lyapunov exponents of the flow.

3. **Dimensions and spectra for Poincaré recurrences.** In this section, we first introduce general spectra following Pesin's framework [17]. Then we proceed with the spectra for Poincaré recurrence, for maps and flows.

3.1. **Preliminaries.** We start by the general Carathéodory-Pesin construction. Let X be a separable metric space satisfying the Vitali condition[1] and τ a set function such that $\tau(S) \in (0, \infty]$ for any $S \subseteq X$. For any $A \subseteq X$, any $\alpha \in \mathbb{R}$ and any $q \in \mathbb{R}$ we define

$$\mathcal{M}^\tau(A, \alpha, q, \varepsilon) \stackrel{\text{def}}{=} \inf_{\substack{(x_i, \varepsilon_i) \\ \varepsilon_i \leq \varepsilon}} \sum_i \exp\big(-q\tau(B(x_i, \varepsilon_i))\big)\varepsilon_i^\alpha \,, \tag{3.8}$$

where the infimum is taken over all finite or countable collections (x_i, ε_i) such that $\bigcup_i B(x_i, \varepsilon_i) \supseteq A$. The limit $\mathcal{M}^\tau(A, \alpha, q) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \mathcal{M}^\tau(A, \alpha, q, \varepsilon)$ exists by monotonicity and we give the following definition

Definition 3.3. *For any non-empty $A \subset X$ and any $q \in \mathbb{R}$,*

$$\alpha_\tau(A, q) \stackrel{\text{def}}{=} \begin{cases} \inf\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = 0\} & \text{if } q \geq 0 \\ \sup\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = \infty\} & \text{if } q < 0 \end{cases} \tag{3.9}$$

is called the spectrum for the set-function τ of the set A.

It is easy to see that whenever $\alpha_\tau(A, q)$ is finite one has

$$\alpha_\tau(A, q) = \inf\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = 0\} = \sup\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = \infty\}.$$

Remark 3.1. *Note that $\alpha_\tau(A, 0)$ is nothing but the Hausdorff dimension of A.*

Now we proceed to the definition of the spectra for measures, following [17].

Definition 3.4 (Spectra for measures)**.** *Let $\alpha_\tau(., q)$ be the spectrum defined above. Let μ be a Borel probability measure on X.*

$$\alpha_\tau^\mu(q) \stackrel{\text{def}}{=} \inf\{\alpha_\tau(Y, q) : Y \subset X,\, \mu(Y) = 1\} \,.$$

The next definition is the local version of these quantities, called pointwise dimensions for the corresponding structure. Inspired by the ideas of [17], we define the following quantities.

Definition 3.5 (Lower pointwise dimension)**.** *The lower q-pointwise dimension of μ at the point x is defined by*

$$d_{\mu,q}^\tau(x) \stackrel{\text{def}}{=} \liminf_{\varepsilon \to 0} \inf_{y \in B(x,\varepsilon)} \frac{\log \mu(B(y, \varepsilon)) + q\tau(B(y, \varepsilon))}{\log \varepsilon} \tag{3.10}$$

This definition is not exactly as in [17]. However, by adopting such a definition we may show directly that when the lower pointwise dimension is essentially a constant, then the spectrum for a measure coincides with that constant.

Proposition 3.1. *Let μ be a probability measure on X and $q \in \mathbb{R}$. If there exists some constant γ such that for μ-almost every x we have $d_{\mu,q}^\tau(x) = \gamma$ then*

$$\alpha_\tau^\mu(q) = \gamma.$$

[1]That is to say, there exists some constant M such that if $\{B(x_i, \varepsilon_i) \colon i \in I\}$ covers the subset $\Lambda \subset X$, and for any $x \in \Lambda$ there exists a sequence $i_1, i_2, \ldots$ with $\varepsilon_{i_n} \to 0$ such that $\cap_n B(x_{i_n}, \varepsilon_{i_n}) = \{x\}$, then there exists a countable subcover with a finite multiplicity bounded by M.

We omit the proof of this proposition because it is direct consequence of the general theorem proved in [9]. (The interested reader should compare our definition and the general theorem just mentioned with the definition p. 24 and Theorem 4.2 p. 28 in [17].)

3.2. **Dimension characteristics for Poincaré recurrence.** Now we apply the general framework to the specific situation where τ refers to the Poincaré recurrence of sets, under the dynamical system under consideration. Namely, if $T : X \to X$ is a map preserving the Borel probability measure μ, $A \subseteq X$ and $q \in \mathbb{R}$, then we denote

$$\begin{aligned} \mathcal{M}^T(A,\alpha,q,\varepsilon) &\stackrel{\text{def}}{=} \mathcal{M}^{\tau_T}(A,\alpha,q,\varepsilon); \\ \alpha_T(A,q) &\stackrel{\text{def}}{=} \alpha_{\tau_T}(A,q); \\ \alpha_T^\mu(q) &\stackrel{\text{def}}{=} \alpha_{\tau_T}^\mu(q); \\ d_{\mu,q}^T(x) &\stackrel{\text{def}}{=} d_{\mu,q}^{\tau_T}(x). \end{aligned}$$

Similarly, if $\Phi : X^\varphi \to X^\varphi$ is a special flow preserving the Borel probability measure $\overline{\mu}$, $\mathbf{A} \subseteq X^\varphi$, $q \in \mathbb{R}$ then we denote

$$\begin{aligned} \mathcal{M}^\Phi(\mathbf{A},\alpha,q,\varepsilon) &\stackrel{\text{def}}{=} \mathcal{M}^{\tau_\Phi}(\mathbf{A},\alpha,q,\varepsilon); \\ \alpha_\Phi(\mathbf{A},q) &\stackrel{\text{def}}{=} \alpha_{\tau_\Phi}(\mathbf{A},q); \\ \alpha_\Phi^{\overline{\mu}}(q) &\stackrel{\text{def}}{=} \alpha_{\tau_\Phi}^{\overline{\mu}}(q); \\ d_{\overline{\mu},q}^\Phi(\mathbf{x}) &\stackrel{\text{def}}{=} d_{\overline{\mu},q}^{\tau_\Phi}(\mathbf{x}). \end{aligned}$$

4. **Poincaré recurrences of cylinders.** In this section, we establish a central result needed to prove existence almost-everywhere of pointwise dimensions for Poincaré recurrences with respect to any ergodic measure of positive entropy. This results tells us that the return time of a cylinder of length n typically grows like n.

