HIGHER-DIMENSIONAL MULTIFRACTAL ANALYSIS

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ABSTRACT. We establish a higher-dimensional version of multifractal analysis for several classes of hyperbolic dynamical systems. This means that we consider multifractal decompositions which are associated to multi-dimensional parameters. In particular, we obtain a conditional variational principle, which shows that the topological entropy of the level sets of pointwise dimensions, local entropies, and Lyapunov exponents can be *simultaneously* approximated by the entropy of ergodic measures. A similar result holds for the Hausdorff dimension. This study allows us to exhibit new nontrivial phenomena absent in the one-dimensional multifractal analysis. In particular, while the domain of definition of a one-dimensional spectrum is always an interval, we show that for higher-dimensional spectra the domain may not be convex and may even have empty interior, while still containing an uncountable number of points. Furthermore, the interior of the domain of a higher-dimensional spectrum has in general more than one connected component.

1. INTRODUCTION

Consider a one-sided subshift of finite type $\sigma\colon X \to X$. This means that there exist a positive integer m, and an $m \times m$ matrix A, called the transfer matrix of σ, whose entries a_{ij} are either 0 or 1, such that X is the set of sequences $(i_1 i_2 \cdots)$ on m symbols satisfying $a_{i_k i_{k+1}} = 1$ for every k. The shift map is defined by $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$. Consider also a continuous function $\varphi\colon X \to \mathbb{R}$. For each $\alpha \in \mathbb{R}$ we define the corresponding level set of the Birkhoff averages of φ by

$$K_\alpha(\varphi) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k x) = \alpha \right\}. \tag{1}$$

By Birkhoff's ergodic theorem, if μ is an ergodic σ-invariant probability measure on X, and $a = \int_X \varphi \, d\mu$, then $\mu(K_a(\varphi)) = 1$. This does not mean that the sets $K_\alpha(\varphi)$ are empty for other values of α. In fact, for several classes of hyperbolic dynamical systems it has been established that:

1. if $K_\alpha(\varphi) \neq \varnothing$, then $K_\alpha(\varphi)$ is a proper dense set;
2. the set $\{\alpha \in \mathbb{R} : K_\alpha(\varphi) \neq \varnothing\}$ is an interval;
3. the function $\alpha \mapsto h(\sigma|K_\alpha(\varphi))$ is analytic and strictly convex, where $h(\sigma|K)$ denotes the topological entropy of $\sigma|K$ (see Section 2 for the definition).

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In particular, these statements imply that there exist uncountably many values of α such that K_α is a proper dense set with positive topological entropy. This observation already reveals an extreme complexity of the structure of Birkhoff averages. We refer the reader to the book [9] for references and full details.

We want to establish a higher-dimensional version of this study. More precisely we want to consider sets which are obtained as the intersection of level sets of Birkhoff averages, such as

$$K_{\alpha,\beta} = K_\alpha(\varphi) \cap K_\beta(\psi), \tag{2}$$

and describe their multifractal properties, including their "size" in terms of topological entropy and of Hausdorff dimension, thus providing a higher-dimensional version of multifractal analysis. The term "higher-dimensional" refers to the multi-dimensional parameter (α, β).

On the other hand, we shall demonstrate that the corresponding higher-dimensional multifractal spectra exhibit several new nontrivial phenomena clearly absent in the one-dimensional case. It turns out that the known approaches to the study of one-dimensional multifractal spectra no longer apply or have to be considerably modified to address this new situation. This is done in this paper. Nevertheless a unifying theme will continue being the use of the thermodynamic formalism.

We shall now illustrate our results with a rigorous statement in the special case of subshifts of finite type. Let $\mathcal{M}(X)$ be the family of σ-invariant Borel probability measures on X, and consider the set

$$\mathcal{D} = \left\{ \left(\int_X \varphi \, d\mu, \int_X \psi \, d\mu \right) \in \mathbb{R}^2 : \mu \in \mathcal{M}(X) \right\}.$$

We denote by $h_\mu(\sigma)$ the metric entropy of a measure $\mu \in \mathcal{M}(X)$, and by $P_X(\varphi)$ the topological pressure of the function φ (see Section 2 for the definition). The following statement provides a conditional variational principle for the sets $K_{\alpha,\beta}$ when σ is topologically mixing, i.e., when $A^k > 0$ for some k, where A is the transfer matrix of σ.

Theorem 1. *Let $\sigma|X$ be a topologically mixing subshift of finite type, and φ and ψ Hölder continuous functions on X. If $(\alpha, \beta) \in \operatorname{int} \mathcal{D}$ then $K_{\alpha,\beta} \neq \varnothing$, and*

$$\begin{aligned} h(\sigma|K_{\alpha,\beta}) &= \sup \left\{ h_\mu(\sigma) : \mu \in \mathcal{M}(X) \text{ and } \left(\int_X \varphi \, d\mu, \int_X \psi \, d\mu \right) = (\alpha, \beta) \right\} \\ &= \inf \left\{ P_X(p(\varphi - \alpha) + q(\psi - \beta)) : (p, q) \in \mathbb{R}^2 \right\}. \end{aligned}$$

Theorem 1 follows from the much more general results formulated in Section 4. In particular we shall consider the intersection of any finite number of level sets of Birkhoff averages, as well as other local quantities such as pointwise dimensions, local entropies, and Lyapunov exponents. We shall also consider the more general class of dynamical systems for which the metric entropy is upper semi-continuous.

The following statement gives further detailed information about the sets $K_{\alpha,\beta}$. In particular it provides a condition under which $\mathcal{D} = \overline{\operatorname{int} \mathcal{D}}$, and thus such that the identities in Theorem 1 hold for an open and dense set of pairs $(\alpha, \beta) \in \mathcal{D}$. In a certain sense this condition is optimal (see Section 4 for a detailed discussion). Recall that two functions φ and ψ are said to be *cohomologous* (with respect to σ) if there exist a constant $c \in \mathbb{R}$ and a continuous function $g\colon X \to \mathbb{R}$ such that $\varphi - \psi = g - g \circ \sigma + c$ on X.

Theorem 2. *If $\sigma|X$ is a topologically mixing subshift of finite type, and φ and ψ are Hölder continuous functions on X, then the following properties hold:*

1. *if $(\alpha, \beta) \notin \mathcal{D}$ then $K_{\alpha,\beta} = \varnothing$;*
2. *if for each $(p, q) \in \mathbb{R}^2$ the function $p\varphi + q\psi$ is cohomologous to no constant, then $\mathcal{D} = \overline{\operatorname{int} \mathcal{D}}$;*
3. *the function $(\alpha, \beta) \mapsto h(\sigma|K_{\alpha,\beta})$ is analytic in $\operatorname{int} \mathcal{D}$;*
4. *there exists an ergodic Gibbs measure $\mu_{\alpha,\beta} \in \mathcal{M}(X)$ with $\int_X \varphi \, d\mu = \alpha$ and $\int_X \psi \, d\mu = \beta$, such that*
$$\mu_{\alpha,\beta}(K_{\alpha,\beta}) = 1 \quad \text{and} \quad h_{\mu_{\alpha,\beta}}(\sigma) = h(\sigma|K_{\alpha,\beta}).$$

The last statement in Theorem 2 says that the topological entropy of the set $K_{\alpha,\beta}$ is in fact fully carried by a special ergodic measure on that same set. Again, Theorem 2 follows from the more general results formulated in the sections below.

Assume now that $\psi > 0$ and set
$$M_\gamma = \left\{ x \in X : \lim_{n\to\infty} \frac{\sum_{k=0}^n \varphi(\sigma^k x)}{\sum_{k=0}^n \psi(\sigma^k x)} = \gamma \right\}.$$
The function $\gamma \mapsto h(\sigma|M_\gamma)$ is called a *mixed* multifractal spectrum (see [2] for a detailed related discussion). Set
$$\underline{\gamma} = \inf \left\{ \frac{\int_X \varphi \, d\mu}{\int_X \psi \, d\mu} : \mu \in \mathcal{M}(X) \right\} \quad \text{and} \quad \overline{\gamma} = \sup \left\{ \frac{\int_X \varphi \, d\mu}{\int_X \psi \, d\mu} : \mu \in \mathcal{M}(X) \right\}.$$

The following statement establishes a precise relationship between the two-dimensional spectrum $(\alpha, \beta) \mapsto h(\sigma|K_{\alpha,\beta})$ and the mixed multifractal spectrum. We write $S_\gamma = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^+ : \alpha/\beta = \gamma\}$.

Theorem 3. *Let $\sigma|X$ be a topologically mixing subshift of finite type, and φ and ψ Hölder continuous functions on X with $\psi > 0$. If $\gamma \in (\underline{\gamma}, \overline{\gamma})$, then*
$$h(\sigma|M_\gamma) = \max\{h(\sigma|K_{\alpha,\beta}) : (\alpha, \beta) \in S_\gamma\}.$$

Notice that $M_\gamma \supset \bigcup_{(\alpha,\beta)\in S_\gamma} K_{\alpha,\beta}$ and that this union is composed of an uncountable number of pairwise disjoint nonempty sets. Theorem 3 shows that surprisingly the topological entropy of M_γ is fully carried by some subset $K_{\alpha,\beta}$. This relationship provides a new insight to the study of mixed multifractal spectra in [2]. A detailed discussion is given in Section 7.

As explained above we introduce in this paper a higher-dimensional version of multifractal analysis. Besides its own interest and source for new phenomena, this study has nontrivial applications to number theory. We emphasize that the one-dimensional multifractal analysis is not sufficient for these applications and that it is crucial to use the full force of the higher-dimensional version introduced in this paper. The reason will become clear shortly below.

We want to consider the base-m representation of real numbers, for a fixed integer $m > 1$. This representation is unique except for countably many points, and since countable sets have zero Hausdorff dimension, the nonuniqueness of the representation does not interfere with our study of dimension. For each $k \in \{0, \ldots, m-1\}$ and $x = 0.x_1x_2 \cdots \in [0,1]$, whenever there exists the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\operatorname{card}\{i \in \{1, \ldots, n\} : x_i = k\}}{n}$$

it is called the *frequency* of the number k in the base-m representation of x. Consider the sets

$$F_m(\alpha_0, \ldots, \alpha_{m-1}) = \{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for } k = 0, \ldots, m-1\},$$

whenever $\alpha_0 + \cdots + \alpha_{m-1} = 1$ with $\alpha_i \in [0,1]$ for each i. One can show that each of these sets is nonempty and hence is dense in $[0,1]$ (note that the limits $\tau_k(x)$ only depend on the tail of the representation). In fact it is straightforward to construct explicitly a point in $F_m(\alpha_0, \ldots, \alpha_{m-1})$. In [5] Eggleston computed the Hausdorff dimension

$$\dim_H F_m(\alpha_0, \ldots, \alpha_{m-1}) = -\frac{\sum_{k=0}^{m-1} \alpha_k \log \alpha_k}{\log m}. \tag{3}$$

It is easy to see that this result is related to multifractal analysis. Observe first that the action of the shift map on the set of sequences in $\{0, \ldots, m-1\}$ can be identified with the action of the map $x \mapsto mx \pmod 1$ on the base-m representation in $[0,1]$. After this identification, when $m = 2$ we have $F_2(\alpha_0, \alpha_1) = K_{\alpha_0}(\varphi)$ (see (1)) for the characteristic function $\varphi = \chi_{[0,1/2)}$. This identity allows one to apply the *one-dimensional multifractal analysis* to obtain a straightforward alternative proof of (3) when $m = 2$. We can also consider the case when $m > 2$. However, it is now essential to use the *higher-dimensional multifractal analysis*. For example, when $m = 3$ we have

$$F_3(\alpha_0, \alpha_1, \alpha_2) = K_{\alpha_0}(\varphi) \cap K_{\alpha_1}(\psi) = K_{\alpha_0,\alpha_1}$$

(see (2)) for the functions $\varphi = \chi_{[0,1/3)}$ and $\psi = \chi_{[1/3,2/3)}$. This observation allows one to apply Theorem 1 to conclude that

$$\dim_H F_3(\alpha_0, \alpha_1, \alpha_2) = \sup\left\{\frac{h_\mu(\sigma)}{\log 3} : \mu\left(\left[0, \frac{1}{3}\right)\right) = \alpha_0, \mu\left(\left[\frac{1}{3}, \frac{2}{3}\right)\right) = \alpha_1\right\}.$$

This readily implies the identity (3) when $m = 3$, since the supremum is always attained at a Bernoulli measure (in this case with probabilities α_0, α_1, and $1 - \alpha_0 - \alpha_1 = \alpha_2$). The appropriate generalization of Theorem 1 for an arbitrary finite number of functions (see Section 4), allows us to obtain a straightforward alternative proof of (3) for an arbitrary m.

This example already illustrates how the higher-dimensional version of multifractal analysis can be used to obtain a very simple proof of a known result. On the other hand our techniques can also be used to study the Hausdorff dimension of several new classes of sets. We now present a few selected examples from our work [3] that illustrate the applications to number theory that we have in mind:

1. Consider $m = 3$, and the set $G_1 = \{x \in [0,1] : \tau_1(x) = 2\tau_0(x)\}$. This is the set of numbers in $[0,1]$ such that its base-3 representation has

a percentage of ones which is two times the percentage of zeros. The percentage of twos is arbitrary. We have

$$G_1 = \bigcup_{\alpha\in[0,1/3]} F_3(\alpha, 2\alpha, 1-3\alpha), \tag{4}$$

and thus

$$\dim_H G_1 \geq \max_{\alpha\in[0,1/3]} \dim_H F_3(\alpha, 2\alpha, 1-3\alpha) = \frac{\log(1+3/\sqrt[3]{4})}{\log 3}. \tag{5}$$

The theory developed in this paper allows us to show that the inequality in (5) is in fact an identity, that is, there exists $\alpha \in [0,1/3]$ such that $F_3(\alpha, 2\alpha, 1-3\alpha) \subset G_1$ and $\dim_H F_3(\alpha, 2\alpha, 1-3\alpha) = \dim_H G_1$. The difficulties have to do with the fact that the disjoint union in (4) is not countable. Further difficulties occur when $m > 3$ since the identity in (4) must be replaced by a proper inclusion (see also the following example).

