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POINCARÉ RECURRENCE FOR OBSERVATIONS

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ABSTRACT. A high dimensional dynamical system is often studied by experimentalists through the measurement of a relatively low number of different quantities, called an observation. Following this idea and in the continuity of Boshernitzan's work, for a measure preserving system, we study Poincaré recurrence for the observation. The link between the return time for the observation and the Hausdorff dimension of the image of the invariant measure is considered. We prove that when the decay of correlations is super polynomial, the recurrence rates for the observations and the pointwise dimensions relatively to the push-forward are equal.

1. INTRODUCTION

The famous Zermelo paradox reveals that the classical Poincaré recurrence theorem has some implications out of physical sense. Indeed, if we start with all the particles in one side of a box, nobody will ever see all the particles coming back in one side of our box at the same time. Nevertheless, if we focus on a few number of these particles, this event will appear after a reasonable time. In the same way, when we study a high dimensional dynamical system we might not know all the aspects of the evolution but only a part or certain quantities of the system. This might be due to the difficulty to study a high dimensional system, but also to the lack of interest of an over-detailed description.

Recently, Ott and York tried to elaborate some Platonic formalism of dynamical systems [11]. The reality, the dynamical system (X, T, μ), is only known through a measurement or observation, that is a function defined on X taking values in (typically) a lower dimensional space. The following result by Boshernitzan [4] about Poincaré recurrence falls in this frame. If we have a measure preserving dynamical system (X, T, μ) and an observable f from X to a metric space (Y, d) then whenever the α-dimensional Hausdorff measure is σ-finite on Y we have

$$\liminf_{n\to\infty} n^{1/\alpha} d\left(f(x), f(T^n x)\right) < \infty \qquad \text{for } \mu\text{-almost every } x. \tag{1}$$

The main aim of this paper is to prove a refinement of (1) and a generalization of [2, 12] for recurrence rates for observations.

In Section 2, we give the precise definition of the recurrence rates for the observations and state an upper bound in term of dimension (Theorem 2 which is proved in Section 3), then under an additional assumption we state our main result (Theorem 5 which is proved in Section 4), and finally, we analyze in the case of the Lebesgue measure the existence of the pointwise dimension for its smooth image (Theorem 9 which is proved in Section 5).

2. STATEMENT OF THE RESULTS

2.1. **Definitions and general inequality.** Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system (m.p.s.) i.e. $\mathcal{A}$ is a σ-algebra, μ is a measure on $(X, \mathcal{A})$ with $\mu(X) = 1$ and μ is invariant by T (i.e $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{A}$) where $T : X \to X$.
Let $f : X \to Y$ be a function, called observable (we will specify the space X and Y later). We introduce the return time for the observation and its associated recurrence rates.

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Definition 1. *Let $f : X \to Y$ be a measurable function, we define for $x \in X$ the return time for the observation:*

$$\tau_r^f(x) := \inf\left\{k \in \mathbb{N}^* : \ f(T^k x) \in B\left(f(x), r\right)\right\}$$

where $B(x,r)$ is the ball centered in x with radius r. We then define the lower and upper recurrence rate for the observation:

$$\underline{R}_i^f(x) := \liminf_{r\to 0} \frac{\log \tau_r^f(x)}{-\log r} \qquad \overline{R}_i^f(x) := \limsup_{r\to 0} \frac{\log \tau_r^f(x)}{-\log r}.$$

We also define for $p \in \mathbb{N}$ the p-non-instantaneous return time for the observation:

$$\tau_{r,p}^f(x) := \inf\left\{k > p : \ f(T^k x) \in B\left(f(x), r\right)\right\}.$$

Then we define the non-instantaneous lower and upper recurrence rates for the observation:

$$\underline{R}^f(x) := \lim_{p\to\infty} \liminf_{r\to 0} \frac{\log \tau_{r,p}^f(x)}{-\log r} \qquad \overline{R}^f(x) := \lim_{p\to\infty} \limsup_{r\to 0} \frac{\log \tau_{r,p}^f(x)}{-\log r}.$$

Whenever $\underline{R}^f(x) = \overline{R}^f(x)$ we denote by $R^f(x)$ the value of the limit.

The lower and upper pointwise or local dimension of a Borel probability measure ν on Y at a point $y \in Y$ are defined by

$$\underline{d}_\nu(y) = \underline{\lim}_{r\to 0} \frac{\log \nu\left(B\left(y,r\right)\right)}{\log r} \qquad \text{and} \qquad \overline{d}_\nu(y) = \overline{\lim}_{r\to 0} \frac{\log \nu\left(B\left(y,r\right)\right)}{\log r}.$$

The pushforward measure $f_*\mu(.) := \mu(f^{-1}(.))$ is a probability measure on Y and we define *the lower and upper pointwise dimension for the observations* with respect to μ at a point $x \in X$ by

$$\underline{d}_\mu^f(x) = \underline{d}_{f_*\mu}(f(x)) \qquad \text{and} \qquad \overline{d}_\mu^f(x) = \overline{d}_{f_*\mu}(f(x)).$$

If they are equal, we denote by $d_\mu^f(x)$ the common value.

Theorem 2. *Let $(X, \mathcal{A}, \mu, T)$ be a m.p.s. Consider a measurable observable $f : X \to Y = \mathbb{R}^N$. Then*

$$\underline{R}^f(x) \le \underline{d}_\mu^f(x) \qquad \text{and} \qquad \overline{R}^f(x) \le \overline{d}_\mu^f(x)$$

for μ-almost every $x \in X$.

This result is satisfactory in the sense that it holds for any dynamical system and observation. Moreover, under natural assumptions we will show that the equality is true. Still, these inequalities may be strict, the caricatural example is when T is the identity map.

Example 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a $\mathbb{P}$-preserving map θ and $Y \subset \mathbb{R}^N$ a Borel set. The family $(F_\omega)_{\omega\in\Omega}$ is called a random transformation, where for each ω, F_ω is a map from Y to Y such that the map $(w, y) \to F_\omega(y)$ is $\mathcal{F} \times \mathcal{B}(\mathbb{R}^N)$-measurable. The map $T : X = \Omega \times Y \to X$ defined by $T(\omega, y) = (\theta\omega, F_\omega(y))$ is called a skew product transformation. Let $\mathcal{M}_\mathbb{P}(X, T)$ be the set of T-invariant probability measure having the marginal $\mathbb{P}$ on Ω. For any $\mu \in \mathcal{M}_\mathbb{P}(X, T)$, Theorem 2 applies with f the projection on Y, and gives an upper bound for the time needed by a typical random orbit $F_{\theta^k\omega} \circ ... \circ F_{\theta\omega} \circ F_\omega(y)$ to come back close to its starting point y.*

2.2. **Poincaré recurrence for observations.** From now on let us assume that X is a metric space and $\mathcal{A}$ is its Borel σ-algebra. We then can introduce the decay of correlations:

Definition 4. *(X, T, μ) has a super-polynomial decay of correlations if, for all ϕ, ψ Lipschitz functions from X to $\mathbb{R}$ and for all $n \in \mathbb{N}^*$, we have:*

$$\left| \int_X \phi \circ T^n \, \psi d\mu - \int_X \phi d\mu \int_X \psi d\mu \right| \le \|\phi\| \|\psi\| \theta_n$$

with $\lim_{n\to\infty} \theta_n n^p = 0$ for all $p > 0$ and where $\|.\|$ is the Lipschitz norm.

