PRODUCTS OF NON-STATIONARY RANDOM MATRICES AND MULTIPERIODIC EQUATIONS OF SEVERAL SCALING FACTORS

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Abstract. Let $\beta > 1$ be a real number and $M : \mathbb{R} \to \mathrm{GL}(\mathbb{C}^d)$ be a uniformly almost periodic matrix-valued function. We study the asymptotic behavior of the product

$$P_n(x) = M(\beta^{n-1}x) \cdots M(\beta x)M(x).$$

Under some condition we prove a theorem of Furstenberg-Kesten type for such products of non-stationary random matrices. Theorems of Kingman and Oseledec type are also proved. The obtained results are applied to multiplicative functions defined by commensurable scaling factors. We get a positive answer to a Strichartz conjecture on the asymptotic behavior of such multiperiodic functions. The case where β is a Pisot–Vijayaraghavan number is well studied.

1. Introduction

Kingman's subadditive ergodic heorem was originally proved in 1968 [Ki1, Ki2]. A more recent proof was given by Katznelson and Weiss in 1982 [KW]. It is one of the most important results in ergodic theory. In this paper we consider the following set-up which resembles a dynamical system without invariant measure and try to get results similar to Kingman's Theorem. Let $\beta > 1$ be a positive real number. Let $\{f_n\}$ be a sequence of uniformly almost periodic functions (i.e. in the sense of Bohr, see Section 2.1) defined on the real line $\mathbb{R}$. Suppose the following subadditivity condition is fulfilled

$$f_{n+m}(x) \le f_n(x) + f_m(\beta^n x) \quad \text{for a.e. } x \text{ and all } n, m.$$

where *a.e.* refers to the Lebesgue measure. We would like to study the almost everywhere convergence of $n^{-1}f_n(x)$. Kingman's Theorem applies in the special case where $\beta > 1$ is an integer and the f_n's are periodic. The typical case in our mind is

$$f_n(x) = \log \|M(\beta^{n-1}x) \cdots M(\beta x)M(x)\| \tag{1.1}$$

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where $M : \mathbb{R} \to \mathrm{GL}(\mathbb{C}^d)$ is a matrix-valued uniformly almost periodic function. We will prove that the limit $\lim_{n\to\infty} n^{-1} f_n(x)$ exists almost everywhere (a.e. for short) with respect to the Lebesgue measure under the condition that the $n^{-1}f_n(x)$ have *joint periods* (see Theorem 2.5). As a consequence, an Oseledec type theorem is proved for the matrix products involved in (1.1) (see Theorem 2.9). It is proved that the condition on the existence of joint periods is satisfied when β is a PV-number (see Section 3).

Our consideration is partially motivated by the study of multiperiodic functions, already investigated by Strichartz *et al.* [JRS], Fan and Lau [FL], and Fan [F]. By a *Multiperiodic function* of one real variable we mean any function $F : \mathbb{R} \to \mathbb{R}$ which is a solution of a functional equation of the following form

$$F(\xi) = f_1\left(\frac{\xi}{\rho_1}\right) F\left(\frac{\xi}{\rho_1}\right) + \cdots + f_d\left(\frac{\xi}{\rho_d}\right) F\left(\frac{\xi}{\rho_d}\right)$$

where $d \geq 1$ is an integer; $\rho_1 > 1, \dots, \rho_d > 1$ are d real numbers, called *scaling factors*; $f_1, \dots, f_d$ are d complex valued functions defined on the real line, called *determining functions*. The equation will be called a *multiperiodic equation*.

We will assume that the determining functions f_j are periodic or almost periodic in the sense of Bohr, as is the case in most applications. We will also assume that the scaling factors ρ_j are commensurable in the sense that ρ_j are powers of some real number $\beta > 1$. Without lost of generality, we assume that $\rho_j = \beta^j$ for $1 \leq j \leq d$. Then the multiperiodic equation becomes

$$F(\xi) = f_1\left(\frac{\xi}{\beta}\right) F\left(\frac{\xi}{\beta}\right) + \cdots + f_d\left(\frac{\xi}{\beta^d}\right) F\left(\frac{\xi}{\beta^d}\right) \tag{1.2}$$

As far as we know, there is few work done for the non–commensurable case which is much more difficult.

In the literature, the case where $d = 1$ and $\beta = 2$ (or an arbitrary integer) has been studied, especially in the theory of wavelets [D]. In fact, the scaling function φ of a wavelet satisfies a scaling equation

$$\varphi(x) = \sum a_n \varphi(2x - n).$$

The Fourier transform of φ satisfies a multiperiodic equation of the form (1.2) with only one scaling factor $\beta = 2$ and only one determining function $f_1(x) = f(x) = \frac{1}{2}\sum_n a_n e^{inx}$.

The scaling functions in wavelets constitute a class of functions sharing a kind of similarity. More generally, multiperiodic functions arise as Fourier transforms of self-similar objects such as Bernoulli convolution measures ($d = 1$, $\beta > 1$ being a real number and f being a trigonometric polynomial), inhomogeneous Cantor measures (d may be greater than 1) or more general self-similar measures produced by iterated function systems (see [S]).

In the case of one scaling factor (i.e. $d = 1$, then we write $f_1(x) = f(x)$), the existence of the solution of the multiperiodic equation (1.2) is simple and is assured by the consistency condition $f(0) = 1$ and a regularity condition, say f is Lipschitz continuous. Actually the solution can be written as an infinite product

$$F(x) = \prod_{n=1}^{\infty} f\left(\frac{x}{\beta^n}\right).$$

For the existence of the general equation (1.2), we have

Theorem A. *Let $d \geq 1$. Suppose that the determining functions $f_1, \cdots, f_d$ are Lipschitz continuous, and satisfy the consistency condition*

$$f_1(0) + \cdots + f_d(0) = 1.$$

Suppose furthermore that $f_1(0), \ldots, f_d(0) \in [0, +\infty)$. Then equation (1.2) admits a unique continuous solution F such that $F(0) = 1$.

The proof of this theorem is postponed to Section 4.2.

Our study of equation (1.2) is converted to that of vector valued equations of the form

$$G(x) = M\left(\frac{x}{\beta}\right) G\left(\frac{x}{\beta}\right) \tag{1.3}$$

where $M : \mathbb{R} \to \mathcal{M}_{d\times d}(\mathbb{C})$ is a matrix valued determining function and $G : \mathbb{R} \to \mathbb{R}^d$ is a vector valued unknown function. Matrix products will be involved in the study of equation (1.3), which produces some difficulties. However, equation (1.3) is a simple recursive relation because it contains only one scaling factor. Equation (1.2) is equivalent to equation (1.3) with $M(x)$ and $G(x)$ equal respectively to

$$\begin{pmatrix} f_1(x) & f_2(\frac{x}{\beta}) & \cdots & f_{d-1}(\frac{x}{\beta^{d-2}}) & f_d(\frac{x}{\beta^{d-1}}) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} F(x) \\ F(\frac{x}{\beta}) \\ \vdots \\ F(\frac{x}{\beta^{d-1}}) \end{pmatrix} \tag{1.4}$$

We would like to know the asymptotic behavior at infinity of the solution G. This is a natural question because G often represents Fourier transform of a function (a measure or a distribution) and the asymptotic behavior at infinity describes quantitatively the regularity of the solution. Unfortunately, there is no closed form formula for G in general and the behavior of G is rather complicated, as is shown by the Fourier transform of the Cantor measure ($d = 1$, $\beta = 3$ and $f(\xi) = \cos\xi$).

Following [JRS], we will study the pointwise asymptotic behaviors of

$$h_n(x) := \frac{1}{n} \log |F(\beta^n x)|$$

as $n \to \infty$. We will prove that, under some conditions, the limit $\lim_{n\to\infty} h_n(x)$ exists and is equal to a constant almost everywhere with respect to Lebesgue measure. This answers partially a questions in [JRS]. More precisely, we have the following results, whose proofs are postponed to Section 4.3.

Theorem B. *Suppose that the conditions in Theorem A are satisfied. Furthermore, suppose that the determining functions $f_1, \cdots, f_d$ are either identically zero or strictly positive and 1-periodic, and that $\beta > 1$ is a Pisot* *number. Let F be the solution of the equation* (1.2). *Then there is a constant $\mathcal{L}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log F(\beta^n x) = \mathcal{L} \quad a.e.$$

The constant $\mathcal{L}$ in the theorem is the leading Liapunov exponent of the matrix $M(x)$ above, defined by

$$\mathcal{L}(M) = \inf_{n\geq 1} \frac{1}{n} \mathbb{M} \log \|M(\beta^{n-1}x)M(\beta^{n-2}x)\cdots M(\beta x)M(x)\|.$$

where $\mathbb{M}f$ denotes the Bohr mean of an almost periodic function f (see Section 2.1 below).