4.1. **Local rate of return time for cylinders.** We now define the local rate of return time for cylinders.

Definition 4.6 ([12]). *Lower and upper local rates of Poincaré recurrences for cylinders are defined respectively for non-invertible and invertible transformations by*

$$\overline{\underline{\mathcal{R}}}_\xi(x) \stackrel{\text{def}}{=} \overline{\underline{\lim}}_{n\to\infty} \frac{\tau(\xi^n(x))}{n} \quad \textit{and} \quad \overline{\underline{\mathcal{R}}}_\xi(x) \stackrel{\text{def}}{=} \overline{\underline{\lim}}_{n+m\to\infty} \frac{\tau(\xi_m^n(x))}{m+n} \,.$$

Weak specification property immediately implies the following result (we omit the proof which is straightforward):

Proposition 4.1. *If the system (X,T) is weakly specified, then $\overline{\mathcal{R}}_\xi(x) \le 1$.*

The following result will be crucial in what follows. We mention that it is established (in the non-invertible case) in [20] by using the notion of Kolmogorov complexity of an orbit and a theorem connecting it to entropy. The proof given here is a direct application of the Shannon-McMillan-Breiman Theorem.

Theorem 4.2. *Let $(X,\mathfrak{B},\mu)$ be a probability space where μ is ergodic with respect to a measurable transformation $T : X \to X$. If ξ is a finite or countable measurable partition with strictly positive entropy $h_\mu(T,\xi)$, then the lower rate of Poincaré* *recurrences for cylinders is almost surely bigger than one, i.e., for μ-a.e. $x \in X$, one has:*

$$\liminf_{n+m\to\infty} \frac{\tau(\xi_m^n(x))}{n+m} \geq 1 \quad \textit{in the case of invertible } T,$$
$$\liminf_{n\to\infty} \frac{\tau(\xi^n(x))}{n} \geq 1 \quad \textit{in the case of non-invertible } T.$$

Proof. For the sake of definiteness, we write the proof for the case of invertible T. The case of non-invertible T can be obtained in a similar way after evident simplifications.

It suffices to prove the theorem for finite partitions, the case of countable ξ will follow easily. More precisely, if $\xi = \{B_1, B_2, \dots\}$ is countable, then for some $m < \infty$ the finite partition $\hat{\xi} = \{B_1, B_2, \dots, B_m, \cup_{\ell>m} B_\ell\}$ will have positive entropy. In addition, ξ is finer than $\hat{\xi}$, hence $\tau(\xi_m^n(x)) \geq \tau(\hat{\xi}_m^n(x))$, and the statement follows.

Assume now that ξ is finite. Observe that $h \stackrel{\text{def}}{=} h_\mu(T, \xi)$ is non-zero and finite. Fix $\varepsilon \in (0, h/3)$. By Shannon-McMillan-Breiman theorem (see [19] for instance) for μ-a.e. x, there exists $N(x)$ such that if $n + m > N(x)$ then

$$\left| \frac{1}{n+m} \log \mu(\xi_m^n(x)) + h \right| \leq \varepsilon.$$

By Egoroff's theorem if $M = M(\varepsilon)$ is sufficiently large then $E_M \stackrel{\text{def}}{=} \{x \in X : N(x) < M\}$ will have a measure $\mu(E_{M(\varepsilon)}) > 1 - \varepsilon$. We can choose c so large that for any $x \in E_{M(\varepsilon)}$ and any positive integers n, m

$$c^{-1} e^{[-(n+m)h-(n+m)\varepsilon]} \leq \mu(\xi_m^n(x)) \leq c e^{[-(n+m)h+(n+m)\varepsilon]}. \tag{4.11}$$

We write now $E = E_{M(\varepsilon)}$. Let $\delta = 1 - \frac{3}{h}\varepsilon$ and set

$$\mathcal{A}_m^n \stackrel{\text{def}}{=} \{x \in E : \tau(\xi_m^n(x)) \leq \delta(n+m)\}.$$

Obviously $\mathcal{A}_m^n = \cup_{k=1}^{\delta(n+m)} R_m^n(k)$ where

$$R_m^n(k) \stackrel{\text{def}}{=} \{x \in E : \tau(\xi_m^n(x)) = k\}.$$

We shall prove that $\sum_{n,m} \mu(\mathcal{A}_m^n) < \infty$. Let n, m positive integers and $0 \leq k \leq n + m$. If the return time of the cylinder $C = [a_m a_{m+1} \cdots a_0 \cdots a_{n-1}] \in \xi_m^n$ is equal to k, i.e. $\tau(C) = k$, then it can be readily checked that $a_{j+k} = a_j$, for all $-m \leq j \leq n - k - 1$. This means that any block made with k consecutive symbols completely determines the cylinder C. In particular, since there exists $p \geq 0$ such that $p \leq m$ and $0 \leq k - p \leq n$, we can choose the cylinder $Z = \xi_p^{k-p}(x) \supset \xi_m^n(x)$. Let

$$\mathcal{Z} = \{\xi_p^{k-p}(x) : x \in R_m^n(k)\}.$$

Because of the structure of cylinders under consideration, for any cylinder $Z \in \mathcal{Z}$ there is a unique cylinder $C_Z \in \xi_m^n$ such that $C_Z \subset Z$ and one has $Z \cap R_m^n(k) \subset C_Z$. This implies

$$\mu(R_m^n(k)) = \sum_{Z \in \mathcal{Z}} \mu(Z \cap R_m^n(k)) \leq \sum_{Z \in \mathcal{Z}} \mu(C_Z).$$

