QUANTITATIVE RECURRENCE IN TWO-DIMENSIONAL EXTENDED PROCESSES

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Abstract. Under some mild condition, a random walk in the plane is recurrent. In particular each trajectory is dense, and a natural question is how much time one needs to approach a given small neighborhood of the origin. We address this question in the case of some extended dynamical systems similar to planar random walks, including $\mathbb{Z}^2$-extension of hyperbolic dynamics. We define a pointwise recurrence rate and relate it to the dimension of the process, and establish a convergence in distribution of the rescaled return times near the origin.

1. Introduction

1.1. **Motivation.** This work was partly motivated by the recurrence properties of the planar Lorentz process. Given an initial condition, say x, we thus know that the process will return back ε close to its starting point x. A basic question is : *when* ? For finite measure preserving dynamical systems this question has some deep relations to the Hausdorff dimension of the underlying invariant measure. Namely, if $\tau_\varepsilon(x)$ represents this time, in many situations

$$\tau_\varepsilon(x) \approx \frac{1}{\varepsilon^{\dim}}$$

for typical points x, where dim is the Hausdorff dimension of the underlying invariant measure. This has been proved for example for interval maps [14] and rapidly mixing systems [1, 13]. Another type of results is the exponential distribution of rescaled return times and the lognormal fluctuations of the return times [8, 4].

In this paper we are dealing with systems where the underlying natural measure is indeed infinite. This typically causes the return times to be non integrable, in contrast with the finite measure case. However, the systems we are thinking about have in common the property that, in some sense, the behaviors at small scale and at large scale are independent. The large scale dynamics being some sort of recurrent random walk, and the small scale dynamics a finite measure preserving system. Although our first motivation was Lorentz process, we will not mention it further in this paper. We instead provide some results related to processes which are in essence similar to it. The first case treated in Section 2 is a toy model designed to give the hint of the general case. Then, in Section 3 we briefly mention the case of planar random walks. Finally, in Section 4 we give a complete analysis of the quantitative behavior of return times in the case of $\mathbb{Z}^2$-extensions of subshifts of finite type.

1.2. **Description of the main result : $\mathbb{Z}^2$-extensions of subshifts of finite type.** In this study of the quantitative behavior of recurrence we choose to work with $\mathbb{Z}^2$-extensions of hyperbolic dynamics. We emphasize that this dimension 2 is at the threshold between recurrent and non recurrent processes, since in higher dimension these processes are neither recurrent (except if degenerate). It makes sense to show how our results behave with respect to the

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dimension. For completeness, we call the non-extended system itself a $\mathbb{Z}^0$-extension. In this non-extended case, the type of results we have in mind (see Table 1) have already been established respectively by Ornstein and Weiss [12], Hirata [8] and Collet, Galves and Schmitt [4]. The case of $\mathbb{Z}^2$-extension is completely done in Section 4. The case of $\mathbb{Z}^1$-extension can be derived easily following the technique used in the present paper. The essential difference is that the local limit theorem has the one-dimensional scaling in $\frac{1}{\sqrt{n}}$, instead of $\frac{1}{n}$ in the two-dimensional case. The following table summarizes the different results as the dimension changes. The first line of results corresponds to Theorem 7, the second to Theorem 8 and the third to Corollary 9. We refer to Section 4 for precise statements.

Dimension	$\mathbb{Z}^0$-extension	$\mathbb{Z}^1$-extension	$\mathbb{Z}^2$-extension
Scale	$\lim_{\varepsilon\to 0}\frac{\log \tau_\varepsilon}{-\log\varepsilon} = d$	$\lim_{\varepsilon\to 0}\frac{\log \tau_\varepsilon}{-\log\varepsilon} = \mathbf{2}d$	$\lim_{\varepsilon\to 0}\frac{\log \mathbf{log}\tau_\varepsilon}{-\log\varepsilon} = d$
Local law	$\nu\big(\nu(B_\varepsilon)\tau_\varepsilon > t\big) \to e^{-t}$	$\nu\big(\nu(B_\varepsilon)^{\mathbf{2}}\tau_\varepsilon > t\big) \to \frac{1}{1+\beta\sqrt{t}}$	$\nu\big(\nu(B_\varepsilon)\mathbf{log}\tau_\varepsilon > t\big) \to \frac{1}{1+\beta t}$
Lognormal fluctuations	$\varepsilon^d\tau_\varepsilon$	$\varepsilon^{\mathbf{2}d}\tau_\varepsilon$	$\varepsilon^d\mathbf{log}\tau_\varepsilon$

TABLE 1. Recurrence for $\mathbb{Z}^k$-extensions. B_ε denotes the ball of radius ε, ν is a Gibbs measure on the base and d is the Hausdorff dimension of ν.

2. A toy model in dimension two

We present a toy model designed to posses a lot of independence. It has the advantage of giving the right formulas with elementary proofs.

2.1. **Description of the model and statement of the results.**

Let us consider two sequences of independent identically distributed random variables $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 0}$ independent one from the other such that :

- the random variable X_1 is uniformly distributed on $\{(1,0),(-1,0),(0,1),(0,-1)\}$;
- the random variable Y_0 is uniformly distributed on $]0;1[^2$.

Let us notice that $S_n := \sum_{k=1}^n X_k$ (with the convention $S_0 := 0$) is the symmetric random walk on $\mathbb{Z}^2$. We study a kind of random walk M_n on $\mathbb{R}^2$ given by $M_n = S_n + Y_n$.

Another representation of our model could be the following. Let $S = \mathbb{R}^d$ and consider the system $\mathbb{Z}^2 \times S$. Attached to each site $i \in \mathbb{Z}^2$ of the lattice, there is a local system which lives on S and σ_n is a i.i.d. sequence of S-valued random variable with some density ρ, independent of the X_n's. Then we look at the random walk (S_n, σ_n), thinking at σ_n as a spin.

We want to study the asymptotic behavior, as ε goes to zero, of the return time in the open ball $B(M_0,\varepsilon)$ of radius ε centered at x (for the euclidean metric). Let

$$\tau_\varepsilon := \min\{m \geq 1\colon |M_m - M_0| < \varepsilon\}.$$

Note that, for all $x \in [0;1[^2$, we have : $\tau_\varepsilon = \min\{m \geq 1 \colon |M_m - x| < \varepsilon\}$. We will prove the following :

Theorem 1. *Almost surely,* $\frac{\log\log \tau_\varepsilon}{-\log \varepsilon}$ *converges to the dimension 2 of the Lebesgue measure on* $\mathbb{R}^2$ *as* ε *goes to zero.*

Theorem 2. *For all* $t \geq 0$ *we have :*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\lambda(B(M_0,\varepsilon)) \log \tau_\varepsilon \leq t\right) = \frac{1}{1+\frac{\pi}{t}}.$$

2.2. **Proof of the pointwise convergence of the recurrence rate to the dimension.** To simplify the exposition we suppose that M_0 is in $]0;1[^2$ and that ε is so small that $B(x,\varepsilon)$ is contained in $[0;1[^2$.

First, let us define $R_1 := \min\{m \geq 1 \colon S_m = 0\}$. According to [6], we know that we have :

$$\mathbb{P}(R_1 > s) \sim \frac{\pi}{\log s} \quad \text{as } s \text{ goes to infinity} \tag{1}$$

We then define for any $p \geq 0$ the p^{th} return time R_p in $[0;1[^2$ by induction :

$$R_{p+1} := \inf\{m > R_p \ : \ S_m = 0\}.$$

Observe that R_p is the p^{th} return time at the origin of the random walk S_n on the lattice, thus the delays between successive return times $R_p - R_{p-1}$, setting $R_0 = 0$, are independent and identically distributed. Consequently :

$$\mathbb{P}(R_p - R_{p-1} > s) = \mathbb{P}(R_1 > s) \tag{2}$$

The proof of Theorem 1 follows from these two lemmas

Lemma 3. *Almost surely,* $\frac{\log\log R_n}{\log n} \to 1$ *as* $n \to \infty$.