The main result of our paper is:

Theorem 5. *Let $(X, \mathcal{A}, \mu, T)$ be a m.p.s with a super-polynomial decay of correlations. Consider a Lipschitz observable $f : X \to Y = \mathbb{R}^N$. Then, we have*

$$\underline{R}^f(x) = \underline{d}_\mu^f(x) \qquad \text{and} \qquad \overline{R}^f(x) = \overline{d}_\mu^f(x)$$

for μ-almost every x such that $\underline{d}_\mu^f(x) > 0$.

Taking the identity function for f, we recover the result of [2] and [12] under weaker assumptions. The main assumption of the theorem about decay of correlations is satisfied in a variety of systems with some hyperbolic behavior and studied in an abundant literature (e.g. [13, 5, 1]).

Definition 6. *We say that a probability measure ν is exact dimensional if there exists a constant $d_\nu \in \mathbb{R}$ such that*

$$\underline{d}_\nu(\cdot) = \overline{d}_\nu(\cdot) = d_\nu \qquad \text{almost everywhere.}$$

It is well known that in this case many notion of dimension coincide (see Section 5.1 for details). In particular the Hausdorff dimension $\dim_H \nu$ satisfies

Proposition 7. *If ν is exact dimensional, then*

$$d_\nu(\cdot) = \dim_H \nu \quad \text{almost everywhere.}$$

Corollary 8. *Let $(X, \mathcal{A}, \mu, T)$ be a m.p.s with a super-polynomial decay of correlations. Consider a Lipschitz observable $f : X \to Y = \mathbb{R}^N$. Then, if $f_*\mu$ is exact dimensional, we have*

$$\underline{R}^f(x) = \overline{R}^f(x) = \dim_H f_*\mu \qquad \text{for } \mu\text{-almost every } x \in X.$$

Remark. *We have the equivalence*

$$f_*\mu \text{ is exact dimensional } \iff \exists d,\, \underline{d}_\mu^f(x) = \overline{d}_\mu^f(x) = d \qquad \text{for } \mu\text{-almost every } x \in X.$$

Proof of Corollary 8. If $\dim_H f_*\mu = 0$, then the conclusion follows from Theorem 2 and Proposition 7. In the general case, it is just a combination of Theorem 5 and Proposition 7. □

Theorem 5 does not apply to those points where $\underline{d}_\mu^f(x) = 0$. When $\overline{d}_\mu^f(x) = 0$ also, this is not a restriction because Theorem 2 applies and gives $\overline{R}^f(x) = \underline{R}^f(x) = 0$. However, the question remains when $\overline{d}_\mu^f(x) \neq \underline{d}_\mu^f(x) = 0$ on a positive measure set. Indeed, the assumptions of Theorem 5 are not strong enough to ensure the almost everywhere existence of the pointwise dimension for the observations. The following result guaranties the existence for a large class of systems.

Theorem 9. *Let $f : \mathbb{R}^M \to \mathbb{R}^N$ be a C^∞ function, let μ be any absolutely continuous measure on $\mathbb{R}^M$. Then, d_μ^f exists and belongs to $\{0, 1, ..., \min\{M, N\}\}$ μ-almost everywhere. More precisely, $d_\mu^f(x) = \operatorname{rank} d_x f$ for μ-almost every $x \in \mathbb{R}^M$.*

This is a non trivial result because the image measure $f_*\mu$ may be quite complicated and rather counter intuitive. Already in the one dimensional case, there exists $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $f(\{f' = 0\})$ is an uncountable set of dimension 0 and $f_*(Leb|_{\{f'=0\}})$ is a non null and non atomic measure. We emphasize that Theorem 9 applies to any C^∞ function, and not only for generic functions. This is essential in applications, where we are mostly interested in particular observables.

Corollary 10. *Let $T : X \subset \mathbb{R}^M \to X$ preserves an absolutely continuous invariant probability measure with super-polynomial decay of correlations. Let $f : \mathbb{R}^M \to \mathbb{R}^N$ be a Lipschitz C^∞ observable. Then R^f exists and belongs to $\{0, 1, ..., \min\{M, N\}\}$ almost everywhere.*

Proof of Corollary 10. We apply Theorem 5 and Theorem 9 when $d_\mu^f > 0$. When $d_\mu^f = 0$, we use Theorem 2. □ □

2.3. **On the necessity of the non-instantaneous recurrence rate.** In this part, we give a simple example which illustrates the utility of non-instantaneous return times.

Let $\Omega := \{0,1\}^{\mathbb{N}}$ and σ be the shift on Ω. Fix some 1-approximable $\alpha \in \mathbb{R}$ (e.g. [6] for a nice perspective) i.e. $\delta(\alpha) = 1$ where

$$\delta(\alpha) = \sup\left\{\delta \geq 1 : |\alpha - \frac{p}{q}| < \frac{1}{q^{1+\delta}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q}\right\}.$$

Let ν be an invariant ergodic probability measure on Ω. Fix some measurable $A \subset \Omega$ such that $1 > \nu(A) > 0$ and set φ:

$$\varphi(\omega) = \begin{cases} 0 & \text{if } \omega \notin A \\ \alpha & \text{if } \omega \in A. \end{cases} \tag{2}$$

Let $\mathbb{T}^1$ denote the 1-dimensional torus and define on $X := \Omega \times \mathbb{T}^1$ the map

$$\begin{array}{rccl} T: & X & \longrightarrow X \\ & (\omega, y) & \longrightarrow (\sigma\omega, y + \varphi(\omega)). \end{array}$$

Let Leb be the Lebesgue measure on $\mathbb{T}^1$. The probability measure $\mu := \nu \otimes Leb$ is T-invariant. We examine below the recurrence rate of the system (X, T, μ) for the observable f given by the projection on the second variable i.e.

$$\begin{array}{rccl} f: & X & \to \mathbb{T}^1 \\ & (\omega, y) & \to y. \end{array}$$

First, we need the following obvious result on the pushforward measure: since $f_*\mu = Leb$ and the local dimension of the Lebesgue measure is one, the measure $f_*\mu$ is exact dimensional and satisfies

$$\forall x \in X, \ d_\mu^f(x) = 1. \tag{3}$$

Proposition 11. *We have $R_i^f \neq d_\mu^f$ on a set of positive measure. More precisely*

$$\forall x = (\omega, y) \in \Omega\backslash A \times \mathbb{T}^1, \ R_i^f(x) = 0.$$

Proof. Let $\omega \in \Omega\backslash A$ and $y \in \mathbb{T}^1$, we have

$$\begin{aligned} f(T(\omega, y)) &= f(\sigma\omega, y) \qquad \text{because } \omega \notin A \\ &= y \\ &= f(\omega, y). \end{aligned}$$

So, for all $r > 0$, $\tau_r^f(x) = 1$ and then $R_i^f(x) = 0$. □

We therefore need to introduce the non-instantaneous return time to avoid this kind of problem.

