RETURN TIME STATISTICS VIA INDUCING

H. BRUIN, B. SAUSSOL, S. TROUBETZKOY, AND S. VAIENTI

ABSTRACT. We prove that return time statistics of a dynamical system do not change if one passes to an induced (i.e. first return) map. We apply this to show exponential return time statistics in i) smooth interval maps with nowhere–dense critical orbits and ii) certain interval maps with neutral fixed points. The method also applies to iii) certain quadratic maps of the complex plane.

1. INTRODUCTION

In the last few years the study of *return* and *hitting* times has become an important ingredient for the statistical characterization of dynamical systems. A historical account of this approach can be found in the review paper [10] or in the introduction of [20], where an extended bibliography is provided. To pose the problem in general terms, suppose that T is a measure preserving transformation of a measure space (X, μ) and $U_x \subset X$ is a neighborhood of a point $x \in X$. The two questions which are the fundamental objects of investigation are:

1. What is the probability distribution of the first hitting time of the set U_x as $\mu(U_x) \to 0$?
2. What is the probability distribution of the first return time for points leaving from U_x as $\mu(U_x) \to 0$?

For large classes of dynamical systems showing some sort of hyperbolic behavior, the answers to those questions are surprisingly easy and universal. The probability distribution function up to a suitable normalization, turns out to be the exponential 1-law $\exp(-t)$: If $\tau_U(x)$ is the smallest integer n such that $T^n(x) \in U$ (so τ stands for first hitting and for first return time), then both

$$\mu(\{x \in X; \tau_U(x) \geq \frac{t}{\mu(U)}\}) \to e^{-t}$$

2000 *Mathematics Subject Classification.* Primary: 37B20 - Secondary: 37D35, 37F10.
Key words and phrases. return time, exponential statistics.
HB was supported by the Royal Dutch Academy of Sciences (KNAW) and the PRODYN program of the European Science Foundation. Also the hospitality of the Centre Physique Théorique, Luminy, is gratefully acknowledged.
BS was supported by FCT's Funding Program and by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, Instituto Superior Técnico, Lisbon, Portugal.

and

$$\frac{1}{\mu(U)}\mu(\{x \in U; \tau_U(x) \geq \frac{t}{\mu(U)}\}) \to e^{-t},$$

as $\mu(U) \to 0$. After the works of Pitskel [36], Hirata [19] and Collet [11] at the beginning of the nineties which focused essentially on uniformly hyperbolic dynamical systems, new developments have brought at least three improvements:

- The possibility to treat larger classes of systems, notably certain non–uniformly hyperbolic systems;
- The possibility to estimate the error to the asymptotic distribution which is closely related to the hyperbolic character of the transformation and to the influence of sets with short recurrence;
- The application of the exponential statistics to the computation of the fluctuations of repetition times in the Ornstein-Weiss formula for the metric entropy.

These improvements were the result of four new approaches. The first approach was originated by Galves and Schmitt [15] and was formulated in a probabilistic setting for systems satisfying a φ-mixing condition. This approach was translated into the dynamical systems language by Haydn for Julia sets [16], by Paccaut [34] for a large class of non-Markovian maps of the interval and by Boubakri [4] for some Collet-Eckmann unimodal maps. Recently Abadi [2] extended it for α-mixing stationary processes. The true probabilistic flavor of this technique led to the discovery of a close connection with the Ornstein-Weiss [33] theorem on metric entropy and resulted in a proof of the log-normal fluctuations of the repetition times for Gibbs measures in [13] (see [22] for related results); see also [34] for further developments.

The second approach is due to Hirata, Saussol and Vaienti [20]. First for measure-preserving dynamical systems it quantifies the error to the asymptotic $\exp(-t)$ distribution giving precise bounds in terms of the mixing properties of the systems and of a sharp control of short recurrences (this last point is in itself an interesting subject suggesting the possibility to formulate a *thermodynamics* of return times [39, 40]). These bounds are then computed for a large class on non-uniformly hyperbolic maps of the interval in [20]. We used this method in the present paper to prove the exponential statistics for the class of Rychlik maps introduced in Section 3. A key observation in [20] allows to link the statistics of hitting and return times quoted above, namely the distribution of the first return time is close to the exponential law if and only if it is close to the distribution of the first hitting time.

The third approach goes back to a probabilistic paper of Sevast'yanov [41]. This approach was already used by Pitskel [36] in the context of Markov chains and Axiom-A diffeomorphisms. The same scheme was applied by Haydn in [16] and [17] for equilibrium states on Julia sets, in the presence of a supremum gap in the Perron-Frobenius operator (namely, if f is an Hölder continuous function on the Julia set of a rational map of degree at least 2, then $P(f) > \sup f$, where $P(f)$ is the topological pressure of f). The original technique of Sevast'yanov only allowed to prove the convergence to the exponential 1-law. A recent work by Haydn and Vaienti [18] quantifies this technique and thus provides bounds for the error estimate in the case of rational maps and parabolic maps of the interval; such bounds improve all other existing bounds for the class of maps considered.

The last approach has been introduced by Collet [12]; it is an application of the Gumbel's law for entrance times [14] to a large class of non–uniformly hyperbolic dynamical systems with exponential decay of correlations for which a tower satisfying Young's conditions can be constructed [48, 49]. Collet has proven that the statistics of closest return to a given point is almost surely asymptotically Poissonian and gave the fluctuations for the nearest return to the starting point.

It is useful to remark at this point that the approaches we are describing also permit to compute the probability distribution of *successive* visits to the set U_x: it gives, in all the cases where the exponential statistics holds, the Poisson distribution $\frac{t^n}{n!}e^{-t}$, where n is the number of visits of U_x.

In this paper we propose a new scheme which allows us to compute the asymptotic distributions for the first and successive return times whenever these distributions are known on a subset endowed with an induced structure. Its interest lies in the fact that several non-uniformly hyperbolic systems admit around almost every point (w.r.t. the invariant measure) a neighborhood where the first return map acts as the induced transformation and exhibits hyperbolic behavior. The first application of our approach is for C^2 interval maps; we prove the exponential return time statistics under the additional hypothesis that the map preserves a conformal measure (see Section 4 for details) and that the closure of the orbit of the critical points has zero measure. Our result complements those given Boubakri [4] and by Collet [12], in the sense that, contrarily to them, the growth rate of the derivatives along the critical orbit plays no role in our theorem. It applies for example to the well known Fibonacci map [28].

As a second example, we improve the exponential statistics for the parabolic map studied in [20]. In both cases, the first return map of the induced systems belongs to a class of piecewise monotonic maps of the interval (with countably many pieces), previously investigated by Rychlik; the induced measure shows exponential decay of correlations with respect to the Rychlik map and this is the kind of hyperbolic behavior which is sufficient to prove the exponential statistics for such maps.

In most of the approaches quoted above, the set U_x shrinking to $\{x\}$ was chosen in the class of cylinders generated by some partition of the space. The exponential return time statistics around cylinders is what is needed to compute the fluctuations in the Ornstein-Weiss theorem [39].

Instead of cylinders one can also use balls for the sets U_x. The use of balls has a twofold interest. Firstly, it enlarges the class of sets where to check the recurrence of rare events, such as entrance time (particularly from the perspective of numerical simulations in physical situations, see for example [9] and [50]), Second, the behavior of return time in balls is strongly related to other statistical indicators, like local dimension and Lyapunov exponents, as shown in [40].

Balls instead of cylinders were used by Haydn [17] and Collet [12]. In this article we prove exponential statistics for Rychlik maps around balls. This improves the work of [34], who considered a particular class of Rychlik maps (the covering weighted systems introduced in [25]), and only cylinders around points.

The general scheme proposed in this paper could in principle be applied to a wide class of dynamical systems for which the induced transformation enjoys an exponential statistics; we think in particular to rational maps of the Riemann sphere (an example is given in Chapter 5) and of billiards for which the induced structures, although not yet studied, seem however more accessible than the original systems.

As a final observation, we would like to point out the robustness of the exponential statistics, which persists in the non-hyperbolic systems covered in this paper. This confirms its central role in the characterization of recurrence for ergodic systems and motivate further analysis for larger class of transformations especially in dimension greater than one and exhibiting singularities.

2. Statistics via inducing: an abstract theorem

Suppose that (T, X, μ) is an ergodic measure preserving transformation of a smooth Riemannian manifold X. Let $\hat{X} \subset X$ be an open set. Furthermore let $\hat{T} : \hat{X} \to \hat{X}$ be the first return map. We denote the induced measure by $\hat{\mu}$: the measure preserving transformation $(\hat{T}, \hat{X}, \hat{\mu})$ is therefore ergodic.[1] For $z \in X$ we denote by $U_r(z)$ the ball of radius r centered at z and by $\tau_{U_r(z)}$ (resp. $\hat{\tau}_{U_r(z)}$) the first return time of $U_r(z)$ for T (resp. $\hat{T}$). For a μ (resp. $\hat{\mu}$) measurable set A we denote by μ_A (resp. $\hat{\mu}_A$) the induced measure on the set A. We suppose that $(\hat{T}, \hat{X}, \hat{\mu})$ has return time statistics $\hat{f}(t)$: i.e. for $\hat{\mu}$-a.e. $z \in \hat{X}$, $\exists \varepsilon_z(r) > 0$ with $\varepsilon_z(r) \to 0$ as $r \to 0$ such that

$$\sup_{t \geq 0} \left| \hat{\mu}_{U_r(z)} \left(x \in U_r(z) : \hat{\tau}_{U_r(z)}(x) > \frac{t}{\hat{\mu}(U_r(z))} \right) - \hat{f}(t) \right| < \varepsilon_z(r). \tag{1}$$

Our main theorem of this section is that on $\hat{X}$ the map $\hat{T}$ obeys the "same" statistical law as T:

Theorem 2.1. *If $\hat{f}$ is continuous at 0, then there exists $f : \mathbb{R}^+ \to [0,1]$ such that $N \stackrel{\text{def}}{=} \{x : \ f \neq \hat{f}\}$ is countable and for μ–a.e. $z \in \hat{X}$ and $t \notin N$,*

[1] We could weaken our assumptions by demanding that the map $\hat{T}$ is ergodic instead of T; the ergodicity of T is however recovered when $X = \cup_{n=0}^{\infty} T^{-n} \hat{X}$

there exists $\delta_{z,t}(r) > 0$ with $\delta_{z,t}(r) \to 0$ uniformly in t as $r \to 0$ such that

$$\left|\mu_{U_r(z)}\Big(x \in U_r(z) : \tau_{U_r(z)}(x) > \frac{t}{\mu(U_r(z))}\Big) - f(t)\right| < \delta_{z,t}(r). \tag{2}$$

In addition, if $\hat{f}$ is continuous then the convergence is uniform in t, which means there exists $\delta_z(r) \to 0$ as $r \to 0$ such that for all $t \in \mathbb{R}^+$ and $r > 0$, $\delta_{z,t}(r) \leq \delta_z(r)$.