2. Consider $m = 4$, and the set $G_2 = \{x \in [0,1] : (\tau_0(x)+\tau_1(x))/2 = \alpha\}$. We can show that

$$\dim_H G_2 = \log_4 \left[2\left(\frac{2\alpha}{1-2\alpha}\right)^{1-2\alpha} + 2\left(\frac{1-2\alpha}{2\alpha}\right)^{2\alpha} \right]. \tag{6}$$

Now

$$G_2 \supset \bigcup_{\beta\in[0,2\alpha]} \bigcup_{\gamma\in[0,1-2\alpha]} F_4(\beta, 2\alpha-\beta, \gamma, 1-2\alpha-\gamma),$$

and the inclusion is proper (contrarily to what happens in (4)). Again, the easy part is to show that the right-hand side in (6) is a lower bound for $\dim_H G_2$.

3. Consider $m = 2$, and the set G_3 of points $x \in [0,1]$ such that the base-2 representation has a frequency of the block 00 equal to that of the block 111. We can show that $\dim_H G_3 = 0.26078\cdots$ (in base 10) and this number can be computed with an arbitrary precision, although we are not aware of an explicit expression. Contrarily to what may be though, this example illustrates another advantage of our approach: the value of the Hausdorff dimension does not need to be guessed a priori. In some works this a priori knowledge is required in order to construct a certain measure sitting on the set, which is then used to show that the guess was correct.

These examples are particular cases of the general theory presented in [3].

We can also consider the representation of real numbers with continued fractions. Similar methods lead to related results concerning the frequencies of a finite number of digits in the continued fraction representation. Furthermore, we believe that this study can be generalized to an infinite number of digits as long as the involved potentials (representing the functional relations between frequencies) are sufficiently "well behaved".

The structure of the paper is as follows. Section 2 briefly recalls the notions of topological pressure and u-dimension. In Section 3 we formulate our results in the case of repellers. This provides a model for more general situations with the advantage of avoiding extra technical details. Section 4 establishes a conditional variational principle for higher-dimensional multifractal spectra when the entropy is upper semi-continuous. This work is used in Section 5 to study the regularity and nondegeneracy of multifractal spectra, and in Section 6 to study the associated irregular sets. In Section 7 we look at the finer structure of multifractal spectra, and, in particular, show that mixed spectra can be expressed in terms of higher-dimensional "non-mixed" spectra. The proofs are based on the thermodynamic formalism and are collected in Section 8.

2. Topological pressure and u-dimension

2.1. **Topological pressure.** Let $f\colon X \to X$ be a continuous map of the compact metric space X, and $\mathcal{U}$ a finite open cover of X. We denote by $\mathcal{W}_n(\mathcal{U})$ the collection of words $\mathbf{U} = (U_0, \ldots, U_n) \in \mathcal{U}^{n+1}$ of length $m(\mathbf{U}) = n$, and define the open set

$$X(\mathbf{U}) = \{x \in X : f^k x \in U_k \text{ for } k = 0, \ldots, n\}.$$

Let $\varphi\colon X \to \mathbb{R}$ be a continuous function. Given $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ with $X(\mathbf{U}) \neq \varnothing$ set

$$\varphi(\mathbf{U}) = \sup_{x \in X(\mathbf{U})} \sum_{k=0}^{m(\mathbf{U})} \varphi(f^k x).$$

For each set $Z \subset X$ and each real number α, we define

$$M(Z, \alpha, \varphi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U \in \Gamma} \exp(-\alpha m(U) + \varphi(\mathbf{U})),$$

where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ such that $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$. The *topological pressure* of φ on the set Z (with respect to f) is defined by

$$P_Z(\varphi) \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\varphi, \mathcal{U}),$$

where

$$P_Z(\varphi, \mathcal{U}) = \inf\{\alpha : M(Z, \alpha, \varphi, \mathcal{U}) = 0\}.$$

We call $h(f|Z) = P_Z(0)$ the *topological entropy* of f on Z.

2.2. **The notion of u-dimension.** We recall a Carathéodory dimension characteristic introduced by Barreira and Schmeling in [4]. Let $u\colon X \to \mathbb{R}$ be a continuous function with $u > 0$. For each set $Z \subset X$ and each real number α, we define

$$M(Z, \alpha, u, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-\alpha u(\mathbf{U})), \tag{7}$$

where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ such that $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$. Set

$$\dim_{u,\mathcal{U}} Z = \inf\{\alpha\colon M(Z, \alpha, u, \mathcal{U}) = 0\}.$$

The limit

$$\dim_u Z \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathcal{U}\to 0} \dim_{u,\mathcal{U}} Z$$

exists, and is called the *u-dimension* of Z. For example, if $u = 1$, then $\dim_u Z$ coincides with the topological entropy of f on Z.

The following result expresses a relation between the u-dimension and the topological pressure, and follows easily from the definitions.

Proposition 4. *We have* $\dim_u Z = \alpha$, *where* α *is the unique root of the equation* $P_Z(-\alpha u) = 0$.

For every Borel probability measure μ on X, let

$$\dim_{u,\mathcal{U}} \mu = \inf\{\dim_{u,\mathcal{U}} Z \colon \mu(Z) = 1\}.$$

The limit

$$\dim_u \mu \stackrel{\text{def}}{=} \lim_{\operatorname{diam} \mathcal{U} \to 0} \dim_{u,\mathcal{U}} \mu$$

exists, and is called the *u-dimension* of μ. When $\mu \in \mathcal{M}(X)$ is ergodic, one can show that (see [4])

$$\dim_u \mu = h_\mu(f) / \int_X u \, d\mu. \tag{8}$$

3. Repellers

3.1. **Preliminaries.** Let $f \colon M \to M$ be a C^1 map of a smooth Riemannian manifold. We assume that f is a local diffeomorphism at each point of some f-invariant subset $X \subset M$. Let also μ be an f-invariant probability measure on M. We shall consider several quantities of local nature:

1. The *(top) Lyapunov exponent* of the point $x \in M$ is given by
$$\lambda(x) \stackrel{\text{def}}{=} \lim_{n \to +\infty} \frac{1}{n} \log \|d_x f^n\| \tag{9}$$
whenever the limit exists.
2. The *pointwise dimension* of μ at the point $x \in M$ is defined by
$$d_\mu(x) \stackrel{\text{def}}{=} \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{10}$$
whenever the limit exists, where $B(x,r) \subset M$ denotes the ball of radius r centered at x.
3. For each finite measurable partition ξ of M, we define the *μ-local entropy* of f at the point $x \in M$ (with respect to ξ) by
$$h_\mu(x) \stackrel{\text{def}}{=} h_\mu(f, \xi, x) \stackrel{\text{def}}{=} \lim_{n \to \infty} -\frac{1}{n} \log \mu(\xi_n(x)) \tag{11}$$
whenever the limit exists, where $\xi_n(x)$ is the atom of the partition $\bigvee_{k=0}^n f^{-k}\xi$ which contains x (which is well-defined mod 0).

By Kingman's sub-additive ergodic theorem and the Shannon–McMillan–Breiman theorem, the functions λ and h_μ are well-defined μ-almost everywhere. For hyperbolic measures invariant under a $C^{1+\alpha}$ diffeomorphism on a compact manifold it was shown in [1] that the function d_μ is well-defined μ-almost everywhere. In the case of repellers the corresponding statement is established in [13]. One can easily verify that each of the functions in (9), (10), and (11) has an f-invariant domain, and that they are f-invariant on the respective domain of definition. In addition, if ξ is a generating partition of M (i.e., a partition such that $\bigvee_{k=0}^\infty f^{-k}\xi$ generates the Borel σ-algebra of M) and μ is ergodic, then $h_\mu(f) = h_\mu(f, \xi, x)$ for μ-almost every $x \in M$, where $h_\mu(f)$ is the measure-theoretic entropy of f (with respect to μ).

We shall consider intersections of level sets of the functions in (9), (10), and (11). The following results describe the "size" of these intersections in terms of topological entropy and Hausdorff dimension. Given a subset $Z \subset M$ we denote by $\dim_H Z$ the Hausdorff dimension of the set Z, and by $h(f|Z)$ the topological entropy of f on Z.

We briefly recall the notion of Hausdorff dimension. Let X be a metric space and consider a subset $Z \subset X$. Given $\alpha > 0$, we set

$$m(Z, \alpha) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all finite or countable cover $\mathcal{U}$ of Z by sets of diameter at most δ. There exists a unique value of α at which $m(Z, \alpha)$ jumps from $+\infty$ to 0. This value is called *Hausdorff dimension* of Z and is denoted by $\dim_H Z$. We have

$$\dim_H Z = \inf\{\alpha : m(Z, \alpha) = 0\}.$$

3.2. **Formulation of the results.** Consider a compact f-invariant set $X \subset M$. We say that f is *expanding* on X, and that X is a *repeller* of f if there exist constants $c > 0$ and $\beta > 1$ such that $\|d_x f^n u\| \geq c\beta^n \|u\|$ for all $x \in X$, $u \in T_x M$, and $n \geq 1$.

In order to define the local entropies we shall always consider a Markov (and thus generating) partition ξ of X (with respect to f) of sufficiently small diameter.

Let $\mathcal{M}(X)$ denote the family of f-invariant probability measures on X. Given $\mu_1, \ldots, \mu_d \in \mathcal{M}(X)$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we consider the set

$$E_\alpha = \bigcap_{i=1}^{d} \{x \in X : h_{\mu_i}(x) = \alpha_i\},$$

and given Hölder continuous functions $\varphi_1, \ldots, \varphi_d$ on X we consider the vector

$$\mathcal{E}_d(\mu) = \left(-\int_X \varphi_1 \, d\mu, \ldots, -\int_X \varphi_d \, d\mu\right)$$

for each $\mu \in \mathcal{M}(X)$. The following statement describes the higher-dimensional spectrum $\alpha \mapsto h(f|E_\alpha)$.

Theorem 5. *Let X be a repeller of a topologically mixing $C^{1+\alpha}$ expanding map f, for some $\alpha > 0$, and let $\mu_1, \ldots, \mu_d$ be the equilibrium measures of Hölder continuous functions $\varphi_1, \ldots, \varphi_d$ on X such that $P_X(\varphi_1) = \cdots = P_X(\varphi_d) = 0$. Then the following properties hold:*

1. *if $\alpha \notin \mathcal{E}_d(\mathcal{M}(X))$ then $E_\alpha = \varnothing$;*
2. *if $\alpha \in \operatorname{int} \mathcal{E}_d(\mathcal{M}(X))$ then $E_\alpha \neq \varnothing$, and*
$$h(f|E_\alpha) = \max\{h_\mu(f) : \mathcal{E}_d(\mu) = \alpha\}; \tag{12}$$
3. *the map $\alpha \mapsto h(f|E_\alpha)$ is analytic in $\operatorname{int} \mathcal{E}_d(\mathcal{M}(X))$;*
4. *if the functions $1, \varphi_1, \ldots, \varphi_d$ are linearly independent as cohomology classes then*
$$\mathcal{E}_d(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{E}_d(\mathcal{M}(X))}.$$

The identity in (12) provides a higher-dimensional conditional variational principle for the local entropies of $\mu_1, \ldots, \mu_d$.

We now consider the sets

$$D_\beta = \bigcap_{i=1}^{d} \{x \in X : d_{\mu_i}(x) = \beta_i\} \quad \text{and} \quad L_\gamma = \{x \in X : \lambda(x) = \gamma\},$$

where $\beta = (\beta_1, \ldots, \beta_d)$. Write $u(x) = \log\|d_x f\|$. Let also

$$\mathcal{D}_d(\mu) = \left(-\frac{\int_X \varphi_1 \, d\mu}{\int_X u \, d\mu}, \ldots, -\frac{\int_X \varphi_d \, d\mu}{\int_X u \, d\mu}\right)$$

and $\mathcal{L}(\mu) = \int_X u \, d\mu$ for each $\mu \in \mathcal{M}(X)$. We say that f is *conformal* on X if $d_x f$ is a multiple of an isometry for every $x \in X$. We shall now obtain further conditional variational principles in the case of conformal maps, for the topological entropy and for the Hausdorff dimension.