Proof. It suffices to prove that for any $0 < \alpha < 1$, almost surely, $e^{n^{1-\alpha}} \leq R_n \leq n e^{n^{1+\alpha}}$ provided n is sufficiently large. By independence and equation (2) we have

$$\mathbb{P}(\log R_n \leq n^{1-\alpha}) \leq \mathbb{P}(\forall p \leq n, \log(R_p - R_{p-1}) \leq n^{1-\alpha}) = \mathbb{P}(\log R_1 \leq n^{1-\alpha})^n.$$

According to the asymptotic formula (1), for n sufficiently large

$$\mathbb{P}(\log R_1 \leq n^{1-\alpha})^n \leq \left(1 - \frac{\pi}{2n^{1-\alpha}}\right)^n \leq e^{-\pi\frac{n^\alpha}{2}}.$$

The first inequality follows then from Borel Cantelli lemma.

Moreover, according to formulas (2) and (1), we have $\sum_{n\geq 1} \mathbb{P}(\log(R_n - R_{n-1}) > n^{1+\alpha})) < +\infty$. Hence, by Borel Cantelli lemma, we know that almost surely, for n sufficiently large, we have $R_n - R_{n-1} \leq e^{n^{1+\alpha}}$. From this we get the second inequality. □

Observe that $\tau_\varepsilon = R_{T_\varepsilon}$ with $T_\varepsilon := \min\{\ell \geq 1 \ : \ |Y_{R_\ell} - Y_0| < \varepsilon\}$.

Lemma 4. *Almost surely,* $\frac{\log T_\varepsilon}{-\log \lambda(B(Y_0,\varepsilon))} \to 1$ *as* $\varepsilon \to 0$.

Proof. By independence of the Y_ℓ, the random variable T_ε has a geometric distribution with parameter $\lambda_\varepsilon := \lambda(B(Y_0,\varepsilon)) = \pi\varepsilon^2$. For any $\alpha > 0$ we have the simple decomposition

$$\mathbb{P}\left(\left|\frac{\log T_\varepsilon}{-\log \lambda_\varepsilon} - 1\right| > \alpha\right) = \mathbb{P}\left(T_\varepsilon > \lambda_\varepsilon^{-1-\alpha}\right) + \mathbb{P}\left(T_\varepsilon < \lambda_\varepsilon^{-1+\alpha}\right).$$

The first term is directly handle by Markov inequality :

$$\mathbb{P}\left(T_\varepsilon > \lambda_\varepsilon^{-1-\alpha}\right) \le \lambda_\varepsilon^\alpha,$$

while the second term may be computed using the geometric distribution :

$$\begin{aligned}
\mathbb{P}(T_\varepsilon < \lambda_\varepsilon^{-1+\alpha}) &= 1-(1-\lambda_\varepsilon)^{(\lambda_\varepsilon^{-1+\alpha})}\\
&= 1-\exp\left[\lambda_\varepsilon^{-1+\alpha}\log(1-\lambda_\varepsilon)\right]\\
&\le -\lambda_\varepsilon^{-1+\alpha}\log(1-\lambda_\varepsilon)\\
&= O(\lambda_\varepsilon^\alpha).
\end{aligned}$$

Let us define $\varepsilon_n := n^{-1/\alpha}$. According to the Borel-Cantelli lemma, $\lambda_{\varepsilon_n}T_{\varepsilon_n}$ converges almost surely to the constant 1. The conclusion follows from the facts that $(\varepsilon_n)_{n\ge1}$ is a decreasing sequence of real numbers satisfying $\lim_{n\to+\infty}\varepsilon_n = 0$ and $\lim_{n\to+\infty}\frac{\varepsilon_n}{\varepsilon_{n+1}} = 1$, and T_ε is monotone in ε. □

Proof of Theorem 1. The theorem follows from Lemma 3 and Lemma 4 since

$$\frac{\log\log\tau_\varepsilon}{-\log\varepsilon} = \frac{\log\log R_{T_\varepsilon}}{\log T_\varepsilon}\frac{\log T_\varepsilon}{-\log\lambda_\varepsilon}\frac{\log\lambda_\varepsilon}{\log\varepsilon} \to 1\times1\times2$$

almost surely as $\varepsilon\to0$. □

2.3. **Proof of the convergence in distribution of the rescaled return time.**

Proof of Theorem 2. Let $t>0$. By independence of T_ε and the R_n we have

$$F_\varepsilon(t) := \mathbb{P}(\lambda(B(Y_0,\varepsilon)\log\tau_\varepsilon \le t) = \sum_{n\ge1}\mathbb{P}(T_\varepsilon = n)\mathbb{P}\left(\log R_n \le \frac{t}{\lambda_\varepsilon}\right).$$

Since T_ε has a geometric law with parameter λ_ε, $F_\varepsilon(t)$ is equal to $G_{\lambda_\varepsilon}(t)$ with :

$$G_\delta(t) := \sum_{n\ge1}\delta(1-\delta)^{n-1}\mathbb{P}(\log R_n \le \frac{t}{\delta}).$$

First, we notice that the independence of the successive returns gives for any $u>0$ that

$$\mathbb{P}\left(R_n\le u\right) \le \mathbb{P}\left(\max_{k=1,\dots,n}R_k - R_{k-1}\le u\right) = \mathbb{P}\left(R_1\le u\right)^n$$

Let $\alpha<1$. Using the inequality above and the equivalence relation (1) we get that for any $\delta>0$ sufficiently small,

$$G_\delta(t) \le \sum_{n\ge1}\delta(1-\delta)^{n-1}\left(1-\alpha\frac{\pi\delta}{t}\right)^n = \frac{1}{1+\alpha\dfrac{\pi}{t}} + O(\delta).$$

This implies that $\limsup_{\delta\to0}F_\epsilon(t) \le \frac{1}{1+\frac{\pi}{t}}$.

Fix $A>0$ and keeping the same notations observe that we have $F_\varepsilon(t)\ge H_{\lambda_\varepsilon}(t)$ with :

$$H_\delta(t) := \sum_{1\le n\le A/\delta}\delta(1-\delta)^{n-1}\mathbb{P}\left(\log R_n\le\frac{t}{\delta}\right).$$

Note that the independence gives in addition that for any $u>0$

$$\mathbb{P}\left(R_n\le u\right) \ge \mathbb{P}\left(\max_{k=1,\dots,n}R_k - R_{k-1}\le u/n\right) = \mathbb{P}\left(R_1\le u/n\right)^n.$$

Let $\alpha > 1$. Using the inequality above and the equivalence relation (1) we get that for sufficiently small $\delta > 0$

$$H_\delta(t) \geq \sum_{1\leq n\leq A/\delta} \delta(1-\delta)^{n-1}\left(1-\alpha\frac{\pi}{\frac{t}{\delta}-\log n}\right)^n \geq \sum_{1\leq n\leq A/\delta} \delta(1-\delta)^{n-1}\left(1-\alpha^2\frac{\pi\delta}{t}\right)^n.$$

Evaluating the limit when $\delta \to 0$ of the geometric sum and then letting $A \to \infty$ we end up with $\liminf_{\delta\to 0} H_\delta(t) \geq \frac{1}{1+\frac{\pi}{t}}$. □

3. Random walk on the plane

We now consider a true random walk on $\mathbb{R}^2$, $M_n = X_1 + \cdots + X_n$ where the X_i's are i.i.d. random variables distributed with a law μ of zero mean, with invertible finite covariance matrix Σ^2 and characteristic function $\hat{\mu}(t) = \int e^{it\cdot x} d\mu(x)$. Let τ_ε be the minimal time for the walk to return in the ε-neighborhood of the origin :

$$\tau_\varepsilon := \min\{n \geq 1 \colon |M_n| < \varepsilon\}.$$

Theorem 5. *Assume additionally that the distribution μ satisfies the Cramer condition*

$$\limsup_{|t|\to\infty} |\hat{\mu}(t)| < 1.$$

Then almost surely $\lim_{\varepsilon\to 0} \frac{\log\log \tau_\varepsilon}{-\log\varepsilon} = 2$.