Remark: the function $\hat{f}$ (resp. f) is known to be decreasing and thus continuous everywhere except for an at most countable set of exceptional points. The set N is a subset of the points of discontinuity of $\hat{f}$.

Proof. First of all, if μ has an atom then μ is supported on a periodic orbit, hence the result is trivial because in this case $f(t) = \hat{f}(t) = 1$ for $t \leq 1$ and 0 otherwise. We may assume then that μ has no atoms, consequently $\mu(U_r(z)) \to 0$ as $r \to 0$ for all $z \in \hat{X}$.

At first we suppose that $z \in \hat{X}$ and assume that r is small enough that $U_r(z) \subset \hat{X}$. Note that this implies $\hat{\mu}_{U_r(z)} = \mu_{U_r(z)}$.

For $x \in \hat{X}$ let $n(x) \stackrel{\text{def}}{=} \tau_{\hat{X}}(x)$ be the T–first return time of x to $\hat{X}$. By Kac's Theorem and ergodicity of $\hat{\mu}$ we have

$$A_m(x) \stackrel{\text{def}}{=} \frac{1}{m}\sum_{i=0}^{m-1} n(\hat{T}^i x) \underset{m\to\infty}{\longrightarrow} c \stackrel{\text{def}}{=} \int_{\hat{X}} n(x)\, d\hat{\mu} = \frac{1}{\mu(\hat{X})} \tag{3}$$

for $\hat{\mu}$-a.e. $x \in \hat{X}$. Let $G \stackrel{\text{def}}{=} \{x \in \hat{X} : \lim_{m\to\infty} A_m(x) = c\}$. Clearly $\hat{\mu}(G) = 1$. For all $x \in G$ and for all $\varepsilon > 0$ there exists $m(x,\varepsilon) < \infty$ such that $|A_m(x) - c| < \varepsilon$ for all $m \geq m(x,\varepsilon)$. Let $G_m \stackrel{\text{def}}{=} \{x \in G : m(x,\varepsilon) < m\}$ where we have suppressed the obvious ε dependence on the set G_m. We choose $M \stackrel{\text{def}}{=} M(\varepsilon)$ such that $\mu(G_M) > 1 - \varepsilon$.

By definition $|\sum_{i=0}^{m-1} n(\hat{T}^i x) - cm| < \varepsilon m$ for all $m \geq M$ and all $x \in G_M$. Thus $\hat{T}^m(x) = T^{cm+s}(x)$ for some $s = s(x)$ with $|s| < \varepsilon m$. It immediately follows that $\tau_{U_r(z)}(x) = c\hat{\tau}_{U_r(z)}(x) + s$ with $|s| < \varepsilon\hat{\tau}_{U_r(z)}(x)$ whenever $\hat{\tau}_{U_r(z)}(x) \geq M$ and $x \in G_M$.

Next we define

$$\tilde{G}_M \stackrel{\text{def}}{=} \Big\{z \in G_M : \mu_{U_r(z)}(G_M) \underset{r\to 0}{\longrightarrow} 1\Big\}. \tag{4}$$

By the Lebesgue Density Theorem, $\hat{\mu}(\tilde{G}_M) = \hat{\mu}(G_M)$ and thus $\hat{\mu}(\tilde{G}_M) > 1 - \varepsilon$. For each $z \in G_M$ there exists $r(z,M,\varepsilon) > 0$ such that $\mu_{U_r(z)}(G_M) > 1 - \varepsilon$ for all $r < r(z,M,\varepsilon)$. Thus if $R > 0$ is sufficiently small then $\hat{\mu}(G_{M,R}) > 1 - \varepsilon$, where $G_{M,r} \stackrel{\text{def}}{=} \{z \in \tilde{G}_m : r(z,M,\varepsilon) > r\}$.

Let S be the set of points $z \in \hat{X}$ for which $\varepsilon_z(r) \to 0$ as $r \to 0$. By assumption $\hat{\mu}(S) = 1$. Hence $\hat{\mu}(G_{M,R} \cap S) = \hat{\mu}(G_{M,R}) > 1 - \varepsilon$.

Denote

$$\begin{aligned}\mathcal{F}_{U_r(z)}(t) &\stackrel{\text{def}}{=} \mu_{U_r(z)}(\{x \in U_r(z) : \tau_{U_r(z)} > t/\mu(U_r(z))\}),\\ \hat{\mathcal{F}}_{U_r(z)}(t) &\stackrel{\text{def}}{=} \hat{\mu}_{U_r(z)}(\{x \in U_r(z) : \hat{\tau}_{U_r(z)} > t/\hat{\mu}(U_r(z))\}),\end{aligned}$$

and set $B_{U_r(z)}(M) = \{x \in U_r(z) : \hat{\tau}_{U_r(z)}(x) > M\}$.

Assume additionally that $z \in G_{M,R} \cap S$. The limiting distribution $\hat{f}(t)$ is continuous at 0 and $\hat{f}(0) = 1$, hence if r is sufficiently small

$$\mu_{U_r(z)}(B_{U_r(z)}(M)^c) = \hat{\mu}_{U_r(z)}(B_{U_r(z)}(M)^c) \leq 1 - \hat{f}(M\hat{\mu}(U_r(z))) + \varepsilon_z(r) < \varepsilon.$$

Thus for all $t \in [0, \infty)$,

$$\begin{aligned}\mathcal{F}_{U_r(z)}(t) &\leq \mu_{U_r(z)}\left(G^c_{M,R} \cup B_{U_r(z)}(M)^c\right) +\\ &\quad +\mu_{U_r(z)}\left(G_{M,R} \cap B_{U_r(z)}(M) \cap \left\{x : \tau_{U_r(z)}(x) > \frac{t}{\mu(U_r(z))}\right\}\right)\\ &\leq \mu_{U_r(z)}(G^c_{M,R}) + \mu_{U_r(z)}(B_{U_r(z)}(M)^c) +\\ &\quad +\mu_{U_r(z)}\left(G_{M,R} \cap B_{U_r(z)}(M) \cap \left\{x : \hat{\tau}_{U_r(z)}(x) > \frac{t/(c+\varepsilon)}{\mu(U_r(z))}\right\}\right)\\ &\leq 2\varepsilon + \hat{\mathcal{F}}_{U_r(z)}\left(\frac{t}{1+\varepsilon/c}\right).\end{aligned}$$

A similar computation with $1 - \mathcal{F}_{U_r(z)}(t) = \mu_{U_r(z)}(\tau_{U_r(z)} \leq t/\mu(U_r(z)))$ yields

$$\hat{\mathcal{F}}_{U_r(z)}\left(\frac{t}{1-\varepsilon/c}\right) - 2\varepsilon \leq \mathcal{F}_{U_r(z)}(t) \leq \hat{\mathcal{F}}_{U_r(z)}\left(\frac{t}{1+\varepsilon/c}\right) + 2\varepsilon. \tag{5}$$

For r sufficiently small, $\varepsilon_z(r) < \varepsilon$. Thus inequality (5) implies:

$$-3\varepsilon + \hat{f}\left(\frac{t}{1+\varepsilon/c}\right) \leq \mathcal{F}_{U_r(z)}(t) \leq \hat{f}\left(\frac{t}{1+\varepsilon/c}\right) + 3\varepsilon. \tag{6}$$

Define

$$\delta_t(z) \stackrel{\text{def}}{=} \max\left\{\left|\hat{f}(t) - \hat{f}\left(\frac{t}{1-c^{-1}\varepsilon}\right)\right|, \left|\hat{f}(t) - \hat{f}\left(\frac{t}{1+c^{-1}\varepsilon}\right)\right|\right\}. \tag{7}$$

For each point t of continuity of $\hat{f}$, the function $\delta_t(\varepsilon) \to 0$ as $\varepsilon \to 0$. Combining (6) and (7) yields:

$$|\mathcal{F}_{U_r(z)}(t) - \hat{f}(t)| \leq 3\varepsilon + \delta_t(\varepsilon). \tag{8}$$

We just showed that for any $\varepsilon > 0$, there exists a set $G(\varepsilon)$ with $\hat{\mu}(G(\varepsilon)) > 1 - \varepsilon$ and a real number $R(\varepsilon) > 0$ such that for all $z \in G(\varepsilon)$, Inequality (8) holds whenever $r < R(\varepsilon)$. The conclusion of the theorem for μ-a.e. $z \in \hat{X}$ follows from the remark that μ-almost every $z \in \hat{X}$ is contained in the set of full measure $\cap_{n>0} \cup_{m>n} G(1/m)$.

Finally we want to prove the uniform convergence in t in the case $\hat{f}$ is continuous. Since $\hat{f}(t) \to 0$ as $t \to \infty$, it is uniformly continuous; hence $q(\delta) = \sup_{0 \leq s < t < s+\delta} |\hat{f}(s) - \hat{f}(t)| \to 0$ as $\delta \to 0$.

Moreover, $\hat{f}(t)$ is bounded by $1/t$ by Chebychev's inequality. Hence Equation (7) gives

$$\delta_t(\varepsilon) \leq \min\left(q\left(\frac{t\varepsilon}{c-\varepsilon}\right), 1/t\right) + 3\varepsilon.$$

If $t\varepsilon/(c-\varepsilon) < \sqrt{\varepsilon}$ then $\delta_t(z) \leq q(\sqrt{\varepsilon}) + 3\varepsilon$, while if $t\varepsilon/(c-\varepsilon) \geq \sqrt{\varepsilon}$ then $\delta_t(z) \leq 1/t \leq \sqrt{\varepsilon}/(c-\varepsilon) + 3\varepsilon$. □

Remark: Using the same method it is possible to show the counterpart of Theorem 2.1 for the successive return times and the number of visits.