Remark 1.1. Setting $\beta^n x = y$ in the above theorem gives a growth rate $F(y) \sim y^{\mathcal{L}/\log\beta}$.

Theorem C. *Suppose that the conditions in Theorem A are satisfied. Furthermore, suppose that the determining functions $f_1, \cdots, f_d$ are 1-periodic, Lipschitz, and that $\beta > 1$ is a Pisot number with maximal conjugate of modulus ρ. Let F be the solution of the equation* (1.2). *If*

$$\sup_x \frac{(1 + |f_1(x)| + \cdots + |f_{d-1}(x/\beta^{d-2})|)(|f_1(x)| + \cdots + |f_d(x/\beta^{d-1})|)}{|f_d(x/\beta^{d-1})|} < \rho^{-1}$$

then there is a constant $\lambda \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{d-1} |F(\beta^{n-j}x)| = \lambda \quad a.e.$$

In this case we do not know if the constant λ equals the leading Lyapunov exponent of the matrix.

Theorem A will be proved in Section 4.2 as a special case of a more general result (Theorem 4.1), Theorem B and Theorem C in Section 4.3. Both Theorem B and Theorem C are consequences of our Kingman's Theorem (Theorem 2.5) and Oseledec's Theorem (Theorem 2.9) which are discussed in Section 2. In Section 3, we prove that the joint period condition required in both Kingman's Theorem and Oseledec' Theorem is satisfied when $\beta > 1$ is a Pisot number.

2. Kingman's Theorem and Oseledec' Theorem

2.1. Total Bohr ergodicity and joint ϵ-period.

Let us first recall the definition of *uniformly almost periodic* functions and some of their properties (see [Bo]). Next we will introduce the notions of total Bohr ergodicity and a joint ϵ-period.

Let f be a real or complex valued function defined on the real line. A number τ is called a translation number of f belonging to $\epsilon \geq 0$ (or an ϵ-period) if

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leq \epsilon.$$

We say that f is a uniformly almost periodic (u.a.p.) function if it is continuous and if for any $\epsilon > 0$ the set of its translation numbers belonging to

ϵ is relatively dense (i.e. there exists a number $\ell > 0$ such that any interval of length ℓ contains at least one such translation number). H. Bohr proved that the space of all u.a.p. functions is a closed sub-algebra of the Banach algebra $C_b(\mathbb{R})$ of bounded continuous functions equipped with the uniform norm and that it is the closure of the space of all (generalized) trigonometric polynomials of the form

$$\sum_{\text{finite}} A_n e^{i\Lambda_n x} \qquad (A_n \in \mathbb{C}, \Lambda_n \in \mathbb{R}).$$

For any u.a.p. function f, as is proved by Bohr, the following limit exists

$$\mathbb{M}f = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)dx.$$

It is called the *Bohr mean* of f. For any locally integrable not necessarily u.a.p. f, we define $\mathbb{M}f$ as the limsup instead of the limit.

Definition 2.1. A sequence of real numbers $(u_n)_{n\geq 0}$ is said to be *totally Bohr ergodic* if for any arithmetic subsequence $(u_{am+b})_{m\geq 0}$ ($a \geq 1, b \geq 0$ being fixed) and for any real $p > 0$, the sequence $(u_{am+b}x)_{m\geq 0}$ is uniformly distributed (modulo p) for almost every $x \in \mathbb{R}$ with respect to the Lebesgue measure.

The following is the main property of totally Bohr ergodic sequences that we will use.

Lemma 2.2. *Suppose that $(u_n)_{n\geq 0}$ is a totally Bohr ergodic sequence. Then for any u.a.p. function f and any integers $a \geq 1, b \geq 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} f(u_{an+b}x) = \mathbb{M}f \qquad a.e.$$

Proof. It is a consequence of the fact that $u_{an+b}x$ is uniformly distributed (modulo p) for almost all x, for any real $p > 0$, the fact that f can be uniformly approximated by trigonometric polynomials and the Weyl criterion. □

Remark 2.3. Suppose that $(u_n)_{n\geq 0}$ is such that $\inf_{n\neq m} |u_n - u_m| > 0$, then the sequence $(u_n x)$ is uniformly distributed for almost every point x [KL]. Consequently, the sequence (u_n) is totally Bohr ergodic. A more special case is $u_n = \beta^n$ with $\beta > 1$. This is the most interesting case for us. On the other hand, no bounded sequence can be totally Bohr ergodic.

Definition 2.4. Let $(F_n)_{n\geq 0}$ be a sequence of u.a.p. functions. Let $\epsilon > 0$ and $N \in \mathbb{N}$. A real number τ is called a joint ϵ-translation number for $(F_n)_{n\geq N}$ if

$$\sup_{n\geq N}\sup_{x\in\mathbb{R}} |F_n(x+\tau) - F_n(x)| \leq \epsilon.$$

If for any $\epsilon > 0$ there exists $N(\epsilon)$ such that such joint ϵ-translation numbers for $(F_n)_{n\geq N(\epsilon)}$ are relatively dense, we say that $(F_n)_{n\geq 0}$ has joint periods.

2.2. **Kingman's Theorem.**

Following ideas of Katznelson and Weiss [KW] we prove the following version of Kingman's Theorem. The difficulty in our case is that we have to deal with an infinite measure space. We are also dealing with non–stationary sequences.

Theorem 2.5. *Let $(u_n)_{n\geq 0}$ be a totally Bohr ergodic sequence of real numbers and $(f_n)_{n\geq 0}$ be a sequence of uniformly almost periodic functions. Suppose*

(i) *The sequence $(n^{-1}f_n)$ has joint periods.*

(ii) *The following subadditivity is fulfilled*

$$f_{n+m}(x) \leq f_n(x) + f_m(u_n x) \quad \textit{for a.e. } x \textit{ and all } n, m.$$

(iii) *For any $n \geq 1$*

$$\sup_m \left(f_m(u_n x) - f_m(x)\right) < \infty \quad \textit{for a.e. } x. \tag{2.1}$$

Then the following limit exists and is a constant

$$\lim_{n\to\infty} \frac{1}{n} f_n(x) = \inf_n \frac{1}{n}\mathbb{M}f_n \quad \textit{for a.e. } x.$$

Proof. The proof is a modification of Katznelson-Weiss' proof [KW]. Without loss of generality we assume that $u_0 = 1$.

Let $\gamma = \inf_n \frac{1}{n}\mathbb{M}f_n$. Let us put

$$f^-(x) = \liminf_{n\to\infty} \frac{1}{n} f_n(x), \qquad f^+(x) = \limsup_{n\to\infty} \frac{1}{n} f_n(x).$$

We remark that the subadditivity implies that $f^\pm(x) \leq f^\pm(u_n x)$ for all $n \in \mathbb{N}$ and a.e. $x \in \mathbb{R}$. In a finite measure space this would imply the invariance a.e. In our case it is the boundedness (2.1) which implies that $f^\pm(u_n x) \leq f^\pm(x)$ a.e., what makes the functions $f^\pm$ invariant in the sense that $f^\pm(x) = f^\pm(u_n x)$ a.e. ($\forall n$).

The first part of the proof, i.e. $f^+ \leq \gamma$ a.e., is simple. We just exploit the fact that any (infinite) arithmetical subsequence of $(u_n x)$ is Bohr-uniform distributed. This provides us with a kind of ergodic theorem. In fact, fix an integer N. For any integer n write $n = mN + r$ with $0 \leq r < N$. We have

$$f_n(x) \leq \sum_{k=0}^{m-1} f_N(u_{kN} x) + f_r(u_{mN} x).$$

The N functions f_r ($r = 0, 1, \cdots, N-1$) being bounded, by Lemma 2.2, this readily implies by the Bohr-uniform distribution of $(u_{kN}x)_{k\geq 0}$ that

$$f^+(x) \leq \lim_{m\to\infty} \frac{1}{mN + r} \sum_{k=0}^{m-1} f_N(u_{kN}(x)) = \frac{1}{N}\mathbb{M}f_N$$

for a.e. x. Hence $f^+(x) \leq \gamma$ for a.e. x.