But for each $Z \in \mathcal{Z}$ we have $Z \cap E \neq \emptyset$ and $C_Z \cap E \neq \emptyset$, thus there exists $x \in E$ such that $Z = \xi_p^{k-p}(x)$ and $C_Z = \xi_m^n(x)$. Using (4.11) we get

$$\mu(\xi_m^n(x)) \leq c \exp[-(n+m)h + (n+m)\varepsilon]$$
$$1 \leq c\mu(\xi_p^{k-p}(x)) \exp[kh + k\varepsilon].$$

Multiplying these inequalities we get

$$\mu(C_Z) \le c^2 \exp[-(n+m)h + (n+m)\varepsilon] \exp[kh + k\varepsilon]\mu(Z).$$

Summing up on $Z \in \mathcal{Z}$ we get (recall that $k \le n+m$)

$$\mu(R_m^n(k)) \le c^2 \exp[-(n+m-k)h + 2(n+m)\varepsilon].$$

This implies that

$$\begin{aligned}\mu(\mathcal{A}_m^n) &= \sum_{k=1}^{\delta(n+m)} \mu(R_m^n(k)) \\ &\le c^2 \frac{e^h}{e^h - 1} \exp\left[-(n+m)(h - \delta h - 2\varepsilon)\right].\end{aligned}$$

Since $h - \delta h - 2\varepsilon = h - (1 - \frac{3}{h}\varepsilon)h - 2\varepsilon = \varepsilon > 0$, we get that

$$\sum_{m\ge 1, n\ge 1} \mu(\mathcal{A}_m^n) < +\infty.$$

In view of Borel-Cantelli Lemma, we finally get that for μ-almost every $x \in E$, $\tau(\xi_m^n(x)) \ge (1 - \frac{3}{h}\varepsilon)(n+m)$ except for finitely many pairs of integer (n,m). Since in addition $\mu(E) > 1 - \varepsilon$, the arbitrariness of ε implies the desired result. □

Let us point out that according to Theorem 4.2 the lower rate of return is greater or equal than one independently of the particular choice of a map T, a partition ξ and an ergodic measure μ. This theorem can also be rephrased by saying that if T is a K-automorphism, then the local rate of return is greater or equal than one. The assumption of positiveness of entropy in Theorem 4.2 is unavoidable, as we shall show by some examples in the next section.

By Statement 4 of Proposition 2.2, it is easy to deduce that $\underline{\mathcal{R}}_\xi(x)$ and $\overline{\mathcal{R}}_\xi(x)$ are subinvariant functions so they are actually invariant since $(X, \mathfrak{B}, \mu)$ is a probability space. By ergodicity they are almost everywhere constant: $\underline{\mathcal{R}}_\xi(x) = \underline{r}, \overline{\mathcal{R}}_\xi(x) = \overline{r}$. Moreover, one has $\underline{r} \le \overline{r}$ by definition. Proposition 4.1 and Theorem 4.2 immediately imply that $\overline{r} \le 1$ and $\underline{r} \ge 1$, respectively, that is $\underline{r} = \overline{r} \stackrel{\text{def}}{=} \mathcal{R}_\xi(x) = 1$ almost everywhere. This establishes the following theorem.

Theorem 4.3. *Let μ be an ergodic measure of the dynamical system (X,T) and ξ be the finite partition of X defined in the set-up, such that $\mathrm{h}_\mu(T) > 0$. Assume that (X, T, d_X) is weakly specified. Then*

$$\mathcal{R}_\xi(x) = 1 \quad \mu\text{-a.e.}.$$

Theorem 4.3 shows us that, provided that the system is weakly specified and has positive entropy, then the local rate of return of cylinders is equal to one, so it is independent of the partition ξ, the measure μ and the map T.

4.2. **Two examples with fast recurrence.** We consider two examples which show that positiveness of entropy is essential in the hypothesis of Theorem 4.2 to get the lower limit greater or equal to 1.

Dyadic adding machine. Let $\Omega_2^+ \stackrel{\text{def}}{=} \{0,1\}^{\mathbb{N}}$ be the set of all one-sided infinite sequences, endowed with the usual distance. Denote by $T : \Omega_2^+ \circlearrowleft$ the following map: if $\omega = (1,1,...)$ then $T\omega = (0,0,...)$; if $\omega = (i_0, i_1, ...)$, $i_k = 1$, $k = 0,1,...,j-1$ and $i_j = 0$, then $(T\omega)_k = 0$, $k = 0,1,...,j-1$, $(T\omega)_j = 1$, $(T\omega)_{j+s} = i_{j+s}$, $s = 1,2,...$; if $\omega = (0, i_1, ...)$ then $T\omega = (1, i_1, ...)$. It is simple to see that T is one-to-one and continuous. The dynamical system (Ω_2^+, T) is called the dyadic adding machine. Let ζ^0 be the partition of Ω_2^+ by m-cylinders $[i_0, ..., i_{i_{m-1}}]$ for some $m \geq 1$. Since $T^{-1}([i_0, ..., i_{i_{m-1}}]) = [j_0, ..., j_{i_{m-1}}]$ then $\zeta^n \stackrel{\text{def}}{=} \bigvee_{j=0}^{n-1} T^{-j}\zeta^0 = \zeta^0$. Moreover, $\tau_T([i_0, ..., i_{i_{m-1}}]) = 2^m$. Therefore

$$\frac{\tau_T(\zeta^n(\omega))}{n} = \frac{2^m}{n} \to 0$$

as $n \to \infty$ for any $\omega \in \Omega_2^+$.