We remark that a kind of Cramer's condition on the law is necessary, since there exists some planar recurrent random walks for which the statement of the theorem is false (the return time being even larger than expected). We discovered after the completion of the proof of this theorem that its statement is contained in Theorem 2 of Cheliotis's recent paper [5]. For completeness we describe the strategy of our original proof here but leave most details to the reader. A key point is a uniform version of the local limit theorem. Indeed we need an estimation of the type $\mathbb{P}(|S_n| < \varepsilon) \sim \frac{c\varepsilon^2}{n}$, with some uniformity in ε (for some $c > 0$). One can follow the classical proof of the local limit theorem (see Theorem 10.17 of [3]) to get the following :

Lemma 6. *Let $\delta \in]0; 1/2[$. There exists $c_1 > 0$, $c_2 > 0$, $\varepsilon_0 > 0$, $a > 0$ and an integer N, such that, for any $n > N$, for any $\varepsilon \in]0; 1[$:*

$$\frac{c_1\varepsilon^2}{n} - \frac{\exp(-an^{1-2\delta})}{\varepsilon} \leq \mathbb{P}\left(S_n \in B(0,\varepsilon)\right) \leq \frac{c_2\varepsilon^2}{n} + \frac{\exp(-an^{1-2\delta})}{\varepsilon}.$$

Then, this information on the probability of return is strong enough to estimate the first return time to the ε-neighborhood of the origin.

Proof of Theorem 5. For any $\alpha > \frac{1}{2}$, using $\varepsilon_n = 1/\log^\alpha n$, we get that $\mathbb{P}(|S_n| < \varepsilon_n)$ is summable. By the Borel Cantelli lemma, we have $\tau_{\varepsilon_n}(x) > n$ eventually almost surely. Thus

$$\liminf_{n\to\infty} \frac{\log\log \tau_{\varepsilon_n}}{-\log\varepsilon_n} \geq \liminf_{n\to\infty} \frac{\log\log n}{\log\log^\alpha n} = \frac{1}{\alpha},$$

which implies by monotonicity and the fact that α is arbitrary that $\liminf_{\epsilon\to 0} \frac{\log\log\tau_\varepsilon}{-\log\varepsilon} \geq 2$.

Let $\alpha < \frac{1}{2}$. To control the lim sup, we will take $n = n_\varepsilon = \lceil \varepsilon^{-\frac{1}{\alpha}} \rceil$. We use a similar decomposition to that of Dvoretski and Erdös in [6]. Let $A_k = \{|S_k| < \varepsilon \text{ and } \forall p = k+1, \ldots, n, |S_p - S_k| > 2\varepsilon\}$. The A_k's are disjoint, hence by independence and invariance, and with our choice for n :

$$1 \geq \sum_{k=1}^n \mathbb{P}(A_k) \geq \sum_{k=1}^n \mathbb{P}(|S_k| < \varepsilon)\mathbb{P}(\tau_{2\varepsilon} > n-k) \geq \sum_{k=1}^n \mathbb{P}(|S_k| < \varepsilon)\mathbb{P}(\tau_{2\varepsilon} > n) \geq \sum_{k=N}^n \frac{c\varepsilon^2}{k}\mathbb{P}(\tau_{2\varepsilon} > n).$$

Hence we have $\mathbb{P}(\tau_\varepsilon > n) \le \frac{1}{c\varepsilon^2 \log n} \le c\varepsilon^{\frac{1}{\alpha}-2}$, if n is large enough. Let $\varepsilon_p = p^{\frac{-2}{\alpha-2}}$. By Borel Cantelli lemma we have $\log \tau_{\varepsilon_p} \le \varepsilon_p^{-\alpha}$ eventually almost surely, hence $\limsup_{p\to\infty} \frac{\log\log \tau_{\varepsilon_p}}{-\log \varepsilon_p} \le \alpha$. By monotonicity and the fact that α is arbitrary we get the result. □

4. Case of Euclidean extension of hyperbolic systems

4.1. **Description of the $\mathbb{Z}^2$-extension of a mixing subshift.** Let us fix a finite set $\mathcal{A}$ called alphabet. Let us consider a matrix M indexed by $\mathcal{A}\times\mathcal{A}$ with 0-1 entries. We suppose that there exists a positive integer n_0 such that each component of M^{n_0} is non zero. We define the set of allowed sequences Σ as follows

$$\Sigma := \{\omega := (\omega_n)_{n\in\mathbb{Z}} \;:\; \forall n \in \mathbb{Z},\ M(\omega_n, \omega_{n+1}) = 1\}.$$

We endow Σ with the metric d given by

$$d\left(\omega, \omega'\right) := e^{-m},$$

where m is the greatest integer such that $\omega_i = \omega'_i$ whenever $|i| < m$. We define the shift $\theta : \Sigma \to \Sigma$ by $\theta\left((\omega_n)_{n\in\mathbb{Z}}\right) = (\omega_{n+1})_{n\in\mathbb{Z}}$. For any function $f\colon \Sigma \to \mathbb{R}$ we denote by $S_n f = \sum_{\ell=0}^{n-1} f\circ\theta^\ell$ its ergodic sum. Let us consider an Hölder continuous function $\varphi : \Sigma \to \mathbb{Z}^2$. We define the $\mathbb{Z}^2$-extension F of the shift θ by

$$\begin{aligned} F\colon \Sigma\times\mathbb{Z}^2 &\to \Sigma\times\mathbb{Z}^2 \\ (x,m) &\mapsto (\theta x, m+\varphi(x)). \end{aligned}$$

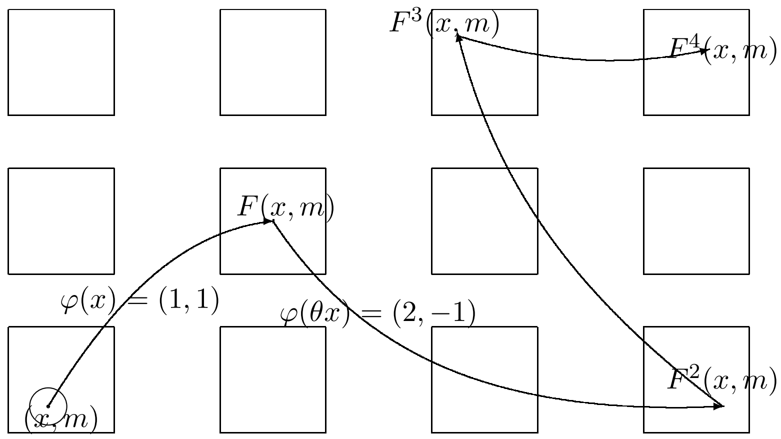

Figure 1. Dynamics of the $\mathbb{Z}^2$-extension F of the shift.