Proposition 12. *We have $R^f = d_\mu^f$ on a set of full measure.*

Proof. For $k \in \mathbb{N}$ and $\omega \in \Omega$, let $q_k(\omega) := \sum_{i=0}^{k-1} \mathbf{1}_A(\sigma^i\omega)$. Let $\varepsilon > 0$ fixed. For $x = (\omega, y)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \tau^f_{\frac{1}{n^{1+\varepsilon}},p}(x) &= \inf\left\{k > p : f(T^k x) \in B\left(f(x), \frac{1}{n^{1+\varepsilon}}\right)\right\} \\ &= \inf\left\{k > p : y + \alpha q_k(\omega) \in B\left(y, \frac{1}{n^{1+\varepsilon}}\right)\right\} \\ &= \inf\left\{k > p : \|\alpha q_k(\omega)\| \leq \frac{1}{n^{1+\varepsilon}}\right\} \end{aligned} \tag{4}$$

where for $q \in \mathbb{Z}$

$$\|q\alpha\| := \min\{|q\alpha - p| : p \in \mathbb{Z}\}.$$

Thanks to the choice of α, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we get $\|k\alpha\| \geq \frac{1}{k^{1+\varepsilon}}$. Taking $n \geq k_0$ we have

$$m_{n,\varepsilon} := \inf\left\{q > k_0 : \|q\alpha\| \leq \frac{1}{n^{1+\varepsilon}}\right\} \geq n. \tag{5}$$

Since ν is ergodic, the Poincaré Recurrence Theorem gives, for ν-almost every $\omega \in \Omega$

$$q_k(\omega) \underset{k\to\infty}{\longrightarrow} +\infty. \tag{6}$$

So, for ν-almost every $\omega \in \Omega$, we can choose p sufficiently large such that $p \geq k_0$ and $q_p(\omega) \geq k_0$. If $k \in \mathbb{N}$ satisfies $p \leq k < m_{n,\varepsilon}$, then $k_0 \leq q_p(\omega) \leq q_k(\omega) \leq k < m_{n,\varepsilon}$ and so $\|\alpha q_k(\omega)\| > \frac{1}{n^{1+\varepsilon}}$. Since this is true for every $k \in [p, ..., m_{n,\varepsilon}]$ we end up with $\tau^f_{\frac{1}{n^{1+\varepsilon}},p}(x) \geq m_{n,\varepsilon} \geq n$. Finally, if $p \geq m_{n,\varepsilon}$, we obviously have $\tau^f_{\frac{1}{n^{1+\varepsilon}},p}(x) \geq m_{n,\varepsilon} \geq n$. Thus for μ-almost every $x = (\omega, y) \in X$, we have

$$\begin{aligned}\underline{R}^f(x) &= \lim_{p\to\infty} \liminf_{n\to+\infty} \frac{\log \tau^f_{\frac{1}{n^{1+\varepsilon}},p}(x)}{-\log \frac{1}{n^{1+\varepsilon}}} \\ &\geq \liminf_{n\to+\infty} \frac{\log m_{n,\varepsilon}}{-\log \frac{1}{n^{1+\varepsilon}}} \\ &\geq \lim_{n\to\infty} \frac{\log n}{\log n^{1+\varepsilon}} = \frac{1}{1+\varepsilon}.\end{aligned}$$

This is true for all $\varepsilon > 0$, thus

$$\underline{R}^f(x) \geq 1.$$

The conclusion follows from Theorem 2 and equation (3). $\square$

Remark. *We point out that indeed our example fulfills the conditions of Corollary 8 when, for example, ν is a Gibbs measure* [7].

3. Upper bound for the recurrence rate in measure preserving systems

The basic strategy of the proof of Theorem 2 follows [2]. We recall the definition of weakly diametrically regular measure:

Definition 13. *A measure μ is weakly diametrically regular (wdr) on the set $Z \subset X$ if for any $\eta > 1$, for μ-almost every $x \in Z$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $r < \delta$ then $\mu(B(x, \eta r)) \leq \mu(B(x, r)) r^{-\varepsilon}$.*

Proof of Theorem 2. Since any probability measure is weakly diametrically regular on $\mathbb{R}^N$ [2], the measure $f_*\mu$ is weakly diametrically regular. We can remark that the function $\delta(f(\cdot), \varepsilon, \eta)$ in the previous definition can be made measurable for every fixed ε and η. Let us fix $\varepsilon > 0$ and $\eta = 4$. We choose $\delta > 0$ sufficiently small to have $\mu(G) > \mu(X) - \varepsilon = 1 - \varepsilon$ where

$$G := \{x \in X : \delta(f(x), \varepsilon, \eta) > \delta\}.$$

For all $r > 0$, $\lambda > 0$, $p \in \mathbb{N}$ and $x \in X$ we define the set

$$A_{r,x} := \left\{y \in X : f(y) \in B(f(x), 4r)\ ,\ \tau^f_{4r,p}(y, x) \geq \lambda^{-1} f_*\mu(B(f(x), 4r))^{-1}\right\}$$

where $\tau^f_{4r,p}(y, x) := \inf\left\{k > p : d\big(f(T^k y), f(x)\big) < 4r\right\}$ for $y \in f^{-1}B(f(x), 4r)$. Markov's inequality gives:

$$\mu(A_{r,x}) \leq \lambda f_*\mu(B(f(x), 4r)) \int_{f^{-1}B(f(x),4r)} \tau^f_{4r,p}(y, x)\, d\mu(y). \tag{7}$$

Since $\tau^f_{4r,p}(y, x)$ is bounded by the p^{th} return time of y in the set $f^{-1}B(f(x), 4r)$, by Kac's lemma we have:

$$\int_{f^{-1}B(f(x),4r)} \tau^f_{4r,p}(y, x)\, d\mu(y) \leq p. \tag{8}$$

Using (7) and (8), we have:

$$\mu(A_{r,x}) \leq p\lambda f_*\mu(B(f(x), 4r)). \tag{9}$$

If $d(f(x), f(y)) < 2r$ then

$$\tau^f_{4r,p}(y, x) f_*\mu(B(f(x), 4r)) \geq \tau^f_{6r,p}(y) f_*\mu(B(f(y), 2r)). \tag{10}$$

Definition 14. *Given $r>0$, a countable set $E\subset F$ is a maximal r-separated set for F if*

(1) $B(x,\frac{r}{2})\cap B(y,\frac{r}{2})=\emptyset$ for any two distinct $x,y\in E$.
(2) $\mu(F\setminus\bigcup_{x\in E}B(x,r))=0$.