Theorem 6. *Let X be a repeller of a topologically mixing $C^{1+\alpha}$ expanding map f, for some $\alpha > 0$, such that f is conformal on X, and let $\mu_1, \ldots, \mu_d$ be the equilibrium measures of Hölder continuous functions $\varphi_1, \ldots, \varphi_d$ on X such that $P_X(\varphi_1) = \cdots = P_X(\varphi_d) = 0$. Then the following properties hold:*

1. *if $\beta \in \operatorname{int} \mathcal{D}_d(\mathcal{M}(X))$ then $D_\beta \neq \varnothing$, and*

$$h(f|D_\beta) = \max\{h_\mu(f) : \mathcal{D}_d(\mu) = \beta\}, \tag{13}$$

$$\dim_H D_\beta = \max\left\{\frac{h_\mu(f)}{\int_X u \, d\mu} : \mathcal{D}_d(\mu) = \beta\right\}; \tag{14}$$

2. *if $(\alpha, \gamma) \in \operatorname{int}(\mathcal{E}_d, \mathcal{L})(\mathcal{M}(X))$ then $E_\alpha \cap L_\gamma \neq \varnothing$, and*

$$h(f|E_\alpha \cap L_\gamma) = \max\{h_\mu(f) : (\mathcal{E}_d, \mathcal{L})(\mu) = (\alpha, \gamma)\}, \tag{15}$$

$$\dim_H(E_\alpha \cap L_\gamma) = \max\left\{\frac{h_\mu(f)}{\int_X u \, d\mu} : (\mathcal{E}_d, \mathcal{L})(\mu) = (\alpha, \gamma)\right\}; \tag{16}$$

3. *if $(\beta, \gamma) \in \operatorname{int}(\mathcal{D}_d, \mathcal{L})(\mathcal{M}(X))$ then $D_\beta \cap L_\gamma \neq \varnothing$, and*

$$h(f|D_\beta \cap L_\gamma) = \max\{h_\mu(f) : (\mathcal{D}_d, \mathcal{L})(\mu) = (\beta, \gamma)\}, \tag{17}$$

$$\dim_H(D_\beta \cap L_\gamma) = \max\left\{\frac{h_\mu(f)}{\int_X u \, d\mu} : (\mathcal{D}_d, \mathcal{L})(\mu) = (\beta, \gamma)\right\}; \tag{18}$$

4. *if $d = 1$ and $(\alpha, \beta) \in \operatorname{int}(\mathcal{E}_1, \mathcal{D}_1)(\mathcal{M}(X))$ then $E_\alpha \cap D_\beta \neq \varnothing$, and*

$$h(f|E_\alpha \cap D_\beta) = \max\{h_\mu(f) : (\mathcal{E}_1, \mathcal{D}_1)(\mu) = (\alpha, \beta)\}, \tag{19}$$

$$\dim_H(E_\alpha \cap D_\beta) = \max\left\{\frac{h_\mu(f)}{\int_X u \, d\mu} : (\mathcal{E}_1, \mathcal{D}_1)(\mu) = (\alpha, \beta)\right\}. \tag{20}$$

We remark that each of the expressions in (13)–(20) is analytic on α, β, γ in the interior of the corresponding domain of definition, as a consequence of more general results formulated below (see Section 5 for details).

One can certainly consider other intersections of level sets besides those in Theorem 6 (see Section 4 for a related discussion), such as $E_\alpha \cap D_\beta$ for $d > 1$. We note that the intersection $E_\alpha \cap D_\beta$ is nonempty if and only if $\alpha = \gamma\beta$ for some $\gamma \in \mathcal{L}(\mathcal{M}(X))$ such that $D_\beta \cap L_\gamma \neq \varnothing$. In this case we have $K_{\gamma\beta} \cap D_\beta = D_\beta \cap L_\gamma$, and thus one can apply Statement 3 in Theorem 6 to the set $\operatorname{int}(\mathcal{E}_d, \mathcal{D}_d)(\mathcal{M}(X))$. However, one can show that $\operatorname{int}(\mathcal{E}_d, \mathcal{D}_d)(\mathcal{M}(X)) = \varnothing$

whenever $d > 1$ (see Section 4 for details) and thus Theorem 6 provides no information in this situation.

We shall now provide sufficient conditions for the interiors considered in Theorem 6 to be dense, and thus such that each of the conditional variational principles in (13)–(20) is valid in an open and dense subset of the corresponding domain. The results in Section 4 indicate that these are in a sense optimal assumptions.

Theorem 7. *Under the hypotheses of Theorem 6 the following properties hold:*

1. *if the functions φ_1, …, φ_d, u are linearly independent as cohomology classes then*
$$\mathcal{D}_d(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{D}_d(\mathcal{M}(X))};$$
2. *if the functions 1, φ_1, …, φ_d, u are linearly independent as cohomology classes then*
$$(\mathcal{E}_d, \mathcal{L})(\mathcal{M}(X)) = \overline{\operatorname{int}(\mathcal{E}_d, \mathcal{L})(\mathcal{M}(X))}$$
and
$$(\mathcal{D}_d, \mathcal{L})(\mathcal{M}(X)) = \overline{\operatorname{int}(\mathcal{D}_d, \mathcal{L})(\mathcal{M}(X))};$$
3. *if $d = 1$ and the functions 1, φ_1, u are linearly independent as cohomology classes then*
$$(\mathcal{E}_1, \mathcal{D}_1)(\mathcal{M}(X)) = \overline{\operatorname{int}(\mathcal{E}_1, \mathcal{D}_1)(\mathcal{M}(X))}.$$

4. Conditional variational principle

4.1. **Preliminaries.** Let now $f \colon X \to X$ be a continuous map on the compact metric space X. We denote by $C(X)$ the space of continuous functions $\varphi \colon X \to \mathbb{R}$. Consider a pair of vectors $(\Phi, \Psi) \in C(X)^d \times C(X)^d$ and write
$$\Phi = (\varphi_1, \dots, \varphi_d) \quad \text{and} \quad \Psi = (\psi_1, \dots, \psi_d).$$
We shall always assume that $\psi_i > 0$ for each $i = 1$, …, d. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ we set
$$K_\alpha = K_\alpha(\Phi, \Psi) = \bigcap_{i=1}^{d} \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} = \alpha_i \right\}, \tag{21}$$
where
$$\varphi_{i,n}(x) = \sum_{k=0}^{n-1} \varphi_i(f^k x) \quad \text{and} \quad \psi_{i,n}(x) = \sum_{k=0}^{n-1} \psi_i(f^k x). \tag{22}$$
We continue to denote by $\mathcal{M}(X)$ the family of f-invariant Borel probability measures on X, and define a continuous function $\mathcal{P} = \mathcal{P}^{(\Phi,\Psi)} \colon \mathcal{M}(X) \to \mathbb{R}^d$ by
$$\mathcal{P}(\mu) = \left(\frac{\int_X \varphi_1 \, d\mu}{\int_X \psi_1 \, d\mu}, \dots, \frac{\int_X \varphi_d \, d\mu}{\int_X \psi_d \, d\mu} \right). \tag{23}$$
Since $\mathcal{M}(X)$ is compact and connected, and $\mathcal{P}$ is continuous, the set $\mathcal{P}(\mathcal{M}(X))$ is also compact and connected.

Given a positive function $u \in C(X)$ we denote by $\dim_u Z$ the u-dimension of the set $Z \subset X$ (see Section 2 for the definition). For example:

1. if $u = 1$, then $\dim_u Z = h(f|Z)$;

2. if $u = \log\|df\|$ for a conformal expanding map on X, then $\dim_u Z = \dim_H Z$ for every $Z \subset X$.

The function $\mathcal{F}_u = \mathcal{F}_u^{(\Phi,\Psi)}$ defined by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha(\Phi, \Psi) \tag{24}$$

is called the *u-dimension spectrum for the pair* (Φ, Ψ).

We denote by $D(X) \subset C(X)$ the family of continuous functions with a unique equilibrium measure. Recall that if the metric entropy is upper semi-continuous, or, more precisely, if the map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:

1. every function $\varphi \in C(X)$ has an equilibrium measure;
2. given $\varphi \in C(X)$, the function $\mathbb{R} \ni t \mapsto P_X(\varphi + t\psi)$ is differentiable at $t = 0$ for each $\psi \in C(X)$ if and only if $\varphi \in D(X)$; in this case the unique equilibrium measure μ_φ of φ is ergodic, and
$$\frac{d}{dt} P_X(\varphi + t\psi)|_{t=0} = \int_X \psi \, d\mu_\varphi; \tag{25}$$
3. if φ, $\psi \in C(X)$ are such that $\operatorname{span}\{\varphi, \psi\} \subset D(X)$, then the function $t \mapsto P_X(\varphi + t\psi)$ is differentiable in $\mathbb{R}$, and is in fact of class C^1 (see [8, Theorem 4.2.11]).

For example, when $f \colon X \to X$ is a one-sided or two-sided topologically mixing subshift of finite type, or an expansive homeomorphism, then the metric entropy is upper semi-continuous. Furthermore, if $f \colon X \to X$ is a one-sided or two-sided topologically mixing subshift of finite type, or an expansive homeomorphism which satisfies specification, and $\varphi \in C_f(X)$, then it has a unique equilibrium measure. Here $C_f(X) \subset C(X)$ is the family of continuous functions $\varphi \colon X \to \mathbb{R}$ for which there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$\left| \sum_{k=0}^{n-1} \varphi(f^k x) - \sum_{k=0}^{n-1} \varphi(f^k y) \right| < \kappa$$

whenever $d(f^k x, f^k y) < \varepsilon$ for every $k = 0, \ldots, n-1$. On the other hand, all β-shifts are expansive, and thus the entropy is upper semi-continuous (see [8] for details), but for β in a residual set of full Lebesgue measure the corresponding β-shift does not satisfy specification (see [11]).

4.2. **Conditional variational principle.** In this section we establish a conditional variational principle for the spectrum $\mathcal{F}_u$. Given vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and $\Phi = (\varphi_1, \ldots, \varphi_d) \in C(X)^d$ we shall write

$$\alpha * \Phi = (\alpha_1\varphi_1, \ldots, \alpha_d\varphi_d) \in C(X)^d \quad \text{and} \quad \langle \alpha, \Phi \rangle = \sum_{i=1}^d \alpha_i \varphi_i \in C(X).$$

We now present the main result of this section.

Theorem 8. *Assume that the metric entropy of f is upper semi-continuous, and that* $\operatorname{span}\{\varphi_1, \psi_1, \ldots, \varphi_d, \psi_d, u\} \subset D(X)$. *If $\alpha \notin \mathcal{P}(\mathcal{M}(X))$ then $K_\alpha = \varnothing$. Furthermore, if $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$ then $K_\alpha \neq \varnothing$ and the following properties hold:*

1. $\mathcal{F}_u(\alpha)$ *satisfies the conditional variational principle*
$$\mathcal{F}_u(\alpha) = \max\left\{\frac{h_\mu(f)}{\int_X u\,d\mu} : \mu \in \mathcal{M}(X) \text{ and } \mathcal{P}(\mu) = \alpha\right\}; \tag{26}$$
2. $\mathcal{F}_u(\alpha) = \inf\{T_u(q) : q \in \mathbb{R}^d\}$, *where* $T_u(q)$ *is the unique number satisfying*
$$P_X(\langle q, \Phi - \alpha * \Psi\rangle - T_u(q)u) = 0;$$
3. *there exists an ergodic equilibrium measure* $\mu_\alpha \in \mathcal{M}(X)$ *with* $\mathcal{P}(\mu_\alpha) = \alpha$ *and* $\mu_\alpha(K_\alpha) = 1$ *such that*
$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u\,d\mu_\alpha} = \mathcal{F}_u(\alpha). \tag{27}$$

When $d = 1$ the statements in Theorem 8 were established by Barreira and Saussol in [2]. In Statement 2 the formula for the spectrum $\mathcal{F}_u$ can be seen as a Legendre-type transform (see [2] for a related discussion in the case $d = 1$).

In particular Theorem 8 implies that:

1. The u-dimension of the set K_α can be arbitrarily approximated by the u-dimension of measures $\mu \in \mathcal{M}(X)$ with $\mathcal{P}(\mu) = \alpha$.
2. There exists an equilibrium measure in $\mathcal{M}(X)$ (the measure μ_α) with the same u-dimension as K_α. The measure μ_α can be obtained in the following manner. It follows from the proof of Theorem 8 that there exists a vector $q(\alpha) \in \mathbb{R}^d$ such that
$$P_X(\langle q(\alpha), \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u) = 0.$$
Then μ_α is the equilibrium measure of $\langle q(\alpha), \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u$.

Furthermore, Theorem 8 gives very detailed information about the multifractal spectrum inside $\operatorname{int}\mathcal{P}(\mathcal{M}(X))$. It is therefore crucial to discuss the properties of this open set, and in particular to establish under which conditions it is nonempty. The following statement provides a characterization of the points in $\operatorname{int}\mathcal{P}(\mathcal{M}(X))$ and in a certain sense it is optimal.

For each q, $\alpha \in \mathbb{R}^d$, we consider the function $S_{q\alpha} \colon \mathbb{R} \to \mathbb{R}$ defined by
$$S_{q\alpha}(t) = P_X(t\langle q, \Phi - \alpha * \Psi\rangle).$$
When $f|X$ is a subshift with the specification property, and $\Phi - \alpha * \Psi \in C_f(X)^d$, the function $S_{q\alpha}$ has the following interpretation. Let
$$\mathcal{E}_{q\alpha}(\beta) = h\left(f|\left\{x \in X : \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\langle q, \Phi - \alpha * \Psi\rangle(f^k x) = \beta\right\}\right).$$
Using (25) we obtain
$$\beta_{q\alpha}(t) \stackrel{\text{def}}{=} -S'_{q\alpha}(t) = -\int_X \langle q, \Phi - \alpha * \Psi\rangle\,d\mu_{t\langle q, \Phi - \alpha*\Psi\rangle},$$
and it follows from work in [4] that
$$\mathcal{E}_{q\alpha}(\beta_{q\alpha}(t)) = S_{q\alpha}(t) + t\beta_{q\alpha}(t)$$
for every t. This shows that $S_{q\alpha}$ is the Legendre transform of $\mathcal{E}_{q\alpha}$.