We want to know the time needed for a typical orbit starting at $(x,m) \in \Sigma\times\mathbb{Z}^2$ to return ε-close to the initial point after iterations of the map F. By translation invariance we can assume that the orbit starts in the cell $m = 0$. More precisely, let

$$\tau_\varepsilon(x) = \min\{n \ge 1\colon F^n(x,0) \in B(x,\varepsilon)\times\{0\}\}.$$

Observe that $F^n(x,m) = (\theta^n x, m + S_n\varphi(x))$, thus

$$\tau_\varepsilon(x) = \min\{n \ge 1\colon S_n\varphi(x) = 0 \text{ and } d(\theta^n x, x) < \varepsilon)\}.$$

Let ν be the Gibbs measure associated to some Hölder continuous potential h, and denote by σ_h^2 the asymptotic variance of h under the measure ν. Recall that σ_h^2 vanishes if and only if h is cohomologous to a constant, and in this case ν is the unique measure of maximal entropy.

We know that there exists a positive integer m_0 such that the function φ is constant on each m_0-cylinders.

Let us denote by σ_φ^2 the asymptotic covariance matrix of φ :

$$\sigma_\varphi^2 = \lim_{n\to+\infty} \mathrm{Cov}_\nu\left(\frac{1}{\sqrt{n}} S_n\varphi\right).$$

We suppose that σ_φ^2 is invertible. Note that if it is not the case then it means that range of $S_n\varphi$ is essentially contained in a one-dimensional lattice; in this case we can reduce our study to the corresponding $\mathbb{Z}$-extension.

We add the following hypothesis of non-arithmeticity on φ : We suppose that, for any $u \in [-\pi;\pi]^2 \setminus \{(0,0)\}$ the only solutions (λ, g), with $\lambda \in \mathbb{C}$ and $g\colon \Sigma \to \mathbb{C}$ measurable with $|g| = 1$, of the functional equation

$$g \circ \sigma \overline{g} = \lambda e^{iu\cdot\varphi}$$

is the trivial one $\lambda = 1$ and $g = const$. Note that if it is not the case then it should mean that the range of $S_n\varphi$ is essentially contained in a sub-lattice; in this case we could just do a change of basis and apply our result to the new twisted $\mathbb{Z}^2$-extension. We emphasize that this non-arithmeticity condition is equivalent to the fact that all the circle extensions T_u defined by $T_u(x,t) = (\theta(x), t + u\cdot\varphi(x))$ are weakly-mixing for $u \in [-\pi;\pi]^2 \setminus \{(0,0)\}$.

In this context, we prove the following results :

Theorem 7. *The sequence of random variables $\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}$ converges almost surely as $\varepsilon \to 0$ to the Hausdorff dimension d of the measure ν.*

Theorem 8. *The sequence of random variables $\nu(B_\varepsilon(\cdot))\log\tau_\varepsilon(\cdot)$ converges in distribution as $\varepsilon \to 0$ to a random variable with distribution function of density $t \mapsto \frac{\beta t}{1+\beta t}\mathbf{1}_{(0;+\infty)}(t)$, with $\beta := \frac{1}{2\pi\sqrt{\det\sigma_\varphi^2}}$.*

Corollary 9. *If the measure ν is not the measure of maximal entropy, then the sequence of random variables $\frac{\log\log\tau_\varepsilon + d\log\varepsilon}{\sqrt{-\log\varepsilon}}$ converges in distribution as $\varepsilon \to 0$ to a centered gaussian random variable of variance $2\sigma_h^2$.*

In the case where ν is the measure of maximal entropy, then the sequence of random variables $\varepsilon^d\log\tau_\varepsilon$ converges in distribution to a finite mixture of the law found in the previous theorem, i.e. there exists some probability vector $\alpha = (\alpha_n)$ and positive constants β_n such that the sequence of random variables $\varepsilon^d\log\tau_\varepsilon$ converges in distribution to a random variable with distribution function of density $\sum_n \alpha_n \frac{\beta_n t}{1+\beta_n t}\mathbf{1}_{(0;+\infty)}(t)$.

Examples 10. *We provide an example where the function $\varphi(x)$ only depends on the first coordinate x_0, i.e. $\varphi(x) = \varphi(x_0)$. On the shift $\Sigma = \{E, N, W, S\}^{\mathbb{Z}}$ the function $\varphi(E) = (1,0)$, $\varphi(N) = (0,1)$, $\varphi(W) = (-1,0)$ and $\varphi(S) = (0,-1)$ fulfill the hypotheses.*

The rest of the section is devoted to the proof of these results. In Subsection 4.2 we recall some preliminary results and prove a uniform conditional local limit theorem. In Subsection 4.3 we prove Theorem 7 and in Subsection 4.4 we prove Theorem 8 and Corollary 9.

4.2. **Spectral analysis of the Perron-Frobenius operator and Local Limit Theorem.** In order to exploit the spectral properties of the Perron-Frobenius operator we quotient out the "past". We define :

$$\hat{\Sigma} := \{\omega := (\omega_n)_{n\in\mathbb{N}} \ : \ \forall n \in \mathbb{N},\ M(\omega_n,\omega_{n+1}) = 1\},$$

$$\hat{d}\left((\omega_n)_{n\geq 0},(\omega'_n)_{n\geq 0}\right) := e^{-\hat{r}(\omega,\omega')}$$

with $\hat{r}((\omega_n)_{n\geq 0},(\omega'_n)_{n\geq 0}) = \inf\{m \geq 0 \ : \ \omega_m \neq \omega'_m\}$ and

$$\hat{\theta}\left((\omega_n)_{n\geq 0}\right) = (\omega_{n+1})_{n\geq 0}.$$

Let us define the canonical projection $\Pi : \Sigma \to \hat{\Sigma}$ defined by $\pi\left((\omega_n)_{n\in\mathbb{Z}}\right) = (\omega_n)_{n\geq 0}$. Let $\hat{\nu}$ be the image probability measure (on $\hat{\Sigma}$) of ν by Π. There exists a function $\psi : \hat{\Sigma} \to \mathbb{Z}^2$ such that $\psi \circ \Pi = \varphi \circ \theta^{m_0}$.

Let us denote by $P : L^2(\hat{\nu}) \to L^2(\hat{\nu})$ the Perron-Frobenius operator such that :

$$\forall f,g \in L^2(\hat{\nu}),\quad \int_{\hat{\Sigma}} Pf(x)g(x)\,d\hat{\nu}(x) = \int_{\Sigma^+} f(x)g\circ\hat{\theta}(x)\,d\hat{\nu}(x).$$

Let $\eta \in]0;1[$. Let us denote by $\mathcal{B}$ the set of bounded η-Hölder continuous function $g : \hat{\Sigma} \to \mathbb{C}$ endowed with the usual Hölder norm :

$$\|g\|_{\mathcal{B}} := \|g\|_\infty + \sup_{x\neq y} \frac{|g(y)-g(x)|}{\hat{d}(x,y)^\eta}.$$

We denote by $\mathcal{B}^*$ the topological dual of $\mathcal{B}$. For all $u \in \mathbb{R}^2$, we consider the operator P_u defined on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ by :

$$P_u(f) := P(e^{iu\psi}f).$$

Note that the hypothesis of non-arithmeticity of φ is equivalent to the following one on ψ : *for any $u \in [-\pi;\pi]^2 \setminus \{(0,0)\}$, the operator P_u has no eigenvalue on the unit circle.*

We will use the method introduced by Nagaev in [10, 11], adapted by Guivarch and Hardy in [7] and extended by Hennion and Hervé in [9]. It is based on the family of operators $(P_u)_u$ and their spectral properties expressed in these two propositions.