Let $C\subset G$ such that $(f(x))_{x\in C}$ is a maximal $2r$-separated set for $f(G)$.

$$\begin{aligned}
D_\varepsilon(r) &:= \mu\left(\left\{y\in G:\, \tau^f_{6r,p}(y)f_*\mu\left(B\left(f(y),2r\right)\right)\geq r^{-2\varepsilon}\right\}\right)\\
&\leq \sum_{x\in C}\mu\left(\left\{y\in f^{-1}B\left(f(x),2r\right):\, \tau^f_{6r,p}(y)f_*\mu\left(B\left(f(y),2r\right)\right)\geq r^{-2\varepsilon}\right\}\right)\\
&\leq \sum_{x\in C}\mu(A_{r,x}) \qquad \text{by (10)}\\
&\leq p\,r^{2\varepsilon}\sum_{x\in C}f_*\mu\left(B(f(x),4r)\right) \qquad \text{with } \lambda=r^{2\varepsilon} \text{ in (9)}\\
&\leq p\,r^{\varepsilon}\sum_{x\in C}f_*\mu\left(B(f(x),r)\right) \qquad \text{since } f_*\mu \text{ is wdr and with } \eta=4\\
&\leq p\,r^{\varepsilon} \qquad \text{according to the definition of } C.
\end{aligned}$$

Finally:

$$\sum_{n,e^{-n}<\delta}D_\varepsilon(e^{-n})=\sum_{n>-\log\delta}D_\varepsilon(e^{-n})\leq p\sum_n e^{-\varepsilon n}<\infty.$$

Then, thanks to the Borel-Cantelli lemma, for μ-almost every $x\in G$

$$\tau^f_{6e^{-n},p}(x)f_*\mu\left(B\left(f(x),2e^{-n}\right)\right)\leq e^{2\varepsilon n}$$

for any n sufficiently large. Then

$$\frac{\log\tau^f_{6e^{-n},p}(x)}{n}\leq 2\varepsilon+\frac{\log f_*\mu(B(f(x),2e^{-n}))}{-n}. \tag{11}$$

Observing that for all $a>0$ we have:

$$\underline{d}^f_\mu(x)=\varliminf_{n\to\infty}\frac{\log f_*\mu\left(B\left(f(x),ae^{-n}\right)\right)}{-n} \quad\text{and}\quad \overline{d}^f_\mu(x)=\varlimsup_{n\to\infty}\frac{\log f_*\mu\left(B\left(f(x),ae^{-n}\right)\right)}{-n}$$

$$\underline{R}^f(x)=\lim_{p\to\infty}\liminf_{n\to\infty}\frac{\log\tau^f_{ae^{-n},p}(x)}{n} \quad\text{and}\quad \overline{R}^f(x)=\lim_{p\to\infty}\limsup_{n\to\infty}\frac{\log\tau^f_{ae^{-n},p}(x)}{n}$$

and since ε can be chosen arbitrarily small , we have the result if we take the limit inferior or the limit superior and then the limit over p in (11). □

4. Recurrence rate and dimension in rapidly mixing systems

Despite some similarities with [12], we emphasize that the proof of Theorem 5 is relatively different. In particular we make no assumption on the entropy of the system.

Let $a>0$, $b>0$, $c>0$ and $\rho>0$. Let $X_a:=\{x\in X\,,\,\underline{d}^f_\mu(x)>a\}$. Set

$$\begin{aligned}
G_1 &= \left\{x\in X_a:\,\forall r\leq\rho,\, f_*\mu\left(B\left(f(x),\frac{r}{2}\right)\right)\leq r^a\right\}\\
G_2 &= \left\{x\in X_a:\,\forall r\leq\rho,\, f_*\mu\left(B\left(f(x),\frac{r}{2}\right)\right)\geq r^{N+b}\right\}\\
G_3 &= \left\{x\in X_a:\,\forall r\leq\rho,\, f_*\mu\left(B\left(f(x),\frac{r}{2}\right)\right)\geq f_*\mu\left(B\left(f(x),4r\right)\right)r^c\right\}.
\end{aligned}$$

We observe that $G(a,b,c,\rho):=G_1\cap G_2\cap G_3$ satisfies

$$\mu(G(a,b,c,\rho))\underset{\rho\to 0}{\longrightarrow}\mu(X_a). \tag{12}$$

Indeed, by definition of $\underline{d}^f_\mu$, we have $\mu(G_1)\to\mu(X_a)$. Moreover, since $\overline{d}^f_\mu\leq N$ $\mu-$almost everywhere, $\mu(G_2)\to\mu(X_a)$, and since the measure $f_*\mu$ is weakly diametrically regular, $\mu(G_3)\to\mu(X_a)$.

Lemma 15. *Under the assumptions of Theorem 5, for any $x \in G(a,b,c,\rho)$, for any $n \in \mathbb{N}^*$ and for any $r \leq \rho$ we have*

$$\mu\left(f^{-1}B(f(x),r) \cap T^{-n}f^{-1}B(f(x),2r)\right) \leq \frac{L^2}{r^2}\theta_n + r^{-2c}f_*\mu(B(f(x),\frac{r}{2}))^2.$$

Proof. Let $\eta_r : [0,+\infty) \to \mathbb{R}$ be the $\frac{1}{r}$-Lipschitz function such that $1_{[0,r]} \leq \eta_r \leq 1_{[0,2r]}$. Set $\psi_{x,r}(y) = \eta_r\left(d(f(x),f(y))\right)$. Since f is L-Lipschitz, $\psi_{x,r}$ is clearly $\frac{L}{r}$-Lipschitz. Using the assumption on the decay of correlations of (X,T,μ), we obtain

$$\begin{aligned}
&\mu\left(f^{-1}B(f(x),r) \cap T^{-n}f^{-1}B(f(x),2r)\right) \leq \int_X \psi_{x,r}(y)\psi_{x,2r}(T^n y)d\mu(y) \\
&\leq \|\psi_{x,r}\|\|\psi_{x,2r}\|\theta_n + \int_X \psi_{x,r}(y)d\mu(y)\int_X \psi_{x,2r}(y)d\mu(y) \\
&\leq \frac{L^2}{r^2}\theta_n + f_*\mu(B(f(x),2r))f_*\mu(B(f(x),4r)) \\
&\leq \frac{L^2}{r^2}\theta_n + r^{-2c}f_*\mu(B(f(x),\frac{r}{2}))^2.
\end{aligned}$$