3. Piecewise monotonic transformations

In this section we show that piecewise monotonic maps of Rychlik's type (even with countably many monotonic pieces) enjoy exponential statistics. Let $X \subset \mathbb{R}$ be a compact set and m a Borel regular probability measure on X. The variation of a function $g : X \to \mathbb{R}$ is defined by

$$\mathrm{Var}\ g \stackrel{\mathrm{def}}{=} \sup\{\sum_{j=0}^{k-1} |g(x_{j+1}) - g(x_j)|\},$$

where the supremum is taken along all finite ordered sequences $(x_j)_{j=1,\dots,k}$ with $x_j \in X$. The norm $\|g\|_{BV} = \sup|g| + \mathrm{Var}\ g$ makes $BV = \{g : X \to \mathbb{R} : \|g\|_{BV} < \infty\}$ into a Banach space. We endow X with the induced topology and denote by $\mathcal{B}(X)$ the Borel σ-algebra of X. We say that $I \subset X$ is an X-interval if there exists some interval $J \subset \mathbb{R}$ such that $I = X \cap J$.

Definition 3.1 ($\mathcal{R}$-map). *Let $T : Y \to X$ be a continuous map, $Y \subset X$ open and dense, and $m(Y) = 1$. Let $S = X \setminus Y$. We call T an* $\mathcal{R}$-map *if the following is true:*

1. *There exists a countable family $\mathcal{Z}$ of closed X-intervals with disjoint interiors (in the topology of X) such that $\cup_{Z\in\mathcal{Z}} Z \supset Y$ and for any $Z \in \mathcal{Z}$ the set $Z \cap S$ consists exactly of the endpoints of Z;*
2. *For any $Z \in \mathcal{Z}$, $T|_{Z\cap Y}$ admits an extension to a homeomorphism from Z to some X-interval;*
3. *There exists a function $g : X \to [0,\infty)$, with $\mathrm{Var}\ g < +\infty$, $g|_S = 0$ such that the operator $P : L^1(m) \to L^1(m)$ defined by*

$$Pf(x) = \sum_{y\in T^{-1}(x)} g(y)f(y)$$

   preserves m. In other words, $m(Pf) = m(f)$ for each $f \in L^1(m)$, that is m is g^{-1}-conformal;
4. *T is expanding: $\sup_{x\in X} g(x) < 1$.*

We first remark that if T is an $\mathcal{R}$-map, then any iterate T^n is also an $\mathcal{R}$-map (see Lemma 2 in [38] and the discussion before for details). Given a weight function g such that $\sup_X g < 1$ and $\mathrm{Var}\ g < \infty$, Liverani, Saussol and

Vaienti have shown in [25] the existence of a conformal measure m which fulfills the hypotheses, under the additional assumption that T is *covering*:

for any interval $I \subset X$, there exists an integer $N > 0$ such that
$$\inf_{x\in Y} P^N \chi_I(x) > 0.$$

Proposition 3.1. *Suppose that $X = [0,1]$, m is Lebesgue measure and T is a piecewise monotonic transformation on X. If*

1. *$T|_{Z\cap X}$ is C^2 for each $Z \in \mathcal{Z}$,*
2. *T is uniformly hyperbolic, i.e. for some $M > 0$, $\inf |(T^M)'| > 1$, the infimum being taken on the subset of X where $(T^M)'$ is defined,*
3. *T has bounded distortion, which means*
$$\sup_{Z\in\mathcal{Z}} \sup_Z \frac{|T''|}{|T'|^2} < \infty \quad \text{and} \quad \sum_{Z\in\mathcal{Z}} \sup_Z \frac{1}{|T'|} < \infty, \tag{9}$$

then T is an $\mathcal{R}$-map with weight function $g = 1/|T'|$ on Y and $g = 0$ on S.

Proof. It is enough to check that the variation of $g = 1/|T'|$ is finite. Since g is C^1 on the interior of $Z \in \mathcal{Z}$, the variation is estimated by
$$\begin{aligned}
\mathrm{Var}\ g &\le \sum_{Z\in\mathcal{Z}} \int_Z |g'(t)|dt + 2\sum_{Z\in\mathcal{Z}} \sup_Z g \\
&\le \sum_{Z\in\mathcal{Z}} \int_Z \frac{|T''(t)|}{|T'(t)|^2} dt + 2\sum_{Z\in\mathcal{Z}} \sup_Z \frac{1}{|T'|} \\
&\le \sup_{Z\in\mathcal{Z}} \sup_Z \frac{|T''|}{|T'|^2} + 2\sum_{Z\in\mathcal{Z}} \sup_Z \frac{1}{|T'|} < \infty.
\end{aligned}$$
□

$\mathcal{R}$-maps possess very strong statistical properties, which we collect below

Theorem 3.1 (Rychlik [38]). *If T is an $\mathcal{R}$-map then there exists an invariant measure μ which is absolutely continuous with respect to the conformal measure m with density $h = \frac{d\mu}{dm} \in BV$. The measures m and μ have no atoms. In addition, there exists a partition (mod m) of X into disjoint open sets $X_{i,j}$, where $i = 1, \dots, k$ and $j = 1, \dots, L_i$ such that each system $(T^{L_i}|_{X_{i,j}}, X_{i,j}, \mu_{X_{i,j}})$ is mixing and has exponential decay of correlations for bounded variation observable, which means that for some $C \in (0,\infty)$ and $\theta \in (0,1)$*
$$\left| \int \varphi \circ T^{nL_i} \psi d\mu_{X_{i,j}} - \int \varphi d\mu_{X_{i,j}} \int \psi d\mu_{X_{i,j}} \right| \le C \|\varphi\|_{L^1(m)} (\|\psi\|_{BV}) \theta^n \tag{10}$$
for any $\varphi \in L^1(m)$ and $\psi \in BV$.

This theorem tells us in particular that X can be partitioned (mod m) into compact sets $X_{i,j}$ where some iterate of T is again an $\mathcal{R}$-map with a mixing measure equivalent to $m|_{X_{i,j}}$ and which has exponential decay of correlations for bounded variation observable.

Theorem 3.2. *Any $\mathcal{R}$-map T with conformal measure m and invariant mixing measure $\mu \ll m$ has exponential return time statistics around balls.*

Proof. In the proof we use the estimate given by Lemma 2.4 in [20] to apply Theorem 2.1 in [20]. We recall the quantities considered there (N is any integer).

$$\begin{aligned}
a_N(U) &\stackrel{\text{def}}{=} \mu_U(\{x : \tau_U(x) \le N\}), \\
b_N(U) &\stackrel{\text{def}}{=} \sup\left\{|\mu_U(T^{-N}V) - \mu(V)| : V \text{ is measurable}\right\}, \\
c(U) &\stackrel{\text{def}}{=} \sup_{k\ge 0} |\mu_U(\{x : \tau_U(x) > k\}) - \mu(\{x : \tau_U(x) > k\})|.
\end{aligned}$$

We denote by h the density of the measure μ with respect to m. Let $U \subset X$ be an interval in X and set $\tau(U) \stackrel{\text{def}}{=} \inf\{\tau_U(x) : x \in U\}$. We now compute an upper bound for a_N and b_N.

Obviously $a_N(U) = \sum_{n=\tau(U)}^{N} \mu_U(\{x : \tau_U(x) = n\})$, and for each $n \ge \tau(U)$

$$\begin{aligned}
\mu_U(\{x : \tau_U(x) = n\}) &\le \frac{1}{\mu(U)} \int 1\!\!1_{T^{-n}U} 1\!\!1_U \, d\mu \\
&= \int \left(\frac{1\!\!1_U}{\mu(U)}\right) \circ T^n \cdot (1\!\!1_U - \mu(U)) d\mu + \mu(U) \\
&\le C \frac{m(U)}{\mu(U)} \, \|1\!\!1_U - \mu(U)\|_{BV} \, \theta^n + \mu(U).
\end{aligned}$$

We used for the last inequality the estimate on decay of correlations (10) given by Theorem 3.1, with $\varphi = 1\!\!1_U/\mu(U)$ and $\psi = 1\!\!1_U - \mu(U)$. Since U is an interval we have $\|1\!\!1_U - \mu(U)\|_{BV} \le 3$, hence the summation on $n = \tau(U), \ldots, N$ gives

$$a_N(U) \le \frac{3C}{1-\theta} \frac{m(U)}{\mu(U)} \theta^{\tau(U)} + N\mu(U). \tag{11}$$

We consider now $b_N(U)$. The decay of correlations given by (10) yields (with $\varphi = 1\!\!1_V$ and $\psi = 1\!\!1_U/\mu(U)$)

$$b_N(U) \le \sup_{V \in \mathcal{B}(X)} C \|1\!\!1_V\|_{L^1(m)} \|1\!\!1_U\|_{BV} \theta^N/\mu(U) \le 3C \frac{\theta^N}{\mu(U)}. \tag{12}$$

Lemma 2.4 in [20] together with (11) and (12) yield (with $N = \frac{2\log\mu(U)}{\log\theta}$)

$$c(U) \le c_1 \frac{m(U)}{\mu(U)} \theta^{\tau(U)} + c_2\mu(U) + c_3\mu(U)|\log\mu(U)|, \tag{13}$$

for some constants c_1, c_2 and c_3 independent of the interval U.

Since μ has no atoms the countable union $W = \cup_{j=0}^{\infty} T^{-j}S$ has zero measure. Hence for μ-a.e. point $z \in X$, the iterates $T^k z$ are well defined and $T^k z \notin S$. One easily sees then that $\tau(U_r(z)) \to +\infty$ as $r \to 0$ provided z is not periodic. Consider the set

$$G = \{z \in X \setminus W : z \text{ is not periodic and } D(z) \stackrel{\text{def}}{=} \overline{\lim_{r\to 0}} \frac{m(U_r(z))}{\mu(U_r(z))} < +\infty\}.$$

The Lebesgue Density Theorem tells us that $D(z)$ is μ-almost everywhere finite because the density $h = \frac{d\mu}{dm} > 0$, μ-a.e. Since μ is aperiodic (it is ergodic with no atoms), we conclude that $\mu(G) = 1$.

Moreover, for all $z \in G$, using (13) we get $c(U_r(z)) \to 0$ as $r \to 0$. We conclude then by Theorem 2.1 in [20] that for μ-almost all $z \in X$,

$$\mu_{U_r(z)}\left(\tau_{U_r(z)} > \frac{t}{\mu(U_r(z))}\right) \underset{r\to 0}{\longrightarrow} \mathrm{e}^{-t}$$

uniformly in $t \in [0, \infty)$. □

4. Statistic of return times for non-hyperbolic interval maps

Here we prove that a large class of interval maps enjoy exponential statistics for the return time. The strategy is to prove that around almost every point, an interval can be found whose first return map is an $\mathcal{R}$-map.