Proposition 11 (Uniform contraction)**.** *There exists $\alpha \in (0;1)$, $C > 0$ such that, for all $u \in [-\pi;\pi]^2 \setminus [-\beta;\beta]^2$ and all integer $n \geq 0$, for all $f \in \mathcal{B}$, we have :*

$$\|P_u^n(f)\|_{\mathcal{B}} \leq C\alpha^n \|f\|_{\mathcal{B}}.$$

This property easily follows from the fact that the spectral radius is smaller than 1 for $u \neq 0$. In addition, since P is a quasicompact operator on $\mathcal{B}$ and since $u \mapsto P_u$ is a regular perturbation of $P_0 = P$, we have :

Proposition 12 (Perturbation result)**.** *There exists $\alpha > 0$, $\beta > 0$, $C > 0$, $c_1 > 0$, $\theta \in]0;1[$ such that : there exists $u \mapsto \lambda_u$ belonging to $C^3([-\beta;\beta]^2 \to \mathbb{C})$, there exists $u \mapsto v_u$ belonging to $C^3([-\beta;\beta]^2 \to \mathcal{B})$, there exists $u \mapsto \phi_u$ belonging to $C^3([-\beta;\beta]^2 \to \mathcal{B}*)$ such that, for all $u \in [-\beta;\beta]^2$, for all $f \in \mathcal{B}$ and for all $n \geq 0$, we have the decomposition :*

$${P_u}^n(f) = {\lambda_u}^n \phi_u(f) v_u + N_u^n(f),$$

with

1. $\|{N_u}^n(f)\|_{\mathcal{B}} \leq C\alpha^n \|f\|_{\mathcal{B}}$,
2. $|\lambda_u| \leq e^{-c_1|u|^2}$ *and* $c_1|u|^2 \leq \sigma_\phi^2 u\cdot u$,
3. *with initial values :* $v_0 = \mathbf{1}$, $\phi_0 = \hat{\nu}$, $\nabla\lambda_{u=0} = 0$ *and* $D^2\lambda_{u=0} = -\sigma_\varphi^2$.

This result is a multidimensional version of IV-8, IV-11, IV-12 of [9], in this context.

Next proposition is essential to our work. It may be viewed as a doubly local version of the central limit theorem : first, it is local in the sense that we are looking at the probability that $S_n\varphi = 0$ while the classical central limit theorem is only concerned with the probability that $|S_n\varphi| \le \varepsilon\sqrt{n}$; second, it is local in the sense that we are looking at this probability conditioned to the fact that we are starting from a set A and landing at a set B on the base.

Proposition 13. *There exists a real number $C_1 > 0$ such that, for all integer $n > k > m_0$ and all $q > 0$, all q-cylinder A of Σ and all measurable subset B of Σ^+, we have :*

$$\left| \nu\left(A \cap \{S_n\varphi = 0\} \cap \theta^{-n}(\theta^k(\Pi^{-1}(B)))\right) - \frac{\nu(A)\hat{\nu}(B)}{2\pi(n-k)\sqrt{det(\sigma_\varphi^2)}} \right| \le C_1 \frac{\hat{\nu}(B)ke^{\eta q}}{(n-k)^{3/2}}.$$

Proof. We want to estimate the measure of the set $Q = A \cap \{S_n\varphi = 0\} \cap \theta^{-n}(\theta^k\Pi^{-1}B)$. Since A is a q-cylinder, $\theta^{-q}A = \Pi^{-1}\hat{A}$ for the cylinder set $\hat{A} = \Pi\theta^{-q}A$. Next, since $\varphi \circ \theta^{m_0} = \psi \circ \Pi$ we have the identity $\{S_n\varphi \circ \theta^{m_0} = 0\} = \{S_n\psi \circ \Pi = 0\}$. Thus using the semi-conjugacy $\hat{\theta} \circ \Pi = \Pi \circ \theta$

$$\begin{aligned}\theta^{-q-m_0}Q &= \theta^{-m_0}(\Pi^{-1}\hat{A}) \cap \{S_n\psi \circ \Pi \circ \theta^q = 0\} \cap \theta^{-n-q+(k-m_0)}(\Pi^{-1}B)) \\ &= \Pi^{-1}(\hat{\theta}^{-m_0}(\hat{A}) \cap \{S_n\psi \circ \hat{\theta}^q = 0\} \cap \hat{\theta}^{-n-q+(k-m_0)}(B))\end{aligned}$$

Since ψ is integer-valued, the relation $1_{\{k=0\}} = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} e^{iu\cdot k}du$ for any $k \in \mathbb{Z}^2$ gives, by invariance of ν,

$$\begin{aligned}\nu(Q) &= \mathbb{E}_{\hat{\nu}}\left(1_{\hat{A}} \circ \hat{\theta}^{m_0} 1_B \circ \hat{\theta}^{q+n-(k-m_0)} \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \exp(iu \cdot S_n\psi \circ \hat{\theta}^q)du\right) \\ &= \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \mathbb{E}_{\hat{\nu}}\left(1_{\hat{A}} \circ \hat{\theta}^{m_0} 1_B \circ \hat{\theta}^{q+n-(k-m_0)} \exp(iu \cdot S_n\psi \circ \hat{\theta}^q)\right) du\end{aligned}$$

We then estimate the expectation $a(u) = \mathbb{E}_{\hat{\nu}}(\cdots)$. Using the fact that the Perron-Frobenius P is the dual of $\hat{\theta}$ we get

$$\begin{aligned}a(u) &= \mathbb{E}_{\hat{\nu}}\left(P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}) \exp(iu \cdot S_n\psi) 1_B \circ \hat{\theta}^{n-(k-m_0)}\right) \\ &= \mathbb{E}_{\hat{\nu}}\left(P_u^n(P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})1_B \circ \hat{\theta}^{n-(k-m_0)}\right) \\ &= \mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B P_u^{n-(k-m_0)} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}))\right).\end{aligned}$$

Let us denote for simplicity $\ell = n - (k - m_0)$. We first show that for large u, the quantity $a(u)$ is negligeable. Using the contraction inequality given in proposition 11 applied to ${P_u}^\ell(\mathbf{1})$, the fact that $\|P^q(\mathbf{1}_{\hat{A}} \circ \hat{\theta}^{m_0})\|_{\mathcal{B}} \le 1 + e^{\eta(q+m_0)}$, and the fact that $|\mathbb{E}_{\hat{\nu}}[{P_u}^{k-m_0}(\mathbf{1}_B g)]| \le \nu_+(B)\|g\|_{\mathcal{B}}$, we get whenever $u \notin [-\beta, \beta]^2$,

$$|a(u)| \le \mathbb{E}_{\hat{\nu}}\left(\mathbf{1}_B P^\ell P^q(\mathbf{1}_{\hat{A}} \circ \hat{\theta}^{m_0})\right) = O(\hat{\nu}(B)\alpha^\ell e^{\eta q}). \tag{3}$$

We then estimate the main term, coming from small values of u. The decomposition given in Theorem 12 gives for any $u \in [-\beta, \beta]^2$

$$a(u) = \underbrace{\lambda_u^\ell \phi_u(P^q(\mathbf{1}_{\hat{A}} \circ \hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}[{P_u}^{k-m_0}(\mathbf{1}_B v_u)]}_{a_1(u)} + \underbrace{\mathbb{E}_{\hat{\nu}}[P_u^{k-m_0}(\mathbf{1}_B N_u^\ell(P^q(\mathbf{1}_{\hat{A}} \circ \hat{\theta}^{m_0})))]}_{a_2(u)}$$