Rotation on the circle. Consider a rotation $f_\omega : x \mapsto x - \omega \bmod 1$ (i.e. $f_\omega^{-1}x = x + \omega \bmod 1$), on the circle $\mathcal{S}^1 = \{x, \bmod 1\}$, where $0 < \omega < 1$ is an irrational number. The number ω can be approximated by rational numbers p/q (p and q are relatively prime) in such a way that

$$\left|\omega - \frac{p}{q}\right| < \frac{1}{q^{\beta+1}} \tag{4.12}$$

for some value β and some pair (p, q). Let $\beta(\omega) \stackrel{\text{def}}{=} \sup \beta$ where the supremum is taken over all β for which inequality (4.12) has infinitely many solutions (p, q) with $q > 0$. Assume that $\beta(\omega) < \infty$, i.e. ω is a Diophantine number. Then for every $\delta \in (0, 1)$ the inequality

$$\left|\omega - \frac{p}{q}\right| < \frac{1}{q^{\beta(\omega)+1-\delta}} \tag{4.13}$$

holds for infinitely many relatively prime pairs (p_i, q_i), with $q_i \to \infty$, as $i \to \infty$. Consider a partition ξ_0 of $\mathcal{S}^1$ made with two closed intervals $[0, \omega]$ and $[\omega, 1]$, and let $\xi_n \stackrel{\text{def}}{=} \bigvee_{i=0}^{n-1} f_\omega^{-i}\xi_0$. Denote by $\xi_n(x)$ an element of ξ_n containing a point x (the definition is correct for all x except for a point belonging to the set of the endpoints of intervals in ξ_n). The rotation is metrically isomorphic to the subshift $\operatorname{clos}(\pi([0, 1))$, where the coding map $\pi : [0, 1] \to \{0, 1\}^{\mathbb{N}}$ is defined in the obvious way by $\pi(x)_n = 0$ is $f_\omega^n(x) \in [0, \omega)$ and $\pi(x)_n = 1$ if $f_\omega^n(x) \in [\omega, 1)$. We now state the following theorem:

Theorem 4.4. *If $\beta(\omega) > 3$ then*

$$\underline{\mathcal{R}}_\xi(x) = \liminf_{n\to\infty} \frac{1}{n}\tau_{f_\omega}(x) = 0 \tag{4.14}$$

for almost every x with respect to the Lebesgue measure on $\mathcal{S}^1$.

Proof. Start by choosing δ to be so small that

$$\beta(\omega) - \delta > 3\,. \tag{4.15}$$

Without loss of generality we may assume that

$$\left|\omega - \frac{p_i}{q_i}\right| = \inf_{p\in\mathbb{Z}} \left|\omega - \frac{p}{q_i}\right|\,.$$

Then because of (4.13), we have

$$\operatorname{dist}(x, f_\omega^{q_i}x) = \inf_{p\in\mathbb{Z}} |x + q_i\omega - x - p| = q_i\left|\omega - \frac{p_i}{q_i}\right| < \frac{1}{q_i^{\beta(\omega)-\delta}}\ \cdot \tag{4.16}$$

Now introduce a number $\alpha > 1$ such that $1 + 2\alpha < \beta(\omega) - \delta$. Let $m_i \stackrel{\text{def}}{=} [q_i^{1+\alpha}]$ the integer part of $q_i^{1+\alpha}$ and let

$$A_{m_i} \stackrel{\text{def}}{=} \#\left\{\xi_{m_i}(x) : \operatorname{diam}\xi_{m_i}(x) \geq \frac{1}{q_i^{1+2\alpha}}\right\},$$

$$B_{m_i} \stackrel{\text{def}}{=} \#\left\{\xi_{m_i}(x) : \operatorname{diam}\xi_{m_i}(x) < \frac{1}{q_i^{1+2\alpha}}\right\}.$$

Since $A_{m_i} + B_{m_i} = 2m_i$ then $A_{m_i} + B_{m_i} \leq 2q_i^{1+\alpha}$. This inequality implies that $B_{m_i} \leq 2q_i^{1+\alpha}$. Moreover, $\mu(\mathcal{B}_{m_i}) \leq 2q_i^{1+\alpha} \cdot q_i^{-(1+2\alpha)} = 2q_i^{-\alpha}$, where

$$\mathcal{B}_{m_i} \stackrel{\text{def}}{=} \bigcup_{\operatorname{diam} \xi_{m_i}(x) < q_i^{-(1+2\alpha)}} \xi_{m_i}(x) ,$$

the union of the elements of the partition ξ_{m_i} of small diameter. In view of Borel-Cantelli Lemma (recall that $\alpha > 1$), we have that μ-almost every point x belongs to the complement of $\mathcal{B}_{m_i}$ provided that m_i is large enough, i.e. $\operatorname{diam} \xi_{m_i}(x) \geq q_i^{-(1+2\alpha)}$.

Now because of (4.16) and the assumption $\beta(\omega) > 3$, we have

$$\tau_{f_\omega}(\xi_{m_i}(x)) \leq q_i .$$

Therefore,

$$\liminf_{n\to\infty} \frac{\tau_{f_\omega}(\xi_n(x)}{n} \leq \lim_{i\to\infty} \frac{q_i}{q_i^{1+\alpha}} = 0 \quad \text{Lebesgue-almost everywhere.}$$

□

5. **Existence of Pointwise dimensions and computation of the spectra for measures.** If (X, T, d) is weakly specified and μ is ergodic, then we can find formulas for pointwise dimensions and the spectrum for the measure.