4.1. **Smooth maps with critical points.** Let $T : [0,1] \to [0,1]$ be a C^2 interval map. Denote the critical set by Crit $\stackrel{\text{def}}{=} \{x \in [0,1] : T'(x) = 0\}$. For this class of systems, the existence of invariant probability measures and their statistical properties have been frequently studied by means of induced maps (jump transformations rather than first return maps) and tower constructions. The situation is best understood for unimodal map with exponential growth of the derivatives along the orbit of the critical value. This is known as the *Collet-Eckmann condition*, but a slightly stronger form (involving a slow recurrence condition of the critical point) has been applied by Benedicks & Carleson [3], in which also the prevalence of such behavior in the quadratic family is proven.

Every Collet-Eckmann unimodal map has an absolutely continuous invariant probability measure μ, μ has exponential decay of correlations (w.r.t. to some iterate of the map), satisfies the Central Limit Theorem, and the Perron-Frobenius operator with suitable weights has a spectral gap [47, 21]. An important quantity is the *tail behavior* of the inducing scheme, i.e. the asymptotic behavior of $m(\{x; R(x) > n\})$, where m is the reference measure (Lebesgue) and R the inducing time (playing the role of the first return time in our paper). Collet-Eckmann maps have exponential tail behavior: $m(\{x; R(x) > n\}) \le Ce^{-\alpha n}$ for some $\alpha, C > 0$. In [7] the tail behavior of induced maps over multimodal maps with arbitrary growth rates on the critical orbits is computed. The correlation decays were computed using Young's framework [48].

The multiple return time statistic for Collet-Eckmann maps has been studied (in the Benedicks-Carleson setting) by Boubakri [4]; he finds that they have a Poissonian distribution, and proves additional fluctuation results. A more general approach is taken by Collet [12], but also his result relies essentially on exponential tail behavior of the inducing structure.

The below theorem covers certain maps that are not Collet-Eckmann, and therefore (cf. [32]) have strictly subexponential decay of correlations. One should think of (unimodal) maps with a so-called *persistently recurrent* critical point, the Fibonacci map being the best known example, see [28]. Such maps frequently admit absolutely continuous invariant probabilities [28, 5], but are never Collet-Eckmann, see [6, Theorem 2]. In this sense, Theorem 4.1 complements Boubakri's and Collet's results. The main importance of the theorem is that it shows that, as long as the measure is mixing, exponential, and in fact, any rate of mixing is not needed to have exponential return time statistics.

We will assume that the orbit of the critical set is nowhere dense. This enables us to use a first return map as induced transformation. The only information on the tail behavior we need is that $\sum_n m(\{x; \tau(x) = n\}) < \infty$. This is equivalent to the existence of an invariant probability measure $\mu \ll m$. Furthermore, we state the result for multimodal maps and for $|T'|^t$-conformal measures m_t.

Theorem 4.1. *Let $T : [0,1] \to [0,1]$ be as above. Assume that*

1. *there exists a non-atomic $|T'|^t$-conformal probability measure m_t for some $t > 0$,*
2. *T preserves an ergodic probability measure $\mu \ll m_t$, and*
3. *$m_t(\overline{\mathrm{orb}(\mathrm{Crit})}) = 0$.*

Then $(\mathrm{supp}(\mu), T, \mu)$ has exponential return time statistics.

Remark: A special case is $t = 1$, i.e. $m = m_t$ is Lebesgue measure. A point $c \in \mathrm{Crit}$ is called *non-flat* if there exists an ℓ, $1 < \ell < \infty$, such that $|f(x) - f(c)| = \mathcal{O}(|x - c|^\ell)$. If each critical point is non-flat, then the condition $m(\overline{\mathrm{orb}(\mathrm{Crit})}) = 0$ follows from

$$\mathrm{orb}(\mathrm{Crit}) \text{ is } \begin{cases} \text{nowhere dense} & \text{if } \#\mathrm{Crit} = 1, \\ \text{a minimal set} & \text{if } 2 \le \#\mathrm{Crit} < \infty. \end{cases}$$

This was shown by Martens [29] (see also [30, Theorem V.1.3']) if $\#\mathrm{Crit} = 1$ and by Vargas [45] if $2 \le \#\mathrm{Crit} < \infty$.

The case $t < 1$ comes into view when T has a periodic attractor and the measure m_t is supported on a repelling Cantor set.

Proof. Let $X = \mathrm{supp}(\mu) \setminus \overline{\mathrm{orb}(\mathrm{Crit})}$. Obviously $\mu(X) = 1$. Let x be any recurrent point in X; by the Poincaré Recurrence Theorem, this concerns μ-a.e. $x \in X$.

Let Y be the component of $[0,1] \setminus \overline{\mathrm{orb}(\mathrm{Crit})}$ containing x and n be any integer such that $T^n(x) \in Y$. As $\mathrm{orb}(\mathrm{Crit}) \cap Y = \emptyset$, there exists a neighborhood U of x such that T^n maps U monotonically onto Y. Let $\hat{T}$ be the first return map to U. By taking n sufficiently large, we can ensure that U is compactly contained in Y and that $T^i(\partial U) \cap U = \emptyset$ for all $i \ge 0$.

Let $W \subset U$ be any maximal interval on which $\hat{T}|_W$ is monotone. Since $\mathrm{orb}(\partial U) \cap U = \emptyset$, it follows that $\hat{T} : W \to U$ is onto, and therefore contains a fixed point, say p_W. The derivative $|\hat{T}'(p_W)|$ is uniformly bounded away from 1, see [30, page 268].

Write $\hat{X} = X \cap U$ and $Z = X \cap W$. Let $\mathcal{Z}$ be the partition of $\hat{X}$ into the sets Z. We will show that $\hat{T} : \cup_{Z \in \mathcal{Z}} Z \to \hat{X}$ is an $\mathcal{R}$-map.

Given intervals $I \subset J$, J is said to contain a *δ-scaled neighborhood* of I if both components of $J \setminus I$ have length $\geq \delta|I|$. Let $\delta_0 > 0$ be the distance between ∂Y and ∂X. For each branch $\hat{T}|_W = T^k|_W$ (where $k \stackrel{\text{def}}{=} \tau_U(W)$ is the return time), there is an interval $W' \supset W$ such that T^k maps W' monotonically onto Y. It follows that $T^k(W')$ contains a $\frac{\delta_0}{|U|}$-scaled neighborhood of $T^k(W)$. More precisely, if $x, y \in W$, then $T^k(W')$ contains a $\delta \stackrel{\text{def}}{=} \frac{\delta_0}{|T^k(x) - T^k(y)|}$-scaled neighborhood of $(T^k(x), T^k(y))$. This allows us to use the Koebe Principle:

Proposition 4.1 (Koebe Principle)**.** *Let $W' \supset W$ be intervals and assume that $T^n : W' \to T(W')$ is a C^2 diffeomorphism. If $T^n(W)$ contains a δ-scaled neighborhood of $T^n(W)$, then there exists K such that the distortion*

$$\sup_{s,t \in (x,y) \subset W} \frac{\hat{T}'(s)}{\hat{T}'(t)} \leq K. \tag{14}$$

<u>Remark 1:</u> Historically, the Koebe Principle was proven under the assumption that T has negative Schwarzian derivative, see [30, Chapter III.6]). Recent work of Kozlovski [23], extended to the multimodal case by Van Strien & Vargas [42] shows that the C^2 assumption suffices. The magnitude of $K = K(\delta) = K_0(\frac{1+\delta}{\delta})^2$, where K_0 depends only on the map.

<u>Remark 2:</u> By taking U small, we can choose K as close to 1 as we want. It follows that $\inf_{Z \in \mathcal{Z}} \inf_Z |\hat{T}'(x)| \geq \frac{1}{K} \inf_{Z \in \mathcal{Z}} |\hat{T}'(p_W)| > 1$, i.e. $\hat{T}$ is expanding.

Since m_t is g^{-1}-conformal,

$$m_t(\hat{X}) = \int_Z |\hat{T}'(x)|^t dm_t \leq \sup_Z |\hat{T}'(x)|^t m_t(Z).$$

By (14),

$$\sup_Z g = \sup_Z \frac{1}{|\hat{T}'|^t} \leq K_0^t \inf_Z \frac{1}{|\hat{T}'|^t} \leq K_0^t \frac{m_t(Z)}{m_t(\hat{X})}.$$

This shows that $\sum_Z \sup_Z g \leq K_0^t < \infty$.

Let $I \supset J$ be intervals such that $T|I$ is monotone, and let L and R be the components of $I \setminus J$. If I is sufficiently small, T *expands the cross ratio* if

$$\frac{|T(I)|}{|T(L)|} \frac{|T(J)|}{|T(R)|} \geq \frac{|I|}{|L|} \frac{|J|}{|R|}. \tag{15}$$

This is a consequence of the Koebe Principle. It follows that $|\hat{T}'|^{-\frac{1}{2}}$ is convex on each branch and therefore $\frac{d}{dx}|\hat{T}'|^{-\frac{1}{2}} = -\frac{1}{2}\hat{T}''|\hat{T}'|^{-\frac{3}{2}}$ is nondecreasing. Hence, each branch domain $(a_W, b_W) \stackrel{\text{def}}{=} W$ contains at most one point r_W

at which $\hat{T}''$ changes sign. We get

$$\begin{aligned}
\mathrm{Var}_Z g &\leq \mathrm{Var}_W g \\
&= \int_{a_W}^{r_W} \frac{|\hat{T}''(s)|}{|\hat{T}'(s)|^{1+t}} ds + \int_{r_W}^{b_W} \frac{|\hat{T}''(s)|}{|\hat{T}'(s)|^{1+t}} ds \\
&\leq \sup_W \frac{1}{|\hat{T}'|^t} \left[\left| \log \frac{\hat{T}'(r_W)}{\hat{T}'(a_W)} \right| + \left| \log \frac{\hat{T}'(b_W)}{\hat{T}'(r_W)} \right| \right] \\
&\leq 2K_0^t \log K_0 \sup_Z g.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Var}_{\hat{X}} g &\leq \sum_{Z\in\mathcal{Z}} \mathrm{Var}_Z g + 2 \sum_{Z\in\mathcal{Z}} \sup_Z g \\
&\leq \sum_{Z\in\mathcal{Z}} 2\left(K_0^t \log K_0 + 1\right) \sup_Z g \leq 2K_0^t \left(K_0^t \log K_0 + 1\right) < \infty.
\end{aligned}$$

Thus $\hat{T}$ is a $\mathcal{R}$-map and it satisfies the assumptions of Proposition 3.1. By Theorems 3.2 and 2.1, the conclusion follows because the set of $\hat{X}$ satisfying the above assumptions have full measure. □

In the proof of this theorem we even get that $\mathcal{Z}$ is a Markov partition for the induced map $\hat{T}$ (in fact, each monotone branch is onto). Note that this is much better than what is necessary, since $\mathcal{R}$-maps need not to satisfy this extremely strong topological property. This supports our belief that this method could be in principle applied to much more general systems, especially those with singularities.