Theorem 9. *Assume that f has finite topological entropy, and that the vectors Φ and Ψ are composed of continuous functions on X. If $\alpha \in \mathcal{P}(\mathcal{M}(X))$ then:*

1. *if $S_{q\alpha}$ is constant for no $q \in \mathbb{R}^d$, then $\alpha \in \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$;*
2. *if $S_{q\alpha}$ is constant for some $q \in \mathbb{R}^d$, then $\alpha \notin \operatorname{int} \mathcal{P}(\mathcal{M}(X))$.*

A noteworthy consequence of Theorem 9 is that if the topological pressure is strictly convex, that is, if for any $q \in \mathbb{R}^d$ and $\alpha \in \mathcal{P}(\mathcal{M}(X))$ the function $S_{q\alpha}$ is strictly convex, then

$$\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}.$$

For example, if $f\colon X \to X$ is a subshift which satisfies specification, and $\varphi_i, \psi_i \in C_f(X)$ for $i = 1, \ldots, d$, then for each $\alpha \in \mathcal{P}(\mathcal{M}(X))$ the following properties are equivalent:

1. the function $q \mapsto P_X(\langle q, \Phi - \alpha * \Psi\rangle)$ is strictly convex;
2. the function $S_{q\alpha}$ is constant for no $q \in \mathbb{R}^d$;
3. the functions $\varphi_i - \alpha_i\psi_i$ for $i = 1, \ldots, d$ are linearly independent as cohomology classes.

For each $q \in S^{2d-1} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^{2d} : \|x\| = 1\}$ we set

$$\Gamma(q) = \partial \left\{\mathcal{P}(\mu_{t\langle q,(\Phi,\Psi)\rangle}) : t \in \mathbb{R}\right\},$$

where ∂A denotes the boundary of the set A.

Theorem 10. *Let $f\colon X \to X$ be a subshift which satisfies specification, and $\varphi_i, \psi_i \in C_f(X)$ for $i = 1, \ldots, d$. Then*

$$\partial\mathcal{P}(\mathcal{M}(X)) \subset \bigcup_{q\in S^{2d-1}} \Gamma(q),$$

and if, in addition, the functions $1, \varphi_1, \ldots, \varphi_d, \psi_1, \ldots, \psi_d$ are linearly independent as cohomology classes, then $\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$.

4.3. **New phenomena observed in higher-dimensional spectra.** In the case $d = 1$, the connectedness of $\mathcal{P}(\mathcal{M}(X))$ implies that only one of the following two exclusive alternatives can occur:

1. The spectrum is *degenerated*: in this case $\mathcal{P}(\mathcal{M}(X)) = \{a\}$ for some $a \in \mathbb{R}$. Furthermore, $K_a = X$ and $K_\alpha = \varnothing$ for every $\alpha \neq a$.
2. The spectrum is *nondegenerated*: in this case $\mathcal{P}(\mathcal{M}(X)) = [\underline{a}, \overline{a}]$ for some real numbers $\underline{a} < \overline{a}$. In particular $\mathcal{P}(\mathcal{M}(X))$ has nonempty interior, and $\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$.

When $d > 1$, that is, in the case of higher-dimensional multifractal spectra, several new phenomena can occur. Namely:

1. $\mathcal{P}(\mathcal{M}(X))$ may not be convex;
2. $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$ may have more than one connected component;
3. $\mathcal{P}(\mathcal{M}(X))$ may have empty interior, but still contain an uncountable number of points.

See Examples 1 and 2 below for explicit constructions. We emphasize that neither of these three situations occurs when $d = 1$.

When $d = 1$ the existence of a cohomology relation between the functions φ and ψ immediately implies that the domain of the spectrum is composed

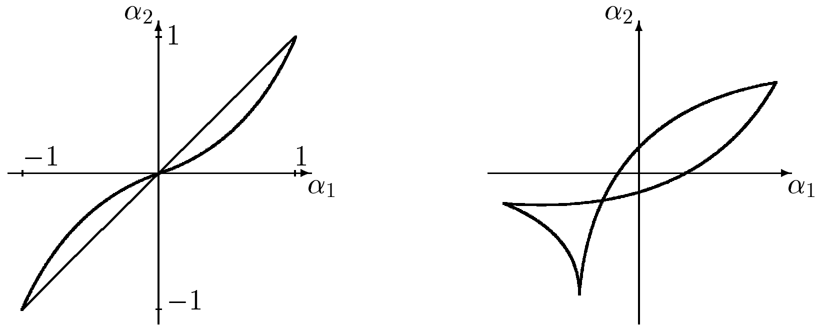

FIGURE 1. Two sets $\mathcal{P}(\mathcal{M}(X))$ for which the interior has two connected components, due to the presence of a cohomology relation. The curves in the pictures represent the boundary of $\mathcal{P}(\mathcal{M}(X))$.

of only one point. In the case of higher-dimensional multifractal spectra this may not be the case, as illustrated in the following example.

Example 1. Consider a subshift of finite type $f\colon X \to X$ and set $d = 2$. Let $\varphi, \psi \in C_f(X)$ be such that φ, ψ, and 1 are linearly independent as cohomology classes. We assume that $\int_X \varphi \, d\mu = 0$ for some measure $\mu \in \mathcal{M}(X)$ and that $\psi > 0$.

Setting $\varphi_1 = \varphi$, $\varphi_2 = \varphi$, $\psi_1 = 1$, and $\psi_2 = \psi$ we obtain $0 \in \mathcal{P}(\mathcal{M}(X))$ (since $\int_X \varphi \, d\mu = 0$) and $(\varphi_1 - 0 \cdot \psi_1) - (\varphi_2 - 0 \cdot \psi_2) = 0$. On the other hand, it is easy to see that $\varphi_1 - \alpha_1\psi_1$ and $\psi_2 - \alpha_2\psi_2$ are linearly independent as cohomology classes whenever $\alpha \neq 0$, and hence $\mathcal{P}^* \stackrel{\text{def}}{=} \mathcal{P}(\mathcal{M}(X)) \setminus \{0\}$ is nonempty (by Theorem 9). In fact, it follows from Theorem 9 that $\mathcal{P}^* \subset \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$. Since $\mathcal{P}(\mathcal{M}(X))$ is closed, we conclude that $\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$. This shows that even though there exists a cohomology relation, the set $\mathcal{P}(\mathcal{M}(X))$ is composed of uncountably many points. Furthermore it has nonempty interior.

The first picture in Figure 1 provides an explicit example when f is the Bernoulli shift on 3 symbols. In this example we took the linear combinations of characteristic functions

$$\varphi = \chi_1 - \chi_2 \quad \text{and} \quad \psi = \chi_1 + \chi_2 + 2\chi_3,$$

where χ_i is the characteristic function of the cylinder C_i of length 1. Observe that in this particular case $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$ has two connected components. Furthermore, the set $\mathcal{P}(\mathcal{M}(X))$ is not convex, but each of the connected components of $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$ is convex.

The second picture in Figure 1 is obtained in a similar manner for the Bernoulli shift on 3 symbols, with the functions

$$\varphi_1 = -4\chi_1 + 4\chi_2 + 8\chi_3 \quad \text{and} \quad \varphi_2 = -6\chi_1 - 3\chi_2 + 5\chi_3,$$

$$\psi_1 = 2\chi_1 + 9\chi_2 + 2\chi_3 \quad \text{and} \quad \psi_2 = 6\chi_1 + \chi_2 + 2\chi_3.$$

Again the set $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$ has two connected components. We note that contrarily to what happens in the previous construction, there exists now a component of $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$ which is not convex. $\square$

We now illustrate that $\mathcal{P}(\mathcal{M}(X))$ may have empty interior, but still contain uncountably many points.

Example 2. Consider the Bernoulli shift on 2 symbols and set $d = 2$. In a similar way to that in Example 1 we define functions

$$\varphi_1 = a_1\chi_1 + b_1\chi_2 \quad \text{and} \quad \varphi_2 = a_2\chi_1 + b_2\chi_2,$$

and $\psi_1 = \psi_2 = u = 1$, where χ_i is the characteristic function of the cylinder C_i of length 1. We assume that $a_1b_2 - b_1a_2 = 1$. The case when $a_1b_2 - b_1a_2 \neq 1$ can be treated in a similar manner. Observe that

$$b_2\varphi_1 - b_1\varphi_2 = \chi_1 \quad \text{and} \quad a_1\varphi_2 - a_2\varphi_1 = \chi_2.$$

Since $\chi_1 + \chi_2 = 1$ we obtain

$$K_{(\alpha_1,\alpha_2)} = K_{b_2\alpha_1 - b_1\alpha_2}(\chi_1) = K_{a_1\alpha_2 - a_2\alpha_1}(\chi_2),$$

and

$$b_2\alpha_1 - b_1\alpha_2 + a_1\alpha_2 - a_2\alpha_1 = 1 \tag{28}$$

for every $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. It follows from Theorem 8 and (28) that

$$\begin{aligned} h(f|K_{(\alpha_1,\alpha_2)}) &= \sup\{h_\mu(f) : \mu(C_1) = b_2\alpha_1 - b_1\alpha_2\} \\ &= -\,(b_2\alpha_1 - b_1\alpha_2)\log(b_2\alpha_1 - b_1\alpha_2) \\ &\quad -(a_1\alpha_2 - a_2\alpha_1)\log(a_1\alpha_2 - a_2\alpha_1). \end{aligned}$$

Furthermore, the domain of the spectrum $(\alpha_1, \alpha_2) \mapsto h(f|K_{(\alpha_1,\alpha_2)})$ is a segment contained in the line defined by (28). $\square$

We remark that in some sense the situation described in Example 2 should be considered degenerated. In fact, Theorem 9 implies that the "degeneracy" in Example 2 is due to the presence of cohomology relations. When this happens one can replace the $2d$ functions in the vectors Φ and Ψ by a maximal set of independent ones, without changing the level sets (up to a change of variables), and in such a way that after the reduction the domain of the spectrum will have nonempty interior with respect to the new functions.

The spectrum itself may not be convex even when $d = 1$ (see [2] for an explicit example).

4.4. **The case of the entropy.** We now consider the particular case of a conditional variational principle for the topological entropy. The following statement is an immediate consequence of Theorem 8 by setting $u = 1$.

Theorem 11. *Assume that the metric entropy is upper semi-continuous, and that* $\operatorname{span}\{\varphi_1, \psi_1, \dots, \varphi_d, \psi_d\} \subset D(X)$. *If* $\alpha \notin \mathcal{P}(\mathcal{M}(X))$ *then* $K_\alpha = \varnothing$. *Furthermore, if* $\alpha \in \operatorname{int}\mathcal{P}(\mathcal{M}(X))$ *then* $K_\alpha \neq \varnothing$ *and the following properties hold:*

1. *we have the conditional variational principle*
$$h(f|K_\alpha) = \max\{h_\mu(f) : \mu \in \mathcal{M}(X) \text{ and } \mathcal{P}(\mu) = \alpha\}; \tag{29}$$
2. $h(f|K_\alpha) = \inf\{P_X(\langle q, \Phi - \alpha * \Psi\rangle) : q \in \mathbb{R}^d\}$;
3. *there exists an ergodic equilibrium measure* $\mu_\alpha \in \mathcal{M}(X)$ *with* $\mathcal{P}(\mu_\alpha) = \alpha$ *and* $\mu_\alpha(K_\alpha) = 1$ *such that* $h_{\mu_\alpha}(f) = h(f|K_\alpha)$.

When $d = 1$ the statement in Theorem 11 was established by Barreira and Saussol in [2]. When $\Psi = (1, \dots, 1)$ and f is a topologically mixing subshift of finite type, Fan, Feng and Wu [6] showed that (29) holds for an arbitrary continuous function Φ. Takens and Verbitskiy recently observed in [14] that their statement also holds when $\Psi = (1, \dots, 1)$ and f satisfies specification (provided that the maximum in (29) is replaced by a supremum). We recall that there exist plenty transformations not satisfying specification for which the entropy is upper semi-continuous. Furthermore, when the entropy is upper semi-continuous the family $D(X)$ is dense in $C(X)$. We refer to Section 4.1 and [2] for a detailed discussion. See also [7] for results of related nature when $\Psi = (1, \dots, 1)$, even though no mention is made to the sets K_α.

5. Regularity and nondegeneracy of the spectrum

We continue to assume that $f\colon X \to X$ is a continuous map on the compact metric space X. For a broad class of dynamical systems we shall now formulate conditions to obtain the regularity and the nondegeneracy of the spectrum, as an application of Theorems 8 and 9. This includes the case of uniformly hyperbolic dynamical systems.

We first study the regularity of the spectrum.

Theorem 12. *Assume that:*

1. *the metric entropy of f is upper semi-continuous;*
2. *the topological pressure of f is of class C^k for some $k \ge 2$.*

If $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$ is such that the second derivative of the function $q \mapsto P_X(\langle q, \Phi - \alpha * \Psi \rangle)$ is a positive definite bilinear form for each $q \in \mathbb{R}^d$, then:

1. *$\mathcal{F}_u$ is of class C^{k-1} in some open neighborhood of α;*
2. *if the topological pressure is analytic then $\mathcal{F}_u$ is analytic in some open neighborhood of α.*

Assume now that $f\colon X \to X$ is topologically mixing, and that it is either a subshift of finite type, an axiom A $C^{1+\varepsilon}$ diffeomorphism, or a $C^{1+\varepsilon}$ expanding map. By Theorem 9 (see also the discussion after Theorem 9), if $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$, and the functions $\varphi_1, \dots, \varphi_d, \psi_1, \dots, \psi_d$ are in $C_f(X)$, then the functions $\varphi_i - \alpha_i \psi_i$ for $i = 1, \dots, d$ are linearly independent as cohomology classes. Therefore, using Ruelle's formula for the second derivative of the topological pressure (see [10]), we conclude that $\partial_q^2 P_X(\langle q, \Phi - \alpha * \Psi \rangle)$ is a positive definitive bilinear form for each q. This readily implies the following statement.