Notice that the second term is, by inequality (1) in Proposition 12, of order

$$a_2(u) = O\left(\hat{\nu}(B)\alpha^\ell e^{\eta q}\right). \tag{4}$$

Moreover, since $u \mapsto v_u$ and $u \mapsto \phi_u$ are C^1-regular with $v_0 = 1$ and $\phi_0 = \hat\nu$, the first term has the estimate

$$\begin{aligned} a_1(u) &= \lambda_u^\ell \hat\nu(\hat A)\mathbb{E}_{\hat\nu}[{P_u}^{k-m_0}(\mathbf{1}_B)] + O\left(\lambda_u^\ell |u| \hat\nu(B) e^{\eta q}\right) \\ &= \lambda_u^\ell \hat\nu(\hat A)\hat\nu(B) + O\left(\lambda_u^\ell |u|\hat\nu(B) k e^{\eta q}\right) \end{aligned}$$

where the second estimate is obtained by reintroducing the unperturbed Perron-Frobenius operator P in P_u, $|\mathbb{E}_{\hat\nu}[{P_u}^{k-m_0}(\mathbf{1}_B)] - \hat\nu(B)| = |\mathbb{E}_{\hat\nu}((e^{iu\cdot S_{k-m_0}\psi} - 1)\mathbf{1}_B)| \le |u|(k-m_0)\|\psi\|_\infty \hat\nu(B)$.

In addition, the intermediate value theorem yields, using C^3 smoothness of λ_u and Theorem 1 (the bounds 2 and initial values 3)

$$\left|\lambda_u^\ell - \exp(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u)\right| \le \ell(\exp -c_1|u|^2)^{\ell-1}|\lambda_u - \exp(-\frac{1}{2}\sigma_\varphi^2 \cdot u)| \le C_0 \ell e^{-c_1\ell|u|^2}|u|^3 = O\left(e^{-c_2\ell|u|^2}|u|\right)$$

for the constant $c_2 = c_1/2$. Thus

$$a_1(u) = \exp(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u)\hat\nu(\hat A)\hat\nu(B) + O\left(e^{-c_2\ell|u|^2}|u|\hat\nu(B)ke^{\eta q}\right).$$

By the classical change of variable $v = u\sqrt{\ell}$ and gaussian integral one easily see that

$$\int_{[-\beta,\beta]^2} \exp(-\frac{\ell}{2}\sigma_\varphi^2 u\cdot u)du = \frac{1}{\ell}\int_{[-\beta\sqrt{\ell},\beta\sqrt{\ell}]^2} \exp(-\frac{1}{2}\sigma_\varphi^2 v\cdot v)dv = \frac{2\pi}{\ell\sqrt{\det\sigma_\varphi^2}} + O\left(\frac{1}{\ell^{3/2}}\right).$$

Proceeding similarly with the error term one gets as well

$$\int_{[-\beta,\beta]^2} |u|e^{-c_2\ell|u|^2}du = \frac{1}{\ell^{3/2}}\int_{[-\beta\sqrt{\ell},\beta\sqrt{\ell}]^2} e^{-c_2|v|^2}dv = O\left(\frac{1}{\ell^{3/2}}\right).$$

Combining these two computations gives by integration of the approximation of $a_1(u)$ obtained above that

$$\int_{[-\beta,\beta]^2} a_1(u)du = \frac{2\pi}{\ell\sqrt{\det\sigma_\varphi^2}}\hat\nu(\hat A)\hat\nu(B) + O\left(\frac{\hat\nu(B)ke^{\eta q}}{\ell^{3/2}}\right).$$

From this main estimate and (3) and (4) it follows immediately that

$$\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} a(u)du = \frac{1}{2\pi\ell\sqrt{\det\sigma_\varphi^2}}\hat\nu(\hat A)\hat\nu(B) + O\left(\frac{\hat\nu(B)ke^{\eta q}}{(n-k)^{3/2}}\right).$$

□

4.3. **Proof of the pointwise convergence of the recurrence rate to the dimension.** Let us denote by $G_n(\varepsilon)$ the set of points for which n is an ε-return :

$$G_n(\varepsilon) := \{x\in\Sigma : S_n\varphi(x) = 0 \text{ and } d(\theta^n(x), x) < \varepsilon\}.$$

Let us consider the first return time in a ε-neighborhood of a starting point $x \in \Sigma$:

$$\tau_\varepsilon(x) := \inf\{m\ge 1 \ : S_m\varphi(x) = 0 \text{ and } d(\theta^m(x), x) < \varepsilon\} = \inf\{m \ge 1 \ : x \in G_m(\varepsilon)\}.$$

Proof of Theorem 7. Let us denote by $\mathcal{C}_k$ the set of k-cylinders of Σ. For any $\delta > 0$ denote by $\mathcal{C}_k^\delta \subset \mathcal{C}_k$ the set of cylinders $C \in \mathcal{C}_k$ such that $\nu(C) \in (e^{-(d+\delta)k}, e^{-(d-\delta)k})$. For any $x\in\Sigma$ let $C_k(x) \in \mathcal{C}_k$ be the k-cylinder which contains x. Since d is twice[1] the entropy of the ergodic measure ν, by the Shannon-McMillan-Breiman theorem, the set $K_N^\delta = \{x \in \Sigma \colon \forall k \ge N, C_k(x) \in \mathcal{C}_k^\delta\}$ has a measure $\nu(K_N^\delta) > 1-\delta$ provided N is taken sufficiently large.

[1]Note that we are working with the two-sided symbolic space Σ.

* Let us prove that, almost surely :

$$\liminf_{\varepsilon\to 0}\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}\geq d.$$

Let $\alpha>\frac{1}{d}$ and $0<\delta<d-\frac{1}{\alpha}$. Let us take $\varepsilon_n:=\log^{-\alpha}n$ and $k_n:=\lceil -\log\varepsilon_n\rceil$. According to Proposition 13, whenever $k_n\geq N$ we have :

$$\begin{aligned}\nu(K_N^\delta\cap G_n(\varepsilon_n)) &= \nu\left(\{x\in K_N^\delta\colon S_n\varphi(x)=0 \text{ and } \theta^n(x)\in C_{k_n}(x)\}\right)\\ &= \sum_{C\in\mathcal{C}_{k_n}^\delta}\nu\left(C\cap\{S_n\varphi=0\}\cap\theta^{-n}\theta^{k_n}(\theta^{-k_n}C)\right)\\ &= \sum_{C\in\mathcal{C}_{k_n}^\delta}\left[\frac{\nu(C)\nu(C)}{n}+O\left(\frac{\nu(C)k_ne^{\eta k_n}}{n^{3/2}}\right)\right].\end{aligned}$$

Observe that for $C\in\mathcal{C}_{k_n}^\delta$ we have

$$\frac{k_ne^{\eta k_n}}{\sqrt{n}}=\frac{\alpha\log\log n\log^{\alpha\eta}n}{\sqrt{n}}=O(\varepsilon_n^{d+\delta})=O(\nu(C)),$$

hence it follows that

$$\nu(K_N^\delta\cap G_n(\varepsilon_n))=O\left(\sum_{C\in\mathcal{C}_{k_n}^\delta}\frac{\nu(C)^2}{n}\right)=O\left(\frac{1}{n(\log n)^{(d-\delta)\alpha}}\right)$$

Hence, by a Borel Cantelli argument, for a.e. $x\in K_N^\delta$, if n is large enough, we have : $\tau_{\varepsilon_n}(x)>n$. This readily implies that :

$$\liminf_{n\to\infty}\frac{\log\log\tau_{\varepsilon_n}}{-\log\varepsilon_n}\geq\frac{1}{\alpha}\quad a.e.,$$

which proves the lower bound on the lim inf since $(\varepsilon_n)_n$ decreases to zero and $\lim_{n\to+\infty}\frac{\varepsilon_n}{\varepsilon_{n+1}}=1$.