4.2. **Maps with neutral fixed points.** Let $\alpha \in (0,1)$ and consider the map T_α defined on $X = [0,1]$ by

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0,1/2), \\ 2x-1 & \text{otherwise.} \end{cases}$$

Let μ_α denote the invariant measure absolutely continuous with respect to Lebesgue (see e.g. [26] for the existence and properties). The system $(X, T_\alpha, \mu_\alpha)$ has exponential return time statistics around cylinders of some naturally associated partition [20]; Here we prove that this is also true if the neighborhoods are balls.

Theorem 4.2. *For any $\alpha \in (0,1)$ the system $([0,1], T_\alpha, \mu_\alpha)$ has exponential return time statistic.*

Proof. Let $\hat{X}_0 \stackrel{\text{def}}{=} (\frac{1}{2}, 1]$ and for $n \geq 1$

$$\hat{X}_n \stackrel{\text{def}}{=} \{x \in X : T^n(x) > \frac{1}{2} \text{ and for } k = 0,1,\ldots,n-1,\ T^k(x) < \frac{1}{2}\}.$$

Fix $n \geq 0$ and let $\hat{X} = \hat{X}_n$. Let $\hat{T} : \hat{X} \to \hat{X}$ be the first return map to $\hat{X}$. We then define a partition of $\hat{X}$ by $\mathcal{Z} = \{Z_p : p = 1,2,\ldots\}$, where $Z_p = \{x \in \hat{X} : \hat{\tau}_{\hat{X}}(x) = p\}$. One easily check that $\mathcal{Z}$ is a partition into intervals, and each branch of $\hat{T} : Z \to \hat{X}$ is monotone and onto. Let m be the normalized Lebesgue measure restricted to $\hat{X}$ and let $g(x) = 1/|\hat{T}'(x)|$ when $x \in \mathrm{int}(Z)$ for some $Z \in \mathcal{Z}$ and $g(x) = 0$ otherwise. In the proof of Proposition 3.3 in [26] it is proved that for some constant K

$$\sup_{x,y \in Z} \frac{\hat{T}'(x)}{\hat{T}'(y)} \le K$$

for all $Z \in \mathcal{Z}$. A straightforward computation shows that

$$\log K \ge \left| \log \frac{\hat{T}'(x)}{\hat{T}'(y)} \right| \ge \left| \int_x^y \frac{\hat{T}''(t)}{\hat{T}'(t)} dt \right|$$

for all $x, y \in Z$ and $Z \in \mathcal{Z}$. Since $\hat{T}$ is increasing and convex on each $Z \in \mathcal{Z}$, we get

$$\int_Z \frac{|\hat{T}''(t)|}{|\hat{T}'(t)|^2} dt \le \sup_Z g \log K.$$

Taking into account that $\hat{T}Z = \hat{X}$ for any $Z \in \mathcal{Z}$, we get

$$\sup_Z g \le K \frac{m(Z)}{m(\hat{T}(Z))} \le K m(Z).$$

Therefore, since g is C^1 in the interior of Z and $g|_{\partial Z} = 0$,

$$\mathrm{Var}\ g \le \sum_{Z \in \mathcal{Z}} \int_Z |g'(t)| dt + 2 \sum_{Z \in \mathcal{Z}} \sup_Z g \le K(2 + \log K) < \infty.$$

Finally, it is obvious that $\sup g < 1$, hence $(\hat{X}, \hat{T}, m)$ is an $\mathcal{R}$-map. By Theorems 3.2 and 2.1 μ_α-almost all points inside $\hat{X}$ have exponential return time statistics, and the conclusion follows since $\cup_{n \ge 0} \hat{X}_n$ has full Lebesgue measure. □

5. Complex quadratic maps

In this section we apply the main framework to certain polynomials on the Riemann sphere $\bar{\mathbb{C}}$. Every rational map T has a $|T'|^t$-conformal measure m_t for some $t \in (0, 2]$, see Sullivan [43]. If T is hyperbolic on the Julia set J, then we can take t equal to the Hausdorff dimension of J, and m_t is equivalent to t-dimensional Hausdorff measure. In general however, m_t can be supported on a proper subset of J; it can even be atomic. This can be an issue if e.g. T has neutral periodic points.

For our results we assume that the orbit of the critical point does not densely fill the Julia set, but we do not require a supremum gap as in e.g. [16].

We start with a complex version of $\mathcal{R}$-map and show (analogous to Theorem 3.2) that under some additional conditions they have exponential return time statistics.

Definition 5.1 (complex Markov maps). *Let $T : Y \to X$ be a continuous map, $Y \subset X$ open subsets of $\bar{\mathbb{C}}$, and $m_t(Y) = 1$ where m_t is a probability measure and $0 < t \le 2$. Let $S = X \setminus Y$. We call T a complex Markov map if the following is true:*

1. *There exists a countable family $\mathcal{Z}$ of pairwise disjoint open discs such that $\cup_{Z\in\mathcal{Z}} Z = Y$;*
2. *For any $Z \in \mathcal{Z}$, $T : Z \to X$ is a conformal diffeomorphism (with $T(Z) = X$), and with bounded distortion:*

$$\sup_{Z\in\mathcal{Z}} \sup_{Z} \frac{|T''|}{|T'|^2} < \infty; \tag{16}$$

3. *The measure m_t is $|T'|^t$-conformal;*
4. *The map T is expanding: $\inf_{y\in Y} |T'(y)| > 1$.*
5. *Let $\mathcal{Z}_k = \mathcal{Z} \vee T^{-1}\mathcal{Z} \vee \cdots \vee T^{-(k-1)}\mathcal{Z}$ is the k-th join of the partition $\mathcal{Z}$. The domains $Z \in \mathcal{Z}_k$ are uniformly* convex-like, *by which we mean*

$$\sup_k \sup_{Z\in\mathcal{Z}_k} \sup_{x\neq y\in Z} \frac{p_Z(x,y)}{|x-y|} < \infty, \tag{17}$$

where $p_Z(x,y)$ denotes infimum of the lengths of the path in Z connecting x and y.

Analogous to Theorem 3.1 one can show that complex Markov maps have an invariant measure with exponential decay of correlations.

Theorem 5.1. *Let T be a complex Markov map as above. There exists an invariant ergodic probability measure μ equivalent to m_t. Moreover (X, T, μ) is mixing with exponential decay of correlations : There exists $C > 0$ and $\rho \in (0, 1)$ such that for any f Lipschitz and $g \in L^1_\mu$*

$$\left| \int f \cdot g \circ T^n d\mu - \int f d\mu \int g d\mu \right| \leq C \|f\|_{Lips} \|g\|_{L^1_\mu} \rho^n, \tag{18}$$

where $\|f\|_{Lips} = \sup |f| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$.

Proof. First let us make the following remark. As T is expanding, a straightforward calculation shows that (16) implies that

$$K := \sup_{n\in\mathbb{N}} \sup_{Z\in\mathcal{Z}_n} \sup_{w\in Z} \frac{|(T^n)''|}{|(T^n)'|^2} < \infty$$

and $\sum_{Z_n\in\mathcal{Z}_n} \sup_Z |(T^n)'|^{-t} < \infty$ uniformly in n. Let P_t be the Perron-Frobenius operator

$$(P_t f)(z) = \sum_{T(y)=z} \frac{f(y)}{|T'(y)|^t}.$$

Let us write $\|f\|_s$ for the seminorm $\sup_{z\neq z'} \frac{|f(z)-f(z')|}{|z-z'|}$. If $\|f\|_{Lips} < \infty$, we have

$$\begin{aligned}
\|P_t^n f\|_s &\leq \sup_{z,z'\in Y} \frac{1}{|z-z'|} |P_t^n(f)(z) - P_t^n(f)(z')| \\
&\leq \sup_{z,z'\in Y} \frac{1}{|z-z'|} \left\{ \sum_y \frac{|f(y)-f(y')|}{|(T^n)'(y)|^t} + \right. \\
&\qquad \left. \sum_{y'} |f(y')| \left| \frac{1}{|(T^n)'(y)|^t} - \frac{1}{|(T^n)'(y')|^t} \right| \right\}
\end{aligned}$$

Here we summed over the pairs y, y' in the same atom $Z_n \in \mathcal{Z}_n$ with $T^n(y) = z$ and $T^n(y') = z'$. As $T : Z \to X$ is onto for every $Z \in \mathcal{Z}$, these pairs are well-defined. The first term in the above expression is bounded by

$$\sum_{Z_n} \sup_{y,y'\in Z_n} \frac{|f(y)-f(y')|}{|y-y'|} \frac{1}{|(T^n)'(y)|^{1+t}} \leq \|f\|_s \|P_t^n \mathbb{1}\|_\infty \sup_{y\in Y} \frac{1}{|(T^n)'(y)|}.$$

Next we use the Mean Value Theorem and (17) to estimate

$$\begin{aligned}
\left| \frac{1}{|(T^n)'(y)|^t} - \frac{1}{|(T^n)'(y')|^t} \right| &\leq \tilde{K} \cdot \left(\frac{1}{|(T^n)'(w)|^t} \right)' |y-y'| \\
&= \frac{\tilde{K}|t|\,|y-y'|}{|(T^n)'(w)|^{t-1}} \cdot \frac{|(T^n)''(w)|}{|(T^n)'(w)|^2},
\end{aligned}$$

for some w. The constant $\tilde{K}$ is an upper bound in (17). This gives for the second term

$$\begin{aligned}
&\sup_{Z\in\mathcal{Z}} \sup_{z,z'\in Z} \sum_{y'} \frac{f(y')}{|z-z'|} \left| \frac{1}{|(T^n)'(y)|^t} - \frac{1}{|(T^n)'(y')|^t} \right| \\
&\quad \leq \|f\|_\infty \sum_{Z_n} \sup_{y,y'\in Z_n} \frac{1}{|y-y'| \cdot |(T^n)'(y)|} \left| \frac{1}{|(T^n)'(y)|^t} - \frac{1}{|(T^n)'(y')|^t} \right| \\
&\quad \leq \|f\|_\infty \sum_{Z_n} \sup_{w\in Z_n} \frac{\tilde{K}|t|}{|(T^n)'(w)|^t} \frac{|(T^n)''(w)|}{|(T^n)'(w)|^2} \\
&\quad \leq 2K\tilde{K} \|f\|_\infty \|P_t^n \mathbb{1}\|_\infty.
\end{aligned}$$