Theorem 13. *Let f be a subshift of finite type, an axiom A $C^{1+\varepsilon}$ diffeomorphism, or a $C^{1+\varepsilon}$ expanding map, which is topologically mixing. If the the functions (Φ, Ψ) and u are Hölder continuous, then $\mathcal{F}_u$ is analytic in* $\operatorname{int} \mathcal{P}(\mathcal{M}(X))$.

We now study the nondegeneracy of the spectrum. We denote by H_θ the set of Hölder continuous functions $\varphi\colon X \to \mathbb{R}$ with exponent θ. The next theorem asserts that typically the spectrum $\mathcal{F}_u$ is nondegenerated for potentials in H_θ.

Theorem 14. *Let f be a subshift of finite type, an axiom A $C^{1+\varepsilon}$ diffeomorphism, or a $C^{1+\varepsilon}$ expanding map, which is topologically mixing. There* *exists a residual subset $\Theta \subset {H_\theta}^d \times {H_\theta}^d$ such that if $(\Phi, \Psi) \in \Theta$ and u are Hölder continuous then:*

1. $\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$*;*
2. $\mathcal{F}_u(\alpha) = 0$ *for every* $\alpha \in \partial\mathcal{P}(\mathcal{M}(X))$.

When $d = 1$ the statement in Theorem 14 was established by Schmeling in [12]. In this case the set $\partial\mathcal{P}(\mathcal{M}(X))$ is composed by either one or two points, respectively if it is degenerated or nondegenerated. On the other hand, when $d > 1$ and thus in the general case considered in Theorem 14 the set $\partial\mathcal{P}(\mathcal{M}(X))$ may consist of uncountably many points (see Examples 1 and 2 for explicit constructions), and indeed by Statement 1 in the theorem this is the generic situation. Correspondingly the second statement in Theorem 14 requires a much more detailed study of the structure of the set $\partial\mathcal{P}(\mathcal{M}(X))$.

6. Irregular sets

We consider the setup of Section 4.1. In particular, given functions $(\Phi, \Psi) \in C(X)^d \times C(X)^d$ we define the sets $K_\alpha = K_\alpha(\Phi, \Psi)$ as in (21). We also define the sets

$$K_{\alpha_i}(\varphi_i, \psi_i) = \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} = \alpha_i \right\},$$

and

$$I(\varphi_i, \psi_i) = \left\{ x \in X : \liminf_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} < \limsup_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} \right\},$$

where $\varphi_{i,n}$ and $\psi_{i,n}$ are as in (22). Set

$$\underline{\alpha}_i = \inf \left\{ \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} : \mu \in \mathcal{M}(X) \right\}$$

and

$$\overline{\alpha}_i = \sup \left\{ \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} : \mu \in \mathcal{M}(X) \right\}.$$

We have

$$X = \bigcup_{\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]} K_{\alpha_i}(\varphi_i, \psi_i) \cup I(\varphi_i, \psi_i) \tag{30}$$

and this union is composed of pairwise disjoint sets.

Let $\mathcal{C}$ be the collection of nonempty subsets of $\{1, \ldots, d\}$ distinct from $\{1, \ldots, d\}$. Intersecting the decompositions in (30) for $i = 1, \ldots, d$ we obtain

$$X = \bigcup_{\alpha \in \mathcal{P}(\mathcal{M}(X))} K_\alpha(\Phi, \Psi) \cup \bigcup_{\alpha \in \mathcal{P}(\mathcal{M}(X)), L \in \mathcal{C}} M_{\alpha,L}(\Phi, \Psi) \cup I(\Phi, \Psi), \tag{31}$$

where

$$M_{\alpha,L}(\Phi, \Psi) = \bigcap_{i \in L} K_{\alpha_i}(\varphi_i, \psi_i) \cap \bigcap_{i \notin L} I(\varphi_i, \psi_i),$$

and $I(\Phi, \Psi) = \bigcap_{i=1}^d I(\varphi_i, \psi_i)$. We remark that the decomposition in (31) is composed of pairwise disjoint sets. We call this decomposition the *multifractal decomposition* associated to the vector (Φ, Ψ). The set $I(\Phi, \Psi)$ is called the *irregular set* associated to the vector (Φ, Ψ).

We want to give a complete description of multifractal decompositions from the point of view of dimension theory. Accordingly, we must consider each of the sets in (31). The sets K_α are considered in the former sections. We now consider the remaining sets in (31).

It is an immediate consequence of Birkhoff's ergodic theorem that the sets $M_{\alpha,L}(\Phi,\Psi)$ and $I(\Phi,\Psi)$ have zero measure with respect to any invariant measure. Nevertheless we shall show that generically, with respect to (Φ,Ψ), they have full u-dimension.

The following statement was established by Barreira and Schmeling.

Theorem 15 ([4]). *Let $f\colon X\to X$ be a topologically mixing subshift of finite type, and $\varphi_1, \ldots, \varphi_d, \psi_1, \ldots, \psi_d, u$ Hölder continuous functions on X. If for each $i=1, \ldots, d$ the function φ_i is not cohomologous to any multiple of ψ_i, then*

$$\dim_u I(\Phi,\Psi)=\dim_u X.$$

See [4, 2] for extensions of this result to more general classes of maps.

Theorem 15 shows that from the point of view of dimension theory (and in particular from the point of view of entropy theory, by setting $u=1$) the irregular sets of multifractal decompositions are as large as the whole space. The following statement shows that the corresponding statement is also valid for each of the sets $M_{\alpha,L}(\Phi,\Psi)$.

Theorem 16. *Let $f\colon X\to X$ be a topologically mixing subshift of finite type, and $\varphi_1, \ldots, \varphi_d, \psi_1, \ldots, \psi_d, u$ Hölder continuous functions on X. If the functions $1, \varphi_1, \ldots, \varphi_d, \psi_1, \ldots, \psi_d$ are linearly independent as cohomology classes, then*

$$\dim_u M_{\alpha,L}(\Phi,\Psi)=\dim_u \bigcap_{i\in L} K_{\alpha_i}(\varphi_i,\psi_i)$$

for every $\alpha\in\mathcal{P}(\mathcal{M}(X))$ and every $L\in\mathcal{C}$.

Observe that $M_{\alpha,L}(\Phi,\Psi)\subset\bigcap_{i\in L}K_{\alpha_i}(\varphi_i,\psi_i)$. Therefore, Theorem 16 shows that the set $M_{\alpha,L}(\Phi,\Psi)$ (which has zero measure with respect to any invariant measure) has full u-dimension in $\bigcap_{i\in L}K_{\alpha_i}(\varphi_i,\psi_i)$. This is a surprising phenomenon since by Theorem 8 the set $\bigcap_{i\in L}K_{\alpha_i}(\varphi_i,\psi_i)$ has full measure with respect to some ergodic equilibrium measure, contrarily to the set $M_{\alpha,L}(\Phi,\Psi)$.

For example, assume that φ and ψ are Hölder continuous functions such that 1, φ, and ψ are linearly independent as cohomology classes. For topologically mixing subshifts of finite type the properties of the sets $K_\alpha(\varphi)\cap K_\beta(\psi)$ are described in the introduction. Let

$$I(\psi)=\left\{x\in X : \liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\psi(f^kx)<\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\psi(f^kx)\right\}.$$

For each $\alpha\in\mathbb{R}$ we have

$$K_\alpha(\varphi)=\bigcup_{\beta\in\mathbb{R}}(K_\alpha(\varphi)\cap K_\beta(\psi))\cup(K_\alpha(\varphi)\cap I(\psi)),$$

and this union is composed of pairwise disjoint sets. It follows from Theorem 16 that

$$\dim_u(K_\alpha(\varphi)\cap I(\psi))=\dim_u K_\alpha(\varphi)$$

for every $\alpha\in\mathbb{R}$ and every Hölder continuous positive function u. This reveals an extreme complexity hidden by Birkhoff's ergodic theorem.

7. Finer structure of the spectrum

We now want to have an even closer look at the fine structure of the level sets K_α in (21). In particular, we shall show that the u-dimension of the level set K_α is entirely carried by a certain level set strictly inside K_α corresponding to a new higher-dimensional parameter, at the expense of considering new vectors Φ and Ψ.

Let $f\colon X \to X$ be a topologically mixing subshift of finite type. Fix Hölder continuous functions φ_i, ψ_i for $i = 1, \ldots, d$, and u on X such that ψ_i for $i = 1, \ldots, d$, and u are positive.

Given $\alpha \in \mathbb{R}^d$ and $\mu \in \mathcal{M}(X)$ we define K_α and $\mathcal{P}(\mathcal{M}(X))$ respectively as in (21) and (23). We consider also the multifractal spectrum $\mathcal{F}_u$ defined by (24). We shall refer to this spectrum as a *mixed* spectrum, due to the noncoincidence in general of the functions ψ_i and u.

For each $(q_1, q_2) \in \mathbb{R}^d \times \mathbb{R}^d$, we consider the unique number $T(q_1, q_2)$ satisfying

$$P_X(\langle q_1, \Phi\rangle + \langle q_2, \Psi\rangle - T(q_1, q_2)u) = 0,$$

and denote by μ_{q_1,q_2} the equilibrium measure of $\langle q_1, \Phi\rangle + \langle q_2, \Psi\rangle - T(q_1, q_2)u$. Set

$$\beta(q_1, q_2) \stackrel{\text{def}}{=} \nabla_{q_1} T(q_1, q_2) \quad \text{and} \quad \gamma(q_1, q_2) \stackrel{\text{def}}{=} \nabla_{q_2} T(q_1, q_2).$$

For each $(\beta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d$, we consider the set $K_{\beta,\gamma}$ of points $x \in X$ such that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n \varphi_i(f^k x)}{\sum_{k=0}^n u(f^k x)} = \beta_i \quad \text{and} \quad \lim_{n\to\infty} \frac{\sum_{k=0}^n \psi_i(f^k x)}{\sum_{k=0}^n u(f^k x)} = \gamma_i$$

for every $i = 1, \ldots, d$. We now establish a precise relationship between the d-dimensional mixed spectrum and the $2d$-dimensional spectrum

$$\mathcal{H}_u(\beta, \gamma) = \dim_u K_{\beta,\gamma}.$$

Theorem 17. *The following properties hold:*

1. $\mu_{q_1,q_2}(K_{\beta(q_1,q_2),\gamma(q_1,q_2)}) = 1$ *and*

$$\begin{aligned}\mathcal{H}_u(\beta(q_1, q_2), \gamma(q_1, q_2)) &= \dim_u \mu_{q_1,q_2}\\ &= T(q_1, q_2) - \langle q_1, \beta(q_1, q_2)\rangle - \langle q_2, \gamma(q_1, q_2)\rangle;\end{aligned}$$

2. *if* $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$, *then there exists* $\gamma \in \mathbb{R}^d$ *such that* $\mathcal{F}_u(\alpha) = \mathcal{H}_u(\alpha * \gamma, \gamma)$.

Observe that $\mathcal{H}_u$ is the Legendre transform of the function T.

Clearly, for each $\gamma \in \mathbb{R}^d$ we have $K_{\alpha*\gamma,\gamma} \subset K_\alpha$. Therefore, the second statement in Theorem 17 says that the u-dimension of the set K_α is fully carried by some subset $K_{\alpha*\gamma,\gamma}$ of K_α (among the uncountable number of pairwise disjoint subsets $K_{\alpha*\gamma,\gamma}$). In particular, the mixed spectrum $\mathcal{F}_u$ can be obtained from the non-mixed (but $2d$-dimensional) spectrum $\mathcal{H}_u$ by

$$\mathcal{F}_u(\alpha) = \max\{\mathcal{H}_u(\alpha * \gamma, \gamma) : \gamma \in \mathbb{R}^d\} \tag{32}$$

for each $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$. This consequence is particularly unexpected since the inclusion $\bigcup_\gamma K_{\alpha*\gamma,\gamma} \subset K_\alpha$ is never an identity, and since the u-dimension of an uncountable union $\bigcup_\gamma I_\gamma$ may in general be strictly larger then $\sup_\gamma \dim_u I_\gamma$.

We now provide an application of Theorem 17.

Theorem 18. *If for some $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$ the maximum in (32) is attained at a point $(\alpha * \gamma, \gamma) \in \mathbb{R}^{2d}$ in the interior of the domain of definition of $\mathcal{H}_u$, then $\mathcal{F}_u$ has at most one local maximum in a neighborhood of α.*

For example, assume that $d = 1$ and that the functions φ_1, ψ_1, and u are linearly independent as cohomology classes. In this case, the function $\mathcal{H}_u$ is strictly convex, and its maximum is attained in the interior of its domain of definition (and coincides with the maximum of $\mathcal{F}_u$). The measure of maximal dimension is the equilibrium measure with potential $-T(0,0)u$. If $d = 1$ and φ_1, ψ_1, and u are linearly independent as cohomology classes, then the spectrum $\mathcal{F}_u$ has only one maximum, and it follows from Theorem 18 that it is strictly convex in an open neighborhood of this maximum. We note that however $\mathcal{F}_u$ may not be convex (everywhere). An example of a nonconvex spectrum is given in [2].

8. Proofs

8.1. **Proofs of the results in Section 4.** We begin with some preparatory lemmas. Let $|q| = |q_1| + \cdots + |q_d|$ be the norm of a vector $q \in \mathbb{R}^d$.