Next, we want to prove that $f^-(x) \geq \gamma$ for a.e. x. For this we assume that $\gamma > -\infty$, otherwise it is trivially true. Adding to each f_n the constant value $-n\|f_1\|$ creates a new subadditive sequence $\tilde{f}_n := f_n - n\|f_1\|$ with $\tilde{f}_n \leq 0$ ($n \geq 1$), $f^- = \tilde{f}^- + \|f_1\|$ and $\gamma_f = \gamma_{\tilde{f}} + \|f_1\|$. So, we may assume that $f_n \leq 0$ ($n \geq 1$). We can furthermore set $f_1 = 0$ (observe that this will not affect the subbadditivity condition since $f_n \le 0$). Then for any $\Delta > 0$ we truncate the function f_n in the way

$$f_{n,\Delta} = \max(f_n, -n\Delta).$$

Note that the sequence $f_{n,\Delta}$ fulfills the assumptions of the theorem. Note that in this case $\gamma_\Delta \ge -\Delta$ and also $f_\Delta^-(x) \ge -\Delta$ for all x. It is clear that

$$f_\Delta^\pm(x) = \max(f^\pm(x), -\Delta).$$

Assume that we proved the theorem for the sequence $f_{n,\Delta}$ for any Δ. Then we claim that $f_\Delta^-(x) \searrow f^-(x)$ for a.e. x as Δ goes to ∞. In fact, if $f_\Delta^-(x) > -\Delta$ for some Δ then $f_\Delta^-(x) = f^-(x)$ and $\gamma_\Delta = \gamma_f > -\infty$. On the other hand if $f_\Delta^-(x) = -\Delta$ for all Δ then obviously $f^-(x) = -\infty$ and $\gamma_\Delta = \gamma_f = -\infty$. This proves the theorem for the sequence f_n.

now on we assume that $f_1 = 0$, $f_n \le 0$ and f_n is truncated and we skip the subscript Δ. Let $\epsilon > 0$. By the hypothesis (i) on the joint periods, there is an integer $N(\epsilon)$ such that the joint ϵ-translation numbers are relatively dense. For these numbers τ we have

$$\left| \frac{f_n(x+\tau)}{n} - \frac{f_n(x)}{n} \right| \le \epsilon \qquad (\forall x \in \mathbb{R}, \forall n \ge N(\epsilon)) \tag{2.2}$$

Notice that there is no loss of generality to suppose that $N(\epsilon)$ increases as ϵ decreases to 0. We define

$$n_\epsilon(x) = \min \left\{ n \ge N(\epsilon) \colon \frac{1}{n} f_n(x) \le f^-(x) + \epsilon \right\}$$

Let $A_K^\epsilon = \{x \colon n_\epsilon(x) > K\}$. Notice that if $\epsilon' < \epsilon''$, we have $N(\epsilon') \ge N(\epsilon'')$ and $n_{\epsilon'}(x) \ge n_{\epsilon''}(x)$, so that $A_K^{\epsilon''} \subset A_K^{\epsilon'}$ for $K > N(\epsilon')$.

We claim that

Lemma 2.6. *For any $\epsilon > 0$, we have*

$$\lim_{K\to\infty} \hat{\mathbb{M}}(A_K^\epsilon) = 0$$

where $\hat{\mathbb{M}}A = \mathbb{M}1_A$ denotes the Bohr mean of the characteristic function of the set A (defined if necessary with the limsup).

In order to prove this Lemma 2.6 we need the following lemma which says that A_K^ϵ is to some extent periodic.

Lemma 2.7. *For any joint ϵ-translation number τ of $(f_n/n)_{n \ge N(\epsilon)}$, we have $A_K^{2\epsilon} + \tau \subset A_K^\epsilon$.*

Let us first prove Lemma 2.7. Suppose $x \in A_K^{2\epsilon} + \tau$, i.e. $x - \tau \in A_K^{2\epsilon}$, then

$$\frac{f_n(x-\tau)}{n} > f^-(x-\tau) + 2\epsilon \qquad (N(2\epsilon) \le n \le K).$$

This, together with the fact that τ is a joint ϵ-translation number for all f_n/n with $n \ge N(\epsilon) (\ge N(2\epsilon))$ (see (2.2)), implies

$$\frac{f_n(x)}{n} \ge \frac{f_n(x-\tau)}{n} - \epsilon > f^-(x-\tau) + \epsilon \qquad (N(\epsilon) \le n \le K).$$

That means $x \in A_K^\epsilon$. Thus we have finished the proof of Lemma 2.7.

Now let us prove Lemma 2.6. Since joint $\frac{\epsilon}{2}$-translation numbers are relatively dense there exists $L = L(\frac{\epsilon}{2}) > 0$ such that any interval of length L

contains such a joint $\frac{\epsilon}{2}$-translation number. Since $\cap_K A_K^{\frac{\epsilon}{2}} = \emptyset$, for any $\eta > 0$ there exists $K_0 > 0$ such that

$$\left| A_K^{\frac{\epsilon}{2}} \bigcap [-L, L] \right| < L\eta \qquad (\forall K \geq K_0) \tag{2.3}$$

where $|\cdot|$ denotes the Lebesgue measure (see the definition of $n_\epsilon(x)$). We claim that $\hat{\mathbb{M}}(A_K^\epsilon) \leq \eta$ $(\forall K \geq K_0)$. Otherwise $\hat{\mathbb{M}}(A_K^\epsilon) > \eta$ for some $K \geq K_0$. Then by the definition of $\hat{\mathbb{M}}(A_K^\epsilon)$ there exists $x_0 \in \mathbb{R}$ such that

$$\int_{x_0}^{x_0+L} \chi_{A_K^\epsilon}(x)dx \geq L\eta.$$

Take a joint $\frac{\epsilon}{2}$-translation number $\tau \in [-x_0 - L, -x_0]$, i.e. $-L \leq x_0 + \tau \leq 0$. Then by Lemma 2.7, we have

$$\begin{aligned}
\left| A_K^{\frac{\epsilon}{2}} \bigcap [-L, L] \right| &\geq \int_{x_0+\tau}^{x_0+\tau+L} \chi_{A_K^{\frac{\epsilon}{2}}}(x)dx \\
&= \int_{x_0}^{x_0+L} \chi_{A_K^{\frac{\epsilon}{2}}}(y+\tau)dy \\
&\geq \int_{x_0}^{x_0+L} \chi_{A_K^\epsilon}(y)dy
\end{aligned}$$

For the first inequality we have used the fact that $[x_0+\tau, x_0+\tau+L] \subset [-L, L]$ and for the last inequality we have used Lemma 2.7. What we have deduced contradicts (2.3). Thus Lemma 2.6 is proved.

We continue our proof of Theorem 2.5. Let $S := \|f^-\|_\infty < \infty$. Let K be such that $\hat{\mathbb{M}}(A_K^\epsilon) \leq \epsilon/S$. We define first

$$g(x) = \begin{cases} f^-(x) & \text{if } x \notin A_K^\epsilon \\ 0 & \text{if } x \in A_K^\epsilon \end{cases} \quad \text{and} \quad m(x) = \begin{cases} n^\epsilon(x) & \text{if } x \notin A_K^\epsilon \\ 1 & \text{if } x \in A_K^\epsilon \end{cases}$$

Lemma 2.6 implies that

$$\mathbb{M}g \leq \mathbb{M}f^- + \epsilon \quad \text{a.e..} \tag{2.4}$$

Moreover by the invariance of f^- we have (remember that $u_0 = 1$)

$$g(x) \leq g(u_k x) \quad \text{for a.e. } x \text{ and all } 0 \leq k \leq m(x) - 1. \tag{2.5}$$

Then we have

$$f_{m(x)}(x) \leq (g(x) + \epsilon)m(x) \leq \sum_{k=0}^{m(x)-1} g(u_k x) + \epsilon m(x). \tag{2.6}$$

We define inductively $m_0(x) = 0$ and

$$m_k(x) = m_{k-1}(x) + m(u_{m_{k-1}(x)}x).$$

Now choose $R > K$ and let $k(x)$ be the maximal k for which $m_k(x) \leq R$. Note that $R - m_{k(x)}(x) < K$. Now we get by the subadditivity and Equation