□

Lemma 16. *Under the assumptions of Theorem 5,*

$$\underline{R}^f(x) > 0 \textit{ for } \mu\textit{-almost every } x \textit{ such that } \underline{d}_\mu^f(x) > 0.$$

Proof. Let $X_+ := \{\underline{d}_\mu^f > 0\}$. Let $1 > \varepsilon > 0$ and let $a > 0$ such that $\mu(X_+) \geq \mu(X_a) > \mu(X_+) - \varepsilon$.
We fix $b > 0$, $c = \frac{a}{4}$ and for $\rho > 0$ we consider the set $G = G(a,b,c,\rho)$ defined above.
Let $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, we have $\varepsilon_n = \frac{1}{n^{4/a}} < \rho$. For $n \geq n_0$ we define

$$A_n := \{x \in X : f(T^n x) \in B(f(x),\varepsilon_n)\}.$$

For any $x \in G$ we have

$$f^{-1}B(f(x),\varepsilon_n) \cap A_n \subset f^{-1}B(f(x),\varepsilon_n) \cap T^{-n}f^{-1}B(f(x),2\varepsilon_n).$$

By Lemma 15 and the definition of G we have

$$\begin{aligned}
\mu\left(f^{-1}B(f(x),\varepsilon_n) \cap A_n\right) &\leq \frac{L^2}{\varepsilon_n^2}\theta_n + \varepsilon_n^{-2c}f_*\mu(B(f(x),\frac{\varepsilon_n}{2}))^2 \\
&\leq f_*\mu\left(B\left(f(x),\frac{\varepsilon_n}{2}\right)\right)\left[L^2(\varepsilon_n)^{-2-N-b}\theta_n + \varepsilon_n^{a/2}\right].
\end{aligned}$$

Let $B \subset G$ such that $(f(x))_{x\in B}$ is a maximal ε_n-separated set for $f(G)$. Since the collection $\left(f^{-1}B(f(x),\varepsilon_n)\right)_{x\in B}$ covers G, we have:

$$\begin{aligned}
\mu(G \cap A_n) &\leq \sum_{x\in B}\mu\left(f^{-1}B(f(x),\varepsilon_n) \cap A_n\right) \\
&\leq \sum_{x\in B} f_*\mu\left(B\left(f(x),\frac{\varepsilon_n}{2}\right)\right)\left[L^2(\varepsilon_n)^{-2-N-b}\theta_n + \varepsilon_n^{a/2}\right] \\
&\leq L^2(\varepsilon_n)^{-2-N-b}\theta_n + \varepsilon_n^{a/2}.
\end{aligned}$$

Since $\sum_{n\in\mathbb{N}^*}\varepsilon_n^{a/2} = \sum_{n\in\mathbb{N}^*}\frac{1}{n^2} < \infty$ and since the decay of correlations is super-polynomial, we obtain:

$$\sum_{n\in\mathbb{N}^*}\mu(G \cap A_n) < +\infty. \tag{13}$$

By Borel-Cantelli lemma and using (12), we have that for μ-almost every $x \in X_a$, there exists $n_1(x)$ such that for every $n \geq n_1(x)$, $f(T^n x) \notin B\left(f(x),\frac{1}{n^{4/a}}\right)$. So, for μ-almost every $x \in X_a$, for $p \geq n_1(x)$ and $n \geq n_1(x)$,

$$\tau^f_{\frac{1}{n^{4/a}},p}(x) > n \tag{14}$$

which gives us

$$\begin{aligned}\underline{R}^f(x) &= \lim_{p\to\infty}\liminf_{r\to 0}\frac{\log\tau^f_{r,p}(x)}{-\log r}\\ &= \lim_{p\to\infty}\liminf_{n\to+\infty}\frac{\log\tau^f_{\frac{1}{n^{4/a}},p}(x)}{-\log\frac{1}{n^{4/a}}}\\ &\geq \lim_{n\to+\infty}\frac{\log n}{\log n^{4/a}}=\frac{a}{4}>0.\end{aligned}$$

Since we can choose ε arbitrarily small, the lemma is proved. □

Lemma 17. *Let $a>0$, $\delta>0$ and $1>\varepsilon>0$. For μ-almost every $x\in X_a$, there exists $r(x)>0$ such that for every $r\in]0,r(x)[$ and for every integer $n\in[r^{-\delta},f_*\mu\left(B(f(x),r)\right)^{-1+\varepsilon}]$, we have $d\left(f(T^nx),f(x)\right)\geq r$.*

Proof. Let $1>\varepsilon>0$. We fix $b>0$, $c=\frac{a\varepsilon}{4}$ and for $\rho>0$ we consider the set $G=G(a,b,c,\rho)$. Let $\delta>0$ and $r\leq\rho$, we define:

$$A_\varepsilon(r):=\left\{y\in X\,:\,\exists n\in[r^{-\delta},f_*\mu\left(B(f(y),3r)\right)^{-1+\varepsilon}]\text{ such that }d\left(f(T^ny),f(y)\right)<r\right\}.$$

For any $x\in G$ we have:

$$\begin{aligned}&f^{-1}B(f(x),r)\cap A_\varepsilon(r)\\ =\ &\left\{y\,:\,f(y)\in B(f(x),r),\exists n\in[r^{-\delta},f_*\mu\left(B(f(y),3r)\right)^{-1+\varepsilon}],d(f(T^ny),f(y))<r\right\}\\ \subset\ &\left\{y\,:\,f(y)\in B(f(x),r),\exists n\in[r^{-\delta},f_*\mu\left(B(f(x),2r)\right)^{-1+\varepsilon}],d(f(T^ny),f(x))<2r\right\}\\ =\ &\bigcup_{r^{-\delta}\leq n\leq f_*\mu(B(f(x),2r))^{-1+\varepsilon}}f^{-1}B(f(x),r)\cap T^{-n}f^{-1}B(f(x),2r).\end{aligned}$$

Let $k>1$ be such that $\delta(k-1)-2\geq N+2b$ and $\rho>0$ such that $n\geq\rho^{-\delta}$ implies $(k-1)(n+1)^{-k}\geq\theta_n$ (which is possible by definition of θ_n). Setting $I_r=[r^{-\delta},f_*\mu\left(B(f(x),2r)\right)^{-1+\varepsilon}]\cap\mathbb{N}$, we have by Lemma 15

$$\begin{aligned}\mu\left(f^{-1}B(f(x),r)\cap A_\varepsilon(r)\right) &\leq \sum_{n\in I_r}\left[\frac{L^2}{r^2}\theta_n+r^{-2c}f_*\mu(B(f(x),\frac{r}{2}))^2\right]\\ &\leq L^2r^{\delta(k-1)-2}+r^{-2c}f_*\mu(B(f(x),\frac{r}{2}))^{1+\varepsilon}\\ &\leq L^2r^{N+2b}+r^{\frac{a\varepsilon}{2}}f_*\mu(B(f(x),\frac{r}{2}))\qquad\text{by definition of }G\\ &\leq f_*\mu(B(f(x),\frac{r}{2}))\left(L^2r^b+r^{\frac{a\varepsilon}{2}}\right).\end{aligned}$$