Lemma 1. *If $\alpha \in \mathcal{P}(\mathcal{M}(X))$ then*

$$\inf_{q \in \mathbb{R}^d} P_X(\langle q, \Phi - \alpha * \Psi \rangle - \mathcal{F}_u(\alpha) u) \geq 0.$$

Proof of Lemma 1. Assume first that $\mathcal{F}_u(\alpha) = 0$. By the definition of $\mathcal{P}$ there exists $\mu \in \mathcal{M}(X)$ such that $\int_X \Phi \, d\mu = \int_X \alpha * \Psi \, d\mu$. Then

$$P_X(\langle q, \Phi - \alpha * \Psi \rangle) \geq h_\mu(f) + \left\langle q, \int_X (\Phi - \alpha * \Psi) \, d\mu \right\rangle = h_\mu(f) \geq 0.$$

We shall now use a modification of an argument in [2] and the notations of Section 2. Assume that $\mathcal{F}_u(\alpha) > 0$. By Proposition 4 the number $\mathcal{F}_u(\alpha)$ is equal to the unique root δ of the equation $P_{K_\alpha}(-\delta u) = 0$. Given $\delta > 0$ and $\tau \in \mathbb{N}$ consider the sets

$$L_{\delta,\tau} = \{x \in X : \|\Phi_n(x) - \alpha \Psi_n(x)\| < \delta n \text{ for every } n \geq \tau\},$$

where

$$\Phi_n = \sum_{k=0}^{n-1} \Phi \circ f^k \quad \text{and} \quad \Psi_n = \sum_{k=0}^{n-1} \Psi \circ f^k.$$

Since $\Psi > 0$ one can easily show that $K_\alpha \subset \bigcap_{\delta > 0} \bigcup_{\tau \in \mathbb{N}} L_{\delta,\tau}$. Let now $\mathcal{U}$ be an open cover of X with sufficiently small diameter such that if n is sufficiently large, $\mathbf{U} \in \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{U})$, and $x \in X(\mathbf{U})$, then

$$\|\Phi(\mathbf{U}) - \Phi_{m(\mathbf{U})}(x)\| \leq \delta m(\mathbf{U}) \quad \text{and} \quad \|\Psi(\mathbf{U}) - \Psi_{m(\mathbf{U})}(x)\| \leq \delta m(\mathbf{U}).$$

Hence, if $\mathbf{U} \in \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{U})$ and $X(\mathbf{U}) \cap L_{\delta,\tau} \neq \varnothing$ then

$$\|\Phi(\mathbf{U}) - \alpha \Psi(\mathbf{U})\| < (2 + |\alpha|) \delta m(\mathbf{U}).$$

Thus

$$P_{L_{\delta,\tau}}(-\mathcal{F}_u(\alpha) u, \mathcal{U}) \leq P_{L_{\delta,\tau}}(\langle q, \Phi - \alpha * \Psi \rangle - \mathcal{F}_u(\alpha) u, \mathcal{U}) + (2 + |\alpha|) \delta |q|.$$

Letting the diameter of $\mathcal{U}$ going to zero yields

$$P_{L_{\delta,\tau}}(-\mathcal{F}_u(\alpha) u) \leq P_X(\langle q, \Phi - \alpha * \Psi \rangle - \mathcal{F}_u(\alpha) u) + (2 + |\alpha|) \delta |q|,$$

and hence,

$$\begin{aligned}0 &\leq P_{\bigcup_{\tau\in\mathbb{N}} L_{\delta,\tau}}(-\mathcal{F}_u(\alpha)u) = \sup_{\tau\in\mathbb{N}} P_{L_{\delta,\tau}}(-\mathcal{F}_u(\alpha)u)\\ &\leq P_X(\langle q, \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u) + (2+|\alpha|)\delta|q|.\end{aligned}$$

Since δ is arbitrary, we obtain

$$\inf_{q\in\mathbb{R}^d} P_X(\langle q, \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u) \geq 0.$$

This completes the proof of the lemma. □

Lemma 2. *If* $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$ *then*

$$\inf_{q\in\mathbb{R}^d} P_X(\langle q, \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u) = 0,$$

and there exists an ergodic equilibrium measure $\mu_\alpha \in \mathcal{M}(X)$ *with* $\mathcal{P}(\mu_\alpha) = \alpha$ *and* $\mu_\alpha(K_\alpha) = 1$ *such that* $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$.

Proof of Lemma 2. Let

$$r = \operatorname{dist}(\mathbb{R}^d \setminus \mathcal{P}(\mathcal{M}(X)), \alpha) > 0$$

(with the distance given by $|\cdot|$). We claim that the infimum over $q \in \mathbb{R}^d$ of the function

$$F(q) = P_X(\langle q, \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u)$$

is attained inside the ball of radius

$$R = \frac{\dim_u X \cdot \sup u + F(0)}{r \min_i \inf \psi_i}.$$

Let $q \in \mathbb{R}^d$ such that $|q| \geq R$. We shall prove that $F(q) \geq F(0)$. Let $a \in (0,1)$ and $\beta \in \mathbb{R}^d$ such that $\beta_i = \alpha_i + ar \operatorname{sgn} q_i$. Clearly $\beta \in \mathcal{P}(\mathcal{M}(X))$, and hence there exists $\mu \in \mathcal{M}(X)$ such that $\int_X \Phi\, d\mu = \int_X \beta * \Psi\, d\mu$. We obtain

$$\begin{aligned}F(q) &\geq h_\mu(f) + \left\langle q, \int_X (\Phi - \alpha * \Psi)\, d\mu\right\rangle - \mathcal{F}_u(\alpha)\int_X u\, d\mu\\ &\geq \left\langle q, \int_X (\beta - \alpha) * \Psi\, d\mu\right\rangle - \dim_u X \cdot \sup u\\ &\geq |q| ar \min_i \inf \psi_i - \dim_u X \cdot \sup u\\ &> a \dim_u X \cdot \sup u + F(0) - \dim_u X \cdot \sup u.\end{aligned}$$

We obtain the claim by letting $a \to 1$.

Since F is of class C^1 its minimum is attained at a point $q = q(\alpha)$ with $|q(\alpha)| < R$ and satisfying $\partial_q F(q(\alpha)) = 0$. Let μ_α be the equilibrium measure of the function $\langle q(\alpha), \Phi - \alpha * \Psi\rangle - \mathcal{F}_u(\alpha)u$. Then

$$\int_X (\Phi - \alpha * \Psi)\, d\mu_\alpha = \partial_q F(q(\alpha)) = 0,$$

and hence $\mathcal{P}(\mu_\alpha) = \alpha$. Furthermore,

$$F(q(\alpha)) = h_{\mu_\alpha}(f) - \mathcal{F}_u(\alpha)\int_X u\, d\mu_\alpha.$$

By Lemma 1 we have $F(q(\alpha)) \ge 0$ and thus

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u\, d\mu_\alpha} \ge \mathcal{F}_u(\alpha).$$

On the other hand, since μ_α is ergodic and $\int_X \Phi\, d\mu_\alpha = \int_X \alpha * \Psi\, d\mu_\alpha$ it follows from Birkhoff's ergodic theorem that $\mu_\alpha(K_\alpha) = 1$. Therefore

$$\mathcal{F}_u(\alpha) \ge \dim_u \mu_\alpha$$

and thus $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$. This completes the proof of the lemma. $\square$

Proof of Theorem 8. Let $\alpha \in \mathbb{R}^d$ with $K_\alpha \ne \varnothing$, and take $x \in K_\alpha$. The sequence of measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}$$

has an accumulation point, say μ, which is invariant. Moreover, for all $i = 1$, …, d we have $\int_X \varphi_i\, d\mu_n / \int_X \psi_i\, d\mu_n \to \alpha_i$ when $n \to \infty$. This implies that

$$\int_X \varphi_i\, d\mu \Big/ \int_X \psi_i\, d\mu = \alpha_i$$

for all $i = 1, \ldots, d$. Hence $\alpha \in \mathcal{P}(\mathcal{M}(X))$, which proves the first statement.

Let now $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$. For any $\mu \in \mathcal{M}(X)$ such that $\mathcal{P}(\mu) = \alpha$ Lemma 2 implies that

$$0 = \inf_{q \in \mathbb{R}^d} P_X(\langle q, \Phi - \alpha * \Psi \rangle - \mathcal{F}_u(\alpha) u) \ge h_\mu(f) - \mathcal{F}_u(\alpha) \int_X u\, d\mu.$$

Therefore

$$h_\mu(f) \Big/ \int_X u\, d\mu \le \mathcal{F}_u(\alpha).$$

On the other hand, again by Lemma 2 there exists an ergodic measure μ_α such that $\mu_\alpha(K_\alpha) = 1$, $\mathcal{P}(\mu_\alpha) = \alpha$, and

$$\mathcal{F}_u(\alpha) = \dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u\, d\mu_\alpha}$$

(using ergodicity and the identity in (8)). This establishes the identities in (26) and (27). Statement 2 is an immediate consequence of Lemma 2. This completes the proof of the theorem. $\square$

Proof of Theorem 9. Changing if necessary Φ by $\Phi - \alpha * \Psi$, we may assume that $\alpha = 0$ without loss of generality. Note that this corresponds to a translation of the set $\mathcal{P}(\mathcal{M}(X))$ by the vector $-\alpha$.

We now establish the first statement. Since $\alpha = 0 \in \mathcal{P}(\mathcal{M}(X))$ there exists a measure $m_0 \in \mathcal{M}(X)$ such that $\int_X \Phi\, dm_0 = 0$. Moreover, the map $m \mapsto \int_X \Phi\, dm$ is affine on the convex set $\mathcal{M}(X)$, and hence

$$\mathcal{M}(\Phi) \stackrel{\text{def}}{=} \left\{ \int_X \Phi\, dm : m \in \mathcal{M}(X) \right\}$$

is convex. We shall show that its interior is nonempty. If $\operatorname{int} \mathcal{M}(\Phi) = \varnothing$ then $\mathcal{M}(\Phi)$ is contained in some hyperplane, and hence there exists $q \in \mathbb{R}^d$

such that $\langle q, \int_X \Phi\, dm\rangle = 0$ for any $m \in \mathcal{M}(X)$. This implies that for any real number t we have

$$\begin{aligned} P_X(t\langle q,\Phi\rangle) &= \sup_{m\in\mathcal{M}(X)} \left(h_m(f) + t\int_X \langle q,\Phi\rangle\, dm\right) \\ &= \sup_{m\in\mathcal{M}(X)} h_m(f) = P_X(0). \end{aligned}$$

This contradicts the hypotheses in the theorem. Therefore $\operatorname{int}\mathcal{M}(\Phi) \neq \varnothing$ and one can find d measures $m_1, \ldots, m_d$ such that the vectors $\int_X \Phi\, dm_1, \ldots, \int_X \Phi\, dm_d$ form a basis of $\mathbb{R}^d$.

Consider the set

$$\Delta = \{p \in \mathbb{R}^d : 0 \le p_i \text{ for each } i = 1, \ldots, d \text{ and } p_1 + \cdots + p_d \le 1\}.$$

For each $p \in \Delta$ let

$$\mu_p = p_1 m_1 + \cdots + p_d m_d + \left(1 - \sum_{i=1}^d p_i\right) m_0 \in \mathcal{M}(X).$$

We define the map $\beta\colon \Delta \to \mathbb{R}^d$ by

$$\beta(p) = \left(\frac{\int_X \varphi_1\, d\mu_p}{\int_X \psi_1\, d\mu_p}, \ldots, \frac{\int_X \varphi_d\, d\mu_p}{\int_X \psi_d\, d\mu_p}\right).$$

Since $\int_X \Phi\, dm_0 = 0$ we have

$$\begin{aligned} \frac{\partial}{\partial p_j}\left(\frac{\int_X \varphi_i\, d\mu_p}{\int_X \psi_i\, d\mu_p}\right)\Big|_{p=0} &= \frac{\int_X \varphi_i\, dm_j - \int_X \varphi_i\, dm_0}{\int_X \psi_i\, d\mu_p}\Big|_{p=0} \\ &\quad - \frac{\left(\int_X \psi_i\, dm_j - \int_X \psi_i\, dm_0\right)\int_X \varphi_i\, d\mu_p}{\left(\int_X \psi_i\, d\mu_p\right)^2}\Big|_{p=0} \\ &= \frac{\int_X \varphi_i\, dm_j}{\int_X \psi_i\, dm_0}. \end{aligned}$$

The map β is of class C^1, and its derivative at $p = 0$ is given by

$$d_0\beta = \begin{bmatrix} \frac{\int_X \varphi_1\, dm_1}{\int_X \psi_1\, dm_0} & \cdots & \frac{\int_X \varphi_1\, dm_d}{\int_X \psi_1\, dm_0} \\ \vdots & \ddots & \vdots \\ \frac{\int_X \varphi_d\, dm_1}{\int_X \psi_d\, dm_0} & \cdots & \frac{\int_X \varphi_d\, dm_d}{\int_X \psi_d\, dm_0} \end{bmatrix}.$$

We denote by $M = (M_{ij})_{ij}$ the $d \times d$ matrix with entries $M_{ij} = \int_X \varphi_j\, dm_i$. Then

$$\det d_0\beta = \left(\prod_{j=1}^d \int_X \psi_j\, dm_0\right)^{-1} \det M.$$

Since the vectors $\int_X \Phi\, dm_1, \ldots, \int_X \Phi\, dm_d$ are linearly independent, the matrix M is invertible, and thus β is a local diffeomorphism at 0. Thus there exist open sets $U \subset \Delta$ and $D = \beta(U)$ such that $0 \in \overline{U}$ and β is a diffeomorphism from U to D. Accordingly, $0 \in \overline{D}$. In particular,

$$\alpha = 0 \in \overline{\operatorname{int}\beta(\Delta)} \subset \overline{\operatorname{int}\mathcal{P}(\mathcal{M}(X))}.$$

We now prove the second statement. We still assume that $\alpha = 0$. There exists $q \in \mathbb{R}^d$ such that $P_X(t\langle q, \Phi\rangle) = P_X(0)$ for any $t \in \mathbb{R}$. We want to show that

$$\{sq : s \in \mathbb{R}\} \cap \mathcal{P}(\mathcal{M}(X)) = \{0\},$$

which immediately implies the statement in the theorem. Let $s \neq 0$. If $sq \in \mathcal{P}(\mathcal{M}(X))$ then there exists μ such that $\int_X \Phi\, d\mu = sq * \int_X \Psi\, d\mu$. Thus for any $t > 0$ we have

$$P_X(0) = P_X(t\langle sq, \Phi\rangle) \geq h_\mu(f) + t\left\langle sq, \int_X \Phi\, d\mu\right\rangle \geq t|sq|^2 \inf_i \inf \psi_i.$$

This gives a contradiction if t is sufficiently large. Therefore $\alpha = 0 \notin \operatorname{int} \mathcal{P}(\mathcal{M}(X))$. This completes the proof of the theorem. $\square$

Proof of Theorem 10. The second statement follows immediately from Theorem 9. Let now

$$\mathcal{E}_{2d}(\mu) = -\left(\int_X \Phi\, d\mu, \int_X \Psi\, d\mu\right)$$

and

$$\widehat{\mathcal{P}} = \left\{\frac{\alpha_i}{\beta_i} : (\alpha, \beta) \in \partial\mathcal{E}_{2d}(\mathcal{M}(X))\right\}.$$

It follows from Theorem 5 that

$$\mathcal{E}_{2d}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{E}_{2d}(\mathcal{M}(X))}.$$

The proof consists of three claims.