Theorem 5.1. *Under the assumptions of Theorem 4.3, for any $q \in \mathbb{R}$ and for μ-a.e. $x \in X$, one has:*

$$d^T_{\mu,q}(x) = \frac{\mathrm{h}_\mu(T) - q}{\chi^+_\mu(T)} \quad \textit{in the non-invertible case, and} \tag{5.17}$$

$$d^T_{\mu,q}(x) = (\mathrm{h}_\mu(T) - q)\left(\frac{1}{\chi^+_\mu(T)} - \frac{1}{\chi^-_\mu(T)}\right) \quad \textit{in the invertible case .} \tag{5.18}$$

Proof. For the sake of definiteness, we write the proof for the case of invertible T. We have to calculate the limit of the ratio in (3.10). Let $x \in X$ be fixed. For any $\varepsilon \in (0,1)$ Lemma 2.1 gives that $B(x,\varepsilon) = \xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x)$. Note that in particular we have $B(y,\varepsilon) = B(x,\varepsilon)$ for every $y \in B(x,\varepsilon)$. In addition, since u and v are bounded, $m_{x,\varepsilon} \to \infty$ and $n_{x,\varepsilon} \to \infty$ as $\varepsilon \to 0$. Hence we only have to calculate the limit of the following ratio

$$\frac{\log \mu(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x)) + q\tau_T(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x))}{\log \varepsilon} .$$

Because of the continuity of u, there exists a sequence $c(k)$ with $c(k) \to 0$ as $k \to \infty$ such that

$$|\mathrm{u}(x) + \mathrm{u}(Tx) + \cdots + \mathrm{u}(T^{n-1}x) - \mathrm{u}(\xi^n_0(x))| \leq c(n)n$$

for any integer $n > 0$. Similarly, we get

$$|\mathrm{v}(x) + \mathrm{v}(T^{-1}x) + \cdots + \mathrm{v}(T^{-m+1}x) - \mathrm{v}(\xi^0_m(x))| \leq c(m)m$$

for any integer $m > 0$. Birkhoff ergodic Theorem and Lemma 2.1 then give

$$\lim_{\varepsilon\to 0} \frac{n_{x,\varepsilon}}{\log \varepsilon} = -\frac{1}{\int \mathrm{u} d\mu} \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{m_{x,\varepsilon}}{\log \varepsilon} = -\frac{1}{\int \mathrm{v} d\mu} \quad \mu\text{-a.e..} \tag{5.19}$$

Shannon-McMillan-Breiman Theorem asserts that

$$\lim_{\varepsilon\to 0} \frac{1}{n_{x,\varepsilon} + m_{x,\varepsilon}} \log \mu(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x)) = -\mathrm{h}_\mu(T) \quad \mu\text{-a.e.} . \tag{5.20}$$

Finally, by Theorem 4.3, we have

$$\lim_{\varepsilon\to 0}\frac{\tau_T(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x))}{n_{x,\varepsilon}+m_{x,\varepsilon}} = 1 \quad \mu\text{-a.e.}. \tag{5.21}$$

We conclude that for any $x \in X$ such that (5.19), (5.20) and (5.21) hold, we have

$$\lim_{\varepsilon\to 0}\frac{\log\mu(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x)) + q\tau_T(\xi^{n_{x,\varepsilon}}_{m_{x,\varepsilon}}(x))}{\log\varepsilon} = (\mathrm{h}_\mu(T)-q)\left(\frac{1}{\int \mathrm{u}\,d\mu}+\frac{1}{\int \mathrm{v}\,d\mu}\right),$$

and the conclusion follows by using (5.19) since these points form a set of full μ-measure (recall (2.2)). □

One could see relations derived in Theorem 5.1 as an analog of Young's formula [21] for dimensions for Poincaré recurrences. It is explicit if $q = 0$. Note also that the spectrum for the set X, $\alpha_T(X, q)$, was obtained for certain subshifts in [4], where it has been shown that it satisfies a non-homogeneous Bowen equation.

Theorem 5.1 and Proposition 3.1 immediately give the expression for the spectrum for the measure:

Theorem 5.2. *Under the assumptions of Theorem 4.3, for any $q \in \mathbb{R}$ one has*

$$\alpha^\mu_T(q) = \frac{\mathrm{h}_\mu(T)-q}{\chi^+_\mu(T)} \quad \textit{in the non-invertible case, and}$$

$$\alpha^\mu_T(q) = (\mathrm{h}_\mu(T)-q)\left(\frac{1}{\chi^+_\mu(T)}-\frac{1}{\chi^-_\mu(T)}\right) \quad \textit{in the invertible case}.$$

Corollary 5.3. *The value of q for which α^μ_T vanishes is equal to $\mathrm{h}_\mu(T) \stackrel{\mathrm{def}}{=} q^T_0$.*

6. **Pointwise dimensions and spectra for special flows.**

In this section, we prove the existence almost-everywhere of pointwise dimensions associated with Poincaré recurrences, and obtain a precise formula, for special flows. This is done with respect to any ergodic measure invariant under the flow constructed with an ergodic measure with positive entropy given on the space X.

The strategy is as follows. We know by (2.5) that a ball in the special space is approximately the product of a ball in the base and an interval. Next, we can relate the return time of the ball to the Birkhoff sum of the roof function. The number of terms in this sum equals the return time of the ball in the base (see (6.25)), thus we can relate precisely return times for maps and special flows.

Proposition 6.1. *Let $\overline{\mu}$ be the measure on X^φ induced by the measure ergodic μ on X where X and μ satisfy hypothesis of Theorem 4.3. We have for $\overline{\mu}$-almost every point $\mathbf{x} \in X^\varphi$ that*

$$\lim_{\varepsilon\to 0}\frac{\tau_\Phi(B(\mathbf{x},\varepsilon))}{-\log\varepsilon} = \left(\frac{1}{\chi^+_{\overline{\mu}}(\Phi)}-\frac{1}{\chi^-_{\overline{\mu}}(\Phi)}\right) \quad \textit{in the case of invertible } \Phi$$

$$\lim_{\varepsilon\to 0}\frac{\tau_\Phi(B(\mathbf{x},\varepsilon))}{-\log\varepsilon} = \frac{1}{\chi^+_{\overline{\mu}}(\Phi)} \quad \textit{in the case of non-invertible } \Phi$$

Proof. As before we only write down the proof in the invertible case.