(2.6)

$$\begin{aligned}
f_R(x) &\le \sum_{k=0}^{k(x)-1} f_{m(u_{m_k(x)}x)}(u_{m_k(x)}x) + \underbrace{f_{R-m_{k(x)}(x)}(u_{m_{k(x)}(x)}x)}_{\le 0} \\
&\le \sum_{k=0}^{k(x)-1} \sum_{j=m_{k-1}(x)}^{m_k(x)} g(u_j x) + (m_k(x) - m_{k-1}(x))\epsilon \\
&\le \sum_{j=0}^{m_{k(x)}(x)} g(u_j x) + m_{k(x)}(x)\epsilon \\
&\le \sum_{j=0}^{R-1} g(u_j x) + R\epsilon + KS.
\end{aligned}$$

Taking the Bohr mean, using that $\mathbb{M}g = \mathbb{M}(g \circ u_j)$ and dividing by R gives

$$\begin{aligned}
\frac{1}{R}\mathbb{M}f_R &\le \mathbb{M}g + \epsilon + \frac{KS}{R} \\
&\le \mathbb{M}f^- + 2\epsilon + \frac{KS}{R}
\end{aligned}$$

by Equation (2.4). Now we let $R \to \infty$ and we get

$$\gamma \le \mathbb{M}f^-.$$

We claim that $f^- \le \gamma$ implies $f^- = \gamma$ for a.e. x. Suppose this was not the case, then one could find $\epsilon > 0$, $\delta > 0$ and an interval $J = (0, L)$ of length $|J| = L = L_\epsilon$ such that $|A \cap J| = \delta > 0$, where

$$A = \{x \colon f^-(x) < \gamma - \epsilon\}$$

By the invariance of f^- we have $u_k A = A$ for all $k \in \mathbb{N}$. Hence, for all $k \in \mathbb{N}$

$$\frac{1}{u_k L}\int_0^{u_k L} \chi_A \, dx = \frac{1}{L}\int_0^L \chi_A \, dx > \frac{\delta}{L} > 0.$$

Since $\limsup_k u_k = +\infty$ (see remark 2.3), we have $\hat{\mathbb{M}}A > \frac{\delta}{L}$, and thus

$$\underline{\mathbb{M}}f^- < \gamma\left(1 - \frac{\delta}{L}\right) + (\gamma - \epsilon)\frac{\delta}{L} < \gamma.$$

$\square$

Remark 2.8. One can prove a similar theorem for more general sequences $(u_n(x))$. In this case it seems to be necessary to assume L^1 Bohr-uniform distribution.

2.3. **Oseledec' Theorem.**

Kingman's Theorem implies the following Oseledec type theorem (see Ruelle [Ru]).

Theorem 2.9. *Let $\beta > 1$ be a real number. Let $M \colon \mathbb{R} \to GL_d(\mathbb{C})$ be a uniformly almost periodic function. Write*

$$M_x^n = M(\beta^{n-1}x) \cdots M(\beta x)M(x).$$

Suppose the q-exterior products $\frac{1}{n}\log\|(M_x^n)^{\wedge q}\|$ *have joint periods, for* $q = 1,\dots,d$. *Then there is* $\Gamma \subset \mathbb{R}$ *with* $\beta\Gamma \subset \Gamma$ *of full Lebesgue measure (in the sense that* $\mathbb{R}\setminus\Gamma$ *has 0 measure) such that if* $x \in \Gamma$ *then*

a) $\lim_{n\to\infty}(M_x^{n*}M_x^n)^{\frac{1}{2n}} = \Lambda_x$ *exists.*

b) Let $\exp\lambda_x^{(1)} < \cdots < \exp\lambda_x^{(s)}$ *be the eigenvalues of* Λ_x *(where* $s = s(x)$ *and the* $\lambda_x^{(r)}$ *are reals), and* $U_x^{(1)},\dots,U_x^{(s)}$ *the corresponding eigenspaces. Let* $m_x^{(r)} = \dim U_x^{(r)}$. *We have* $\lambda_{\beta x}^{(r)} = \lambda_x^{(r)}$ *and* $m_{\beta x}^{(r)} = m_x^{(r)}$ *and*

$$\lim_{n\to\infty}\frac{1}{n}\log\|M_x^n v\| = \lambda_x^{(r)} \quad \textit{when} \quad v \in V_x^{(r)}\setminus V_x^{(r-1)}$$

for $r = 0,\dots,s$ *where* $V_x^{(0)} = \{0\}$ *and* $V_x^{(r)} = U_x^{(1)} + \cdots + U_x^{(r)}$.

c) Moreover $V_x^{(r)}$ *depends measurably on* x *and* $M_x V_x^{(r)} = V_{\beta x}^{(r)}$.

d) In addition the functions $\lambda_x^{(r)}$ *and* $m_x^{(r)}$ *are constant a.e.*

Proof. This theorem follows in the standard way from the a.e. convergence of

$$\lim_{n\to\infty}\frac{1}{n}\log\|\,(M_x^n)^{\wedge q}\,\|$$

for $1 \le q \le d$, which in the classical case is insured by Kingman's Theorem. So we only need to check that for the functions $f_n^{(q)}(x) = \log\|\,(M_x^n)^{\wedge q}\,\|$ and the sequence $u_n = \beta^n$ the assumptions of Theorem 2.5 hold.

First we note that M^{-1} is uniformly almost periodic because M is uniformly almost periodic and $M^{-1}(x) \in GL_d(\mathbb{C})$. Second we note that $M^{\wedge q}$ and $(M^{\wedge q})^{-1}$ are again uniformly almost periodic, since each entry is a rational function of the entries of M and M^{-1}, respectively. Hence,

$$\sup_{x\in\mathbb{R}}\|M(x)^{\wedge q}\| + \sup_{x\in\mathbb{R}}\|(M^{\wedge q})^{-1}(x)\| = W_q < \infty.$$

Subadditivity is obviously fulfilled since $(M_x^{n+m})^{\wedge q} = (M_{\beta^n x}^m)^{\wedge q}(M_x^n)^{\wedge q}$.

Condition (2.1) follows from

$$\|(M_{\beta x}^n)^{\wedge q}\| = \|M(\beta^n x)^{\wedge q}(M_x^n)^{\wedge q}(M(x)^{\wedge q})^{-1}\| \le \|(M_x^n)^{\wedge q}\| + W_q.$$

Finally by Remark 2.3 the sequence β^n is totally Bohr ergodic, so Theorem 2.5 applies. Assertions (a), (b) and (c) follows from Proposition 1.3 (see also the proof of Theorem 1.6) in [Ru].

Now we prove d). By Kingman's Theorem (Theorem 2.5), we have for almost all x

$$\lim_{n\to\infty}\frac{1}{n}\log\|(M_x^n)^{\wedge k}\| = \inf_{n\ge 1}\frac{1}{n}\mathbb{M}\log\|(M_x^n)^{\wedge k}\| \qquad (1 \le k \le s). \tag{2.1}$$

On the other hand, by the properties of exterior product, if we write $k = \sum_{i=1}^{j-1} m_x^{(s-i)} + \ell$ with $0 \le j < s$ and $0 \le \ell \le m_x^{(s-j)}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log\|(M_x^n)^{\wedge k}\| = m_x^{(s)}\lambda_x^{(s)} + \cdots + m_x^{(s-j+1)}\lambda_x^{(s-j+1)} + \ell\lambda_x^{(s-j)}. \tag{2.2}$$

We can solve $\lambda_x^{(r)}$ $(1 \le r \le s)$ from the system (2.1)-(2.2). The solution is independent of x since it depends only on the right hand side terms in (2.1). Consequently $m_x^{(r)}$ is also independent of x. $\square$

3. When β is a Pisot–Vijayaraghavan number

We restrict our attention to the special case where $\beta > 1$ is a Pisot–Vijayaraghavan (PV) number and f_n are defined by (1.1). We will prove that, under some extra condition, the sequence $n^{-1}f_n$ has joint periods and the Kingman's Theorem and the Oseledec' Theorem apply. To do this, we need a distortion lemma and some properties of PV-numbers.