Claim 1. $\partial\mathcal{P}(\mathcal{M}(X)) \subset \widehat{\mathcal{P}}$.

Let $(\alpha, \beta) \in \operatorname{int} \mathcal{E}_{2d}(\mathcal{M}(X))$. This means that $(\alpha, \beta) + \varepsilon \in \mathcal{E}_{2d}(\mathcal{M}(X))$ for all sufficiently small $\varepsilon \in \mathbb{R}^{2d}$, and thus by Theorem 5 there exists an ergodic measure μ_ε with $\mathcal{P}(\mu_\varepsilon) = (\alpha, \beta) + \varepsilon$. Hence for all sufficiently small $\delta \in \mathbb{R}^d$ the δ-neighborhood of $(\alpha_1/\beta_1, \ldots, \alpha_d/\beta_d)$ is entirely contained in $\mathcal{P}(\mathcal{M}(X))$. This establishes the claim.

Claim 2. *The set $\mathcal{E}_{2d}(\mathcal{M}(X))$ is convex.*

The claim follows immediately from the convexity of $\mathcal{M}(X)$ and the convexity of the functional $\mathcal{E}_{2d}$ on this space.

For each $q \in S^{2d-1}$ we set

$$W_q = -\left\{\int_X \langle q, (\Phi, \Psi)\rangle\, d\mu : \mu \in \mathcal{M}(X)\right\}.$$

Claim 3. *For each $(\alpha, \beta) \in \partial\mathcal{E}_{2d}(\mathcal{M}(X))$ there exists a vector $q \in S^{2d-1}$ such that $\langle(\alpha, \beta), q\rangle \in \partial W_q$.*

Since $\mathcal{E}_{2d}(\mathcal{M}(X))$ is a convex set each of its boundary points has a supporting plane. Let $(\alpha, \beta) \in \partial\mathcal{E}_{2d}(\mathcal{M}(X))$, and denote by P the orthogonal projection of $\mathcal{E}_{2d}(\mathcal{M}(X))$ onto the normal to the supporting plane at (α, β). The point (α, β) is mapped by P into a boundary point of the interval $P(\mathcal{E}_{2d}(\mathcal{M}(X)))$. The orthogonal projection of a point (α, β) onto the line in the direction of a normal vector $q \in S^{2d-1}$ is given by $\langle(\alpha, \beta), q\rangle$. This establishes the claim since W_q is the image of $\mathcal{E}_{2d}(\mathcal{M}(X))$ under this projection.

Now we are ready to prove the proposition. Let $(\alpha,\beta)\in\mathbb{R}^{2d}$ be such that $(\alpha_1/\beta_1,\dots,\alpha_d/\beta_d)\in\widehat{\mathcal{P}}$. This means that $(\alpha,\beta)\in\partial\mathcal{E}_{2d}(\mathcal{M}(X))$. Hence there is a point $q\in S^{2d-1}$ such that $\langle(\alpha,\beta),q\rangle\in\partial W_q$. This concludes the proof. ∎

8.2. **Proofs of the results in Section 5.**

Proof of Theorem 12. Let $\alpha\in\operatorname{int}\mathcal{P}(\mathcal{M}(X))$ and put

$$Q(\delta,q,\alpha)=P_X(\langle q,\Phi-\alpha*\Psi\rangle-\delta u).$$

Proceeding as in the proof of Lemma 2 one can show that there exist $q(\alpha)\in\mathbb{R}^d$ and an ergodic equilibrium measure μ_α such that $q\mapsto Q(\mathcal{F}_u(\alpha),q,\alpha)$ attains a minimum at $q=q(\alpha)$, and thus

$$\partial_q Q(\mathcal{F}_u(\alpha),q(\alpha),\alpha)=\int_X(\Phi-\alpha*\Psi)\,d\mu_\alpha=0.$$

By Lemma 2 we have $Q(\mathcal{F}_u(\alpha),q(\alpha),\alpha)=0$.

Consider the system of equations

$$Q(\delta,q,\alpha)=0\quad\text{and}\quad\partial_q Q(\delta,q,\alpha)=0.$$

We want to apply the implicit function theorem to establish the uniqueness of the solution $(\delta,q)=(\mathcal{F}_u(\alpha),q(\alpha))$ for this system, and its regularity in α. In particular this will establish the regularity of the spectrum. Let

$$G(q,\delta,\alpha)=(Q(\delta,q,\alpha),\partial_{q_1}Q(\delta,q,\alpha),\dots,\partial_{q_d}Q(\delta,q,\alpha)).$$

It is enough to show that the matrix

$$(\partial_\delta,\partial_{q_1},\dots,\partial_{q_d})^tG=\begin{bmatrix}\partial_\delta Q & \partial_\delta\partial_{q_1}Q & \cdots & \partial_\delta\partial_{q_d}Q\\ \partial_{q_1}Q & \partial_{q_1}\partial_{q_1}Q & \cdots & \partial_{q_1}\partial_{q_d}Q\\ \vdots & \vdots & \ddots & \vdots\\ \partial_{q_d}Q & \partial_{q_d}\partial_{q_1}Q & \cdots & \partial_{q_d}\partial_{q_d}Q\end{bmatrix}\tag{33}$$

is invertible at $(q(\alpha),\mathcal{F}_u(\alpha),\alpha)$. Denote by $\mu_{q,\delta,\alpha}$ the unique equilibrium measure of the function $\langle q,\Phi-\alpha*\Psi\rangle-\delta u$. For each $i=1,\dots,d$ we have $\partial_{q_i}Q(\mathcal{F}_u(\alpha),q(\alpha),\alpha)=0$, and $\delta=\mathcal{F}_u(\alpha)$. Hence the first column of the matrix in (33) is zero at $(q(\alpha),\mathcal{F}_u(\alpha),\alpha)$, with the exception of the first term which is

$$\partial_\delta Q(\mathcal{F}_u(\alpha),q(\alpha),\alpha)=-\int_X u\,d\mu_{q,\delta,\alpha}<0.$$

Therefore, it suffices to check that the remaining right lower $d\times d$ matrix, say H, is invertible. The second derivative of the pressure at $q(\alpha)$ is a bilinear symmetric form $B\colon\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ and we have

$$\partial_{q_i}\partial_{q_j}Q(\mathcal{F}_u(\alpha),q(\alpha),\alpha)=B(e_i,e_j)$$

where $(e_i)_{i=1,\dots,d}$ denotes the canonical basis in $\mathbb{R}^d$. By hypothesis B is positive definite. If H were not invertible, then some nontrivial linear combination of its columns would be zero, and thus there would exist $\lambda\in\mathbb{R}^d\setminus\{0\}$ such that

$$\sum_{j=1}^d\lambda_jB(e_i,e_j)=0$$

for $i = 1, \ldots, d$. Hence, setting $g = \sum_{i=1}^d \lambda_i e_i$ we would obtain

$$B(g, g) = \sum_{i=1}^d \lambda_i \sum_{j=1}^d \lambda_j B(e_i, e_j) = 0.$$

Since $g \neq 0$ this contradicts the positive definiteness of B. Thus H is invertible. By the implicit function theorem the functions $\delta(\alpha)$ and $q(\alpha)$ must be at least as regular as the function G, which is of class C^{k-1} (or analytic if the pressure is analytic). This completes the proof of the theorem. $\square$

Proof of Theorem 13. Let $G = \Phi - \alpha * \Psi$ and define $F(q) = P_X(\langle q, G\rangle)$. Ruelle's formula for the second derivative of the topological pressure shows that for any $p \in \mathbb{R}^d$ (see [10]) we have

$$\partial_q^2 F(p,p) = \int_X \langle p, G\rangle^2 \, d\mu_q + 2\sum_{n=1}^\infty \int_X \langle p, G\rangle \cdot \langle p, G \circ f^n\rangle \, d\mu_q \geq 0,$$

where μ_q denotes the unique equilibrium measure of the function $\langle q, G\rangle$.

We shall prove that $\partial_q^2 F(p,p) > 0$ whenever $p \neq 0$. Suppose on the contrary that $\partial_q^2 F(p,p)$ is zero. In this case the function $\langle p, G\rangle$ must be cohomologous to some constant c. Since $\alpha \in \mathcal{P}(\mathcal{M}(X))$ there exists a measure $\mu \in \mathcal{M}(X)$ such that $\int_X G \, d\mu = 0$. This implies that $c = 0$. Since $\langle p, G\rangle$ is cohomologous to zero we conclude that $t \mapsto P_X(t\langle p, G\rangle)$ is constant. By Theorem 9 this never happens when $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(X))$ and thus $\partial_q^2 F(p,p) > 0$. The desired statement follows now immediately from Theorem 12. $\square$

We now need an auxiliary statement.

Lemma 3. *The set of vectors $(\varphi_1, \ldots, \varphi_d) \in {H_\theta}^d$ such that $\varphi_1, \ldots, \varphi_d$ are linearly independent as cohomology classes is residual in ${H_\theta}^d$.*

Proof of Lemma 3. Consider d distinct periodic orbits $x_1, \ldots, x_d$ of period respectively $n_1, \ldots, n_d$. We set

$$S_{ij} = \frac{1}{n_i} \sum_{k=0}^{n_i - 1} \varphi_j(f^k x_i).$$

By the Livschitz theorem, if the $d \times d$-matrix with entries S_{ij} has full rank, then the functions $\varphi_1, \ldots, \varphi_d$ are linearly independent as cohomology classes. The desired statement follows now from the fact that this is a generic condition. $\square$

Proof of Theorem 14. Set

$$\underline{u} = \min\{u(x) : x \in X\} \quad \text{and} \quad \overline{u} = \max\{u(x) : x \in X\}.$$

It follows from (7) below that

$$M(Z, \alpha\overline{u}, 1, \mathcal{U}) \leq M(Z, \alpha, u, \mathcal{U}) \leq M(Z, \alpha\underline{u}, 1, \mathcal{U}).$$

Therefore

$$\frac{\dim_{1,\mathcal{U}} Z}{\overline{u}} \leq \dim_{u,\mathcal{U}} Z \leq \frac{\dim_{1,\mathcal{U}} Z}{\underline{u}}$$

and hence $h(f|Z)/\overline{u} \le \dim_u Z \le h(f|Z)/\underline{u}$. This shows that $\mathcal{F}_1(\alpha) = 0$ if and only if $\mathcal{F}_u(\alpha) = 0$. Hence it is sufficient to prove that the topological entropy vanishes at the boundary of $\mathcal{P}(\mathcal{M}(X))$.

We shall reduce our problem to a one-dimensional problem. For each $q, \alpha \in \mathbb{R}^d$ and each $(\Phi, \Psi) \in {H_\theta}^d \times {H_\theta}^d$ we consider the function $\chi_{\Phi\Psi} = \langle q, \Phi - \alpha * \Psi\rangle$. In [12] (in the proof of Proposition 5.3) it is shown that there is an open and dense subset $\Theta^\varepsilon \subset H_\theta$ such that if $\chi \in \Theta^\varepsilon$ then

$$h\left(f \middle| \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi(f^n x) \in \{\underline{\beta}, \overline{\beta}\} \right\}\right) < \varepsilon, \tag{34}$$

where

$$\underline{\beta} = \inf \left\{ \int_X \chi \, d\mu : \mu \in \mathcal{M}(X) \right\} \quad \text{and} \quad \overline{\beta} = \sup \left\{ \int_X \chi \, d\mu : \mu \in \mathcal{M}(X) \right\}.$$

Therefore for each fixed q, $\alpha \in \mathbb{R}^d$ the set

$$\Theta^\varepsilon_{q\alpha} = \{(\Phi, \Psi) \in {H_\theta}^d \times {H_\theta}^d : \chi_{\Phi\Psi} \in \Theta^\varepsilon\} \tag{35}$$

is open and dense in ${H_\theta}^d \times {H_\theta}^d$.