By Fubini's theorem we have $\overline{\mu}(\{\mathbf{x} = (x, s)\colon x \in Y, 0 < s < \varphi(x)\}) = 1$ whenever $\mu(Y) = 1$. We choose Y to be the set of points such that (5.19) and (5.21) in the proof of Theorem 5.1 hold and such that

$$\lim_{n,m\to\infty}\frac{\varphi(T^{-m}x)+\cdots+\varphi(x)+\cdots+\varphi(T^{n-1}x)}{n+m} = \int_X \varphi\,d\mu. \tag{6.22}$$

Let $x \in Y$ and $0 < s < \varphi(x)$. It is clear that if $\varepsilon > 0$ is sufficiently small then (see (2.4))

$$\{\mathbf{y} : d_\pi(\mathbf{x},\mathbf{y}) < \varepsilon\} = B(x,\varepsilon) \times (s-\varepsilon, s+\varepsilon).$$

Accordingly by (2.5) we have for some $\delta > 0$ independent of ε

$$B(x,\delta\varepsilon) \times (s-\delta\varepsilon, s+\delta\varepsilon) \subset B(\mathbf{x},\varepsilon) \subset B(x,\varepsilon/\delta) \times (s-\varepsilon/\delta, s+\varepsilon/\delta). \tag{6.23}$$

Because of (6.23) it is enough to prove the following:

$$\lim_{\varepsilon\to 0} \frac{\tau_\Phi(B(x,\varepsilon)\times[s-\varepsilon,s+\varepsilon])}{-\log\varepsilon} = \left(\frac{1}{\chi_{\mu}^{+}} - \frac{1}{\chi_{\mu}^{-}}\right). \tag{6.24}$$

We first remark that

$$\begin{aligned} \tau_\Phi(B(x,\varepsilon)\times[s-\varepsilon,s+\varepsilon]) &= \tau_\Phi(B(x,\varepsilon)\times\{0\}) - 2\varepsilon \\ &= \inf_{y\in B(x,\varepsilon)} \inf\{t>0 : \Phi_t(y,0) \in \xi_{m_{x,\varepsilon}}^{n_{x,\varepsilon}} \times \{0\}\} \\ &= \inf_{y\in B(x,\varepsilon)} \sum_{k=0}^{\tau_T(y,B(x,\varepsilon))-1} \varphi(T^k y) \stackrel{\text{def}}{=} t(x,\varepsilon). \end{aligned} \tag{6.25}$$

Let us put $\mathrm{var}_k\varphi \stackrel{\text{def}}{=} \sup\{|\varphi(z)-\varphi(z')| : z' \in \xi_k^k(z)\}$. Since φ is continuous we have $\mathrm{var}_k\varphi \to 0$ as $k\to\infty$. Let us remark that $(1/n)\sum_{k=0}^n \mathrm{var}_k\varphi \to 0$ as well, by Césaro's Lemma. Given an integer n and $y \in \xi_0^n(x)$, for any $k \in \{0,\dots,n-1\}$ we have $T^k y \in \xi_{-k}^{n-k}(T^k x)$ and then

$$|\varphi(T^k y) - \varphi(T^k x)| \le \mathrm{var}_{\min(k,n-k)}\varphi, \quad \forall k \in \{0,\dots,n-1\}.$$

The same is true if $y \in \xi_m^0(x)$ for some integer m, namely,

$$|\varphi(T^{-k} y) - \varphi(T^{-k} x)| \le \mathrm{var}_{\min(m-k,k)}\varphi, \quad \forall k \in \{1,\dots,m\}.$$

Notice that by Lemma 2.1 we have $B(x,\varepsilon) = \xi_{m_{x,\varepsilon}}^{n_{x,\varepsilon}}(x)$, thus if $\tau_T(y,B(x,\varepsilon)) = \ell < \infty$ then $y \in \xi_0^{n_{x,\varepsilon}}(x)$ and $T^\ell y \in \xi_{m_{x,\varepsilon}}^0(x)$, which implies that

$$\begin{aligned} \left|\sum_{k=0}^{n_{x,\varepsilon}-1} \varphi(T^k y) - \sum_{k=0}^{n_{x,\varepsilon}-1} \varphi(T^k x)\right| &\le 2\sum_{k=0}^{n_{x,\varepsilon}/2} \mathrm{var}_k(\varphi) = o(n_{x,\varepsilon}), \\ \left|\sum_{k=\ell-m_{x,\varepsilon}}^{\ell-1} \varphi(T^k y) - \sum_{k=-m_{x,\varepsilon}}^{-1} \varphi(T^k x)\right| &\le 2\sum_{k=0}^{m_{x,\varepsilon}/2} \mathrm{var}_k(\varphi) = o(m_{x,\varepsilon}). \end{aligned} \tag{6.26}$$

By (5.21) we have

$$|\tau_T(B(x,\varepsilon)) - n_{x,\varepsilon} - m_{x,\varepsilon}| = o(n_{x,\varepsilon} + m_{x,\varepsilon}). \tag{6.27}$$

In addition, Since $\ell \ge \tau_T(B(x,\varepsilon))$, (6.27) and (6.26) give

$$t(x,\varepsilon) \ge \sum_{k=-m_{x,\varepsilon}}^{n_{x,\varepsilon}-1} \varphi(T^k x) - o(n_{x,\varepsilon} + m_{x,\varepsilon}). \tag{6.28}$$