3.1. **Distortion lemmas.**

Lemma 3.1. *Let $M\colon \mathbb{R} \to GL_d(\mathbb{C})$ such that*

$$D := \sup_{x\in\mathbb{R}} \|M(x)\| \ \|M(x)^{-1}\| < \infty \tag{3.3}$$

Let (x_k) and (y_k) be two sequences in $\mathbb{R}$ and let $\theta_k = \|M(x_k) - M(y_k)\|$. Then for all $n \in \mathbb{N}$ and $0 \neq \mathbf{v} \in \mathbb{C}^d$ we have

$$\frac{|M(x_1)\cdots M(x_{n-1})M(x_n)\mathbf{v}|}{|M(y_1)\cdots M(y_{n-1})M(y_n)\mathbf{v}|} \leq \exp\left(C\sum_{k=1}^{n} D^k\theta_k\right).$$

where $C = (\sup_{x\in\mathbb{R}} \|M(x)\|)^{-1}$.

Proof. Let

$$\begin{array}{rclll} Q_1(x) & = & \mathbb{I}, & Q_k(x) = M(x_1)M(x_2)\cdots M(x_{k-1}) & (1 < k \leq n) \\ Q^n(x) & = & \mathbb{I}, & Q^k(x) = M(x_{k+1})M(x_{k+2})\cdots M(x_n) & (1 \leq k < n). \end{array}$$

We can write

$$\frac{|Q_n(x)\mathbf{v}|}{|Q_n(y)\mathbf{v}|} = \prod_{k=1}^{n} \frac{|Q_k(y)M(x_k)Q^k(x)\mathbf{v}|}{|Q_k(y)M(y_k)Q^k(x)\mathbf{v}|}. \tag{3.4}$$

Setting $E_k = M(x_k)\cdot M^{-1}(y_k) - \mathbb{I}$ we get the following estimate for the numerator of the general term in the above product

$$\begin{aligned} |Q_k(y)M(x_k)Q^k(x)\mathbf{v}| &= |Q_k(x)(\mathbb{I}+E_k)M(y_k)Q^k(x)\mathbf{v}| \\ &= |Q_k(x)(\mathbb{I}+E_k)Q_k(x)^{-1}Q_k(x)M(y_k)Q^k(x)\mathbf{v}| \\ &= |(\mathbb{I}+\tilde{E}_k)Q_k(x)M(y_k)Q^k(x)\mathbf{v}| \end{aligned}$$

where $\tilde{E}_k = Q_k(x)E_kQ_k(x)^{-1}$. It follows that

$$\frac{|Q_k(y)M(x_k)Q^k(x)\mathbf{v}|}{|Q_k(y)M(y_k)Q^k(x)\mathbf{v}|} \leq \|\mathbb{I}+\tilde{E}_k\|. \tag{3.5}$$

It is obvious that

$$\|\tilde{E}_k\| \leq D^{k-1}\|E_k\|. \tag{3.6}$$

On the other hand

$$\|E_k\| \leq \sup_x \|M(x)^{-1}\|\theta_k. \tag{3.7}$$

By combining (3.4), (3.5), (3.6) and (3.7), we obtain

$$\frac{|M(x_1)\cdots M(x_{n-1})M(x_n)\mathbf{v}|}{|M(y_1)\cdots M(y_{n-1})M(y_n)\mathbf{v}|} \leq \prod_{k=0}^{n-1}\left(1+CD^k\theta_k\right).$$

□

If $M(x)$ is non-negative, the next lemma shows that condition (3.3) is not needed for positive vectors

Lemma 3.2. *Let $M\colon \mathbb{R} \to GL_d(\mathbb{R})$ be such that the entries of $M(x)$ are either identically zero or bounded from below by a positive number $\delta > 0$ (independent of entries). Then for any sequences (x_k) and (y_k) in $\mathbb{R}$ and for any non-negative vector $\mathbf{v}$ we have*

$$\frac{|M(x_1)\cdots M(x_{n-1})M(x_n)\mathbf{v}|}{|M(y_1)\cdots M(y_{n-1})M(y_n)\mathbf{v}|} \le \exp\left(\frac{1}{\delta}\sum_{k=1}^{n}\theta_k\right),$$

where $\theta_k = \|M(x_k) - M(y_k)\|$ and the norm $|v| = \sum_{i=1}^{d}|v_i|$ on $\mathbb{R}^d$ is chosen.

Proof. We may write

$$\begin{aligned} & |M(x_1)\cdots M(x_{n-1})M(x_n)\mathbf{v}| \\ = & \sum_{i_0,i_1\cdots,i_n} M(x_1)_{i_0,i_1}M(x_2)_{i_1,i_2}\cdots M(x_n)_{i_{n-1},i_n}v_{i_n} \end{aligned}$$

We have a similar expression for $|M(y_1)\cdots M(y_{n-1})M(y_n)\mathbf{v}|$. Now compare the two expressions term by term. By the hypothesis, both quantities $M(x_1)_{i_0,i_1}$ and $M(y_1)_{i_0,i_1}$ are either zero or larger than δ. So, using the trivial inequality $x/y \le e^{x/y-1}$ we have

$$M(x_1)_{i_0,i_1} \le M(y_1)_{i_0,i_1}e^{\delta^{-1}\theta_k}.$$

The same estimates hold for other pairs $M(x_k)_{i_{k-1},i_k}$ and $M(y_k)_{i_{k-1},i_k}$. The desired inequality follows. $\square$

3.2. **Two properties of PV–numbers.**

Let $\beta > 1$ be a PV–number of order r. We denote its conjugates by $\beta'_1, \cdots, \beta'_{r-1}$. Then for $n \ge 1$, denote

$$F_n = \beta^n + \beta'^n_1 + \cdots + \beta'^n_{r-1}.$$

Lemma 3.3. *The number F_n is an integer and we have*

$$|\beta^n - F_n| \le (r-1)\rho^n \qquad (\forall n \ge 1)$$

where $\rho = \max_{1\le j\le r-1}|\beta'_j| < 1$.

Given any real number $\beta > 1$ (not necessarily integral), we can expand each number $x \in [0,1)$ in a canonical way into its *β-expansion* [Re] (see also [P, Bl]):

$$x = \sum_{n=1}^{\infty}\frac{\epsilon_n(x)}{\beta^n}$$

where $(\epsilon_n(x))_{n\ge1}$ is a uniquely determined sequence in $\{0,1,\cdots,[\beta]\}^{\mathbb{N}}$. We may also call $(\epsilon_n(x))_{n\ge1}$ the β-expansion of x. We note that not all sequences in $\{0,1,\cdots,[\beta]\}^{\mathbb{N}}$ are β-expansions. Let D_β be the set of all possible β-expansions of numbers in $[0,1)$. A finite sequence $\epsilon = (\epsilon_1,\cdots,\epsilon_n)$ (of length n) in $\{0,1,\cdots,[\beta]\}^n$ is said to be *admissible* if it is the prefix of the β-expansion of some number x. For such an admissible sequence, we define

$$I(\epsilon_1,\cdots,\epsilon_n) = \{x \in [0,1) : \epsilon_1(x) = \epsilon_1,\cdots,\epsilon_n(x) = \epsilon_n\}.$$

It is known that if D_β is endowed with the lexicographical order, the map which associates x to its β-expansion is strictly increasing. The set $I(\epsilon) = I(\epsilon_1,\cdots,\epsilon_n)$ is an interval, called a *β-interval* of level n. Its length is denoted by $|I(\epsilon)|$.

Lemma 3.4. *Suppose $\beta > 1$ is a PV–number. There is a constant $C > 0$ such that*

$$C^{-1}\beta^{-n} \le |I(\epsilon_1, \cdots, \epsilon_n)| \le C\beta^{-n}$$

for any integer $n \ge 1$ and any β-interval $I(\epsilon_1, \cdots, \epsilon_n)$.

See [F] for proofs of Lemma 3.3 and Lemma 3.4.

3.3. **Existence of joint periods.**

Definition 3.5. Let $\beta > 1$ be a positive real number and let $M : \mathbb{R} \to \mathcal{M}_{d\times d}(\mathbb{C})$. If the entries of M are functions of the form $f(\beta^n x)$ $(n \in \mathbb{Z})$ where f is 1–periodic continuous, we say that M is *β–adapted u.a.p.*

Remark 3.6. The matrix $M(x)$ defined by (1.4) associated to a multiperiodic function is β–adapted.