Let $\theta = \sup |T'|^{-1} \in (0,1)$. Then

$$\frac{\|P_t^n f\|_s}{\|P_t^n \mathbb{1}\|_\infty} \leq \theta^n \|f\|_s + 2K\tilde{K}\|f\|_\infty.$$

Obviously $\|P_t^n f\|_\infty \leq \|f\|_\infty \|P^n \mathbb{1}\|_\infty$. Therefore also

$$\frac{\|P_t^n f\|_{Lips}}{\|P_t^n \mathbb{1}\|_\infty} \leq \theta^n \|f\|_{Lips} + (2K\tilde{K}+1)\|f\|_\infty.$$

By construction $\|P_t^n \mathbb{1}\|_\infty = \sum_{Z\in\mathcal{Z}} \sup_Z \frac{1}{|(T^n)'|^t} < \infty$ uniformly in n. This allows us to use the Tulcea-Ionescu & Marinescu theorem, which shows that P_t is a quasicompact operator. Since each branch is onto, 1 is a simple eigenvalue with Lipschitz eigenvector $h > 0$, and is the unique eigenvalue on the unit circle. Consequently, for any Lipschitz function f we have

$$\|P_t^n(fh) - h\int f d\mu\|_{Lips} \leq C_0\|fh\|_{Lips}\rho^n,$$

for some $C_0 > 0$ and $\rho \in (0,1)$. It follows that the correlations between Lipschitz functions f and L^1_μ functions decays exponentially fast: if $g \in L^1_\mu$ we find

$$\begin{aligned}
\left|\int f \cdot g \circ T^n d\mu - \int f d\mu \int g d\mu\right| &= \left|\int (P_t^n(fh) - h\int f d\mu) \cdot g dm_t\right| \\
&\leq \|P_t^n(fh) - h\int f d\mu\|_\infty \int |g| dm_t \\
&\leq C\|f\|_{Lips}\|g\|_{L^1_\mu}\rho^n,
\end{aligned}$$

for $C = C_0\|h\|_{Lips}\|1/h\|_\infty$. □

Theorem 5.2. *Any complex Markov map T as defined above with $|T'|^t$-conformal measure m_t for some $t > 1$ admits an invariant mixing measure μ equivalent to m_t with exponential return time statistics around balls.*

The proof of Theorem 5.2 is similar to that of Theorem 3.2. The additional difficulty is that the decay of correlation is given for Lipschitz function and not characteristic function of balls. The usual way to overcome this problem is to approximate balls by an union of small cylinders. This is the object of the next lemmas. Note that if we restrict ourselves to cylinder sets, then the theorem is valid for all $t > 0$.

As before, let $\mathcal{Z}_k = \mathcal{Z} \vee T^{-1}\mathcal{Z} \vee \cdots \vee T^{-k+1}\mathcal{Z}$ denote the k-dynamical partition and $\mathcal{B}_k$ the σ-algebra generated by $\mathcal{Z}_k$.

Lemma 5.1. *Let T be a complex Markov map. There exists some constants Γ and $\gamma > 0$ such that for any set $A \in \mathcal{B}_k$ and Borel set B we have*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \Gamma\mu(A)\mu(B)\exp(-\gamma n). \tag{19}$$

Proof. We can rewrite (19) as

$$\left|\int h^{-1} \cdot P_t^k(h 1\!\!1_A) \cdot 1\!\!1_B \circ T^n d\mu - \mu(A)\mu(B)\right| \leq \Gamma\mu(A)\mu(B)\exp(-\gamma n).$$

where $h = \frac{d\mu}{dm_t}$ is Lipschitz and bounded away from below (see [46]), and P_t is the Perron-Frobenius operator as defined in the previous proof. Recall also that $K \stackrel{\text{def}}{=} \sup_{k\in\mathbb{N}} \sup_{Z\in\mathcal{Z}_k} \sup_Z \frac{|(T^k)''|}{|(T^k)'|^2} < \infty$. Since $T^k : A \to X$ is one-to-one and onto when $A \in \mathcal{Z}_k$, the above facts imply that

$$\sup_{k\in\mathbb{N}} \sup_{A\in\mathcal{Z}_k} \frac{\|h^{-1}P_t^k(h 1\!\!1_A)\|_{Lips}}{\mu(A)} < \infty.$$

The lemma follows now from (18) in Theorem 5.1 by taking $f = P_t^k(1\!\!1_A)/\mu(A)$ and $g = \mu(A)1\!\!1_A$. □

Given $S \subset \mathbb{C}$ we denote by $\mathcal{B}_k(S)$ the smallest element of the σ-algebra $\mathcal{B}_k$ containing $S \cap J$.

Lemma 5.2. *There exists $\alpha > 0$ such that for any $z \in Y$, $r > 0$ and $k > 0$ we have*

$$\mu\left(\mathcal{B}_k(U_r(z)) \cap \mathcal{B}_k(U_r(z)^c)\right) \leq \exp(-\alpha k), \tag{20}$$

where $U_r(z)$ denotes the ball of radius r about the point z.

Proof. Let us assume for simplicity that $\mathrm{diam}(Y) \leq 1$. We denote by $\mathrm{int}_J S$ the interior of a subset $S \subset J$ in the induced topology. Since T is expanding, there exists $a \in (0,1)$ such that $\mathrm{diam}(Z) < a^k$ for any $Z \in \mathcal{Z}_k$ and integer k. By the Markov property, the bounded distortion and the conformality of the map T we can find some constant $c > 0$ such that the following property holds: For any integer k and cylinder $Z \in \mathcal{Z}_k$ there exists $p_Z \in \mathrm{int}_J Z$ and $r_Z > c \cdot \mathrm{diam}(Z)$ such that $U_{r_Z}(p_Z) \cap \mathrm{int}_J Z' = \emptyset$ for any $Z \neq Z' \in \mathcal{Z}_k$ different from Z. Given $Z \neq Z' \in \mathcal{Z}_k$, we have $d(p_Z, p_{Z'}) > \max(r_Z, r_{Z'})$; in particular, $U_{r_Z/2}(p_Z) \cap U_{r'_Z/2}(p_{Z'}) = \emptyset$.

Let $x \in Y$, $r > 0$ and consider the following partition

$$\mathcal{P} = \{Z \in \mathcal{Z}_k : Z \subset \mathcal{B}_k(U_r(z))\}.$$

Let $\mathcal{P}_n = \{Z \in \mathcal{P} : a^n < \mathrm{diam}(Z) \leq a^{n-1}\}$. Any $Z \in \mathcal{P}_n$ is a subset of the annulus $S_n(z,r) \stackrel{\mathrm{def}}{=} U_{r+a^n}(z) \setminus U_{r-a^n}(z)$. Thus there exists $\mathrm{card}(\mathcal{P}_n)$ disjoint balls or radius at least $a^n/2$ inside $S_n(z,r)$. Since the area of $S_n(z,r)$ is equal to $4\pi r a^n$ when $a^n \leq r$ we get

$$\mathrm{card}(\mathcal{P}_n) \leq \frac{4\pi r a^n}{c^2 a^{2n}/4} = \frac{8\pi}{c^2} r a^{-n} \leq \frac{8\pi}{c^2} a^{-n}.$$

Obviously when $a^n > r$ we also have $\mathrm{card}(\mathcal{P}_n) \leq \frac{8\pi}{c^2} \leq \frac{8\pi}{c^2} a^{-n}$.

Since the measure m_t is $|T'|^t$-conformal and the map itself is conformal and Markov we have for some constant c_1, $m_t(Z) < c_1 \mathrm{diam}(Z)^t$ for any $Z \in \mathcal{Z}_k$ for some k. The previous inequalities imply (observe that $\mathcal{P}_n = \emptyset$ if $n \leq k$ and recall that $t > 1$)

$$\begin{aligned}
\mu\Big(\bigcup_{Z \in \mathcal{P}} Z\Big) &= \sum_{n>k} \mu\Big(\bigcup_{Z \in \mathcal{P}_n} Z\Big) \\
&\leq \sum_{n>k} \max\{\mu(Z) : Z \in \mathcal{P}_n\} \cdot \mathrm{card}(\mathcal{P}_n) \\
&\leq \sum_{n>k} c_1 a^{-t} a^{tn} \frac{8\pi}{c^2} a^{-n} \\
&= \frac{8\pi c_1}{(a - a^t)c^2} a^{(t-1)k}.
\end{aligned}$$

Taking $\alpha \in (0, (1-t)\log a)$ sufficiently small gives the result. $\square$

Proof of Theorem 5.2. The proof closely follows the one of Theorem 3.2, we use the same notation a_N and b_N. Let $U = U_r(z)$ and $k, N \in \mathbb{N}$ to be chosen later on. By Lemma 5.1 and Lemma 5.2

$$\begin{aligned} a_N(U) &\leq \sum_{n=\tau(U)}^{N} \frac{1}{\mu(U)} \mu(\mathcal{B}_n(U) \cap T^{-n}U) \\ &\leq (1+\Gamma) \sum_{n=\tau(U)}^{N} \mu(\mathcal{B}_n(U)) \\ &\leq \frac{1+\Gamma}{1-\exp(-\alpha)} \left[N\mu(U) + \exp(-\alpha\tau(U)) \right]. \end{aligned}$$

Similarly, we get by Lemmas 5.1 and (twice) Lemma 5.2

$$\begin{aligned} b_N(U) &\leq \Gamma\mu(\mathcal{B}_k(U)) \exp[-\gamma(N-k)] + \frac{1}{\mu(U)} \mu(\mathcal{B}_k(U) \cap \mathcal{B}_k(U^c)) \\ &\leq \Gamma \exp[-\gamma(N-k)] + \frac{1}{\mu(U)} \left(\Gamma \exp[-\alpha k - \gamma(N-k)] + \exp[-\alpha k] \right), \end{aligned}$$

for all $k \leq N$. Taking $k = -\frac{2}{\alpha} \log \mu(U)$ and $N = 2k$ gives

$$b_N(U) \leq \Gamma\mu(U)^{\frac{2\gamma}{\alpha}} + \Gamma\mu(U)^{1+\frac{2\gamma}{\alpha}} + \mu(U).$$

For all non periodic points $z \notin \cup_{k \in \mathbb{N}} \partial \mathcal{Z}_k$ we have $\tau(U_r(z)) \to \infty$, which implies $c(U_r(z)) \to 0$ as $r \to 0$. Since this concerns μ-almost all points in J, the theorem is proved. □

We will apply these results to quadratic maps on $\mathbb{C}$. Induced systems have been used before for rational maps, notably by Aaronson et al. [1]. They consider parabolic rational maps (i.e. rational maps whose Julia sets contain no critical point but rationally indifferent periodic points), and establish the existence of an invariant measure $\mu \ll m_t$, where m_t is a t-conformal measure with $t = \mathrm{HD}(J)$. Moreover, μ is finite if and only if $t \cdot \min_p \frac{a(p)+1}{a(p)} > 2$. Here the minimum is taken over all parabolic points p and $a(p)$ is such that $T^q(z) = z + \alpha(z-p)^{a(p)} + \dots$ for the appropriate iterate q. It follows that μ is finite only if $t > 1$, which is the hypothesis in Theorem 5.2. It is to be expected therefore that parabolic rational maps with a finite invariant measure $\mu \ll m_t$ have exponential return time statistics on balls.