Next, if y is such that $\tau_T(y,B(x,\varepsilon)) = \tau_T(B(x,\varepsilon))$ then by (6.27) and (6.26) we find out that

$$t(x,\varepsilon) \le \sum_{k=-m_{x,\varepsilon}}^{n_{x,\varepsilon}-1} \varphi(T^k x) + o(n_{x,\varepsilon} + m_{x,\varepsilon}). \tag{6.29}$$

By (6.22) and the estimates (6.28) and (6.29) we get

$$t(x,\varepsilon) = (n_{x,\varepsilon} + m_{x,\varepsilon}) \int \varphi d\mu + o(n_{x,\varepsilon} + m_{x,\varepsilon}).$$

The relation (6.24) is now a consequence of this equation together with (5.19) and (6.25). This finishes the proof of the proposition. □

Lemma 6.2. *Let $\overline{\mu}$ be the measure on X^φ induced by the measure μ on X. There exists some constant $c > 0$ such that for any point $\mathbf{x} = (x,s) \in X^\varphi$ and $\varepsilon > 0$,*

$$c\mu(B(x,c\varepsilon)) \cdot \varepsilon \leq \overline{\mu}(B(\mathbf{x},\varepsilon)) \leq c^{-1}\mu(B(x,c^{-1}\varepsilon)) \cdot \varepsilon. \tag{6.30}$$

Proof. Since the flow is Lipschitz and preserves the measure $\overline{\mu}$, we can always assume that the ball is centered at some point $\mathbf{x} = (x,s)$ far away from the roof. Then the conclusion follows from (6.23) in the proof of Proposition 6.1 and formula (2.6). □

Theorem 6.3. *Let $\overline{\mu}$ be the measure on X^φ induced by the ergodic measure μ on X, where X and μ satisfy hypothesis of Theorem 4.3. For all $q \in \mathbb{R}$, the lower pointwise dimension is $\overline{\mu}$-a.e. $\mathbf{x} \in X^\varphi$ is equal to*

$$d^{\Phi}_{\overline{\mu},q}(\mathbf{x}) = 1 + \frac{\mathrm{h}_{\overline{\mu}}(\Phi) - q}{\chi^{+}_{\overline{\mu}}(\Phi)} \text{ in the non-invertible case, and}$$

$$d^{\Phi}_{\overline{\mu},q}(\mathbf{x}) = 1 + (\mathrm{h}_{\overline{\mu}}(\Phi) - q)\left(\frac{1}{\chi^{+}_{\overline{\mu}}(\Phi)} - \frac{1}{\chi^{-}_{\overline{\mu}}(\Phi)}\right) \text{ in the invertible case.}$$

Proof. For the sake of definiteness, we write the proof for the case of invertible T. The non-invertible case follows straightforwardly. We will compute the limit of infimum of the ratio in (3.10). By definition it is obvious that for any $\mathbf{x} \in X^\varphi$

$$d_{\overline{\mu}}(\mathbf{x}) \leq \liminf_{\varepsilon \to 0} \frac{\log \overline{\mu}(B(\mathbf{x},\varepsilon)) + q\tau_\Phi(B(\mathbf{x},\varepsilon))}{\log \varepsilon}.$$

Let $(x,s) = \mathbf{x}$ with $0 < s < \varphi(x)$. For any $\mathbf{y} \in B(\mathbf{x},\varepsilon)$ we have $B(\mathbf{y},\varepsilon) \subset B(\mathbf{x},2\varepsilon)$ hence

$$\frac{\log \overline{\mu}(B(\mathbf{y},\varepsilon))}{\log \varepsilon} \geq \frac{\log \overline{\mu}(B(\mathbf{x},2\varepsilon))}{\log \varepsilon}$$

provided $\varepsilon < 1$. We now consider the two different cases $q \leq 0$ and $q > 0$:

$\underline{q \leq 0}$: In this case we have $q/\log\varepsilon \geq 0$ if $\varepsilon < 1$, hence

$$\frac{q\tau_\Phi(B(\mathbf{y},\varepsilon))}{\log \varepsilon} \geq \frac{q\tau_\Phi(B(\mathbf{x},2\varepsilon))}{\log \varepsilon} \tag{6.31}$$

because $\tau_\Phi(B(\mathbf{y},\varepsilon)) \geq \tau_\Phi(B(\mathbf{x},2\varepsilon))$.

$\underline{q > 0}$: For any $\mathbf{y} = (y,t)$ we have by construction $B(y,\varepsilon) \times \{t\} \subset B(\mathbf{y},\varepsilon)$. Thus

$$\begin{aligned}\tau_\Phi(B(\mathbf{y},\varepsilon)) &\leq \tau_\Phi(B(y,\varepsilon) \times \{t\}) \\ &\leq \sum_{k=0}^{\tau_T(B(y,\varepsilon))} \varphi(T^k z),\end{aligned}$$

where $z \in B(y,\varepsilon)$ is such that $\tau_T(z, B(y,\varepsilon)) = \tau_T(B(y,\varepsilon))$. By Lemma 2.1 there exists $m_{y,\varepsilon}$ and $n_{y,\varepsilon}$ such that $B(y,\varepsilon) = \xi^{n_{y,\varepsilon}}_{m_{y,\varepsilon}}(y)$ and by weak-specification we have $\tau_T(\xi_{m_{y,\varepsilon}}^{n_{y,\varepsilon}}(y)) \leq m_{y,\varepsilon} + n_{y,\varepsilon} + o(m_{y,\varepsilon} + n_{y,\varepsilon})$. Proceeding as in the proof of Proposition 6.1 we find out that

$$\tau_\Phi(B(\mathbf{y},\varepsilon)) \leq \sum_{k=-m_{y,\varepsilon}}^{n_{y,\varepsilon}} \varphi(T^k y) + o(m_{y,\varepsilon} + n_{y,\varepsilon}).$$