Proposition 3.7. *Let $\beta > 1$ be a PV–number and let $M : \mathbb{R} \to GL_d(\mathbb{C})$ be β–adapted and α–Hölder continuous. Suppose that*

$$D\rho^\alpha < 1 \tag{3.8}$$

where ρ is the maximal modulus of the conjugates of β and D is the same as in the distortion lemma (Lemma 3.1). Then for any $1 \le q \le d$ the sequence $n^{-1} f_n^{(q)}(x)$ has joint periods, where

$$f_n^{(q)}(x) = \log \|(M(\beta^{n-1}x) \cdots M(\beta x)M(x)))^{\wedge q}\|.$$

Proof. Since M is β–adapted, the entries of $M(\beta^k x)$ are all of the form $h_{i,j}(\beta^{\ell_{i,j}} x)$ with 1–periodic function $h_{i,j}(x)$ and integer $\ell_{i,j} \ge 0$ for sufficiently large k. So, if necessary, we consider $\log \|(M(\beta^{n-1}x) \cdots M(\beta^{k_0}x))^{\wedge q}\|$ for some sufficiently large but fixed $k_0 \ge 0$.

Consider $\tau = \beta^m \eta_m + \cdots + \beta\eta_1 + \eta_0$ where $m \ge 1$ and $0 \le \eta_i \le \beta$ are integers. We are going to show that all such τ are joint ϵ-translation numbers for $n^{-1} f_n^{(n)}(x)$ with $n \ge N(\epsilon)$, where $N(\epsilon)$ depending on ϵ is an integer to be determined.

By Lemma 3.3, we have

$$\inf_{j\in\mathbb{Z}} |\beta^k \tau - j| \le C' \rho^k$$

for all k and some constant C' independent of k and τ. For $k \ge 0$ each entry of $M^{\wedge q}(\beta^{k+k_0}x)$ is a degree q polynomial in d^2 variables of the form $h(\beta^{\ell+k}x)$, with h α-Hölder, 1–periodic, and $\ell \ge 0$. Notice that we have $h(\beta^{\ell+k}(x+\tau)) = h(\beta^{\ell+k}x) + O(\rho^{k\alpha})$, hence

$$\|M(\beta^{k+k_0}(x+\tau))^{\wedge q} - M(\beta^{k+k_0}(x))^{\wedge q}\| = C_q \rho^{k\alpha}$$

for some constant C_q. By the distortion Lemma 3.1 and the above estimate we have

$$\begin{aligned} & |f_n^{(q)}(x+\tau) - f_n^{(q)}(x)| \\ = & \left| \log \frac{\|(M(\beta^{n-1}x + \beta^{n-1}\tau) \cdots M(\beta x + \beta\tau)M(x+\tau))^{\wedge q}\|}{\|(M(\beta^{n-1}x) \cdots M(\beta x)M(x))^{\wedge q}\|} \right| \\ = & C_q C' \sum_{k=1}^{n} D^k \rho^{(k-k_0)\alpha} \le \frac{CC'D\rho^{-k_0}}{1 - D\rho^\alpha} =: \mathcal{C} \end{aligned}$$

So, we may choose $N(\epsilon) = \mathcal{C}/\epsilon$. In order to finish the proof, it suffices to notice that Lemma 3.4 implies that all these τ form a subset with bounded gap in $\mathbb{R}$. $\square$

Proposition 3.8. *Let $\beta > 1$ be a PV–number and let $M : \mathbb{R} \to GL_d(\mathbb{C})$ be β–adapted and α–Hölder continuous. Suppose that the entries of $M(x)$ are either identically zero or larger than a constant $\delta > 0$. Then $n^{-1} f_n(x)$ has joint periods, where*

$$f_n^{(q)}(x) = \log \|(M(\beta^{n-1}x) \cdots M(\beta x) M(x))^{\wedge q})\|.$$

Proof. The proof is the same as the last proposition. But we use Lemma 3.2 instead of Lemma 3.1. $\square$

4. Multiperiodic functions

As we pointed out in the introduction and as we will see in Section 4.2, our scalar equation (1.2) can be converted to the vector equation (1.3). So, we first study the vector equation (1.3).

4.1. Equation $G(x) = M(x/\beta)G(x/\beta)$.

Let $M : \mathbb{R} \to \mathcal{M}_{d\times d}(\mathbb{C})$ be a matrix valued function. We consider the following vector valued equation

$$G(x) = M\left(\frac{x}{\beta}\right) G\left(\frac{x}{\beta}\right)$$

where the unknown $G : \mathbb{R} \to \mathbb{C}^d$ is a vector valued function.

Theorem 4.1. *Let $\beta > 1$ be a real number and $M(x)$ be a complex matrix valued function. Suppose that M is Lipschitzian and that $M(0)$ is non negative and has 1 as a simple eigenvalue with a corresponding strictly positive eigenvector $\mathbf{v}$. Then there exists, up to a multiplicative constant, a unique continuous solution $G(0) \neq 0$ of the equation $G(x) = M(x/\beta)G(x/\beta)$. The solution can be defined by*

$$G(x) = \lim_{n\to\infty} M\left(\frac{x}{\beta}\right) M\left(\frac{x}{\beta^2}\right) \cdots M\left(\frac{x}{\beta^n}\right) \mathbf{v}$$

where the convergence is uniform on every compact subset in $\mathbb{R}$.

Proof. Write $\mathbf{v} = (v_1, \cdots, v_d)^t$. We introduce the following norm for $\mathbb{C}^d$

$$\|z\| = \max_{1\le j\le d} \frac{|z_j|}{v_j} \qquad (z = (z_j)_{1\le j\le d} \in \mathbb{C}^d).$$

Then a matrix $A = (a_{i,j}) \in \mathcal{M}_{d\times d}(\mathbb{C})$, considered as an operator on the normed space $(\mathbb{C}^d, \|\cdot\|)$, admits its operator norm

$$\|A\| = \max_{1\le i\le d} \frac{1}{v_i} \sum_{j=1}^{d} |a_{i,j}| v_j.$$

Notice that $\|M(0)\| = 1$ because $M(0)\mathbf{v} = \mathbf{v}$.

Since the eigenvalue 1 of $M(0)$ is simple (and isolated), and $M(x)$ is Lipschitz continuous, by the perturbation theory of matrices, there is a neighborhood of 0, say $[-\delta, \delta]$ $(\delta > 0)$, such that for any $x \in [-\delta, \delta]$, $M(x)$ has a simple eigenvalue $\lambda(x)$ and a corresponding eigenvector $v(x)$ satisfying

$$|\lambda(x) - 1| \leq C|x|, \qquad \|v(x) - \mathbf{v}\| \leq C|x| \qquad (x \in [-\delta, \delta]) \tag{4.1}$$

for some constant $C > 0$. We claim that the limit

$$G(x) = \lim_{n \to \infty} M\left(\frac{x}{\beta}\right) M\left(\frac{x}{\beta^2}\right) \cdots M\left(\frac{x}{\beta^n}\right) \mathbf{v} \tag{4.2}$$

exists (uniformly on any compact set). It is clear that the limit function is a solution.

Denote

$$Q_n(x) = M\left(\frac{x}{\beta}\right) M\left(\frac{x}{\beta^2}\right) \cdots M\left(\frac{x}{\beta^n}\right).$$

The proof of the existence of the limit in (4.2) is based on the following lemma.

Lemma 4.2. *For any $\delta > 0$, there exists a constant $D > 0$ such that for any $n \geq 1$ and any $x \in [-\delta, \delta]$ we have*

$$\|Q_n(x)\| \leq D \qquad \|Q_n(x)\mathbf{v} - \mathbf{v}\| \leq D|x|.$$

To get the boundedness of $\|Q_n(x)\|$, it suffices to notice that

$$\|Q_n(x)\| \leq \prod_{j=1}^{n} f\left(\frac{x}{\beta^j}\right)$$

where the scalar function $f(x) = \|M(x)\|$ is Lipschitzian and $f(0) = 1$ (we have used our choice of the norm of $\mathbb{C}^d$), and that the products converge uniformly on $[-\delta, \delta]$ to a continuous function [FL]. Now we prove that

$$\|Q_n(x)\mathbf{v} - Q_{n-1}(x)\mathbf{v}\| \leq C' \frac{|x|}{\beta^n} \tag{4.3}$$

where $C' > 0$ is some constant. In fact, since $M(x)v(x) = \lambda(x)v(x)$, we have

$$M\left(\frac{x}{\beta^n}\right)\mathbf{v} - \mathbf{v} = M\left(\frac{x}{\beta^n}\right)\left[\mathbf{v} - v\left(\frac{x}{\beta^n}\right)\right] + \left[\lambda\left(\frac{x}{\beta^n}\right) v\left(\frac{x}{\beta^n}\right) - \mathbf{v}\right].$$