Let $B\subset G$ such that $(f(x))_{x\in B}$ is a maximal r-separated set for $f(G)$. Since the collection $\left(f^{-1}B\left(f(x),r\right)\right)_{x\in B}$ covers G, we have:

$$\begin{aligned}\mu\left(G\cap A_\varepsilon(r)\right) &\leq \sum_{x\in B}\mu\left(f^{-1}B\left(f(x),r\right)\cap A_\varepsilon(r)\right)\\ &\leq \sum_{x\in B}f_*\mu\left(B\left(f(x),\frac{r}{2}\right)\right)\left(L^2r^b+r^{\frac{a\varepsilon}{2}}\right)\\ &\leq L^2r^b+r^{a\varepsilon/2}.\end{aligned}$$

Then

$$\sum_{k\in\mathbb{N}}\mu\left(G\cap A_\varepsilon(e^{-k})\right)<+\infty$$

thus, by Borel-Cantelli lemma, we have for μ-almost every $y\in G$, there exists $n_1(y)$ such that for every $k\geq n_1(y)$, $y\notin A_\varepsilon(e^{-k})$. So, for r sufficiently small there exists $k\in\mathbb{N}$ such that $e^{-k-1}<r\leq e^{-k}\leq e^{-n_1(y)}$. In addition, since $e^{\delta k}\leq r^{-\delta}$ and $3e^{-m}<3er$, there does not exist any integer $n \in [r^{-\delta}, f_*\mu\left(B(f(y),3r)\right)^{-1+\varepsilon}]$ such that $d(f(T^n y), f(y)) < r$. Since $f_*\mu$ is weakly diametrically regular the factor $3e$ is irrelevant and the lemma is proved. □

Proof of Theorem 5. Let $\zeta > 0$. Since $\underline{R}^f(x) > 0$ for μ-almost every $x \in X_+ = \{\underline{d}_\mu^f > 0\}$ by Lemma 16, there exists $a > 0$ such that $\mu(X_+) \geq \mu(\{\underline{R}^f > a\}) > \mu(X_+) - \zeta$. For any $x \in \{\underline{R}^f > a\}$, for p sufficiently large and r sufficiently small, we have

$$\tau_{r,p}^f(x) \geq r^{-a}.$$

Thanks to Lemma 17 with $\delta = a$ and $\varepsilon > 0$, for μ-almost every $x \in \{\underline{R}^f > a\}$, if r is sufficiently small and p sufficiently large, then $\tau_{r,p}^f(x) \geq f_*\mu\left(B(f(x),r)\right)^{-1+\varepsilon}$. Thus, $\underline{R}^f \geq (1-\varepsilon)\underline{d}_\mu^f$ and $\overline{R}^f \geq (1-\varepsilon)\overline{d}_\mu^f$ μ-almost everywhere on $\{\underline{R}^f > a\}$. The theorem is proved choosing $\varepsilon > 0$ arbitrarily small and then $\zeta > 0$ arbitrarily small. □

5. Dimensions of the smooth image of Lebesgue measure

5.1. **Hausdorff and packing dimensions.** In this section, we recall the notion of Hausdorff dimension, packing dimension and pointwise dimension and the link between each other (see [9] for more details).

Let (X, d) be a metric space. Let U be a non-empty set, its diameter is

$$\operatorname{diam} U := \sup\{d(x,y) \,:\, x, y \in U\}.$$

Given $\delta > 0$, a collection $\{U_i\}_{i\in I}$ is called a countable δ-cover of a set E if I is countable, $E \subset \cup_{i\in I} U_i$ and for all $i \in I$, $0 < \operatorname{diam} U_i \leq \delta$.

Let E be a subset of X and $s \geq 0$, for $\delta > 0$, we define:

$$\mathcal{H}_\delta^s(E) = \inf\left\{\sum_{i\in I} (\operatorname{diam} U_i)^s \,:\, \{U_i\}_{i\in I} \text{ is a countable } \delta\text{-cover of } E\right\}. \tag{15}$$

We then define the *Hausdorff s-dimensional outer measure of E* as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(E). \tag{16}$$

There exists a unique t such that $\mathcal{H}^s(E) = \infty$ if $s < t$ and $\mathcal{H}^s(E) = 0$ if $s > t$ which is called the *Hausdorff dimension of E* i.e.

$$\dim_H E = \inf\{s \,:\, \mathcal{H}^s(E) < \infty\} = \sup\{s \,:\, \mathcal{H}^s(E) > 0\}. \tag{17}$$

If μ is a probability measure on X, we define the *Hausdorff dimension of μ*

$$\dim_H \mu = \inf\{\dim_H Y \,:\, \mu(Y^c) = 0\}. \tag{18}$$

Remark. *We warn the reader that this definition of the Hausdorff dimension of a measure differs from the one given by Falconer* [9] *but it is the most used in Ergodic Theory.*

Given $\varepsilon > 0$, the collection $\{B(x_i, r_i)\}_{i\in I}$ is called a ε-packing of E if I is a finite or countable set, for all $i \in I$ we have $x_i \in E$, $r_i \leq \varepsilon$ and the balls are disjoints. For $s \geq 0$, we write

$$\mathcal{P}_\varepsilon^s(E) = \sup\left\{\sum_{i\in I} (r_i)^s \,:\, \{B(x_i, r_i)\}_{i\in I} \text{ is a } \varepsilon\text{-packing of } E\right\}$$

and

$$\mathcal{P}_0^s(E) = \lim_{\varepsilon\to 0} \mathcal{P}_\varepsilon^s(E).$$

We then introduce the *s-dimensional packing outer measure*

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i=1}^\infty \mathcal{P}_0^s(E_i) \,:\, E \subset \bigcup_{i=1}^\infty E_i\right\} \tag{19}$$

and the *packing dimension of E* is defined like the Hausdorff dimension

$$\dim_P E = \inf\{s \,:\, \mathcal{P}^s(E) < \infty\} = \sup\{s \,:\, \mathcal{P}^s(E) > 0\}. \tag{20}$$

For a probability measure μ, we also have a *packing dimension of* μ

$$\dim_P \mu = \inf\left\{\dim_P Y \,:\, \mu(Y^c) = 0\right\}. \tag{21}$$

There is a link between Hausdorff dimension, packing dimension and pointwise dimension:

Proposition 18. *Assume that $X \subset \mathbb{R}^N$ for some N,*

$$\dim_H \mu = \textit{ess-sup}\ \underline{d}_\mu \tag{22}$$

and

$$\dim_P \mu = \textit{ess-sup}\ \overline{d}_\mu. \tag{23}$$

5.2. **Existence of the pointwise dimension.** Bates and Moreira proved a generalization of the classical Morse-Sard Theorem for Hausdorff measures. Unfortunately, in view of (22), this is not enough to get an upper bound for the upper pointwise dimension. A key ingredient of their proof is the following generalized Morse decomposition. Given a differentiable f from $\mathbb{R}^M$ to $\mathbb{R}^N$, for $\kappa \in \{0, 1, ..., \min\{M, N\}\}$, we define $C_\kappa := \{x \in \mathbb{R}^M : \operatorname{rank}(d_x f) = \kappa\}$.