Whenever $\mathbf{y} = (y,t) \in B(\mathbf{x},\varepsilon)$, since $0 < s < \varphi(x)$ (2.5) implies that (see also (2.4)) if $\varepsilon > 0$ is sufficiently small then $y \in B(x,\delta\varepsilon)$. Proceeding again as in the proof of Proposition 6.1 (and also Theorem 5.1) we find out that the continuity of the functions φ, u and v implies

$$|n_{x,\varepsilon} - n_{y,\varepsilon}| = o(n_{x,\varepsilon}),$$
$$|m_{x,\varepsilon} - m_{y,\varepsilon}| = o(m_{x,\varepsilon}),$$
$$\left| \sum_{k=-m_{y,\varepsilon}}^{n_{y,\varepsilon}} \varphi(T^k y) - \sum_{k=-m_{x,\varepsilon}}^{n_{x,\varepsilon}} \varphi(T^k x) \right| = o(m_{x,\varepsilon} + n_{x,\varepsilon}).$$

Since we have $q/\log\varepsilon < 0$ if $\varepsilon < 1$, we get

$$\frac{q\tau_\Phi(B(\mathbf{y},\varepsilon))}{\log\varepsilon} \geq \frac{q}{\log\varepsilon}\left(\sum_{k=-m_{x,\varepsilon}}^{n_{x,\varepsilon}} \varphi(T^k x)\right) - o(m_{x,\varepsilon} + n_{x,\varepsilon}). \tag{6.32}$$

Whatever the sign of q is, the lower bound in (6.31) and (6.32) have $\overline{\mu}$-a.e. the same limit (given by Proposition 6.1) that is equal to

$$q\left(\frac{1}{\chi_{\overline{\mu}}^+(\Phi)} - \frac{1}{\chi_{\overline{\mu}}^-(\Phi)}\right) .$$

By Lemma 6.2 we also have

$$\lim_{\varepsilon\to 0} \frac{\log\overline{\mu}((x,s),\varepsilon))}{\log\varepsilon} = 1 + \lim_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon} .$$

This last quantity converges for μ-a.e. $x \in X$ to

$$1 + h_\mu(T)\left(\frac{1}{\chi_\mu^+(T)} - \frac{1}{\chi_\mu^-(T)}\right) = 1 + h_{\overline{\mu}}(\Phi)\left(\frac{1}{\chi_{\overline{\mu}}^+(\Phi)} - \frac{1}{\chi_{\overline{\mu}}^-(\Phi)}\right)$$

by Abramov formula (2.7). This finishes the proof of our theorem. □

Now we can state the following theorem :

Theorem 6.4. *Under the conditions of Theorem 6.3, the spectrum for the measure* $\overline{\mu}$ *is given for all* $q \in \mathbb{R}$ *by*

$$\alpha_\Phi^{\overline{\mu}}(q) = 1 + \frac{\mathrm{h}_{\overline{\mu}}(\Phi) - q}{\chi_{\overline{\mu}}^+(\Phi)} \text{ in the non-invertible case, and}$$

$$\alpha_\Phi^{\overline{\mu}}(q) = 1 + (\mathrm{h}_{\overline{\mu}}(\Phi) - q)\left(\frac{1}{\chi_{\overline{\mu}}^+(\Phi)} - \frac{1}{\chi_{\overline{\mu}}^-(\Phi)}\right) \text{ in the invertible case.}$$

Proof. By Theorem 6.3 we are in the condition to apply Proposition 3.1. This proves the theorem. □

The following Corollary is straightforward but useful:

Corollary 6.5. *Under assumptions of Theorems 5.2 and 6.4, one has*

$$\alpha_\Phi^{\overline{\mu}}\left(\frac{q}{\int\varphi\,d\mu}\right) = 1 + \alpha_T^\mu(q) .$$

Proof. Abramov formula and Theorems 5.2 and 6.4 give the relation. □

Corollary 6.6. *The value of q for which $\alpha_\Phi^{\overline{\mu}}$ vanishes is equal to*

$$\mathrm{h}_{\overline{\mu}}(\Phi) + \chi_{\overline{\mu}}^{+}(\Phi) \stackrel{\text{def}}{=} q_0^\Phi \text{ in the non-invertible case, and}$$

$$\mathrm{h}_{\mu}(\Phi) + \frac{1}{\frac{1}{\chi_{\overline{\mu}}^{+}(\Phi)} - \frac{1}{\chi_{\overline{\mu}}^{-}(\Phi)}} \stackrel{\text{def}}{=} q_0^\Phi \text{ in the invertible case.}$$

We can express q_0^Φ as a function of q_0^T by observing that $q_0^T = \mathrm{h}_\mu(T)$ to get (in the non-invertible case):

$$q_0^\Phi = \frac{q_0^T}{\int \varphi\, d\mu} + \chi_{\overline{\mu}}^{+}(\Phi) \cdot$$

7. **Closing note.** Our results may be generalized in different directions. First of all, it would be worthwhile to extend our study to conformal Axiom A diffeomorphisms and flows. We mention the work [18] in which Hausdorff dimension in the case of conformal Axiom A flows is studied. Secondly, multifractal analysis for Poincaré recurrences seems to be feasible: Theorems 5.1 and 6.3 can be viewed as the first steps in this direction. Furthermore, we did not study both capacities and box dimensions associated with Poincaré recurrences (see [17] for general definitions). We did not investigate the problem of existence of measures of full dimensions (for Poincaré recurrences), either. As in [4] for the case of maps, we can expect to derive some nonhomogeneous Bowen equations for special flows.

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E-mail address: `valentin@cactus.iico.usalp.mx`

E-mail address: `jeanrene@cpht.polytechnique.fr`

URL: `http://www.cpht.polytechnique.fr/cpht/chazottes/`

E-mail address: `benoit.saussol@u-picardie.fr`

URL: `http://www.mathinfo.u-picardie.fr/saussol/`