Multiplying both sides by $Q_{n-1}(x)$, we get

$$\begin{aligned} &\|Q_n(x)\mathbf{v} - Q_{n-1}(x)\mathbf{v}\| \\ \leq\ &\left\|Q_n(x)\left(\mathbf{v} - v\left(\frac{x}{\beta^n}\right)\right)\right\| + \left\|Q_{n-1}(x)\left(\mathbf{v} - \lambda\left(\frac{x}{\beta^n}\right) v\left(\frac{x}{\beta^n}\right)\right)\right\| \end{aligned}$$

Notice that

$$\|\lambda(x)v(x) - \mathbf{v}\| \leq \|\lambda(x) - 1\| \ \|v(x)\| + \|v(x) - \mathbf{v}\|.$$

Using the last inequality, the estimates in (4.1) and that we have just proved $\|Q_n(x)\| \leq D$, we obtain (4.3). Then for $n > m$

$$\|Q_n(x)\mathbf{v} - Q_m(x)\mathbf{v}\| \leq \sum_{k=m+1}^{n} \|Q_k(x)\mathbf{v} - Q_{k-1}(x)\mathbf{v}\| \leq \frac{C'|x|}{\beta^{m-1}(\beta - 1)}.$$

That means $Q_n(x)\mathbf{v}$ is a Cauchy sequence in the space $C([-\delta,\delta])$ of continuous functions equipped with uniform norm. Since for any fixed integer n_0, we have

$$\lim_{n\to\infty} Q_n(x)\mathbf{v} = Q_{n_0}(x)\cdot \lim_{n\to\infty} Q_n\left(\frac{x}{\beta^{n_0}}\right)\mathbf{v},$$

it follows that the uniform convergence of $Q_n(x)\mathbf{v}$ on $[-\delta,\delta]$ implies its uniform convergence on any compact set.

The uniqueness of solution is easy. Let $G \neq 0$ be a solution. First notice that $G(0)$ is an eigenvector of $M(0)$ associated to the simple eigenvalue 1. Hence we may assume that $G(0) = \mathbf{v}$. By iterating the equation, we get

$$G(x) = Q_n(x)G\left(\frac{x}{\beta^n}\right) = Q_n(x)\mathbf{v} + Q_n(x)\left(G\left(\frac{x}{\beta^n}\right) - \mathbf{v}\right)$$

The last term converges to zero (uniformly on any compact set) because of $\|Q_n(x)\| \leq D$. So, $G(x)$ must be the limit of $Q_n(x)\mathbf{v}$. □

Remark 4.3. In the theorem, neither the almost periodicity of $M(x)$ nor the positivity of $M(x)$ is required, but only the positivity of $M(0)$. That 1 is an eigenvalue of $M(0)$ is necessary for the equation (1.3) to have a solution $G(x)$ such that $G(0) \neq 0$.

Remark 4.4. The Lipschitz continuity is not really necessary. Hölder continuity or even Dini continuity is sufficient.

Remark 4.5. If the entries of M are (real) analytic, then the solution G is also analytic. Because, for any $x_0 \in \mathbb{R}$, there is a disk on the complex plane centered at x_0 on which $Q_n(x)\mathbf{v}$ (as functions of complex variable x) uniformly converges.

4.2. **Existence of multiperiodic functions.**

Here we give a proof of Theorem A based on Theorem 4.1.

Let $M(x)$ be as in (1.4). It is easy to see that the characteristic polynomial of $M(0)$ takes the form

$$P(u) = u^d - f_1(0)u^{d-1} - f_2(0)u^{d-2} - \cdots - f_{d-1}(0)u - f_d(0)$$

The consistency condition implies that 1 is an eigenvalue of $M(0)$. Notice that

$$P'(1) = f_1(0) + 2f_2(0)\cdots + (d-1)f_{d-1}(0) + df_d(0) > 0.$$

So, the eigenvalue 1 is simple. By Theorem 4.1, there is a unique solution of $G(x) = M(x/\beta)G(x/\beta)$. Let

$$G(x) = (G_1(x), G_2(x), \cdots, G_d(x))^t$$

Then $G_1(x)$ is a solution of (1.2). If F is a solution of (1.2). Let

$$\tilde{G}_1(x) = F(x), \tilde{G}_2(x) = F(x/\beta), \cdots, \tilde{G}_d(x) = F(x/\beta^{d-1}).$$

Then $\tilde{G} = (\tilde{G}_1, \cdots, \tilde{G}_d)^t$ is a solution of $G(x) = M(x/\beta)G(x/\beta)$. Thus the uniqueness of the solution of Equation (1.3) implies that of Equation (1.2).

4.3. **Asymptotic behavior of multiperiodic functions.**

Let us consider the asymptotic behavior of a multiperiodic function, or more generally the asymptotic behavior of a solution G of Equation (1.3) provided it exists (the existence may be guaranteed by Theorem 4.1).

Theorem 4.6. *Let $\beta > 1$ be a PV–number whose maximal conjugate has modulus ρ. Let $M : \mathbb{R} \to \mathrm{GL}_d(\mathbb{C})$ be a β–adapted u.a.p. Hölder function of order $\alpha > 0$. Suppose that G is a solution of $G(x) = M(x/\beta)G(x/\beta)$. Suppose furthermore that one of the following conditions is satisfied*

(i) $D\rho^\alpha < 1$ where $D = \sup_{x\in\mathbb{R}} \|M(x)\|\|M(x)^{-1}\|$ (NB. β must be Pisot).

(ii) The entries of $M(x)$ are either identically zero or larger than a constant $\delta > 0$.

Then for a.e. $x \in \mathbb{R}$ the limit

$$h(x) = \lim_{n\to\infty} \frac{1}{n} \log |G(\beta^n x)|$$

exists and is independent of x.

Proof. We first consider the case (i). By Proposition 3.7, Theorem 2.9 applies. Hence for a.e. x, if we denote by $r(x)$ the integer such that the vector $G(x) \in V_x^{(r)} \setminus V_x^{(r-1)}$ we get

$$\lim_{n\to\infty} \frac{1}{n} \log |G(\beta^n x)| = \lim_{n\to\infty} \frac{1}{n} \log |M_x^n G(x)| = \lambda_x^{(r(x))}.$$

But $G(\beta x) = M(x)G(x)$ hence $G(\beta x) \in V_{\beta x}^{(r)} \setminus V_{\beta x}^{(r-1)}$, from what follows $r(\beta x) = r(x)$, i.e. r is invariant. Hence constant a.e. because of the total Bohr ergodicity of the sequence β^n.

Case (ii). We use the notation of Proposition 3.8. Since $n^{-1}f_n$ has joint periods, Theorem 2.5 applies (see the proof of Theorem 2.9 for details). Hence the following limit exists a.e.

$$\lim_{n\to\infty} \frac{1}{n} f_n(x) = \mathcal{L},$$

where $\mathcal{L} = \inf_n \frac{1}{n}\mathbb{M}(f_n)$. In view of $G(\beta^n x) = M(\beta^{n-1}x)\cdots M(x)G(x)$ the positivity of M and G gives

$$\lim_{n\to\infty} \frac{1}{n} \log |G(\beta^n x)| = \mathcal{L} \quad \text{a.e.}$$

□

Note that when G is the solution of equation (1.3) with M and G given by (1.4) we have

$$|G(x)| = |F(x)| + |F(x/\beta)| + \cdots + |F(x/\beta^{d-1})|, \tag{4.4}$$

thus the asymptotic behavior of $\frac{1}{n}\log|G(\beta^n x)|$ and $\frac{1}{n}\log\sum_{j=0}^{d-1}|F(\beta^{n-j}x)|$ are the same. Thus Theorem C follows as an immediate corollary of Theorem 4.6. This partially answers a question in [JRS] (Conjecture 4.1., p. 263).

We prove now Theorem B. By the primitivity of $M(0)$ and the hypothesis, there exists an ingeter $\tau \geq 1$ such that $\tilde{M}(x) := M(x/\beta^{\tau-1})\cdots M(x/\beta)M(x)$

has all its entries strictly positive (even larger than $c\delta^\tau$ for some constant $c > 0$). Consider the equation

$$G(x) = \tilde{M}(x/\beta)G(x/\beta^\tau).$$

We examine the first entries of both sides. We can find two constants $0 < c_1 < c_2$ such that we get

$$c_1 F(x) \le F(x/\beta^{\tau+1}) + \cdots + F(x/\beta^{\tau+d}) \le c_2 F(x).$$

Thus Theorem B follows from Theorem 4.6.