Lemma 19 ([3]). *Let $f \in C^k(\mathbb{R}^M, \mathbb{R}^N)$ with $k \geq 2$. Let $\kappa \leq M$ be an integer. Let $\eta > 0$. There is a decomposition $\{A_i\}_{i\in\mathbb{N}}$ of C_κ such that for each $i \in \mathbb{N}$ there exist two subspaces E'_i and E''_i which satisfy $\mathbb{R}^M = E'_i \oplus E''_i$, $\dim E'_i \leq \kappa$ and if $S \subset \mathbb{R}^M$:*

$$\operatorname{diam}\left(f(S \cap A_i)\right) \leq \left(\|f|_{A_i}\|_{C^1} + \eta\right) \operatorname{diam}(\pi_{E'_i} S) + \eta(\operatorname{diam} S)^k. \tag{24}$$

This decomposition will be instrumental to prove an analogue result but for the packing dimension.

Lemma 20. *If $f \in C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ then the packing dimension of the critical set satisfies*

$$\dim_P f(C_\kappa) \leq \kappa.$$

Proof. Let $k \geq 2$, since f is of class C^∞, f is of class C^k. Let $\{A_i\}_{i\in\mathbb{N}}$ be the decomposition of C_κ given by Lemma 19 with $\eta = 1$. Let $K \subset \mathbb{R}^M$ be a compact set. Let $i \in \mathbb{N}$. Let d_i be the distance in $\mathbb{R}^M$ such that for $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^M$, $d_i(x, y) = d(\pi_{E'_i} x, \pi_{E'_i} y) + d(\pi_{E''_i} x, \pi_{E''_i} y)$. Let $\varepsilon > 0$ and $\{B(f(x_j), r_j)\}_{j\in J}$ a ε-packing of $f(A_i \cap K)$. Let $j \in J$ and $l \in J$. Taking $S := \{x_j, x_l\}$ in (24) gives

$$\begin{aligned} r_j + r_l &\leq d_i(f(x_j), f(x_l)) = \operatorname{diam}(f(S \cap A_i)) \\ &\leq Cd(\pi_{E'_i} x_j, \pi_{E'_i} x_l) + \left(d(\pi_{E'_i} x_j, \pi_{E'_i} x_l) + d(\pi_{E''_i} x_j, \pi_{E''_i} x_l)\right)^k \end{aligned}$$

where $C := \|f\|_{C^1} + 1$. This implies that

- either $Cd(\pi_{E'_i} x_j, \pi_{E'_i} x_l) \geq \frac{1}{2}(r_j + r_l)$
- or $Cd(\pi_{E'_i} x_j, \pi_{E'_i} x_l) \leq \frac{1}{2}(r_j + r_l)$ and then

  $$\left(d(\pi_{E'_i} x_j, \pi_{E'_i} x_l) + d(\pi_{E''_i} x_j, \pi_{E''_i} x_l)\right)^k \geq \frac{1}{2}(r_j + r_l).$$

  Taking ε so small that $\frac{\varepsilon}{C} \leq (1 - 2^{-1/k})\varepsilon^{1/k}$ we get

  $$\frac{1}{2C}(r_j + r_l) \leq \left(1 - 2^{-1/k}\right)\left(\frac{1}{2}(r_j + r_l)\right)^{1/k}$$

  thus

  $$d(\pi_{E''_i} x_j, \pi_{E''_i} x_l) \geq \left(\frac{1}{2}\right)^{1/k}\left(\frac{1}{2}(r_j + r_l)\right)^{1/k}.$$

For $j \in J$, let $S_j = B_{E'_i}(\pi_{E'_i} x_j, \frac{1}{4C} r_j) \times B_{E''_i}(\pi_{E''_i} x_j, \frac{1}{2}\left(\frac{1}{4} r_j\right)^{1/k}) \subset E'_i \times E''_i$. If $l \neq j$, we have $S_j \cap S_l = \emptyset$ since

- either $d(\pi_{E'_i} x_j, \pi_{E'_i} x_l) \geq \frac{1}{2C}(r_j + r_l) > \frac{1}{4C} r_j + \frac{1}{4C} r_l$
- or $d(\pi_{E''_i} x_j, \pi_{E''_i} x_l) \geq \left(\frac{1}{4}(r_j + r_l)\right)^{1/k} > \frac{1}{2}\left(\frac{1}{4} r_j\right)^{1/k} + \frac{1}{2}\left(\frac{1}{4} r_l\right)^{1/k}$.

There exists a constant δ_i such that

$$\operatorname{diam} S_j \leq \delta_i \varepsilon^{1/k}.$$

The rectangles S_j are disjoints and have non empty intersection with K, thus

$$\sum_{j\in J} \operatorname{Vol}(S_j) \leq \operatorname{Vol}(K + B(0, \delta_i \varepsilon^{1/k})). \tag{25}$$

Let $p = \dim E'_i \leq \kappa$. There exists a constant γ_i such that the volume of each S_j is

$$\operatorname{Vol}(S_j) = \gamma_i \left(\frac{1}{4C} r_j\right)^p \times \left[\frac{1}{2}\left(\frac{1}{4} r_j\right)^{1/k}\right]^{M-p}.$$

This implies together with (25)

$$\sum_{j\in J} (r_j)^{p+(M-p)/k} \leq c(i,k,K) < \infty$$

where $c(i,k,K)$ is a finite constant depending on i,k and K.
Now, by definition of the s-dimensional packing measure (19), computed with the particular metric d_i, we obtain:

$$\mathcal{P}^{M/k+p(1-1/k)}(f(A_i \cap K)) \leq c(i,k,K) < \infty.$$

This inequality holds (possibly with another constant) for the packing measure computed with the euclidean metric d since d and d_i are equivalent. Therefore

$$\dim_P f(A_i \cap K) \leq \frac{M}{k} + p(1 - \frac{1}{k}) \leq \frac{M}{k} + \kappa.$$

Finally, taking a sequence of compacts K_n such that $\mathbb{R}^M = \cup_{n\in\mathbb{N}} K_n$, we obtain:

$$\begin{aligned}
\dim_P f(C_\kappa) &= \dim_P f(\bigcup_{i,n\in\mathbb{N}} A_i \cap K_n) \\
&= \dim_P \bigcup_{i,n} f(A_i \cap K_n) \\
&= \sup_{i,n} \dim_P f(A_i \cap K_n) \qquad \text{see [9]} \\
&\leq \frac{M}{k} + \kappa.
\end{aligned}$$

Since k is arbitrarily large we get

$$\dim_P f(C_\kappa) \leq \kappa. \tag{26}$$

□

Without loss of generality, we prove Theorem 9 on $\mathbb{R}^M$ with μ equal to the Lebesgue measure λ. The general case can be deduced from it easily.