* Let us prove that, almost surely :

$$\limsup_{\varepsilon\to 0}\frac{\log\log\tau_\varepsilon}{-\log\varepsilon}\leq d.$$

Let $0<\alpha<\frac{1}{d}$ and $\delta>0$ such that $1-\alpha d-\alpha\delta>0$. Let us take $\varepsilon_n:=\log^{-\alpha}n$ and $k_n:=\lceil-\log\varepsilon_n\rceil$. For all $\ell=1,...,n$, we define :

$$A_\ell(\varepsilon):=G_\ell(\varepsilon)\cap\theta^{-\ell}\{\tau_\varepsilon>n-\ell\}$$

Let us take $L_n:=\lceil\log^a n\rceil$, with $a>2\alpha(d+\delta+\eta)$. The sets $A_\ell(\varepsilon)$ are pairwise disjoint thus :

$$1\geq\sum_{\ell=0}^n\nu(A_\ell(\varepsilon_n))\geq\sum_{\ell=L_n}^n\sum_{C\in\mathcal{C}_{k_n}^\delta}\nu(C\cap A_\ell(\varepsilon_n)).$$

According to Proposition 13, we have

$$\begin{aligned}\nu(C\cap A_\ell(\varepsilon_n)) &= \nu\left(C\cap\{S_\ell\varphi=0\}\cap\theta^{-\ell}\left(C\cap\{\tau_\varepsilon>n-\ell\}\right)\right)\\ &= \left[\frac{\nu(C)}{2\pi\sqrt{\det\sigma^2}}+O(\frac{k_ne^{\eta k_n}}{\sqrt{\ell-k_n}})\right]\frac{1}{\ell-k_n}\nu\left(C\cap\{\tau_\varepsilon>n\}\right)\\ &\geq c\varepsilon_n^{d+\delta}\frac{1}{\ell-k_n}\nu\left(C\cap\{\tau_\varepsilon>n\}\right)\end{aligned}$$

for any $C \in \mathcal{C}_{k_n}^\delta$ provided $k_n \geq N$; indeed, the error term is negligible since :

$$\frac{k_n e^{\eta k_n}}{\sqrt{\ell - k_n}} = O\left(\frac{(\log\log n)\log^{\alpha\eta} n}{\log^{a/2}(n)}\right) = o({\varepsilon_n}^{d+\delta}),$$

since $a > 2\alpha(d+\delta+\eta)$. This chain of inequalities gives

$$\nu(K_N^\delta \cap \{\tau_\varepsilon > n\}) \leq \sum_{C \in \mathcal{C}_{k_n}^\delta} \nu(C \cap \{\tau_\varepsilon > n\}) \leq \left(\varepsilon_n^{d+\delta} \log \frac{n-k_n}{L_n - k_n}\right)^{-1} = O\left(\frac{1}{\log^{1-\alpha d - \alpha\delta} n}\right).$$

Now let us take $n_p := \lfloor \exp(p^{2/(1-\alpha d-\alpha\delta)}) \rfloor$. We have :

$$\sum_{p \geq 1} \nu(K_N^\delta \cap \{\tau_{\varepsilon_{n_p}} > n_p\}) < +\infty.$$

Hence, by the Borel-Cantelli lemma, almost surely $x \in K_N^\delta$, we have :

$$\limsup_{p \to +\infty} \frac{\log\log \tau_{2\varepsilon_{n_p}}}{-\log \varepsilon_{n_p}} \leq \frac{1}{\alpha}.$$

This gives the estimate lim sup since $(\varepsilon_{n_p})_p$ decreases to zero and since $\lim_{p\to+\infty} \frac{\varepsilon_{n_p}}{\varepsilon_{n_{p+1}}} = 1$. □

4.4. **Fluctuations of the rescaled return time.** Recall that $C_k(x) = \{y \in \Sigma \colon d(x,y) < e^{-k}\}$. Let $R_k(y) = \min\{n \geq 1 \colon \theta^n(y) \in C_k(y)\}$ denotes the first return time of a point y into its k-cylinders $C_k(y)$, or equivalently the first repetition time of the first k symbols of y. There have been a lot of studies on this quantity, among all the results we will use the following.

Proposition 14 (Hirata [8]). *For ν-almost every point $x \in \Sigma$, the return time into the cylinders $C_k(x)$ are asymptotically exponentially distributed in the sense that*

$$\lim_{k\to\infty} \nu_{C_k(x)}\left(R_k(\cdot) > \frac{t}{\nu(C_k(x))}\right) = e^{-t}$$

for a.e. x, where the convergence is uniform in t.

Lemma 15. *Let x be such that $\lim_{k\to\infty} \nu_{C_k(x)}\left(R_k(\cdot) > \frac{t}{\nu(C_k(x))}\right) = e^{-t}$ for all $t > 0$. Then, for all $t > 0$, we have :*

$$\lim_{k\to+\infty} \nu\left(\tau_{e^{-k}} > \exp\left(\frac{t}{\nu(C_k(x))}\right) \middle| C_k(x)\right) = \frac{1}{1+\beta t},$$

with $\beta := \frac{1}{2\pi\sqrt{det \sigma^2}}$.

Proof. We are inspired by the method used by Dvoretzky and Erdös in [6]. Let $k \geq m_0$ and n be some integers. We make a partition of a cylinder $C_k(x)$ according to the value $\ell \leq n$ of the last passage in the time interval $0, \ldots, n$ of the orbit of $(x, 0)$ by the map F into $C_k(x) \times \{0\}$. This gives the following equality :

$$\nu(C_k(x)) = \sum_{\ell=0}^{n} \nu\left(C_k(x) \cap \{S_\ell = 0\} \cap \theta^{-\ell}\left(C_k(x) \cap \{\tau_{e^{-k}} > n - \ell\}\right)\right). \tag{5}$$

<u>Upper bound.</u> Let $n_k = \left\lfloor e^{\frac{t}{\nu(C_k(x))}} \right\rfloor$. First we claim that :

$$\limsup_{k\to+\infty} \nu\left(\{\tau_{e^{-k}} > n_k\} | C_k(x)\right) \leq \frac{1}{1+\beta t}.$$

Let $a > 2\eta$ and $L_k = e^{ak}$. According to the decomposition (5) and to Proposition 13, there exists $C_1' > 0$ such that we have :

$$\begin{aligned}\nu(C_k(x)) &\geq \nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k\}\right) + \sum_{\ell=L_k}^{n_k}\beta\frac{\nu(C_k(x))\nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k\}\right)}{\ell-k}\\ &\quad -C_1\sum_{\ell=L_k}^{n_k}\frac{ke^{\eta k}\nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k-\ell\}\right)}{(\ell-k)^{\frac{3}{2}}}\\ &\geq \nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k\}\right)\left(1+\beta\nu(C_k(x))\sum_{\ell=L_k}^{n_k}\frac{1}{\ell-k}\right) - C_1'\nu(C_k(x))ke^{\eta k}e^{\frac{-ak}{2}}.\end{aligned}$$

Hence, we get :

$$\frac{\nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k\}\right)}{\nu(C_k(x))} \leq \frac{1-C_1'ke^{k\left(\eta-\frac{a}{2}\right)}}{1+\beta\nu(C_k(x))\sum_{\ell=L_k}^{n_k}\frac{1}{\ell-k}}.$$

The claim follows from the fact that $a > 2\eta$.