Theorem 4.7. *Under the same conditions as Theorem 4.6, for any $q \in \mathbb{R}^+$, the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \|M(\beta^{n-1}x)\cdots M(\beta x)M(x)\|^q dx.$$

Proof. Write

$$Z_n = \int_0^1 P_n(x)^q dx \quad \text{with} \quad P_n(x) = \|M(\beta^{n-1}x)\cdots M(\beta x)M(x)\|.$$

It suffices to show that there is a constant $C > 0$ such that

$$Z_{n+m} \le C Z_n Z_m \qquad (n \ge 1, m \ge 1).$$

We assume that $q = 1$, just for simplicity. We will use the fact that there is a constant $L > 0$ such that

$$\|M(x) - M(y)\|_2 \le L|x-y|^\alpha \qquad (\forall x, y \in \mathbb{R}).$$

We use the notation $\Pi_{i=0}^n M_i = M_n M_{n-1} \cdots M_1 M_0$ for the (noncommutative) product of the matrices $M_0, \dots, M_n$. Write

$$\begin{aligned} Z_{n+m} &= \sum_\epsilon \int_{I(\epsilon)} \left\| \prod_{k=0}^{m-1} M(\beta^{n+k}x) \cdot \prod_{j=0}^{n-1} M(\beta^j x) \right\| dx \\ &\le \sum_\epsilon \int_{I(\epsilon)} \left\| \prod_{j=0}^{n-1} M(\beta^j x) \right\| \cdot \left\| \prod_{k=0}^{m-1} M(\beta^{n+k}x) \right\| dx \end{aligned}$$

where the sum is taken over all β-intervals $I(\epsilon)$ of level n (see Lemma 3.4). Let a_ϵ be the left endpoint of $I(\epsilon)$. The integral in the last sum , after the change of variables $\beta^n(x - a_\epsilon) = y$, becomes

$$\beta^{-n} \int_0^{\beta^n|I(\epsilon)|} \left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon + \beta^{-n+j}y) \right\| \cdot \left\| \prod_{k=0}^{m-1} M(\beta^k y + \beta^{n+k}a_\epsilon) \right\| dy.$$

Notice that if $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$, then

$$\beta^{n+k}a_\epsilon = \beta^{n+k}\left(\frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_n}{\beta^n}\right) = \beta^{n+k-1}\epsilon_1 + \cdots + \beta^k \epsilon_n.$$

So, by Lemma 3.3, there is an integer n_ϵ such that

$$|\beta^{n+k}a_\epsilon - n_\epsilon| = O(\rho^k + \rho^{k+1} + \cdots + \rho^{n+k-1}) = O(\rho^k) \tag{4.5}$$

By the distortion lemma (Lemma 3.1, Lemma 3.2), we have

$$\left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon + \beta^{-n+j} y) \right\| \le C \left\| \prod_{j=0}^{n-1} M(\beta^j a_\epsilon) \right\|$$

$$\left\| \prod_{k=0}^{m-1} M(\beta^k y + \beta^{n+k} a_\epsilon) \right\| \le C \left\| \prod_{k=0}^{m-1} M(\beta^k y) \right\|$$

Therefore, we get

$$Z_{n+m} \le C\beta^{-n} \sum_{\epsilon} P_n(a_\epsilon) \le C' Z_n Z_m.$$

□

Corollary 4.8. *Let F be the multiperiodic function defined by* (1.2)*. Suppose that $\beta > 1$ is a PV–number and that $f_1, \cdots, f_d$ are either identically zero or larger than a constant $\delta > 0$. Suppose further that $M_x^\ell := M(\beta^{\ell-1}x)\cdots M(\beta x)M(x)$ has strictly positive entries for some integer $\ell > 0$. Then for any $q \in \mathbb{R}^+$, the following limit exists*

$$\lim_{T\to\infty} \frac{1}{\log T} \int_0^T F(x)^q dx.$$

Proof. Without loss of generality, we may only consider the subsequence $T_n = \beta^n$. Since $|G(x)| = \sum_{j=0}^{d-1} |F(x/\beta^j)|$ where G is the solution of the associated vector equation (1.3), we have only to show the existence of the limit

$$\lim_{n\to\infty} \frac{1}{n} \int_0^{\beta^n} |G(x)|^q dx.$$

Making the change of variables $x = \beta^n y$, we are led to prove the existence of the limit

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 |G(\beta^n x)|^q dx.$$

Notice that $G(\beta^n x) = M_x^n G(x)$. Notice also that $G(x)$ has strictly positive entries by the hypothesis on M_x^ℓ. So, for the non negative matrix M_x^n we have

$$C^{-1}\|M_x^n\| \le |G(\beta^n x)| \le C\|M_x^n\| \qquad (\forall x \in [0,1])$$

for some constant $C > 0$. By the proof of the last theorem, $\log \int_0^1 |G(\beta^n x)|^q dx$ is subadditive. □

Remark 4.9. Let $M(x)$ be the matrix defined by (1.4). Let $\tilde{M}$ be the numerical matrix obtained by replacing $f_j(x)$ in $M(x)$ by 0 or 1 according to $f_j(x) \equiv 0$ or not. Then $\tilde{M}^\ell > 0$ implies $M_x^\ell > 0$. In particular, $\tilde{M}^d > 0$ if $f_j(x)$ are all strictly positive.

Example 4.10. Let $\beta > 1$ be a PV–number. Let $f_1(x)$ and $f_2(x)$ be two strictly positive 1–periodic Hölder continuous functions such that $f_1(0) + f_2(0) = 1$. There is a unique multiperiodic function F defined by

$$F(x) = f_1\left(\frac{x}{\beta}\right) F\left(\frac{x}{\beta}\right) + f_2\left(\frac{x}{\beta^2}\right) F\left(\frac{x}{\beta^2}\right)$$

For almost every $x \in \mathbb{R}$, $n^{-1} \log F(\beta^n x)$ has a limit as $n \to \infty$; for any $q \in \mathbb{R}^+$, $(\log T)^{-1} \int_0^T F(x)^q dx$ has a limit as $T \to \infty$.

Example 4.11. Let $\beta > 1$ and $a, b \in \mathbb{Z}$. Consider the contractive transformations on $\mathbb{R}$ defined by

$$S_1 x = \frac{x+a}{\beta}, \qquad S_1 x = \frac{x+b}{\beta^2}.$$

For any $0 < p < 1$, there exists a unique probability measure μ with compact support such that

$$\mu = p\ \mu \circ S_1^{-1} + (1-p)\ \mu \circ S_2^{-1}.$$

It is a self-similar measure. Its Fourier transform satisfies the equation

$$\widehat{\mu}(x) = f_1(x/\beta)\widehat{\mu}(x/\beta) + f_2(x/\beta^2)\widehat{\mu}(x/\beta^2)$$

with $f_1(x) = pe^{2\pi i a x}$ and $f_2(x) = qe^{2\pi i b x}$ with $q = 1 - p$. This is a special case of the equation (1.2). The corresponding matrix defined by (1.4) and its inverse are respectively equal to

$$M(x) = \begin{pmatrix} pe^{2\pi i a x} & qe^{2\pi i b x/\beta} \\ 1 & 0 \end{pmatrix}, \quad M(x)^{-1} = q^{-1} e^{-2\pi i b x/\beta} \begin{pmatrix} 0 & qe^{2\pi i b x/\beta} \\ 1 & -pe^{2\pi i a x} \end{pmatrix}$$

If we take the norm $|v| = \max(|v_1|, |v_2|)$ on $\mathbb{C}^2$, the operator norms for $M(x)$ and $M(x)^{-1}$ are respectively $\|M(x)\| = 1$ and $\|M(x)^{-1}\| = \frac{1+p}{1-p}$. So, when β is a PV–number, under the condition $\frac{1+p}{1-p} < \frac{1}{\rho}$, for almost all $x \in \mathbb{R}$ the following limit exists and does not depend on x

$$\lim_{n\to\infty} \frac{1}{n} \log\left(|\widehat{\mu}(\beta^n x)| + |\widehat{\mu}(\beta^{n-1} x)|\right).$$

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