Proof of Theorem 9. Let $\kappa \in \{0, ..., \min\{M, N\}\}$.
1. If $A \subset C_\kappa$ and $\lambda(A) > 0$ then $\dim_H f_*(\lambda|_A) \geq \kappa$:
Indeed, let $B \subset C_\kappa$ with $\lambda(B) > 0$, there exist V open with $\lambda(B \cap V) > 0$ and $\pi : \mathbb{R}^N \to \mathbb{R}^\kappa$ a linear map such that $d_x\pi \circ f$ is of maximal rank κ for every $x \in V$ and so $f_\kappa := \pi \circ f$ satisfies $Jf_\kappa \neq 0$ for every $x \in V$ (where J is the Jacobian i.e. $Jf_\kappa = \sqrt{\det(d_x f_\kappa)(d_x f_\kappa)^t}$). Since π is Lipschitz, it is known [9] that:

$$\dim_H \pi(f(B)) \leq \dim_H f(B). \tag{27}$$

Using the coarea formula (e.g. [8], in fact we could have worked directly with f using [10]):

$$\int_B Jf_\kappa d\lambda = \int_{f_\kappa(B)} H^{M-\kappa}\left(B \cap f_\kappa^{-1}(\{y\})\right) d\lambda_\kappa(y) \tag{28}$$

where λ_κ is the Lebesgue measure on $\mathbb{R}^\kappa$. Since $\lambda(B\cap V)>0$ and $Jf_\kappa(x)\neq 0$ for every $x\in B\cap V$, the left-hand side of (28) does not vanish and therefore, neither does the right-hand side. Then $\lambda_\kappa(f_\kappa(B))>0$ and so $\dim_H f_\kappa(B)\geq\kappa$ which gives, using (27):

$$\kappa\leq\dim_H f(B). \tag{29}$$

Let $A\subset C_\kappa$ with $\lambda(A)>0$. We recall

$$\dim_H f_*(\lambda|_A):=\inf\{\dim_H Y\,:\,(f_*(\lambda|_A))(Y^c)=0\}. \tag{30}$$

Let Y be such that $(f_*(\lambda|_A))(Y^c)=0$. Since $f\left(A\cap f^{-1}(Y)\right)\subset Y$ we have

$$\dim_H Y\geq\dim_H f\left(A\cap f^{-1}(Y)\right).$$

Moreover, since $\lambda(f^{-1}(Y)\cap A)=\lambda(A)>0$ and $f^{-1}(Y)\cap A\subset C_\kappa$, we can choose $B=f^{-1}(Y)\cap A$ in the previous consideration and (29) gives:

$$\dim_H Y\geq\dim_H f\left(A\cap f^{-1}(Y)\right)\geq\kappa$$

and then

$$\dim_H f_*(\lambda|_A)\geq\kappa. \tag{31}$$

We define $\nu:=f_*\lambda$ and $\nu_\kappa:=f_*(\lambda|_{C_\kappa})$.
<u>2. Let us prove that $d_\nu=\kappa$ ν_κ-almost everywhere:</u>
• Firstly, since ν_κ is supported by $f(C_\kappa)$, by Lemma 20 we have

$$\dim_P\nu_\kappa\leq\dim_P f(C_\kappa)\leq\kappa.$$

Since the packing dimension satisfies the relation (23) we get

$$\overline{d}_{\nu_\kappa}(x)\leq\kappa \text{ for } \nu_\kappa\text{-almost every } x\in\mathbb{R}^N.$$

Since, for every $x\in\mathbb{R}^N$ and every $\varepsilon>0$, $\nu\left(B(x,\varepsilon)\right)\geq\nu_\kappa\left(B(x,\varepsilon)\right)$, we have

$$\overline{d}_\nu(x)\leq\overline{d}_{\nu_\kappa}(x)$$

and then

$$\overline{d}_\nu(x)\leq\kappa \text{ for } \nu_\kappa\text{-almost every } x\in\mathbb{R}^N. \tag{32}$$

• Let $K\subset\mathbb{R}^N$ be a compact subset. Let $\rho<\kappa$. Let $Z:=\{\underline{d}_\nu\leq\rho\}\cap K$. If $\nu_\kappa(Z)>0$ then $\lambda(C_\kappa\cap f^{-1}(Z))>0$, thus by (31) we obtain $\dim_H f_*(\lambda|_{C_\kappa\cap f^{-1}(Z)})\geq\kappa$. By (30) and since $f_*(\lambda|_{C_\kappa\cap f^{-1}(Z)})(Z^c)=0$, we obtain $\dim_H Z\geq\kappa$.

On the other hand, by definition of $\underline{d}_\nu$, $\forall x\in Z$, $\exists J_x\subset\mathbb{R}^+$ with $0\in\overline{J_x}$, such that $\forall r\in J_x$, $\nu\left(B(x,r)\right)\geq r^\rho$. Let $\delta>0$. We notice that $\{B(x,r)\,,\,x\in Z\,,\,r\in J_x\cap[0,\delta]\}$ cover Z so, by Besicovitch covering Theorem, there exists a subcovering $\{B(x_i,r_i)\}_{i\in I}$ with I countable and m_0 a constant depending only on N such that $Z\subset\cup_{i\in I}B(x_i,r_i)$ and the multiplicicty of the subcovering is bounded by m_0. By the definition (15)

$$\mathcal{H}^\rho_\delta(Z)\leq\sum_{i\in I}r_i^\rho\leq\sum_{i\in I}\nu\left(B(x_i,r_i)\right)\leq m_0\nu(K).$$

This implies $\dim_H Z\leq\rho$. But this is in contradiction with the fact that $\dim_H Z\geq\kappa$. Then, for all $\rho<\kappa$ and for all compact K, $\nu_\kappa(\{\underline{d}_\nu\leq\rho\}\cap K)=0$. Thus

$$\underline{d}_\nu(x)\geq\kappa \text{ for } \nu_\kappa\text{-almost every } x\in\mathbb{R}^N. \tag{33}$$

<u>3. Conclusion:</u> Using (32) together with (33) implies that $d_\nu=\kappa$ ν_κ-almost everywhere. The theorem follows from $\mathbb{R}^M=C_0\cup C_1\cup...\cup C_{\min\{M,N\}}$. □

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