Theorem 5.3. *Let $T(z) = z^2 + c$ be a quadratic map on $\mathbb{C}$ such that T is not infinitely renormalizable (see discussion below) and its Julia set J contains no Cremer points. Suppose that also for some $t > 1$*

- *J supports a non-atomic $|T'|^t$-conformal measure m_t,*
- *$m_t(\overline{\mathrm{orb}(\mathrm{Crit})}) = 0$, and*
- *T preserves a probability measure μ equivalent to m_t.*

Then $(\mathrm{supp}(\mu), T, \mu)$ has exponential return time statistics on disks.

The Hausdorff dimension of the Julia set $\mathrm{HD}(J) > 1$ for each parameter c in the Mandelbrot set $\mathcal{M}$ (see Zdunik, [51]); the two exceptions $c = 2$ and $c = 0$ are easy real one-dimensional cases. So assuming that these parameter values allow a t-conformal measure with $t = \mathrm{HD}(J)$, the condition $t > 1$ is no restriction. By continuity, there are many parameters close to $\mathcal{M}$ that give $t > 1$ as well.

Note that we allow T to have parabolic points or Siegel disks. The map T is called *renormalizable* if there exist open disks W_0 and W_1, $0 \in W_0 \subset W_1$, such that $T^n : W_0 \to W_1$ is a two-fold covering map for some $n \geq 2$ and $T^{in}(0) \in W_1$ for all $i \geq 1$. If there are infinitely many integers n such that this is possible, T is *infinitely renormalizable.* We assume that T is not infinitely renormalizable and has no Cremer point to be able to use Yoccoz' puzzle construction. In particular the result that for each $z \in J$, the puzzle pieces $P_n(z)$ containing z shrink to z as $n \to \infty$ is important. For an exposition of Yoccoz' puzzles and the proof of these statements we refer to [31].

Yoccoz' results have been extended to certain infinitely renormalizable polynomials by Lyubich and Levin & van Strien, [27, 24]. For reasons of simplicity, we have not tried to extend Theorem 5.3 to these cases; we prefer to work with a single set of initial puzzle pieces P_0. Prado [37] has shown that in all of the above cases, the conformal measure m_t is ergodic.

Proof. Let $z \in \text{supp}(\mu) \setminus \overline{\text{orb}(\text{Crit})}$ be arbitrary, and let $U \supset V \ni z$ be open disks such that $U \cap \overline{\text{orb}(\text{Crit})} = \emptyset$ and V is compactly contained in U. Let $\log \delta = \text{mod}(U \setminus V) > 0$ be the modulus of $U \setminus V$. Assume also that V is *convex-like* in the sense that $\sup_{x \neq y \in V} p_V(x,y)/|x-y| < \infty$, where p_V is as in (17).

The strategy is to find a subset $\hat{X}$ of V such that $T^n(\partial \hat{X}) \cap \hat{X} = \emptyset$ for all $n \geq 0$, and then we can invoke Theorem 5.2.

If J is a Cantor set, we can assume that ∂V is contained in the Fatou set F, which is the basin of ∞ in this case. Moreover, there are no neutral or stable periodic orbits. Thus each point in F, in particular ∂V, converges to ∞. It follows that $T^n(\partial V)$ intersects V for at most finitely many $n \geq 0$. Let $\hat{X}$ be the component of $V \setminus \cup_{n \geq 0} T^n(\partial V)$ containing z. Then $T^n(\partial \hat{X}) \cap \hat{X} = \emptyset$ for all $n \geq 0$ as required. Since $\hat{X}$ consists of the intersection and difference of at most finitely many, convex-like disks, we find $\sup_{x \neq y \in \hat{X}} p_{\hat{X}}(x,y)/|x-y| \leq C(\hat{X}) < \infty$.

Assume now that J is connected and by Yoccoz' results also locally connected. Let F_i, $i \geq 0$, be the periodic components of the Fatou set, with F_0 the basin of ∞. Since T is a polynomial (with exceptional point ∞), $J = \partial F_0$. There are only finitely many such components, and by Sullivan's Theorem (see e.g. [44]), every $z' \in F$ is eventually mapped into $\cup_i F_i$. We construct a special forward invariant subset G of $\overline{\cup_i F_i}$.

1. Consider the renormalization $T^n : W_0 \to W_1$ of the highest possible period n. (If T is not renormalizable, then we just take $T : \mathbb{C} \to \mathbb{C}$.) It is known that W_1 contains an n-periodic point p with at least two external rays, say A_0 and A_0'. The existence of such external rays

(and the arcs A_i defined later on) is guaranteed by results initiated by Douady, see [35] and references therein.) Let $G_0 = \{z \in \mathbb{C}; |z| > 10\}$.

2. If F_i, $i \geq 1$, contains a stable periodic point, let G_i be a disk compactly contained in F_i such that $T^{per(F_i)}(G_i) \subset G_i$ and orb(Crit $\cap F_i) \subset G_i$. There is at least one $per(F_i)$-periodic point p_i in the boundary of F_i. Let A_i be a smooth compact arc connecting p_i and G_i such that $T^{per(F_i)}(A_i) \subset A_i \cup G_i$. Since $p_i \in \partial F_0$, there is also an external ray A'_i landing at p_i.
3. If F_i contains a parabolic point p_i in its boundary such that each $z \in F_i$ converges to p_i, let $G_i \subset F_i$ be a disk such that $T^{per(F_i)}(G_i) \subset G_i$, orb(Crit) $\cap F_i \subset G_i$, and $\partial G_i \cap \partial F_i = \{p_i\}$. Let $A'_i \subset F_0$ be an external ray landing at p_i.

Let

$$G = \bigcup_{i \geq 0} (G_i \cup \{p_i\} \cup A_i \cup A'_i).$$

Then G is connected, and for some $N \in \mathbb{N}$, $\cup_{j=1}^N T^j G$ is forward invariant. We start a Yoccoz puzzle construction by putting, for $n \geq 0$,

$$P_n = \{\text{components of } \bar{\mathbb{C}} \setminus T^{-n} G\}$$

For each $n \geq 1$, T maps any element of of P_n into an element of P_{n-1}. Using the arguments in [31], one can show that the diameters of the elements $Y \in P_i$ tend to 0 as $i \to \infty$, unless Y eventually intersects a Siegel disk.

We can assume that the point $z \notin \cup_n T^{-n}(G)$. Moreover, since orb(Crit) densely fills the boundary of any Siegel disk, and $m_t(\overline{\text{orb(Crit)}}) = 0$, we can assume that z does not lie on the boundary of a Siegel disk. Find n so large that the element Y of P_n containing z is contained in V. Let $\hat{X} = V \cap Y$. Then $T^n(\partial \hat{X}) \cap \hat{X} = \emptyset$ for all $n \geq 0$. Note also that $T^n(Y)$ is bounded by finitely many smooth curves of ∂G_i, A_i and A'_i. At worst these curves end in a logarithmic spiral, namely as they approach the hyperbolic periodic points p_i. Therefore also $T^n(\hat{X})$ is convex-like, and obviously simply connected. It follows that $\sup_{x \neq y \in \hat{X}} p_{\hat{X}}(x,y)/|x-y| \leq C(\hat{X}) < \infty$.

The rest of the argument works for both J locally connected and J a Cantor set. Let $\hat{T} : \hat{X} \to \hat{X}$ be the first return map to $\hat{X}$. Then $\hat{T}$ is defined on a countable collection $\mathcal{Z}$ of disks Z. The modulus mod$(U \setminus \hat{X}) \geq \log \delta$, and for each branch $\hat{T} = T^\tau : Z \to \hat{X}$ there exists a disk $Z' \supset Z$ such that T^τ maps Z' univalently onto U. It follows from the Koebe $\frac{1}{4}$-Theorem (e.g. [8, Theorem 1.4]) that the distortion of $\hat{T}|_Z$ is uniformly bounded:

$$\sup_{x,y \in Z} \frac{|\hat{T}'(x)|}{|\hat{T}'(y)|} \leq K = K(\delta) \tag{21}$$

More precisely, take $x \in Z$ and let $[x,y] \subset Z$ be a straight arc containing x such that $\hat{T}''/\hat{T}'$ varies little on A. Then

$$\frac{|\hat{T}''(x)|}{|\hat{T}'(x)|} \leq \frac{2}{|y-x|} \left| \int_x^y \frac{\hat{T}''(u)}{\hat{T}(u)} du \right| \leq \frac{2}{|y-x|} \log \frac{|\hat{T}'(y)|}{|\hat{T}(x)|}.$$

To estimate this, let B be the maximal round disk centered at x contained in Z'. Let δ_0 be the radius of B; we have $\delta_0|\hat{T}'(x)| = \mathcal{O}(\delta)$. Define $f(w) = \frac{\hat{T}(x+\delta_0 w)-\hat{T}(x)}{\delta_0 \hat{T}'(x)}$. Then f is a univalent map on the unit disk with $f'(0) = 1$. By Theorem 1.6. of [8], we obtain

$$\begin{aligned}
\frac{|\hat{T}'(y)|}{|\hat{T}'(x)|^2} &= \frac{1}{|\hat{T}'(x)|} \log \frac{|f'(\frac{|y-x|}{\delta_0})|}{|f'(0)|} \\
&\leq \frac{1}{|\hat{T}'(x)|} \left[\log(1+\frac{|y-x|}{\delta_0}) - 3\log(1-\frac{|y-x|}{\delta_0}) \right] \\
&\leq \frac{5|y-x|}{\delta_0 |\hat{T}'(x)|},
\end{aligned}$$

provided y is close to x. Therefore $\frac{|\hat{T}''(x)|}{|\hat{T}'(x)|^2} \leq \frac{10}{\delta_0|\hat{T}'(x)|}$, proving that

$$\sup_{Z\in\mathcal{Z}} \sup_{Z} \frac{|\hat{T}''|}{|\hat{T}'|^2} < \infty. \tag{22}$$

The distortion bound $K(\delta)$ applies by the same argument also to iterates $\hat{T}^n$ of the induced map. Therefore each domain $Z \in \mathcal{Z}_n$ is not much less convex than $\hat{X}$:

$$\sup_k \sup_{Z\in\mathcal{Z}_k} \sup_{x\neq y\in Z} \frac{p_Z(x,y)}{|x-y|} \leq C$$

for some $C = C(\delta, \hat{X}) < \infty$.