It follows from Lemma 3 that by changing $\Theta^\varepsilon_{q\alpha}$ slightly but leaving it still open and dense we may assume that there is no cohomology relation between the functions $\varphi_i - \alpha_i\psi_i$ for $i = 1, \ldots, d$. This implies that there is an open neighborhood $U(q, \alpha) \subset \mathbb{R}^d \times \mathbb{R}^d$ of (q, α) such that there is an open and dense subset $\Theta^\varepsilon_{q\alpha}$ of ${H_\theta}^d \times {H_\theta}^d$ without cohomology relations such that $(\Phi, \Psi) \in \Theta^\varepsilon_{q'\alpha'}$ for every $(q', \alpha') \in U(q, \alpha)$. Now we choose a sequence (q_n, α_n) such that $\bigcup_{n=1}^\infty U(q_n, \alpha_n) = \mathbb{R}^d \times \mathbb{R}^d$ and set

$$\Theta = \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \Theta^{1/m}_{q_n\alpha_n}. \tag{36}$$

By construction the set Θ is residual and for every $(\Phi, \Psi) \in \Theta$ there are no cohomology relations (see Theorem 9 and the discussion after this theorem). This establishes the first assertion of the theorem.

By Theorem 10, for each $(\Phi, \Psi) \in \Theta$ the boundary points of $\mathcal{P}(\mathcal{M}(X))$ are contained in $\bigcup_{q\in S^{2d-1}} \Gamma(q)$. By construction of the set Θ (see (34)–(36)) the spectrum vanishes at these points. This completes the proof of the theorem. □

8.3. **Proofs of the results in Section 6.**

Proof of Theorem 16. The following is an immediate consequence of a result of Barreira and Schmeling (see [4, Theorem 7.2]).

Lemma 4. *For a subshift $f : X \to X$ with the specification property, and an f-invariant set $K \subset X$, if for each $i \notin L$ there exist measures μ_i^1, $\mu_i^2 \in \mathcal{M}_E(K)$ such that*

$$\frac{\int_X \varphi_i \, d\mu_i^1}{\int_X \psi_i \, d\mu_i^1} \ne \frac{\int_X \varphi_i \, d\mu_i^2}{\int_X \psi_i \, d\mu_i^2},$$

then

$$\dim_u \left(K \cap \bigcap_{i \notin L} I(\varphi_i, \psi_i) \right) \geq \min\{\dim_u \mu_i^1, \dim_u \mu_i^2 : i \notin L\}.$$

We want to apply Lemma 4 with $K = \bigcap_{i \in L} K_{\alpha_i}(\varphi_i, \psi_i)$. By Theorem 8 there exists an ergodic measure $\mu \in \mathcal{M}_E(K)$ with $\mu(K) = 1$ and $\dim_u \mu = \dim_u K$. By Birkhoff's ergodic theorem, for each $i = 1, \ldots, d$ there exists a constant β_i such that

$$\lim_{n \to \infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} \to \beta_i$$

for μ-almost every $x \in K$. Clearly $\beta_i = \alpha_i$ for each $i \in L$, and $M_{\alpha,L}(\Phi, \Psi) = M_{\beta,L}(\Phi, \Psi)$ where $\beta = (\beta_1, \ldots, \beta_d)$. Furthermore,

$$\dim_u K \geq \dim_u K_\beta \geq \dim_u \mu = \dim_u K,$$

and thus $\dim_u K_\beta = \dim_u K$.

By Theorem 12 the spectrum $\mathcal{F}_u$ is analytic, and thus the function $\gamma_i \mapsto \mathcal{F}_u(\gamma)$ is also analytic for each $i \notin L$, where $\gamma_j = \alpha_j$ for every $j \in L$ and $\gamma_j = \beta_j$ for every $j \notin L \cup \{i\}$. By Theorem 10 we have $\mathcal{P}(\mathcal{M}(X)) = \overline{\operatorname{int} \mathcal{P}(\mathcal{M}(X))}$. This implies that for each $\varepsilon > 0$ and each $i \notin L$, there exists γ_i sufficiently close to β_i (but different from β_i) and an ergodic measure μ_i such that $\mu_i(K_\gamma) = 1$, $\mathcal{P}(\mu_i) = \gamma$, and

$$\dim_u \mu_i > \dim_u K_\beta - \varepsilon = \dim_u K - \varepsilon.$$

Since $\mu_i(K) \geq \mu_i(K_\gamma)$ and

$$\frac{\int_X \varphi_i \, d\mu_i}{\int_X \psi_i \, d\mu_i} = \gamma_i \neq \beta_i = \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu},$$

it follows from Lemma 4 that

$$\dim_u \left(K \cap \bigcap_{i \notin L} I(\varphi_i, \psi_i) \right) \geq \dim_u K - \varepsilon.$$

The arbitrariness of ε implies the desired result. □

8.4. **Proofs of the results in Section 7.**

Proof of Theorem 17. With a straightforward modification of the standard one-dimensional multifractal analysis (or applying Theorem 8 with $\Psi = (1, \ldots, 1)$), we have

$$\mu_{q_1,q_2}(K_{\beta(q_1,q_2),\gamma(q_1,q_2)}) = 1 \quad \text{and} \quad \mathcal{H}_u(\beta(q_1, q_2), \gamma(q_1, q_2)) = \dim_u \mu_{q_1,q_2}.$$

Furthermore

$$\beta(q_1, q_2) = \frac{\int_X \Phi \, d\mu_{q_1,q_2}}{\int_X u \, d\mu_{q_1,q_2}} \quad \text{and} \quad \gamma(q_1, q_2) = \frac{\int_X \Psi \, d\mu_{q_1,q_2}}{\int_X u \, d\mu_{q_1,q_2}}.$$

We obtain

$$\begin{aligned}\dim_u \mu_{q_1,q_2} &= \frac{h_{\mu_{q_1,q_2}}(f)}{\int_X u\, d\mu_{q_1,q_2}}\\ &= \frac{-\int_X(\langle q_1,\Phi\rangle + \langle q_2,\Psi\rangle - T(q_1,q_2)u)\, d\mu_{q_1,q_2}}{\int_X u\, d\mu_{q_1,q_2}}\\ &= T(q_1,q_2) - \langle q_1,\beta(q_1,q_2)\rangle - \langle q_2,\gamma(q_1,q_2)\rangle.\end{aligned}$$

This completes the proof of the first statement.

We now establish the second statement. By Theorem 8 there exists a measure of maximal u-dimension μ_α on K_α. Then for μ_α-almost every $x \in X$ there exist the limits

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n \Phi(f^k x)}{\sum_{k=0}^n u(f^k x)} = \beta(\alpha) = \frac{\int_X \Phi\, d\mu_\alpha}{\int_X u\, d\mu_\alpha},$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n \Psi(f^k x)}{\sum_{k=0}^n u(f^k x)} = \gamma(\alpha) = \frac{\int_X \Psi\, d\mu_\alpha}{\int_X u\, d\mu_\alpha},$$

and $\beta(\alpha) = \alpha * \gamma(\alpha)$. Therefore

$$\mu_\alpha(K_{\beta(\alpha),\gamma(\alpha)}) = \mu_\alpha(K_{\alpha*\gamma(\alpha),\gamma(\alpha)}) = 1.$$

This implies that

$$\dim_u K_{\alpha*\gamma(\alpha),\gamma(\alpha)} \geq \dim_u \mu_\alpha = \dim_u K_\alpha.$$

On the other hand $K_{\alpha*\gamma,\gamma} \subset K_\alpha$ for every $\gamma \in \mathbb{R}^d$, and thus $\dim_u K_{\alpha*\gamma,\gamma} \leq \dim_u K_\alpha$. We conclude that

$$\dim_u K_\alpha = \sup\{\dim_u K_{\alpha*\gamma,\gamma} : \gamma\} = \dim_u K_{\alpha*\gamma(\alpha),\gamma(\alpha)}.$$

This completes the proof of the second statement. ☐

Proof of Theorem 18. Fix $\alpha_0 \in \mathbb{R}^d$ in the domain of definition of $\mathcal{F}_u$, and let $(\alpha_0 * \gamma_0, \gamma_0)$ be a point in the interior of the domain of definition of $\mathcal{H}_u$ such that

$$\mathcal{H}_u(\alpha_0 * \gamma_0, \gamma_0) = \sup\{\mathcal{H}_u(\alpha_0 * \gamma, \gamma) : \gamma \in \mathbb{R}^d\}.$$

Then there exists a ball $B = B((\alpha_0 * \gamma_0, \gamma_0), r) \subset \mathbb{R}^{2d}$ such that whenever $|\alpha - \alpha_0|$ is sufficiently small the maximum of $\gamma \mapsto \mathcal{H}_u(\alpha * \gamma, \gamma)$ is attained at a point γ such that $(\alpha * \gamma, \gamma) \in B$. Since $\mathcal{H}_u$ is analytic and strictly convex, the maxima will be attained on a smooth d-dimensional submanifold Γ transversal to the family of d-dimensional planes $\{(\alpha * \gamma, \gamma) : \gamma \in \mathbb{R}^d\}$. These planes foliate the space $\mathbb{R}^d$. Furthermore each of them is tangent to the level set $\mathcal{H}_u = \mathcal{F}_u(\alpha)$ and to no other level set (see Figure 2). If $c < \max \mathcal{H}_u$ then the corresponding level set $\mathcal{H}_u = c$ is an analytic compact convex $(2d-1)$-dimensional submanifold. This level set divides the space $\mathbb{R}^{2d}$ into the sets

$$\mathcal{H}_{\text{int}} = \{x \in \mathbb{R}^{2d} : \mathcal{H}_u(x) > c\} \quad \text{and} \quad \mathcal{H}_{\text{ext}} = \{x \in \mathbb{R}^{2d} : \mathcal{H}_u(x) < c\}.$$

Let α_c be such that $\mathcal{F}_u(\alpha_c) = c$. Since the curve Γ is transversal to the plane $\{(\alpha_c * \gamma, \gamma) : \gamma \in \mathbb{R}^d\}$ at the point $(\alpha_c * \gamma_c, \gamma_c)$ of tangency to $\mathcal{H}_c$, any neighborhood of $(\alpha_c * \gamma_c, \gamma_c)$ in Γ contains points in $\mathcal{H}_{\text{int}}$ as well as in $\mathcal{H}_{\text{ext}}$. This yields that there cannot be a local maximum at α_c. This proves the theorem. ☐

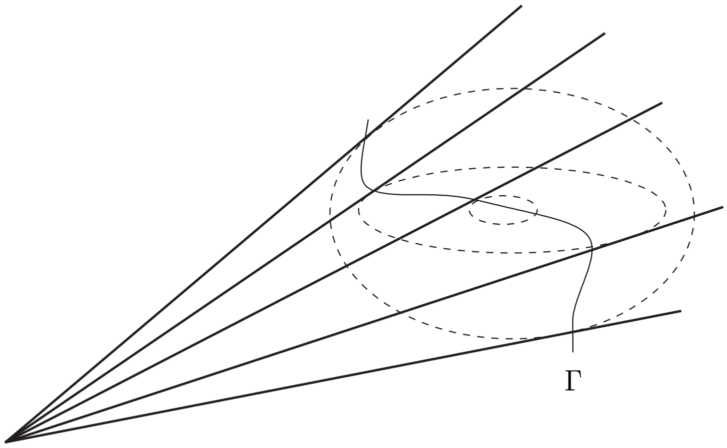

FIGURE 2. Looking for the maxima of mixed spectra. Each plane $\gamma \mapsto (\alpha * \gamma, \gamma)$ is tangent to the level set of $\mathcal{H}_u$ given by $\mathcal{H}_u = \mathcal{F}_u(\alpha)$. The maxima are attained in the curve Γ.

8.5. **Proofs of the results in Sections 1 and 3.** Let μ be a Gibbs measure of the function φ with respect to the dynamical system $f|X$. Without loss of generality we can assume that $P_X(\varphi) = 0$. Then the pointwise dimension and local entropy can be written respectively as

$$d_\mu(x) = \lim_{n\to\infty} -\frac{\sum_{k=0}^{n-1} \varphi(f^k x)}{\sum_{k=0}^{n-1} \log\|d_{f^k x} f\|}$$

and

$$h_\mu(x) = \lim_{n\to\infty} -\frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k x).$$

Furthermore, if f is differentiable and conformal on X, then

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log\|d_{f^k x} f\|.$$

Therefore, the results in Theorems 1–3 and Theorems 5–7 can be reformulated using the notions introduced in Sections 2 and 4.

Proof of Theorem 1. Considering the vectors $\Phi = (\varphi, \psi)$ and $\Psi = (1, 1)$, and the function $u = 1$, the desired statement follows immediately from Theorem 8. □

Proofs of Theorem 2. Proceeding as in the proof of Theorem 1, the desired statements are immediate consequences of Theorems 8, 9, and 13. □

Proof of Theorem 3. The theorem is an immediate consequence of the second statement in Theorem 17. □

Proof of Theorems 5, 6, and 7. Considering $u = 1$ in the case of the topological entropy, and $u = a$, where $a = \log\|df\|$, in the case of the Hausdorff dimension, the desired statements are immediate consequences of Theorems 8, 9, and 13, by making an appropriate choice of the vectors Φ and Ψ in Section 4:

1. $\Phi = -(\varphi_1, \ldots, \varphi_d)$ and $\Psi = (1, \ldots, 1)$ for $\mathcal{E}_d$;
2. $\Phi = -(\varphi_1, \ldots, \varphi_d)$ and $\Psi = (a, \ldots, a)$ for $\mathcal{D}_d$;
3. $\Phi = (-\varphi_1, \ldots, -\varphi_d, a)$ and $\Psi = (1, \ldots, 1, 1)$ for $(\mathcal{E}_d, \mathcal{L})$;
4. $\Phi = (-\varphi_1, \ldots, -\varphi_d, a)$ and $\Psi = (a, \ldots, a, 1)$ for $(\mathcal{D}_d, \mathcal{L})$;
5. $\Phi = -(\varphi_1, \varphi_1)$ and $\Psi = (1, a)$ for $(\mathcal{E}_1, \mathcal{D}_1)$;

This completes the proof of the theorem. □

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