<u>Lower bound.</u> Let $b = \liminf \frac{-1}{k}\log\nu(C_k(x)) > 0$. Without loss of generality we assume that the Hölder exponent η is such that $b > 2\eta$. Let $q_k = \left\lfloor e^{\frac{t}{\nu(C_k(x))}}\right\rfloor$, $n_k = \lfloor q_k\log(q_k)\rfloor$, $m_k = n_k - q_k$ and choose $\delta > 0$ such that $2\eta < b(1-\delta)$. We now claim that :

$$\liminf_{k\to+\infty}\nu\left(\{\tau_{e^{-k}} > q_k\}|C_k(x)\right) \geq \frac{1}{1+\beta t}.$$

Let us denote by $A_\ell(k,x)$ the sets involved in the decomposition (5) :

$$A_\ell(k,x) := C_k(x)\cap\{S_\ell = 0\}\cap\theta^{-\ell}\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k-\ell\}\right).$$

For $\ell = 0$ we have

$$\nu(A_0(k,x)) \leq \nu(C_k(x)\cap\{\tau_{e^{-k}} > q_k\}). \tag{6}$$

Let $M_k = \lfloor\nu(C_k(x)^{-1+\delta}\rfloor$. We first show that the contribution from small ℓ is negligible. According to the exponential statistics for return times, there exists ε_k, with $\lim_{k\to+\infty}\varepsilon_k = 0$, such that we have (remember that the $A_\ell(k,x)$ are disjoints) :

$$\begin{aligned}\sum_{\ell=1}^{M_k}\nu(A_\ell(k,x)) &\leq \nu\left(C_k(x)\cap\{R_k\leq M_k\}\right)\\ &\leq \nu(C_k(x))\left(1-\exp\left(\nu(C_k(x))^\delta\right)+\varepsilon_k\right)\\ &\leq o(\nu(C_k(x))).\end{aligned}\tag{7}$$

We now estimate the measure of $A_\ell(k,x)$ for large values of ℓ. According to our local limit theorem (Proposition 13), for all $\ell = M_k+1,\dots,n_k$, we have :

$$\nu(A_\ell(k,x)) \leq \beta\underbrace{\frac{\nu(C_k(x))\nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k-\ell\}\right)}{\ell-k}}_{\text{main term}} + C_1\underbrace{\frac{ke^{\eta k}\nu\left(C_k(x)\cap\{\tau_{e^{-k}} > n_k-\ell\}\right)}{(\ell-k)^{\frac{3}{2}}}}_{\text{error term}}. \tag{8}$$

Observe that the error term is controlled, for some constant $C_2 > 0$, by

$$\sum_{\ell\geq M_k+1}\frac{ke^{\eta k}\nu\left(C_k(x)\right)}{(\ell-k)^{\frac{3}{2}}} \leq C_2\nu(C_k(x))ke^{\eta k}(\nu(C_k(x))^{-1+\delta}-k)^{-\frac{1}{2}} = o(\nu(C_k(x))). \tag{9}$$

On the other hand the main term may be estimated for non extremal values of ℓ by :

$$\sum_{\ell=M_k+1}^{m_k}\frac{\nu(C_k(x))\nu\left(C_k(x)\cap\{\tau_{e^{-k}}>n_k-\ell\}\right)}{\ell-k}\le\nu(C_k(x))\nu\left(C_k(x)\cap\{\tau_{e^{-k}}>q_k\}\right)\sum_{\ell=M_k+1}^{m_k}\frac{1}{\ell-k} \tag{10}$$

and for extremal values of ℓ the simple bound below holds :

$$\begin{aligned}\sum_{\ell=m_k+1}^{n_k}\frac{\nu(C_k(x))\nu\left(C_k(x)\cap\{\tau_{e^{-k}}>n_k-\ell\}\right)}{\ell-k} &\le \nu(C_k(x))^2\sum_{\ell=m_k+1}^{n_k}\frac{1}{\ell-k}\\ &\le C_3\nu(C_k(x))^2\log\left(\frac{n_k-k}{m_k-k}\right)\\ &= o(\nu(C_k(x))). \end{aligned}\tag{11}$$

Using the decomposition (5) and putting together formulas (6), (7), (8), (9), (10), (11), we get :

$$\begin{aligned}\nu(C_k(x)) &\le \nu(C_k(x)\cap\{\tau_{e^{-k}}>q_k\})\left(1+\beta\nu(C_k(x))\sum_{\ell=M_k+1}^{n_k}\frac{1}{\ell-k}\right)+o(\nu(C_k))\\ &\le \nu(C_k(x)\cap\{\tau_{e^{-k}}>q_k\})\left(1+\beta\nu(C_k(x))\log n_k\right)+o(\nu(C_k(x)).\end{aligned}$$

This proves the claim, which achieves the proof of the lemma. □

Proof of Theorem 8. Since the exponential statistics of return time holds a.e. by Proposition 14, Lemma 15 applies a.e. and by integration, using Lebesgue dominated convergence theorem, we get that

$$\lim_{k\to\infty}\nu\left(\log\tau_{e^{-k}}(\cdot)>\frac{t}{\nu(C_k(\cdot))}\right)=\frac{1}{1+\beta t}$$

for all $t\ge 0$. □

Proof of Corollary 9. Nonzero variance. Let us write :

$$Y_k:=\frac{\log\log\tau_{e^{-k}}(\cdot)-kd}{\sqrt{k}}.$$

In this case ν is a Gibbs measure with a non degenerate hölder potential h. The logarithm of the measure of the k-cylinder about x is, up to some constants, given by the birkhoff sum $\sum_{j=-k}^{k}h\circ\sigma^k(x)$ of h on the orbit of x. It is well known that such sums follow a central limit theorem (e.g. [2]). This readily implies that $X_k=\frac{\log(\nu(C_k(\cdot))+kd}{\sqrt{k}}$ converges in distribution to a centered gaussian random variable of variance $2\sigma_h^2$. It is enough to prove that Y_k+X_k converges in probability to 0. This will be true if Y_k+X_k converges in distribution to 0. This follows from Theorem 8 and from the formula :

$$Y_k+X_k=\frac{\log\log\tau_{e^{-k}}(\cdot)+\log(\nu(C_k))}{\sqrt{k}}.$$

Zero variance. In this case the potential is cohomologous to a constant and the measure ν is the measure of maximal entropy, which is a Markov measure. Denote by π the transition matrix and by p the left eigenvector such that $p\pi=p$. The measure of a cylinder $C_k(x)$ is equal to $p_{x_{-k}}\prod_{j=-k}^{k-1}\pi_{x_jx_{j+1}}$. Since the function $\log\pi_{x_0x_1}$ has to be cohomologous to the entropy, the measure of a cylinder $C_k(x)$ simplifies down to

$$\nu(C_k(x))=Q_{x_{-k}x_k}e^{-(2k+1)\frac{d}{2}},$$

where $Q = (Q_{ij})$ is a (constant) matrix. Proceeding as in the proof of Theorem 8, we get that

$$\lim_{k\to\infty} \nu\left(e^{-kd}\log\tau_{e^{-k}} > t\right) = \sum_{i,j\in\mathcal{A}} \lim_{k\to\infty} \int_{\{x_{-k}=i,x_k=j\}} \mathbf{1}_{\{e^{-kd}\log\tau_{e^{-k}}>t\}} d\nu = \sum_{i,j\in\mathcal{A}} p_i p_j \frac{1}{1+\beta Q_{ij} t}.$$

□

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