We can also assume that V and hence $\hat{X}$ is so small that $\inf_{\cup Z} |\hat{T}'(x)| > 1$. Thus $\hat{T}$ is hyperbolic. Recall that m_t is the $|T'|^t$-conformal measure of T. As $\hat{T}$ is a first return map, it is straightforward to show that $\hat{m}_t = \frac{m_t}{m_t(\hat{X})}$ is a $|\hat{T}'|^t$-conformal map for $\hat{T}$. In fact, $\hat{m}_t$ can also be constructed using Sullivan's techniques [43]. Now we can invoke Theorem 5.2 with the invariant measure $\hat{\mu} = \frac{1}{\mu(\hat{X})}\mu$. □

References

[1] J. Aaronson, M. Denker, M. Urbanski, *Ergodic theory for Markov fibred systems and parabolic rational maps,* Trans. Amer. Math. Soc. **337** (1993) 495–548

[2] M. Abadi, *Exponential approximation for hitting times in mixing stochastic processes,* to appear in Math. Phys. Elec. J.

[3] M. Benedicks, L. Carleson, *On iterations of $x \mapsto 1-ax^2$ on $(-1,1)$,* Ann. Math. **122** (1985) 1–25

[4] A. Boubakri, *Distribution asymptotique des temps d'entrée pour une classe d'application unimodales de l'intervalle,* Thèse Université de Dijon, (1996)

[5] H. Bruin, *Topological conditions for the existence of absorbing Cantor sets,* Trans. Amer. Math. Soc. **350** (1998) 2229–2263

[6] H. Bruin, *Quasi-symmetry of conjugacies between interval maps,* Nonlinearity **9** (1996) 1191–1207

[7] H. Bruin, S. Luzzatto, S. van Strien, *Decay of correlations in one-dimensional dynamics*, Preprint Warwick (1999)

[8] L. Carleson, W. Gamelin, *Complex dynamics,* Springer (1995)

[9] B. V. Chirikov, D. L. Shepelyansky, *Asymptotic statistics of Poincaré recurrences in Hamiltonian systems with divided phase space*, Phys. Rev. Lett., **82**, (1999) 528

[10] Z. Coelho, *Asymptotic laws for symbolic dynamical systems*, Topics in Dynamical Systems and Applications, Eds. F. Blanchard, A. Maass, A. Nogueira, LMS Lect. Notes **279** Cambridge Univ. Press, (2000) 123–165
[11] P. Collet, *Some ergodic properties of maps of the interval*, Dynamical Systems (Temuco, 1991/1992), Travaux en Cours, **Vol. 52**, Hermann Paris, (1996) 55-91
[12] P. Collet, *Statistics of closest return times for some non uniformly hyperbolic systems,* Ergod. Th. & Dyn. Sys. **21** (2001) 401–420
[13] P. Collet, A. Galves, B. Schmitt, *Repetition times for Gibbsian sources*, Nonlinearity, **12**,(1999), 1225–1237
[14] J. Galambos, *The asymptotic theory of extreme order statistics*, Wiley Series in Probability and Mathematical Statistics, New York (1978)
[15] A. Galves, B. Schmitt *Inequalities for hitting time in mixing dynamical systems*, Random Comput. Dynam. **5** (1997) 319–331
[16] N. Haydn, *The distribution of the first return time for rational maps,* J. Statist. Phys. **94** (1999) 1027–1036
[17] N. Haydn, *Statistical properties of equilibrium states for rational maps*, Ergod. Th. & Dyn. Sys., **94**, (1999) 657–699
[18] N. Haydn, S. Vaienti *The limiting distribution and error terms for return time of hyperbolic maps*, in preparation
[19] M. Hirata, *Poisson limit law for Axiom-A diffeomorphisms*, Ergod. Th. & Dyn. Sys., **13**, (1993) 533–556
[20] M. Hirata, B. Saussol, S. Vaienti, *Statistics of return times: a general framework and new applications,* Commun. Math. Phys. **206** (1999) 33–55
[21] G. Keller, T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps,* Commun. Math. Phys. **149** (1992) 31–69
[22] I. Kontoyiannis, *Asymptotic Recurrence and waiting times for stationary processes*, J. Theoret. Probab., **11**, (1998) 795–811
[23] O. Kozlovski, *Structural stability in one-dimensional dynamics* Ph.D. Thesis Amsterdam (1998).
[24] G. Levin, S. van Strien, *Local connectivity of Julia sets of real polynomials,* Stony Brook Preprint **5** (1995)
[25] C. Liverani, B. Saussol, S. Vaienti, *Conformal measure and decay of correlations for covering weighted systems,* Ergod. Th. & Dyn. Sys. **18** (1998) 1399–1420
[26] C. Liverani, B. Saussol, S. Vaienti, *A probabilistic approach to intermittency,* Ergod. Th. & Dyn. Sys. **19** (1999) 671–685
[27] M. Lyubich, *Geometry of quadratic polynomials: moduli, rigidity and local connectivity,* Stony Brook Preprint **9** (1993)
[28] M. Lyubich, J. Milnor, *The Fibonacci unimodal map,* J. Amer. Math. Soc. **6** (1993) 425-457.
[29] M. Martens, *Interval dynamics,* Ph.D. Thesis, Delft (1990)
[30] W. de Melo, S. van Strien, *One-dimensional dynamics,* Springer (1993)
[31] J. Milnor, *Dynamics in one complex variable. Introductory lectures,* Friedr. Vieweg & Sohn, Braunschweig, (1999)
[32] T. Nowicki, D. Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps,* Invent. Math. **132** (1998) 633-680
[33] D. Ornstein, B. Weiss, *Entropy and data compression*, IEEE Trans. Inf. Theory, **39** (1993) 78–83
[34] F. Paccaut, *Statistics of return times for weighted maps of the interval*, Ann. Inst. Henri Poincaré, Proba & Stat **36**, 3 (2000) 339–366
[35] C. Petersen, *On the Pommerenke-Levin-Yoccoz inequality,* Ergod. Th. & Dyn. Sys. **13** (1993), 785–806
[36] B. Pitskel, *Poisson limit law for Markov chains*, Ergod. Th. & Dyn. Sys., **11** (1991) 501–513
[37] E. Prado, *Ergodicity of conformal measures for unimodal polynomials,* Conform. Geom. Dyn. **2** (1998) 29–44
[38] M. Rychlik, *Bounded variation and invariant measures,* Studia Math. **76** (1983) 69–80

[39] B. Saussol, *On fluctuations and the exponential statistics of return times,* Nonlinearity **14** (2001) 179–191
[40] B. Saussol, S. Troubetzkoy, S. Vaienti, *Recurrence, dimensions and Lyapunov exponents,* Journ. Stat. Phys. **106** (2002) to appear.
[41] B. A. Sevast'yanov, *Poisson limit law for a scheme of sums of independent random variable*, Th. Prob. Appl., **17**, (1972) 695–699
[42] S. van Strien, E. Vargas, *Real bounds, ergodicity and negative Schwarzian for multimodal maps*, Preprint (2000) and (2001)
[43] D. Sullivan, *Conformal dynamical systems,* Lect. Notes Math. **1007** (1983) 725–752
[44] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains,* Ann. of Math. **122** (1985) 401–418
[45] E. Vargas, *Measure of minimal sets of polymodal maps,* Ergod. Th. & Dyn. Sys.**16** (1996) 159–178
[46] P. Walters, *Invariant measures and equilibrium states for some mappings which expand distance,* Trans. Amer. Math. Soc., **236**, (1978) 121–153
[47] L.S. Young, *Decay of correlations of certain quadratic maps,* Commun. Math. Phys. **146** (1992) 123–138
[48] L.S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math., **147**, (1998) 585–650,
[49] L.S. Young, *Recurrence times and rates of mixing,* Isr. J. Math. **110** (1999) 153–188
[50] G. M. Zaslavsky, M. K. Tippet, *Connection between recurrence-time statistics and anomalous transport*, Phys. Rev. Lett., **67** (1991) 3251
[51] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps,* Invent. Math. **99** (1990) 627–649

Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, the Netherlands

E-mail address: `bruin@math.rug.nl`

URL: `http://www.math.rug.nl/~bruin`

Laboratoire Amiénois de Mathémathiques Fondamentales et Appliquées, Université de Picardie–Jules Verne, 33 rue St Leu, 80039 Amiens Cedex 1, France

E-mail address: `benoit.saussol@mathinfo.u-picardie.fr`

URL: `http://www.mathinfo.u-picardie.fr/saussol/`

Centre de Physique Théorique et Institut de Mathématiques de Luminy, FRUMAM, Féderation de Recherche des Unites de Mathematiques de Marseille, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: `serge@cpt.univ-mrs.fr`

E-mail address: `troubetz@iml.univ-mrs.fr`

URL: `http://iml.univ-mrs.fr/~troubetz/`

Centre de Physique Théorique, FRUMAM, Féderation de Recherche des Unites de Mathematiques de Marseille, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: `vaienti@cpt.univ-mrs.fr`

URL: `http://www.cpt.univ-mrs.fr/`